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Spaces of differentiable functions on compact groups

Dedicated to Professor Hisaaki YOSHIZAWA on his 60th birthday

By

Takashi EDAMATSU

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Introduction

This work concerns differentiation on compact groups. We obtain here natural C^n-classes ($n=\infty$, 1, 2,...), i.e., classes of n-times continuously differentiable functions, on compact groups by the use of one-parameter subgroups, and determine their fundamental structures. Besides, we make some observations on the differential structure on such groups.

Let G be a compact group, and $R(G)$ the totally of its one-parameter subgroups. We define the right (resp. left) derivative $d_\alpha^{(r)}f$ (resp. $d_\alpha^{(l)}f$) of a function f on G along $\alpha \in R(G)$ by $d_\alpha^{(r)}f(x)=\frac{d}{dt}f(x\alpha(t))|_{t=0}$ (resp. $d_\alpha^{(l)}f(x)=\frac{d}{dt}f(\alpha(-t)x)|_{t=0}$) $(x\in G)$. And for each n, we define the "right C^n-class" $\mathscr{E}_n^{(r)}(G)$ on G as the set of all *continuous* functions f having the continuous right derivative $d_{\alpha_k}^{(r)}\cdots d_{\alpha_1}^{(r)}f$ of higher order for any $\alpha_1,\dots,\alpha_k\in R(G)$ with $1\leq k<n+1$. The "left C^n-class" $\mathscr{E}_n^{(l)}(G)$ is defined similarly using left derivatives. Needless to say, if G is a Lie group, $\mathscr{E}_1^{(r)}(G)$ and $\mathscr{E}_1^{(l)}(G)$ coincide with each other for each n and give the usual C^n-class on G. Our basically important result, Theorem 2.1, asserts that this coincidence remains true for any compact group G.

The essential part of Theorem 2.1 consists in the equality $\mathscr{E}_1^{(r)}(G)=\mathscr{E}_1^{(l)}(G)$. For $x\in G$ and $\alpha\in R(G)$, let $x\alpha x^{-1}$ denote the member of $R(G)$ such that $(x\alpha x^{-1})(t)=x\alpha(t)x^{-1}$ for all real t. Then $x\alpha(t)=(x\alpha x^{-1})(t)x$. Hence we see immediately that $d_\alpha^{(r)}f$ exists for every $\alpha\in R(G)$ if and only if so with $d_\alpha^{(l)}f$. For such an f, we have $d_\alpha^{(r)}f(x)=-d_{x\alpha x^{-1}}^{(l)}f(x)$ $(\alpha\in R(G),\ x\in G)$. But it is not so easy to know whether, for such an f, $d_\alpha^{(r)}f$ is *continuous* for every $\alpha\in R(G)$ if and only if so with $d_\alpha^{(l)}f$. The equality $\mathscr{E}_1^{(r)}(G)=\mathscr{E}_1^{(l)}(G)$ is no other than the affirmative answer to this question under the assumption of continuity of f itself. Here we ought to remark that, unlike the Lie group case, the continuity of f is not necessarily assured even if the continuity of all $d_\alpha^{(r)}f$ or $d_\alpha^{(l)}f$ $(\alpha\in R(G))$ is assumed. In fact, it is possible that for a discontinuous function on a connected compact abelian group, all derivatives of every order of it exist and are continuous (cf. [6], p. 56). Let us sketch the method of proving the above equality.

In 1979, by the aid of Tannaka duality, K. Mckennon [4] showed that $R(G)$ is in one-to-one correspondence to a certain Lie algebra $\Lambda(G)$ (infinite dimensional in general), which we present in §1.1 in a form convenient to us and with a natural locally convex topology. Namely, in our presentation, $\Lambda(G)$ conisits of certain matrix fields over $\hat{G}$, the dual object of G, with the coordinatewise algebraic operations and topology. It behaves as our basic machine in this paper. On the other hand, by the structure theorem, G is isomorphic to an inverse limit of Lie or finite groups G_λ: $G \cong \varprojlim G_\lambda$. This induces an inverse system $\{\mathfrak{g}_\lambda\}$ of the Lie algebras of G_λ's, and so a Lie algebra $\varprojlim \mathfrak{g}_\lambda$, denoted by $\mathfrak{g}(G)$. Here, of course, $\{\mathfrak{g}_\lambda\}$ can be viewed also as an inverse system of finite dimensional locally convex linear spaces. So we equip $\mathfrak{g}(G)$ with its limit locally convex topology. Through §1 we shall show that $\Lambda(G)$ is no other than a realization of $\mathfrak{g}(G)$ including the topology. This observation enables us in particular to see that $\Lambda(G)$ is a Baire space, and so, also a barrelled space (Lemma 1.13).

Now we regard $R(G)$ as a Lie algebra and also as a locally convex space isomorphic to $\Lambda(G)$ under the above correspondence. Then we have in particular the following two facts: for each $f \in \mathscr{E}_1^{(r)}(G)$ (resp. $\mathscr{E}_1^{(l)}(G)$), the function $R(G) \times G \ni (\alpha, x) \mapsto d_\alpha^{(r)} f(x)$ (resp. $d_\alpha^{(l)} f(x)$) is continuous (Lemma 2.9); and second, the map $R(G) \times G \ni (\alpha, x) \mapsto x\alpha x^{-1} \in R(G)$ is continuous (Lemma 2.10). Here, for verification of the former, the property of $R(G)$, being Baire and barrelled, plays the key role (see Proofs of Lemmas 2.8 and 2.9). Since $d_\alpha^{(r)} f(x) = -d_{x\alpha x^{-1}}^{(l)} f(x)$ for $\alpha \in R(G)$, $x \in G$ and $f \in \mathscr{E}_1^{(r)}(G) \cup \mathscr{E}_1^{(l)}(G)$, the equality $\mathscr{E}_1^{(r)}(G) = \mathscr{E}_1^{(l)}(G)$ follows from the above two facts at once.

The present paper consists of four sections. In §1 the necessary facts concerning the Lie algebras and one-parameter subgroups of compact groups are prepared. In §2 we establish Theorem 2.1 and, by virtue of it, define each C^n-class on G to be the identical sets $\mathscr{E}_n^{(r)}(G)$ and $\mathscr{E}_n^{(l)}(G)$. §3 is devoted to a study of structures of our C^n-classes. Here the locally convex structure of $R(G)$ mentioned above is needed again. And also, Lemmas 1.10 and 1.15, which we owe again to Mckennon [4], are essential. §4 is, as a continuation of §1, concerned with the differential structure on G.

Here we give a summary of results in §§3 and 4. Let $\mathscr{E}_n(G)$ be our C^n-classes ($n = \infty, 1, 2, \ldots$), and $\mathscr{D}(G)$ the space of regular functions on G in Bruhat's sense ([2], Definition 1). For a closed normal subgroup N of G, put $\mathscr{E}_n(G, N) = \{f \in \mathscr{E}_n(G);\ f(xy) = f(x)\ (x \in G,\ y \in N)\}$. This space can be identified with the space $\mathscr{E}_n(G/N)$ in the obvious way (by Lemma 1.10). Denote by $\boldsymbol{H}_0(G)$ (resp. $\boldsymbol{H}_1(G)$) the totality of closed normal subgroups N of G such that the quotient group G/N is Lie or finite (resp. finite dimensional and separable). Then $\boldsymbol{H}_0(G) \subseteq \boldsymbol{H}_1(G)$, and $\mathscr{D}(G) = \cup\{\mathscr{E}_\infty(G, N);\ N \in \boldsymbol{H}_0(G)\}$ by its definition.

Our Theorem 3.1 asserts that $\mathscr{E}_n(G) = \cup\{\mathscr{E}_n(G, N);\ N \in \boldsymbol{H}_1(G)\}$ ($n = \infty, 1, 2, \ldots$) (see Corollary to Theorem 3.1). This theorem determines the basic structure of the spaces $\mathscr{E}_n(G)$, and constitutes the core of this paper together with Theorem 2.1. It exhibits in particular the difference between $\mathscr{D}(G)$ and $\mathscr{E}_\infty(G)$. Here, as to when

$\mathscr{E}_\infty(G)$ coincides with $\mathscr{D}(G)$, criteria are given by Corollary to Proposition 3.2. It is the case if and only if G is locally connected. As a result of Theorem 3.1, each $\mathscr{E}_n(G)$ obtains a certain natural inductive limit topology, denoted by $\boldsymbol{\tau}_*$ without regard to n (Definition 3.4). The rest of §3 concerns topological aspects of the spaces $\mathscr{E}_n(G)$ bearing $\boldsymbol{\tau}_*$. Some results, such as their completeness, a generalization of the classical Weierstrass approximation theorem (Proposition 3.5) etc., are obtained.

§4 contains three results which generalize elementary facts in Lie group theory to the case of compact groups. Let $R(G)^c$ be the complexification of the Lie algebra $R(G)$, and $\boldsymbol{U}(G)$ the universal enveloping algebra of $R(G)^c$. Theorem 4.1 states that the map $\alpha+\sqrt{-1}\,\beta \mapsto d_\alpha^{(r)}+\sqrt{-1}\,d_\beta^{(r)}$ (α, $\beta \in R(G)$) is an isomorphism of $R(G)^c$ onto the Lie algebra formed of all $\boldsymbol{\tau}_*$-continuous and left invariant derivations on the algebra $\mathscr{E}_\infty(G)$. Theorem 4.2 asserts that this isomorphism extends to an algebra isomorphism of $\boldsymbol{U}(G)$ onto $\boldsymbol{D}_l(G)$, the algebra of left invariant differential operators on G (cf. Definition 4.2 and Lemma 4.3). Lastly, Theorem 4.3 describes the center of $\boldsymbol{U}(G)$ by means of the "adjoint representation" of G.

This work originates in our desire to generalize Riss' theory of differentiation to the non-abelian case, which was introduced in [6] for locally compact abelian groups using one-parameter subgroups. His theory is based on Potryagin duality and the general structure theorem for such groups. Our present work extends Riss' treatment to the case of compact groups on the bases of Tannaka duality and the structure theorem for such groups. As for treating the general locally compact groups (LC groups), we can grasp the one-parameter subgroups of such groups through Tatsuuma's duality theory (see, for instance, [8]). But there, problems remain to be solved.

While, in [2, $n°12$], *F.* Bruhat defined differential operators and n-times continuously differentiable functions ($n=\infty$, 1, 2,...) on LC groups in terms of his distributions that were formulated in [2] on LC groups depending solely on the structure theorem of LC groups. But, if the group is not locally connected, his discussions do not elucidate what such functions really are. Our present theory enables us to understand his differentiable functions on G, a compact group, no matter G is locally connected or not. In fact, combined with the discussions given there, our theory exhibits that for each n, the n-times continuously differentiable functions on G in Bruhat's sense just coincide with ours.

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Notation. Unless otherwise stated, G denotes an arbitrary compact Hausdorff group with unity e and Haar measure d_G such that $\int_G d_G=1$. $\hat{G}$ denotes the dual of G, i.e., the set of all equivalence classes of continuous irreducible (hence finite dimensional) unitary representations of G. The dimension of $\sigma\in\hat{G}$ is denoted by d_σ. As a representative of each class $\sigma\in\hat{G}$, we choose a unitary matrix representation $U(\sigma)$ and fix it once for all.

$\boldsymbol{R}$ and $\boldsymbol{C}$ designate the fields of reals and compexes, respectively, with the usual topologies. For any topological group G, $c(G)$ denotes the connected component of the unity, and $R(G)$ the totality of one-parameter subgroups of G, where a one-parameter subgroup means a continuous homomorphism of the additive group $\boldsymbol{R}$ into G. For $x \in G$ and $\alpha \in R(G)$, $x^{-1}\alpha x$ denotes the member of $R(G)$ defined as $x^{-1}\alpha x(t) = x^{-1}\alpha(t)x$ $(t \in \boldsymbol{R})$. For a $\boldsymbol{C}$-valued function f on G and $x_0 \in G$, the functions ${}_{x_0}f$, f_{x_0}, $\check{f}$ and $\bar{f}$ are defined as follows: ${}_{x_0}f(x) = f(x_0^{-1}x)$, $f_{x_0}(x) = f(xx_0)$, $\check{f}(x) = f(x^{-1})$ and $\bar{f}(x) = \overline{f(x)}$ (complex-conjugate) $(x \in G)$.

Let M be a topological space. $C(M)$ denotes the space of all $\boldsymbol{C}$-valued continous functions on M. For $f \in C(M)$, supp(f) denotes the support of f; and for a subset $\mathscr{S} \subseteq C(M)$, $\mathscr{S}^+$ the totality of non-negative $\boldsymbol{R}$-valued functions in $\mathscr{S}$. If M is a C^∞-manifold, $C^n(M)$ $(n = \infty, 1, 2,\dots)$ denotes the usual class of $\boldsymbol{C}$-valued, n-times (infinitely if $n = \infty$) continuously differentiable functions on M. For any topological linear spaces E and F over $\boldsymbol{C}$, $L(E, F)$ denotes the space of all continuous linear mappings of E into F.

§ 1. Lie algebra and one-parameter subgroups

1.1. In this paragraph, following Mckennon [4], arranged suitably for our purpose, we associate a Lie algebra with each compact group G, and set up a bijection of it onto $R(G)$.

Definition 1.1. For $\sigma \in \hat{G}$, let $\mathfrak{M}(d_\sigma, \boldsymbol{C})$ be the totality of complex matrices of order d_σ. Let $\Sigma(G)$ denote the set of all matrix fields $T = (T(\sigma))_{\sigma \in \hat{G}}$ on $\hat{G}$ such that $T(\sigma) \in \mathfrak{M}(d_\sigma, \boldsymbol{C})$ for each $\sigma \in \hat{G}$.(1) We regared $\Sigma(G)$ as involutive algebra over $\boldsymbol{C}$ under the coordinatewise usual algbraic operations of matrices and the involution $T \mapsto T^* = (T(\sigma)^*)_{\sigma \in \hat{G}}$, where $T(\sigma)^*$ denotes the hermite conjugate of $T(\sigma)$. The The element $I = (I(\sigma))_{\sigma \in \hat{G}}$, where $I(\sigma)$ is the identity matrix in $\mathfrak{M}(d_\sigma, \boldsymbol{C})$, is the identity of the algebra $\Sigma(G)$. For $\sigma \in G$, let $\| \ \|_\sigma$ denote the Hilbert-Schmidt norm on $\mathfrak{M}(d_\sigma, \boldsymbol{C})$, i.e., $\|A\|_\sigma = (\mathrm{tr}\,(A^*A))^{1/2}$ $(A \in \mathfrak{M}(d_\sigma, \boldsymbol{C}))$, and P_σ the seminorm on $\Sigma(G)$, viewed as vector space, defined by $P_\sigma(T) = \|T(\sigma)\|_\sigma$ $(T \in \Sigma(G))$. We equip $\Sigma(G)$ with the locally convex Hausdorff topology defined by $\{P_\sigma;\ \sigma \in G\}$.

Let the decomposition of the tensor product $\sigma \otimes \sigma'$ of σ, $\sigma' \in \hat{G}$ into irreducible components be given by $\sigma \otimes \sigma' = \sigma_1 \oplus \cdots \oplus \sigma_m$ $(\sigma_1,\dots, \sigma_m \in \hat{G})$. If we use the representatives, this is expressed as

$$U(\sigma) \otimes U(\sigma') = V^{-1}(U(\sigma_1) \oplus \cdots \oplus U(\sigma_m))V, \tag{1.1}$$

where $\otimes$ means the Kronecker product of matrices, $\oplus$ the direct sum of matrices in the conventional sense, and V is a unitary matrix of order $d_\sigma d_{\sigma'}$.

(1) If the representative $U(\sigma)$ of a $\sigma \in \hat{G}$ is exchanged for another, say, $VU(\sigma)V^{-1}$ (V a unitary matrix in $\mathfrak{M}(d_\sigma, \boldsymbol{C})$), then the σ-th coordinate $T(\sigma)$ of every $T \in \Sigma(G)$ ought to be considered as transformed to $VT(\sigma)V^{-1}$. But, since we have fixed the representative $U(\sigma)$ of each $\sigma \in \hat{G}$, this respect disappears, and $\Sigma(G)$ can be viewed simply as the cartesian product of $\mathfrak{M}(d_\sigma, \boldsymbol{C})$'s.

Definition 1.2. Let $T=(T(\sigma))_{\sigma\in\hat{G}}$ be a member of $\Sigma(G)$. Assume that if a decomposition of $\sigma\otimes\sigma'$ (σ, $\sigma'\in\hat{G}$) into irreducible components is given by (1.1), then

$$T(\sigma)\otimes T(\sigma')=V^{-1}(T(\sigma_1)\oplus\cdots\oplus T(\sigma_m))V$$

holds. In this case T is said to satisfy the condition (C1). While, if

$$T(\sigma)\otimes I(\sigma')+I(\sigma)\otimes T(\sigma')=V^{-1}(T(\sigma_1)\oplus\cdots\oplus T(\sigma_m))V$$

holds under (1.1), T is said to satisfy the condition (C2).

It is obvious that the totality of the members of $\Sigma(G)$ satisfying the condition (C1) (resp. (C2)) is stable under the multiplication (resp. linear operations) and the involution.

For $T\in\Sigma(G)$, we denote by $\exp T$ the member of $\Sigma(G)$ with σ-th coordinate $\exp T(\sigma)$ ($\sigma\in\hat{G}$), where each $\exp T(\sigma)$ is defined by the Taylor series $\sum_{n=0}^{\infty}\frac{1}{n!}T(\sigma)^n$.

Lemma 1.1. *For $T=(T(\sigma))_{\sigma\in\hat{G}}\in\Sigma(G)$, the following two statements are equivalent.*

(*a*) *T satisfies the condition* (C2).

(*n*) $\exp tT$ *satisfies the condition* (C1) *for every $t\in\boldsymbol{R}$.*

Proof. Take any σ, $\sigma'\in\hat{G}$ and assume (1.1). Put, for $t\in\boldsymbol{R}$,

$$S(t)=\exp tT(\sigma),\quad S'(t)=\exp tT(\sigma'),$$
$$S_k(t)=\exp tT(\sigma_k)\quad(k=1,\dots,m),$$

and define

$$A(t)=S(t)\otimes S'(t)-V^{-1}(S_1(t)\oplus\cdots\oplus S_m(t))V.$$

Then

$$\begin{aligned}(1.2)\qquad \frac{dA(t)}{dt}&=T(\sigma)S(t)\otimes S'(t)+S(t)\otimes T(\sigma')S'(t)\\&\quad-V^{-1}(T(\sigma_1)S_1(t)\oplus\cdots\oplus T(\sigma_m)S_m(t))V\\&=(T(\sigma)\otimes I(\sigma')+I(\sigma)\otimes T(\sigma'))(S(t)\otimes S'(t))\\&\quad-V^{-1}(T(\sigma_1)\oplus\cdots\oplus T(\sigma_m))VV^{-1}(S_1(t)\oplus\cdots\oplus S_m(t))V.\end{aligned}$$

If we assume (a), (1.2) can be written as

$$\frac{dA(t)}{dt}=(T(\sigma)\otimes I(\sigma')+I(\sigma)\otimes T(\sigma'))A(t).$$

Since $A(0)=0$, this demands that $A(t)\equiv 0$. Hence $\exp tT$ satisfies the condition (C1) for every $t\in\boldsymbol{R}$. Conversely if we assume (b), then $A(t)\equiv 0$. Therefore, from (1.2),

$$T(\sigma)\otimes I(\sigma')+I(\sigma)\otimes T(\sigma')-V^{-1}(T(\sigma_1)\oplus\cdots\oplus T(\sigma_m))V=0.$$

This shows that T satisfies the condition (C2). q.e.d.

Definition 1.3. $\hat{G}$ denotes the set of all T in $\Sigma(G)$ having the following two properties:

(a) T satisfies the condition (C1), and

(b) each coordinate $T(\sigma)$ of T is unitary, i.e. $T^*T=I$.

$\hat{\hat{G}}$ is called the bidual of G.

Plainly $\hat{\hat{G}}$ becomes a topological group under the multiplication and topology in $\Sigma(G)$, i.e., the coordinatewise ones. The element $I=(I(\sigma))_{\sigma\in\hat{G}}$ is the unity of this group. For each $x\in G$, put $U_x=(U_x(\sigma))_{\sigma\in\hat{G}}$ ($\in\Sigma(G)$). Let i_G denote the map $x\mapsto U_x$ of G into $\Sigma(G)$. The next lemma is well known as Tannaka duality theorem (cf. [3], (30.5)).

Lemma 1.2. *i_G is a topological group isomorphism of G onto $\hat{\hat{G}}$.*

Definition 1.4. $\Lambda(G)$ denotes the set of all H in $\Sigma(G)$ having the following two properties:

(a) H satisfies the condition (C2), and

(b) each coordinate $H(\sigma)$ of H is skew-hermite, i.e. $H^*=-H$.

Lemma 1.3. *The set $\Lambda(G)$ is stable under the $\boldsymbol{R}$-linear operations and the commutator product $[H, H']=HH'-H'H$. $\Lambda(G)$ becomes a real Lie algebra under these operations.*

Proof. It suffices to check that $\Lambda(G)$ is stable under the commutator product. Take any $H, H'\in\Lambda(G)$. Obviously we have $[H, H']^*=-[H, H']$. Next, for $\sigma, \sigma'\in\hat{G}$, assume (1.1). Then

$$\begin{aligned}&V^{-1}([H, H'](\sigma_1)\oplus\cdots\oplus[H, H'](\sigma_m))V\\&=[V^{-1}(H(\sigma_1)\oplus\cdots\oplus H(\sigma_m))V, V^{-1}(H'(\sigma_1)\oplus\cdots\oplus H'(\sigma_m))V]\\&=[H(\sigma)\otimes I(\sigma')+I(\sigma)\otimes H(\sigma'), H'(\sigma)\otimes I(\sigma')+I(\sigma)\otimes H'(\sigma')]\\&=[H, H'](\sigma)\otimes I(\sigma')+I(\sigma)\otimes[H, H'](\sigma').\end{aligned}$$

This shows that $[H, H']$ satisfies the condition (C2). Hence $[H, H']\in\Lambda(G)$.

q. e. d.

The next lemma is clear from Lemma 1.1.

Lemma 1.4. *A member H of $\Sigma(G)$ belongs to $\Lambda(G)$ if and only if* $\exp tH\in\hat{\hat{G}}$ *for every $t\in\boldsymbol{R}$.*

Lemma 1.5. *The map $\Lambda(G)\ni H\mapsto$esp tH ($t\in\boldsymbol{R}$) is a bijection of $\Lambda(G)$ onto $R(\hat{\hat{G}})$, the set of all one-parameter subgroups of $\hat{\hat{G}}$.*

Proof. By Lemma 1.4, the map $\boldsymbol{R}\ni t\mapsto\exp tH$ belongs to $R(\hat{\hat{G}})$ for every $H\in\Lambda(G)$. The injectivity of the map $\Lambda(G)\ni H\mapsto\{\exp tH\}\in R(\hat{\hat{G}})$ is evident. Each member of $R(\hat{\hat{G}})$ has the form $U_{\alpha(t)}$ for some $\alpha\in R(G)$ (Lemma 1.2), and each coordinate $U_{\alpha(t)}(\sigma)$ of it is of the form $\exp tH(\sigma)$ with some skew-hermite matrix $H(\sigma)$. Put $H=(H(\sigma))_{\sigma\in\hat{G}}$. Then, by Lemma 1.4, $H\in\Lambda(G)$. Therefore the above map is surjective. q.e.d.

Since G and $\hat{\hat{G}}$ are isomorphic under i_G, Lemma 1.5 can be viewed as setting up a bijection of $\Lambda(G)$ onto $R(G)$.

Definition 1.5. h_G denotes the inverse map of the bijection of $\Lambda(G)$ onto $R(G)$ just stated. Thus

$$i_G(\alpha(t))=\exp th_G(\alpha) \quad (\alpha\in R(G),\ t\in \boldsymbol{R}). \tag{1.3}$$

Lemma 1.6. *Assume that the compact group G is a Lie group with Lie algebra* $\mathfrak{g}$. *For* $\sigma\in\hat{G}$, *let* $\partial U(\sigma)$ *denote the infinitesimal representation of* $\mathfrak{g}$ *induced from* $U(\sigma)$. *Then the map*

$$\iota:\ \mathfrak{g}\ni X\longmapsto H_x=(\partial U(\sigma)X)_{\sigma\in\hat{G}}\in\Sigma(G)$$

is a Lie algebra isomorphism of $\mathfrak{g}$ *onto* $\Lambda(G)$.

Proof. Since

$$\begin{aligned}\exp tH_X &= (\exp t(\partial U(\sigma)X))_{\sigma\in\hat{G}} \\ &= (U_{\exp tX}(\sigma))_{\sigma\in\hat{G}}=U_{\exp tX} \quad (t\in\boldsymbol{R}),\end{aligned} \tag{1.4}$$

the map $\boldsymbol{R}\ni t\mapsto\exp tH_X$ belongs to $R(\hat{\hat{G}})$, and hence $H_X\in\Lambda(G)$ by Lemma 1.4. Since the map $\mathfrak{g}\ni X\mapsto U_{\exp tX}$ is bijective to $R(\hat{\hat{G}})$, (1.4) together with Lemma 1.5 shows that ι is a bijection of $\mathfrak{g}$ onto $\Lambda(G)$. Since each $\partial U(\sigma)$ is a Lie algebra homomorphism, ι is a Lie algebra isomorphism. q.e.d.

By the above lemma we can call $\Lambda(G)$ the Lie algebra of G, any compact group, compatibly with the case of Lie groups. Besides, we see from (1.4) that the map $\Lambda(G)\ni H\mapsto\exp H\in\hat{\hat{G}}$ generalizes the exponential mapping in Lie group theory. $\Lambda(G)$ is not necessarily finite-dimensional. In fact, a direct product of infinitely many compact Lie groups gives such an example. If the compact group G is abelian, then, as is seen from Definition 1.4, the Lie lagebra $\Lambda(G)$ is commutative and consists of all pure-imaginary characters v (i.e., homomorphisms into the additive group $\sqrt{-1}\boldsymbol{R}$) of G^*, the Pontryagin dual group of G. And $\exp v$ is the unitary character $\zeta\mapsto e^{v(\zeta)}$ of G^*.

1.2. Lie algebra of a closed normal subgroup. For σ, $\sigma'\in\hat{G}$, we denote by $\sigma\times\sigma'$ the set of all irreducible components ($\in\hat{G}$) of $\sigma\otimes\sigma'$, and by $\bar{\sigma}$ the element of $\hat{G}$ conjugate to σ, i.e., the equivalence class containing the complex conjugate representation $G\in x\mapsto\overline{U_x(\sigma)}$ to $U(\sigma)$. A non-void subset of $\hat{G}$ stable under the operations $\times$ and conjugation is called a ring in $\hat{G}$. For a subset Δ of $\hat{G}$, $[\Delta]$ denotes the smallest ring in $\hat{G}$ containing Δ. We put, furthermore, $A(G,\Delta)=\{x\in G;\ U_x(\sigma)=I(\sigma)$ for all $\sigma\in\Delta\}$ and, for a closed normal subgroup N of G, $A(\hat{G},N)=\{\sigma\in\hat{G};\ U_x(\sigma)=I(\sigma)$ for all $x\in N\}$.

Lemma 1.7. ([3], (28.9)). (i) *For any subset Δ of $\hat{G}$, $A(G, \Delta)$ is a closed normal subgroup of G, and $A(\hat{G}, A(G, \Delta))=[\Delta]$.*

(ii) *For any closed normal subgroup N of G, $A(\hat{G}, N)$ is a ring in $\hat{G}$, and $A(G, A(\hat{G}, N))=N$.*

Definition 1.6. For each closed normal subgroup N of G, an ideal $\Lambda_N(G)$ of the Lie algebra $\Lambda(G)$ is defined as

$$\Lambda_N(G)=\{H\in\Lambda(G);\ H(\sigma)=0 \text{ for all } \sigma\in A(\hat{G}, N)\}.$$

Before proceeding to the next lemma, note that $R(G_0)$, where G_0 is a topological subgroup of G, consists of all members of $R(G)$ with orbit in G_0.

Lemma 1.8. *Let N be a closed normal subgroup of G.*

(i) $i_G(N)=\{T\in\hat{\hat{G}};\ T(\sigma)=I(\sigma)$ *for all* $\sigma\in A(\hat{G}, N)\}$.

(ii) $R(N)=\{\alpha\in R(G);\ h_G(\alpha)\in\Lambda_N(G)\}$.

(iii) $\Lambda(N)\cong\Lambda_N(G)$ (as Lie algebras).

Proof. Since $A(G, A(\hat{G}, N)=N$, (i) is obvious. A member α of $R(G)$ belongs to $R(N)$ if and only if $\exp th_G(\alpha)\in i_G(N)$ for all $t\in\boldsymbol{R}$. By (i), this is equivalent to that $h_G(\alpha)\in\Lambda_N(G)$. Hence (ii). Fianlly we prove (iii) by giving explicitly an isomorphism of $\Lambda(N)$ onto $\Lambda_N(G)$. Let $\hat{N}$ be the dual of N. Choose a unitary matrix representation $V(\tau)$ from each $\tau\in\hat{N}$. For $\sigma\in\hat{G}$, let $U(\sigma)|_N$ denote the restriction of $U(\sigma)$ to N, and

$$U(\sigma)|_N=W_\sigma^{-1}(V(\tau_1(\sigma))\oplus\cdots\oplus V(\tau_{m(\sigma)}(\sigma)))W_\sigma$$

be its irreducible decomposition, where W_σ is a unitary matrix of order d_σ. We define a map φ of $\Sigma(N)$ into $\Sigma(G)$ as

$$\varphi(S)=(W_\sigma^{-1}(S(\tau_1(\sigma))\oplus\cdots\oplus S(\tau_{m(\sigma)}(\sigma))W_\sigma)_{\sigma\in\hat{G}}\quad (S\in\Sigma(N)).$$

Then, plainly, $\varphi(i_N(y))=i_G(y)$ $(y\in N)$. Therefore, for $\alpha\in R(N)$ and $t\in\boldsymbol{R}$, $\varphi(i_N(\alpha(t)))=i_G(\alpha(t))$, i.e., $\varphi(\exp th_N(\alpha))=\exp th_G(\alpha)$. Hence $\varphi(h_N(\alpha))=h_G(\alpha)$ $(\alpha\in R(N))$. This together with (ii) shows that φ maps $\Lambda(N)$ onto $\Lambda_N(G)$ in a one-to-one way. Furthermore, we see from its definition that φ, restricted to $\Lambda(N)$, is a Lie algebra homomorphism, hence isomorphism. q.e.d.

1.3. Lie algebra of a quotient group. Let N be a closed normal subgroup of G. We can identify the dual $(G/N)^\wedge$ of G/N, the quotient group, with the ring $A(\hat{G}, N)$ in $\hat{G}$ naturally, including the ring operations $\times$ and conjugation. As a representative of each σ in $(G/N)^\wedge$, identified with $A(\hat{G}, N)$, the representation $G/N\ni xN\mapsto U_x(\sigma)$ $(x\in G)$ can be taken. Throughout the paper we keep these conventions. Then, $\Sigma(G/N)$ consists of matrix fields on $A(\hat{G}, N)$. In particular, the bidual $(G/N)^{\hat{\wedge}}$ of G/N consists of all unitary matrix fields on $A(\hat{G}, N)$ satisfying (C1) on it. Its group operations and topology are defined coordinatewise. While, the Lie algebra $\Lambda(G/N)$ consists of all skew-hermite fields on $A(\hat{G}, N)$ satisfying (C2), with algebraic operations coordinatewise.

Definition 1.7. Let N be a closed normal subgroup of G. r_N denotes the restriction map $\Sigma(G) \ni T \mapsto T|_{A(\hat{G},N)} \in \Sigma(G/N)$. π_N denotes the natural homomorphism of G onto G/N, and $\bar{\pi}_N$ the map $R(G) \in \alpha \mapsto \pi_N \circ \alpha \in R(G/N)$.

Lemma 1.9. *Let N be a closed normal subgroup of G.*

(i) *r_N maps $\hat{\hat{G}}$ into $(G/N)\hat{}$, and the following diagram is commutative.*

$$\begin{array}{ccc} G & \xrightarrow{\pi_N} & G/N \\ {\scriptstyle i_G}\big\downarrow & & \big\downarrow{\scriptstyle i_{G/N}} \\ \hat{\hat{G}} & \xrightarrow[r_N]{} & (G/N)\hat{} \end{array}$$

(ii) *The map $\Lambda(G) \in H \mapsto r_N(H)$ is a Lie algebra homomorphism of $\Lambda(G)$ into $\Lambda(G/N)$, and the following diagram is commutative.*

$$\begin{array}{ccc} R(G) & \xrightarrow{\bar{\pi}_N} & R(G/N) \\ {\scriptstyle h_G}\big\downarrow & & \big\downarrow{\scriptstyle h_{G/N}} \\ \Lambda(G) & \xrightarrow[r_N]{} & \Lambda(G/N) \end{array}$$

Proof. Plainly r_N maps $\hat{\hat{G}}$ (resp. $\Lambda(G)$) into $(G/N)\hat{}$ (resp. $\Lambda(G/N)$), preserving the algebraic operations. The commutativity of the diagram in (i) is obvious. Next, for $\alpha \in R(G)$ and $t \in \boldsymbol{R}$, we have

$$\begin{aligned} \exp t r_N(h_G(\alpha)) &= r_N(\exp t h_G(\alpha)) = r_N(i_G(\alpha(t))) \\ &= i_{G/N}(\pi_N(\alpha(t))) \quad \text{(by (i))} \\ &= i_{G/N}(\bar{\pi}_N(\alpha)(t)) = \exp t h_{G/N}(\bar{\pi}_N(\alpha)) . \end{aligned}$$

Hence the commutativity of the diagram in (ii) follows. q. e. d.

The next lemma is important.

Lemma 1.10 ([4], Theorem 4). *Let N be the same as above. Then $\bar{\pi}_N$ carries $R(G)$ onto $R(G/N)$. Equivalently, r_N carries $\Lambda(G)$ onto $\Lambda(G/N)$.*

Since the kernel of r_N, restricted to $\Lambda(G)$, is $\Lambda_N(G)$, we have the following

Corollary. $\Lambda(G/N) \cong \Lambda(G)/\Lambda_N(G)$ *(as Lie algebras).*

1.4. Structure of $\Lambda(G)$.

Definition 1.8. $H_0(G)$ denotes the totality of closed normal subgroups N of G such that G/N is a Lie or finite group.

Lemma 1.11. (i) *A closed normal subgroup N of G belongs to $\boldsymbol{H}_0(G)$ if and only if $N = A(G, \Delta)$ for some finite subset $\Delta \subseteq \hat{G}$.*

(ii) $\hat{G} = \cup \{A(\hat{G}, N);\ N \in \boldsymbol{H}_0(G)\}$.

(iii) *For any neighbourhoods V of e in G and $\mathscr{V}$ of 0 in $\Lambda(G)$, there exists $N \in \boldsymbol{H}_0(G)$ such that $N \subseteq V$ and $\Lambda_N(G) \subseteq \mathscr{V}$. Here $\Lambda(G)$ is equipped with the relative topology of $\Sigma(G)$.*

Proof. For proof of (i), we have only to recall that a Lie (or finite) group has a faithful, finite-dimensional, continuous, unitary representation. The easy detail is omitted. For $\sigma \in \hat{G}$, put $N = A(G, \{\sigma\})$. Then $N \in \boldsymbol{H}_0(G)$ (by (i)), and $A(\hat{G}, N) = [\{\sigma\}]$. Hence (ii). Since G is isomorphic with $\hat{G}$ under i_G, and the topologies of $\hat{\hat{G}}$ and $\Lambda(G)$ are coordinatewise, we can choose a finite subset $\Delta \subseteq \hat{G}$ and $\varepsilon > 0$ so that

$$V_1 = \{x \in G;\ \|U_x(\sigma) - U_e(\sigma)\|_\sigma < \varepsilon\ (\sigma \in \Delta)\} \subseteq V \quad \text{and}$$

$$\mathscr{V}_1 = \{H \in \Lambda(G);\ \|H(\sigma)\|_\sigma < \varepsilon\ (\sigma \in \Delta)\} \subseteq \mathscr{V}.$$

Put $N = A(G, \Delta)$. Then, $N \subseteq V_1$ and $\Lambda_N(G) \subseteq \mathscr{V}_1$. Hence (iii). q.e.d.

It is seen from (i) and (iii) of the above lemma that $\boldsymbol{H}_0(G)$ is lower directed under inclusion, and has the intersection $\{e\}$. An element $\sigma \in \hat{G}$ is said to be torsion if $N = A(G, \{\sigma\})$ is open in G, i.e., the group G/N is finite. The totality of torsion elements in $\hat{G}$ coincides with $A(\hat{G}, c(G))$ ([3], (28.18)). Therefore, if G is totally disconnected, every $\sigma \in \hat{G}$ is torsion, and hence, G/N is finite for every $N \in \boldsymbol{H}_0(G)$. While, if otherwise, $\hat{G}$ contains at least one non-torsion element and so, as is seen through an ordinary isomorphism theorem, all such N's that G/N is a Lie group form a cofinal subfamily of $\boldsymbol{H}_0(G)$. For $N, N' \in \boldsymbol{H}_0(G)$ such that $N \subseteq N'$, denote by $\pi_{N'N}$ the canonical homomorphism of G/N onto G/N', and by $r_{N'N}$ the restriction map $\Lambda(G/N) \in H \mapsto H|_{A(\hat{G}, N')} \in \Lambda(G/N')$. Then we have the projective systems $\{G/N, \pi_{N'N}\}$ of Lie or finite groups and $\{\Lambda(G/N), r_{N'N}\}$ of Lie algebras. Evidently, the limit of the former is isomorphic with G.

Lemma 1.12. (i) *The map* ψ: $\Lambda(G) \in H \mapsto (r_N(H))_{N \in \boldsymbol{H}_0(G)}$ *gives a Lie algebra isomorphism of* $\Lambda(G)$ *onto the limit of* $\{\Lambda(G/N), r_{N'N}\}$.

(ii) *In case* G/N *and* G/N' *are Lie groups,* $r_{N'N}$ *is the differential of* $\pi_{N'N}$.

Proof. (i). Put $\Lambda_1 = \varprojlim \{\Lambda(G/N), r_{N'N}\}$. Obviously ψ is a Lie algebra homomorphism of $\Lambda(G)$ into Λ_1. The injectivity of ψ is clear from Lemma 1.11, (ii). Take any $(H_N)_{N \in \boldsymbol{H}_0(G)} \in \Lambda_1$, where $H_N \in \Lambda(G/N)$. We can well define the union H of all H_N's, viewing them as matrix-valued functions. Then $H \in \Lambda(G)$ and $\psi(H) = (H_N)_{N \in \boldsymbol{H}_0(G)}$. Hence ψ is surjective. (ii). For $H \in \Lambda(G/N)$, there exists $\alpha \in R(G)$ such that $r_N(h_G(\alpha)) = H$ (Lemma 1.10) and so, $r_{N'N}(H) = r_{N'}(h_G(\alpha))$. Hence, in view of (1.3) and Lemma 1.9, we have for $t \in \boldsymbol{R}$,

$$\begin{aligned}
\pi_{N'N}(i_{G/N}^{-1}(\exp\ tH)) &= \pi_{N'N} i_{G/N}^{-1}\, r_N(\exp\ th_G(\alpha)) \\
&= \pi_{N'N}\pi_N\, i_G^{-1}\ (\exp\ th_G(\alpha)) = \pi_{N'}(\alpha(t)) = \overline{\pi_{N'}}(\alpha)\ (t) \\
&= i_{G/N'}^{-1}(\exp\ tr_{N'}(h_G(\alpha))) = i_{G/N'}^{-1}(\exp\ tr_{N'N}(H)).
\end{aligned}$$

This shows that $r_{N'N}$ is just the differential of $\pi_{N'N}$. q.e.d.

Definition 1.9. $R(G)$ is regarded as a real Lie algebra isomorphic with $\Lambda(G)$ under the map h_G. That is, for $\alpha, \beta \in R(G)$ and $a, b \in \boldsymbol{R}$, $a\alpha + b\beta$ and $[\alpha, \beta]$ are the elements of $R(G)$ corresponding to $ah_G(\alpha) + bh_G(\beta)$ and $[h_G(\alpha), h_G(\beta)]$ under h_G, respectively. Furthermore, $\Lambda(G)$ is equipped with the relative topology of $\Sigma(G)$, and then, $R(G)$ is topologized homeomorphically with $\Lambda(G)$ under h_G. (They are, as real vector spaces, locally convex and Hausdorff.)

Lemma 1.13. *The locally convex space $\Lambda(G)$ (i.e., $R(G)$) is isomorphic with $\boldsymbol{R}^I$, a direct product of the straight line. Hence it is a Baire space, and so, also barrelled.*

Proof. We can regard $\{\Lambda(G/N), r_{N'N}\}$ as a projective system of finite-dimensional, locally convex, Hausdorff, real vector spaces. Let Λ_1 be its limit space. Since the topologies of $\Lambda(G)$ and each $\Lambda(G/N)$ are coordinatewise, the map ψ given in Lemma 1.12 is obviously homeomorphic from $\Lambda(G)$ onto Λ_1. While, Λ_1 is a closed linear subspace of the product space of all $\Lambda(G/N)$'s ($N \in \boldsymbol{H}_0(G)$) which is evidently isomorphic with $\boldsymbol{R}^J$. Hence Λ_1 is isomorphic with some $\boldsymbol{R}^I$. This proves the first assertion of the lemma. Then, as a direct product of complete metric spaces, $\Lambda(G)$ is a Baire space, and so, barrelled ([1], Chap. 3, §1). q.e.d.

In view of Lemma 1.12, the next lemma is rather well known. The proof is omitted (cf. [5], §47).

Lemma 1.14. *The dimension (finite or infinite) of $\Lambda(G)$ coincides with the usual covering dimension of the compact space G. It is finite if and only if $\boldsymbol{H}_0(G)$ contains a totally disconnected member.*

Before concluding this section, we quote one more lemma.

Lemma 1.15. ([4], Theorem 5). *The union of the orbits of the members of $R(G)$ is dense in $c(G)$.*

Since $G/c(G)$ is totally disconnected, and so, $R(G/c(G))$ is trivial, we have, by Lemma 1.8, (iii) and Corollary to Lemma 1.10, $\Lambda(G)=\Lambda_{c(G)}(G)\cong\Lambda(c(G))$. The Lie algebra $\Lambda(G)$ is commutative if and only if $c(G)$ is abelian. Indeed, if $\Lambda(G)$) ($\cong\Lambda(c(G))$) is commutative, we see by Lemma 1.15 that $c(G)^\wedge$ is abelian. Conversely, if $c(G)$ is abelian, $\Lambda(c(G))$ is commutative.

§2. Continuous differentiability of functions

2.1. Definition of continuous differentiability.

Definition 2.1. A $\boldsymbol{C}$-valued function f on G is said to be right (resp. left) differentiable with respect to $\alpha \in R(G)$ at $x \in G$ if the function $f(x\alpha(t))$ (resp. $f(\alpha(-t)x)$) of real variable t is differentiable at $t=0$. In this case we define

$$d_\alpha^{(r)}f(x)=\frac{d}{dt}f\,(x\alpha(t))|_{t=0}$$

$$\left(\text{resp. } d_\alpha^{(l)}f(x)=\frac{d}{dt}f(\alpha(-t)x)|_{t=0}\right),$$

and call this value the right (resp. left) differential coefficient of f with respect to α

at x. In particular, if f is right (resp. left) differentiable with respect to α at every $x \in G$, then the function $x \mapsto d_\alpha^{(r)} f(x)$ (resp. $x \mapsto d_\alpha^{(l)} f(x)$) defined on G is called the right (resp. left) derivative of f with respect to α, and denoted by $d_\alpha^{(r)} f$ (resp. $d_\alpha^{(l)} f$).

Definition 2.2. $\mathscr{E}_0(G)$ denotes the set of all $\boldsymbol{C}$-valued continuous functions on G. For each $n=1, 2, 3,\dots$, $\mathscr{E}_n^{(r)}(G)$ (resp. $\mathscr{E}_n^{(l)}(G)$) denotes the set of all those $f \in \mathscr{E}_0(G)$ for which the successive right (resp. left) derivatives

$$d_{\alpha_1}^{(r)} f,\ d_{\alpha_2}^{(r)} d_{\alpha_1}^{(r)} f, \dots,\ d_{\alpha_n}^{(r)} \cdots d_{\alpha_2}^{(r)} d_{\alpha_1}^{(r)} f$$

$$(\text{resp. } d_{\alpha_1}^{(l)} f,\ d_{\alpha_2}^{(l)} d_{\alpha_1}^{(l)} f, \dots,\ d_{\alpha_n}^{(l)} \dots d_{\alpha_2}^{(l)} d_{\alpha_1}^{(l)} f)$$

exist and belong to $\mathscr{E}_0(G)$ for any $\alpha_1, \alpha_2,\dots, \alpha_n \in R(G)$.(2) (Obviously $\mathscr{E}_1^{(r)}(G) \supseteq \mathscr{E}_2^{(r)}(G) \supseteq \mathscr{E}_3^{(r)}(G) \supseteq \cdots$, and $\mathscr{E}_1^{(l)}(G) \supseteq \mathscr{E}_2^{(l)}(G) \supseteq \mathscr{E}_3^{(l)}(G) \supseteq \cdots$.) $\mathscr{E}_\infty^{(r)}(G)$ (resp. $\mathscr{E}_\infty^{(l)}(G)$) denotes the intersection of all $\mathscr{E}_n^{(r)}(G)$'s (resp. $\mathscr{E}_n^{(l)}(G)$'s) ($n=1, 2, 3,\dots$).

Suppose G is a Lie group. Then, of course, $\mathscr{E}_n^{(r)}(G)$ and $\mathscr{E}_n^{(l)}(G)$ coincide with each other, and give the class of n-times (infinitely if $n=\infty$) continuously differentiable functions on G, which can be defined by using local coordinates, too. The aim of this section is to prove the following theorem for any compact group G.

Theorem 2.1. *For each $n=\infty, 1, 2,\dots$, the sets $\mathscr{E}_n^{(r)}(G)$ and $\mathscr{E}_n^{(l)}(G)$ coincide with each other.*

Since $f(x\alpha(t)) = f((x\alpha x^{-1})(t)x)$ ($x \in G$, $\alpha \in R(G)$), we see that $d_\alpha^{(r)} f$ exists on G for every $\alpha \in R(G)$ if and only if so with $d_\alpha^{(l)} f$. But it is not so easy to see that $d_\alpha^{(r)} f$ is continuous on G for every $\alpha \in R(G)$ if and only if so with $d_\alpha^{(l)} f$. Paragraph 2.3 will be devoted to proving this theorem. For the moment we make the following

Definition 2.3. for $n=\infty, 1, 2,\dots$, we set $\mathscr{E}_n(G) = \mathscr{E}_n^{(r)}(G) \cap \mathscr{E}_n^{(l)}(G)$, and call its elements the n-times (infinitely if $n=\infty$) continuously differentiable functions on G.

Lemma 2.1. (i) *Suppose that $f, g \in \mathscr{E}_1^{(r)}(G)$. Then $af+bg$ ($a, b \in \boldsymbol{C}$), fg, $\bar{f}$, ${}_{x_0}f$ and f_{x_0} ($x \in G$) belong to $\mathscr{E}_1^{(r)}(G)$. And, for $\alpha \in R(G)$, the following hold:*

$$d_\alpha^{(r)}(af+bg) = a d_\alpha^{(r)} f + b d_\alpha^{(r)} g,$$

$$d_\alpha^{(r)}(fg) = (d_\alpha^{(r)} f) g + f(d_\alpha^{(r)} g),$$

$$d_\alpha^{(r)} \bar{f} = \overline{d_\alpha^{(r)} f},$$

$$d_\alpha^{(r)}({}_{x_0}f) = {}_{x_0}(d_\alpha^{(r)} f),$$

$$d_\alpha^{(r)}(f_{x_0}) = (d_{x_0^{-1}\alpha x_0}^{(r)} f)_{x_0}.$$

The cooresponding assertions hold also for functions in $\mathscr{E}_1^{(l)}(G)$.

(2) Here the requirement for the continuity of f itself is not excessive. Indeed, the existence and continuity of all successive right and left derivatives of a function f on G does not necessarily imply the continuity of f, even if G is connected (cf. [6], page 56).

(ii) *A* $\boldsymbol{C}$*-valued function* f *on* G *belongs to* $\mathscr{E}_1^{(r)}(G)$ *if and only if* $\check{f}\in\mathscr{E}_1^{(l)}(G)$. *In this case, for* $\alpha\in R(G)$,

$$d_\alpha^{(l)}\check{f}=(d_\alpha^{(r)}f)\check{}\,.$$

Proof. It suffices to check the listed formulas. The first three are obvious. The others are also clear from the following: ${}_{x_0}f(x\alpha(t))=f(x_0^{-1}x\alpha(t))$, $f_{x_0}(x\alpha(t))=f(xx_0x_0^{-1}\alpha(t)x_0)$, and $\check{f}(\alpha(-t)x)=f(x^{-1}\alpha(t))$. q.e.d.

Corollary. *Each* $\mathscr{E}_n(G)$ $(n=\infty, 1, 2,\dots)$ *is an algebra over* $\boldsymbol{C}$ *under the usual algebraic operations of functions, and stable under the left and the right translations, the inversion and the complex conjugation.*

Proof. Easily seen from Lemma 2.1. q.e.d.

2.2. Regular functions.

Definition 2.4. Let N be *a* closed normal subgroup of G. For each $n=\infty$, $1, 2,\dots$, we define a subalgebra $\mathscr{E}_n(G, N)$ of $\mathscr{E}_n(G)$ as

$$\mathscr{E}_n(G, N)=\{f\in\mathscr{E}_n(G);\, f(xy)=f(x) \text{ for all } x\in G \text{ and } y\in N\}\,.$$

$\mathscr{E}_n(G, N)$ is evidently stable under the left and the right translations, the inversion and the complex conjugation.

Lemma 2.2. *Let* N *be a closed normal subgroup of* G. *A* $\boldsymbol{C}$*-valued function* g *on* G/N *belongs to* $\mathscr{E}_1^{(r)}(G/N)$ (resp. $\mathscr{E}_1^{(l)}(G/N)$) *if and only if* $g\circ\pi_N\in\mathscr{E}_1^{(r)}(G)$ (resp. $\mathscr{E}_1^{(l)}(G)$). *In this case, for* $\alpha\in R(G)$,

$$d_\alpha^{(r)}(g\circ\pi_N)=(d_{\bar{\pi}_N(\alpha)}^{(r)}g)\circ\pi_N \tag{2.1}$$

$$(\text{resp. } d_\alpha^{(l)}(g\circ\pi_N)=(d_{\bar{\pi}_N(\alpha)}^{(l)}g)\circ\pi_N).$$

Proof. This lemma is clear from the equality $(g\circ\pi_N)(x\alpha(t))=g(\pi_N(x)\bar{\pi}_N(\alpha)(t))$ (resp. $(g\circ\pi_N)(\alpha(-t)x)=g(\bar{\pi}_N(\alpha)(-t)\pi_N(x))$) and Lemma 1.10. q.e.d.

Corollary. *Let* N *be the same as above. For each* $n=\infty, 1, 2,\dots$, *the map* $\mathscr{E}_n(G/N)\ni g\mapsto g\circ\pi_N$ *is an algebra isomorphism of* $\mathscr{E}_n(G/N)$ *onto* $\mathscr{E}_n(G, N)$.

Proof. Easily seen from Lemma 2.2. q.e.d.

In particular, each $\mathscr{E}_\infty(G, N)$ with N in $\boldsymbol{H}_0(G)$ can be regarded as the class of infinitely continuously differentiable functions on the Lie or finite group G/N.

Definition 2.5. Define a subalgebra $\mathscr{D}(G)$ of $\mathscr{E}_\infty(G)$ as

$$\mathscr{D}(G)=\cup\{\mathscr{E}_\infty(G, N);\, N\in\boldsymbol{H}_0(G)\}\,.$$

Following [2], each member of $\mathscr{D}(G)$ is called a regular function on G.

Let $\mathscr{T}(G)$ denote the set of all trigonometric polynomials on G, i.e., finite $\boldsymbol{C}$-linear combinations of coordinate functions $u_{jk}^{(\sigma)}$ $(j, k=1,\dots, d_\sigma, \sigma\in\hat{G})$ of the representations $U(\sigma)$. For $\alpha\in R(G)$, put $H_\alpha=h_G(\alpha)$. Then, for each $\sigma\in\hat{G}$ and $x\in G$, $U_{x\alpha(t)}(\sigma)=U_x(\sigma)\exp tH_\alpha(\sigma)$ and $U_{\alpha(-t)x}(\sigma)=(\exp -tH_\alpha(\sigma))U_x(\sigma)$. Hence we see that the coordinate functions $u_{jk}^{(\sigma)}$ of $U(\sigma)$ belong to $\mathscr{E}_1(G)$ with derivatives

$$d_\alpha^{(r)}U(\sigma)=U(\sigma)H_\alpha(\sigma) \quad \text{and} \tag{2.2}$$

$$d_\alpha^{(l)}U(\sigma)=-H_\alpha(\sigma)U(\sigma) \quad (\alpha\in R(G)), \tag{2.2'}$$

where the left hand sides mean to take derivatives coordinatewise. So it is also clear that each $u_{jk}^{(\sigma)}$ belongs to $\mathscr{E}_\infty(G)$, and therefore, to $\mathscr{E}_\infty(G, N)$ with $N=A(G, \{\sigma\})$. Thus we have seen that $\mathscr{T}(G)$ is included in $\mathscr{D}(G)$. Evidently $\mathscr{T}(G)$ is a subalgebra of $\mathscr{D}(G)$ stable under the left and the right translations, the inversion and the complex conjugation.

Partial integration formula. Suppose $f, g\in\mathscr{E}_1(G)$ and $\alpha\in R(G)$. Then

$$\int_G d_\alpha^{(r)}f(x)d_Gx=\int_G d_\alpha^{(r)}f(x\alpha(t))d_Gx$$
$$=\frac{d}{dt}\int_G f(x\alpha(t))d_Gx=\frac{d}{dt}\int_G f(x)d_Gx=0.$$

Replacing f by fg, we have

$$\int_G (d_\alpha^{(r)}f)(x)g(x)d_Gx=-\int_G f(x)(d_\alpha^{(r)}g)(x)d_Gx.$$

The same holds also for the left differentiation.

Example 1. Let $\boldsymbol{Q}$ be the discrete additive group of rational numbers and G its Pontryagin dual. Identify $\hat{G}$ and $\boldsymbol{Q}$ canonically, and denote by χ_r the unitary character of G corresponding to $r\in\boldsymbol{Q}$, i.e., $(x, \chi_r)=(r, x)$ $(x\in G)$. The commutative Lie algebra $\Lambda(G)$ consists of all pure-imaginary characters of $\boldsymbol{Q}$, and so, 1-dimensional. Each $\alpha\in R(G)$ is of the form $(r, \alpha(t))=e^{tvr}$ $(r\in\boldsymbol{Q}, t\in\boldsymbol{R})$ with some $v\in\sqrt{-1}\boldsymbol{R}$. Hence

$$d_\alpha^{(r)}\chi_r(=-d_\alpha^{(l)}\chi_r)=vr\chi_r.$$

Now, for $p=1, 2,\dots$, put $N_p=A(G, \{1/p!\})$. Then $(G/N_p)^\wedge=A(\hat{G}, N_p)=[\{1/p!\}]$, $N_p=A(G, [\{1/p!\}])$ and $[\{1/p!\}]=\{q/p!;\ q=0, \pm1, \pm2,\dots\}$. Hence $\{N_p;\ p=1, 2,\dots\}$ is cofinal in $\boldsymbol{H}_0(G)$ and so, $\mathscr{D}(G)=\bigcup_{p=1}^\infty\mathscr{E}_\infty(G, N_p)$. Furthermore, since $(G/N_p)^\wedge=\{q/p!;\ q=0, \pm1, \pm2,\dots\}$, $d_\alpha^{(r)}\chi_{q/p!}=v\dfrac{q}{p!}\chi_{q/p!}$ and the partial integration is valid, we can show in the same way as in the classical case that each $\mathscr{E}_\infty(G, N_p)$ consists of all functions on G that permit Fourier expansion by $\{\chi_{q/p!};\ q=0, \pm1, \pm2,\dots\}$ with rapidly decreasing coefficients. Thus the functions in $\mathscr{D}(G)$ have been characterized. On the other hand, take any absolutely convergent series $\sum_{p=1}^\infty c_p$ of non-zero complex numbers and define

$$f(x)=\sum_{p=1}^\infty c_p\chi_{1/p!}(x) \quad (x\in G).$$

Then the termwise differentiation shows immediately that f is infinitely continuously differentiable. But f is not regular, since it belongs to none of $\mathscr{E}_\infty(G, N_p)$.

By the way, the following should be noted here: G is connected; each N_p is totally disconnected and G/N_p is isomorphic with the 1-dimensional torus; G is locally isomorphic with $(G/N_p)\times N_p$ for any p, but not isomorphic.

Example 2. Suppose G is finite dimensional (see Lemma 1.14) and separable. In this case let us construct a function $f\in\mathscr{E}_\infty(G)$ such that $f=f_y$ for $y\in G$ implies $y=e$. If G is neither Lie nor finite, such f is not regular. Take a linear base $\{\alpha_1,\dots,\alpha_n\}$ of $R(G)$ (see Definition 1.9) and put $H_j=h_G(\alpha_j)$ $(j=1,\dots,n)$. As $\hat{G}$ is countable by assumption, put $\hat{G}=\{\sigma_i;\ i=1,2,\dots\}$. For each i, take a non-singular matrix $C(\sigma_i)\in\mathfrak{M}(d_{\sigma_i}, \boldsymbol{C})$ and $m_i>1$ so that m_i is larger than any of the absolute values of coordinates of $H_j(\sigma_i)$ $(j=1,\dots,n)$ and $C(\sigma_i)$. By (2.2) and (2.2)′ we have, for any $\beta_1=\sum_{j_1=1}^n a_{1j_1}\alpha_{j_1},\dots,\beta_p=\sum_{j_p=1}^n a_{pj_p}\alpha_{j_p}\in R(G)$ $(p=1,2,\dots)$,

$$\begin{aligned}&d_{\beta_1}^{(r)}\cdots d_{\beta_p}^{(r)}\operatorname{Tr}(U(\sigma_i)C(\sigma_i))\\&\quad=\sum_{1\le j_1,\dots,j_p\le n}a_{1j_1}\cdots a_{pj_p}\operatorname{Tr}(U(\sigma_i)H_{j_1}(\sigma_i)\cdots H_{j_p}(\sigma_i)C(\sigma_i)),\\&d_{\beta_1}^{(l)}\cdots d_{\beta_p}^{(l)}\operatorname{Tr}(U(\sigma_i)C(\sigma_i))\\&\quad=\sum_{1\le j_1,\dots,j_p\le n}(-1)^p a_{1j_1}\cdots a_{pj_p}\operatorname{Tr}(H_{j_1}(\sigma_i)\cdots H_{j_p}(\sigma_i)U(\sigma_i)C(\sigma_i)).\end{aligned}$$

The coefficients of the $d_{\sigma_i}^2$ terms of each of these trigonometric polynomials do not exceed $n^p a^p d_{\sigma_i}^p m_i^{p+1}$ in absolute value, where $a=\max\{|a_{kj_k}|;\ j_k=1,\dots,d_{\sigma_i}, k=1,\dots,p\}$. Hence, if we define

$$f(x)=\sum_{i=1}^\infty i^{-2}d_{\sigma_i}^{-2}(nid_{\sigma_i}m_i)^{-i}\operatorname{Tr}(U_x(\sigma_i)C(\sigma_i))\quad(x\in G),$$

the termwise differentiation shows that $f\in\mathscr{E}_\infty(G)$. Here, if $n=0$ (i.e., G is totally disconnected), we replace it by 1. Now we show that this f satisfies the required condition. Suppose $f=f_y$. Then, by uniqueness of Fourier expansion, $U_y(\sigma_i)C(\sigma_i)=C(\sigma_i)$ for all i. Since each $C(\sigma_i)$ is non-singular, this implies that $U_y(\sigma_i)=I(\sigma_i)$ $(i=1,2,\dots)$. Hence $y=e$.

2.3. Proof of Theorem 2.1.

Lemma 2.3. *For each neighbourhood V of e in G, there exists a function $\theta\in\mathscr{D}(G)^+$ such that* $\operatorname{supp}(\theta)\subseteq V$ *and* $\int_G\theta d_G=1$.

Proof. Take a neighbourhood W of e and $N\in\boldsymbol{H}_0(G)$ so that $\overline{W}N\subseteq V$. Since G/N is Lie or finite, there exists $g\in\mathscr{E}_\infty(G/N)^+$ such that $\operatorname{supp}(g)\subseteq\pi_N(W)$ and $\int_{G/N}g d_{G/N}\neq 0$. Put $f=g\circ\pi_N$. Then $f\in\mathscr{E}_\infty(G,N)^+$, $\operatorname{supp}(f)\equiv V$ and $\int_G f d_G\neq 0$. Therefore, suitably normalizing f, we obtain θ. q.e.d.

Lemma 2.4. *Let m be a complex Radon measure on G and $f\in\mathscr{E}_1^{(r)}(G)$* (resp. $\mathscr{E}_1^{(l)}(G)$). *Then $m*f\in\mathscr{E}_1^{(r)}(G)$* (resp. $f*m\in\mathscr{E}_1^{(l)}(G)$) *and, for $\alpha\in R(G)$,* (2.3) $$d_\alpha^{(r)}(m*f) = m*(d_\alpha^{(r)} f)$$

$$(\text{resp. } d_\alpha^{(l)}(f*m) = (d_\alpha^{(l)} f)*m).$$

Proof. The differentiation of $\int_G f(y^{-1}x\alpha(t))dm(y)$ and $\int_G f(\alpha(-t)xy^{-1})\,dm(y)$ in t under the integral sign proves the lemma. q.e.d.

From Lemma 2.4 we see that if $f \in \mathscr{E}_\infty(G)$, then $m*f \in \mathscr{E}_\infty^{(r)}(G)$ and $f*m \in \mathscr{E}_\infty^{(l)}(G)$. Now suppose $f \in \mathscr{E}_\infty(G, N)$ with $N \in \boldsymbol{H}_0(G)$. Then, $m*f$ and $f*m$ are constant on each coset of N, and so, can be viewed as functions on G/N. By Lemma 2.2, they belong to $\mathscr{E}_\infty^{(r)}(G, N)$ and $\mathscr{E}_\infty^{(l)}(G, N)$ respectively. But, since G/N is Lie or finite, $\mathscr{E}_\infty^{(r)}(G/N)$, $\mathscr{E}_\infty^{(l)}(G/N)$ and $\mathscr{E}_\infty(G/N)$ coincide with one another. Hence both $m*f$ and $f*m$ belong to $\mathscr{E}_\infty(G, N)$. Thus, we have shown that if $f \in \mathscr{D}(G)$, then both $m*f$ and $f*m$ belong to $\mathscr{D}(G)$.

Definition 2.6. We denote by P_0 the norm on the $\boldsymbol{C}$-linear space $\mathscr{E}_0(G)$ defined as $P_0(f) = \sup_{x \in G} |f(x)|$ $(f \in \mathscr{E}_0(G))$, and topologize $\mathscr{E}_0(G)$ by it. For any $\alpha_1, \dots, \alpha_k \in R(G)$ $(k = 1, 2, \dots)$, $P_{\alpha_1 \cdots \alpha_k}$ (resp. $P^{\alpha_1 \cdots \alpha_k}$) denotes the seminorm on the $\boldsymbol{C}$-linear space $\mathscr{E}_k^{(r)}(G)$ (resp. $\mathscr{E}_k^{(l)}(G)$) defined as

$$P_{\alpha_1 \cdots \alpha_k}(f) = \sup_{x \in G} |d_{\alpha_1}^{(r)} \cdots d_{\alpha_k}^{(r)} f(x)| \quad (f \in \mathscr{E}_k^{(r)}(G))$$

$$(\text{resp. } P^{\alpha_1 \cdots \alpha_k}(f) = \sup_{x \in G} |d_{\alpha_1}^{(l)} \cdots d_{\alpha_k}^{(l)} f(x)| \quad (f \in \mathscr{E}_k^{(l)}(G))).$$

For $n = \infty, 1, 2, \dots$, we denote by $\mathfrak{F}_n^{(r)}$ (resp. $\mathfrak{F}_n^{(l)}$) the family

$$\{P_0, P_{\alpha_1 \cdots \alpha_k};\ 1 \le k < n+1,\ \alpha_1, \dots, \alpha_k \in R(G)\}$$

$$(\text{resp. } \{P_0, P^{\alpha_1 \cdots \alpha_k};\ 1 \le k < n+1,\ \alpha_1, \dots, \alpha_k \in R(G)\}),$$

and topologize $\mathscr{E}_n^{(r)}(G)$ (resp. $\mathscr{E}_n^{(l)}(G)$) by it. Thus the $\boldsymbol{C}$-linear spaces $\mathscr{E}_n^{(r)}(G)$ and $\mathscr{E}_n^{(l)}(G)$ $(n = \infty, 1, 2, \dots)$ are locally convex and Hausdorff.

Lemma 2.5. *Let $\{V_\nu\}_{\nu \in \mathscr{N}}$ be a neighbourhood base at e in G. For each $\nu \in \mathscr{N}$, choose a function $\theta_\nu \in \mathscr{D}(G)^+$ so that* $\operatorname{supp}(\theta_\nu) \subseteq V_\nu$ *and* $\int_G \theta_\nu d_G = 1$ *(Lemma* 2.3), *and regard $\mathscr{N}$ as directed by defining $\nu \ge \nu'$ if $V_\nu \subseteq V_{\nu'}$. Then, for each $f \in \mathscr{E}_n^{(r)}(G)$* (resp. $\mathscr{E}_n^{(l)}(G)$) $(n = \infty, 1, 2, \dots)$, *the net* $\{\theta_\nu * f\}_{\nu \in \mathscr{N}}$ (resp. $\{f*\theta_\nu\}_{\nu \in \mathscr{N}}$) *lies in $\mathscr{D}(G)$ and converges to f in* $\mathscr{E}_n^{(r)}(G)$ (resp. $\mathscr{E}_n^{(l)}(G)$).

Proof. $\theta_\nu \in \mathscr{D}(G)$ implies $\theta_\nu * f \in \mathscr{D}(G)$, as remarked above. Take any $\alpha_1, \dots, \alpha_k \in R(G)$ $(0 \le k < n+1)$. Let $D_{\alpha_1 \cdots \alpha_k}$ denote the map $f \mapsto d_{\alpha_1}^{(r)} \cdots d_{\alpha_k}^{(r)} f$ of $\mathscr{E}_n^{(r)}(G)$ into $\mathscr{E}_{n-k}^{(r)}(G)$, where we put $\mathscr{E}_0^{(r)}(G) = \mathscr{E}_0(G)$, and agree that if $k = 0$, $D_{\alpha_1 \cdots \alpha_k}$ means the identity map on $\mathscr{E}_n^{(r)}(G)$. Then, by (2.3) and the uniform continuity of $D_{\alpha_1 \cdots \alpha_k} f$ on G,

$$|(D_{\alpha_1 \cdots \alpha_k}(\theta_\nu * f))(x) - (D_{\alpha_1 \cdots \alpha_k} f)(x)|$$

$$= |(\theta_\nu * D_{\alpha_1 \cdots \alpha_k} f)(x) - (D_{\alpha_1 \cdots \alpha_k} f)(x)|$$

$$= \left| \int_G \theta_\nu(y)\{(D_{\alpha_1\cdots\alpha_k}f)(y^{-1}x) - (D_{\alpha_1\cdots\alpha_k}f)(x)\} d_G y \right|$$

$$\leq \sup_{y\in V_\nu} |(D_{\alpha_1\cdots\alpha_k}f)(y^{-1}x) - (D_{\alpha_1\cdots\alpha_k}f)(x)|$$

$$\longrightarrow 0 \ (\nu \to 0), \text{ uniformly w.r.t. } x \in G.$$

Hence the assertion for $f \in \mathscr{E}_n^{(r)}(G)$. The other can be checked similarly. q.e.d.

Before proceeding to the following Lemmas 2.6–2.10, recall Definition 1.9 and Lemma 1.13.

Lemma 2.6. *Let* $f \in \mathscr{E}_1^{(r)}(G)$ (resp. $\mathscr{E}_1^{(l)}(G)$). *Then the map*

$$\alpha \longmapsto d_\alpha^{(r)} f \quad (\text{resp. } d_\alpha^{(l)} f)$$

of $R(G)$ *into* $\mathscr{E}_0(G)$ *is* $\boldsymbol{R}$*-linear.*

Proof. We prove the assertion for $f \in \mathscr{E}_1^{(r)}(G)$. First suppose $f \in \mathscr{D}(G)$, and take $N \in \boldsymbol{H}_0(G)$ and $g \in \mathscr{E}_\infty(G/N)$ so that $f = g \circ \pi_N$. Then, since the map $\bar{\pi}_N$ of $R(G)$ into $R(G/N)$ is linear (Lemma 1.9, (ii)) and G/N is a Lie or finite group, we have, for $a, b \in \boldsymbol{R}$ and $\alpha, \beta \in R(G)$,

$$\begin{aligned} d_{a\alpha+b\beta}^{(r)} f &= (d_{\bar{\pi}_N(a\alpha+b\beta)}^{(r)} g) \circ \pi_N && (\because (2.1)) \\ &= (a d_{\bar{\pi}_N(\alpha)}^{(r)} g + b d_{\bar{\pi}_N(\beta)}^{(r)} g) \circ \pi_N \\ &= a d_\alpha^{(r)} f + b d_\beta^{(r)} f && (\because (2.1)). \end{aligned}$$

Next, for any $f \in \mathscr{E}_1^{(r)}(G)$, there exists a net $\{f_\lambda\}_\lambda$ in $\mathscr{D}(G)$ converging to f in $\mathscr{E}_1^{(r)}(G)$ (Lemma 2.5). Then, for each $\alpha \in R(G)$, $\{d_\alpha^{(r)} f_\lambda\}_\lambda$ converges to $d_\alpha^{(r)} f$ in $\mathscr{E}_0(G)$. So the linearity of the map $\alpha \mapsto d_\alpha^{(r)} f$ follows from that of every map $\alpha \mapsto d_\alpha^{(r)} f_\lambda$.
q.e.d.

Lemma 2.7. *Let* $\{\alpha_\lambda\}_\lambda$ *be a net in* $R(G)$. *If this net converges to* α *in* $R(G)$, *then* $\{\alpha_\lambda(t)\}_\lambda$ *converges to* $\alpha(t)$ *in* G *for each* $t \in \boldsymbol{R}$.

Proof. Put $h_G(\alpha_\lambda) = (H_\lambda(\sigma))_{\sigma\in\hat{G}}$ and $h_G(\alpha) = (H(\sigma))_{\sigma\in\hat{G}}$. Then, by assumption, $\|H_\lambda(\sigma) - H(\sigma)\|_\sigma \to 0$ $(\lambda \to 0)$ for each $\sigma \in \hat{G}$, and hence, $\|\exp tH_\lambda(\sigma) - \exp tH(\sigma)\|_\sigma \to 0$ $(\lambda \to 0)$. This proves the lemma. q.e.d.

Lemma 2.8. *For each* $f \in \mathscr{E}_1^{(r)}(G)$ (resp. $\mathscr{E}_1^{(l)}(G)$) *and* $x \in G$, *the map*

$$\alpha \longrightarrow d_\alpha^{(r)} f(x) \quad (\text{resp. } d_\alpha^{(l)} f(x))$$

of $R(G)$ *into* $\boldsymbol{C}$ *is* $\boldsymbol{R}$*-linear and continuous.*

Proof. We prove the assertion for $f \in \mathscr{E}_1^{(r)}(G)$. The $\boldsymbol{R}$-linearity of this map is clear from Lemma 2.6. We demonstrate its continuity. For $\varepsilon > 0$, put $B_\varepsilon = \{z \in \boldsymbol{C}; |z| \leq \varepsilon\}$, and for $n = 1, 2, \ldots$, define

$$g_n(\alpha) = n\left\{f\left(x\alpha\left(\frac{1}{n}\right)\right) - f(x)\right\} \quad (\alpha \in R(G)).$$

Then, for each $\alpha\in R(G)$, $\lim_{n\to\infty} g_n(\alpha)=d_\alpha^{(r)}f(x)$ and so, the set $\{g_n(\alpha)\}_n$ is bounded. Therefore, for $\alpha\in R(G)$, there exists a positive integer k_α such that $\alpha\in\bigcap_{n=1}^{\infty} g_n^{-1}(B_{k_\alpha\varepsilon})$. Hence $\bigcup_{k=1}^{\infty}(\bigcap_{n=1}^{\infty} g_n^{-1}(B_{k\varepsilon}))=R(G)$. On the other hand, by Lemma 2.7, each g_n is continuous on $R(G)$ and so, each $\bigcap_{n=1}^{\infty} g_n^{-1}(B_{k\varepsilon})$ is closed in $R(G)$. Since $R(G)$ is a Baire space, it follows that, for some k_0, $\bigcap_{n=1}^{\infty} g_n^{-1}(B_{k_0\varepsilon})$ contains a non-void open subset O of $R(G)$. Hence it follows that $d_\alpha^{(r)}f(x)\in B_{k_0\varepsilon}$ for all $\alpha\in O$. Now take any $\alpha_0\in O$ and put $O_1=\frac{1}{2k_0}(O-\alpha_0)$. Then O_1 is a neighbourhood of 0 in $R(G)$ and, by $\boldsymbol{R}$-linearity of the map $\alpha\mapsto d_\alpha^{(r)}f(x)$, we have $d_\alpha^{(r)}f(x)\in B_\varepsilon$ for all $\alpha\in O_1$. This proves the continuity in question. q.e.d.

Lemma 2.9. *Let* $f\in\mathscr{E}_1^{(r)}(G)$ (resp. $\mathscr{E}_1^{(l)}(G)$). *Then the map*

$$(\alpha, x)\longmapsto d_\alpha^{(r)}f(x) \quad (\text{resp. } d_\alpha^{(l)}f(x))$$

of $R(G)\times G$ *into* $\boldsymbol{C}$ *is continuous.*

Proof. As $\mathscr{E}_1^{(r)}(G)$ and $\mathscr{E}_1^{(l)}(G)$ are stable under the complex conjugation, we can assume without loss of generality that f is real valued. For $x\in G$, let φ_x denote the map $\alpha\mapsto d_\alpha^{(r)}f(x)$ of $R(G)$ into $\boldsymbol{R}$. By Lemma 2.8 each φ_x belongs to $R(G)'$, the dual space of the locally convex space $R(G)$. The set $\{\varphi_x;\, x\in G\}$ is plainiy bounded in $R(G)'$ relative to the weak topology $\sigma(R(G)', R(G))$, and hence equl-continuous because $R(G)$ is barrelled ([1], Chap. 3, §3, Theorem 2). This together with the inequality $|d_\alpha^{(r)}f(x)-d_{\alpha_0}^{(r)}f(x_0)|\le|d_\alpha^{(r)}f(x)-d_{\alpha_0}^{(r)}f(x)|+|d_{\alpha_0}^{(r)}f(x)-d_{\alpha_0}^{(r)}f(x_0)|$ shows that the map $(\alpha, x)\mapsto d_\alpha^{(r)}f(x)$ is continuous. The continuity of the other map can be proved similarly. q.e.d.

Lemma 2.10. *The map* $(\alpha, x)\mapsto x^{-1}\alpha x$ *of* $R(G)\times G$ *into* $R(G)$ *is continuous.*

Proof. Put $h_G(\alpha)=H_\alpha=(H_\alpha(\sigma))_{\sigma\in\hat{G}}$. Then $h_G(x^{-1}\alpha x)=U_{x^{-1}}H_\alpha U_x=(U_{x^{-1}}(\sigma)H_\alpha(\sigma)U_x(\sigma))_{\sigma\in\hat{G}}$. Hence the assertion. q.e.d.

Lemma 2.11. *A* $\boldsymbol{C}$*-valued function* f *on* G *belongs to* $\mathscr{E}_1^{(r)}(G)$ *if and only if it belongs to* $\mathscr{E}_1^{(l)}(G)$. *And in this case,*

$$d_\alpha^{(r)}f(x)=-d_{x\alpha x^{-1}}^{(l)}f(x) \quad (\alpha\in R(G),\ x\in G).$$

Proof. Since $f(x\alpha(t))=f(x\alpha(t)x^{-1}x)$ $(x\in G,\ \alpha\in R(G),\ t\in\boldsymbol{R})$, this lemma follows immediately from Lemmas 2.9 and 2.10. q.e.d.

Lemma 2.12. *A function* f *belongs to* $\mathscr{E}_2^{(r)}(G)$ *if and only if it belongs to* $\mathscr{E}_2^{(l)}(G)$. *In this case,*

$$d_\beta^{(r)}d_\alpha^{(l)}f=d_\alpha^{(l)}d_\beta^{(r)}f \quad (\alpha,\ \beta\in R(G)).$$

Proof. Assume that $f\in\mathscr{E}_2^{(r)}(G)$. Then, by Lemma 2.11, f and $d_\beta^{(r)}f$ $(\beta\in R(G))$ belongs to $\mathscr{E}_1^{(l)}(G)$. Now take any $x\in G$ and α, $\beta\in R(G)$. Define, for $t\in\boldsymbol{R}$,

$g(t)=d_\alpha^{(l)}f(x\beta(t))$, and

$$g_n(t)=n\left\{f\left(\alpha\left(-\frac{1}{n}\right)x\beta(t)\right)-f(x\beta(t))\right\} \quad (n=1, 2,\dots).$$

Then, as n tends to infinity, $g_n(t)$ converges to $g(t)$ for each $t\in \boldsymbol{R}$. On the other hand, since $d_\beta^{(r)}f\in \mathscr{E}_1^{(l)}(G)$, we have by the mean value theorem

$$\frac{d}{dt}g_n(t)=n\left\{d_\beta^{(r)}f\left(\alpha\left(-\frac{1}{n}\right)x\beta(t)\right)-d_\beta^{(r)}f(x\beta(t))\right\}$$

$$=d_\alpha^{(l)}(d_\beta^{(r)}f)\ (\alpha(s_n)x\beta(t)) \quad \left(-\frac{1}{n}<s_n<0\right).$$

As n tends to infinity, this converges to $d_\alpha^{(l)}d_\beta^{(r)}f(x\beta(t))$ uniformly in t, because $d_\alpha^{(l)}d_\beta^{(r)}f$ is uniformly continuous on G. Hence the limit function $g(t)$ is differentiable and

$$\frac{d}{dt}g(t)=d_\alpha^{(l)}d_\beta^{(r)}(x\beta(t)).$$

Therefore, in particular, $d_\beta^{(r)}(d_\alpha^{(l)}f)(x)$ exists and is equal to $d_\alpha^{(l)}d_\beta^{(r)}f(x)$. Since x and β are arbitrary and $d_\alpha^{(l)}d_\beta^{(r)}f$ is continuous, this shows that $d_\alpha^{(l)}f\in\mathscr{E}_1^{(r)}(G)$. Since α is arbitrary and $\mathscr{E}_1^{(r)}(G)=\mathscr{E}_1^{(l)}(G)$ by Lemma 2.11, we have $f\in\mathscr{E}_2^{(l)}(G)$. In the same way we can prove the reverse assertion that a function in $\mathscr{E}_2^{(l)}(G)$ belongs to $\mathscr{E}_2^{(r)}(G)$. q.e.d.

We are now in the position to complete

Proof of Theorem 2.1. It suffices to prove the assertion for $n=1, 2, 3,\dots$. But the cases $n=1, 2$ have already been verified in Lemmas 2.11 and 2.12. So we make the proof for any n by induction. It suffices only to show that $\mathscr{E}_n^{(r)}(G)\subseteq\mathscr{E}_n^{(l)}(G)$ $(n\geq 3)$, because the reverse inclusion can be shown similarly. Take any $n\geq 3$, and assume that $\mathscr{E}_{n-1}^{(r)}(G)\subseteq\mathscr{E}_{n-1}^{(l)}(G)$. Under this assumption we show that $\mathscr{E}_n^{(r)}(G)\subseteq\mathscr{E}_n^{(l)}(G)$. Suppose $f\in\mathscr{E}_n^{(r)}(G)$. Then, by repeated use of Lemma 2.12, we have for any $p\leq n-1$ and any $\alpha, \beta_1,\dots, \beta_p\in R(G)$,

$$d_{\beta_1}^{(r)}\cdots d_{\beta_p}^{(r)}\ (d_\alpha^{(l)}f)=d_\alpha^{(l)}(d_{\beta_1}^{(r)}\cdots d_{\beta_p}^{(r)}f).$$

Since $d_{\beta_1}^{(r)}\cdots d_{\beta_p}^{(r)}f\in\mathscr{E}_1^{(r)}(G)=\mathscr{E}_1^{(l)}(G)$, the right hand side of this equality is continuous on G. Hence $d_\alpha^{(l)}f\in\mathscr{E}_{n-1}^{(r)}(G)$ and so, by our assumption of induction, $d_\alpha^{(l)}f\in\mathscr{E}_{n-1}^{(l)}(G)$. Since α is arbitrary, this shows that $f\in\mathscr{E}_n^{(l)}(G)$. Hence $\mathscr{E}_n^{(r)}(G)\subseteq\mathscr{E}_n^{(l)}(G)$. q.e.d.

By Theorem 2.1, $\mathscr{E}_n(G)=\mathscr{E}_n^{(r)}(G)=\mathscr{E}_n^{(l)}(G)$ $(n=\infty, 1, 2,\dots)$. Thus we see that it is perfectly natural to call the members of $\mathscr{E}_n(G)$ the n-times (infinitely if $n=\infty$) continuously differentiable functions on G. It should be noticed that Theorem 2.1 permits us to make mixed derivatives $d_{\alpha_1}^{(r)}\cdots d_{\alpha_p}^{(r)}d_{\beta_1}^{(l)}\cdots d_{\beta_q}^{(l)}f$, for $f\in\mathscr{E}_n(G)$ $(n=\infty, 1, 2,\dots)$ and $0\leq p+q<n+1$. Here the 'operators' $d_\alpha^{(r)}$ and $d_\beta^{(l)}$ $(\alpha, \beta\in R(G))$ commute with each other on $\mathscr{E}_2(G)$, as was shown in Lemma 2.12.

§3. Structure of the spaces $\mathscr{E}_n(G)$ $(n=\infty, 1,2,\ldots)$

3.1. Structure of $\mathscr{E}_n(G)$'s as sets.

Definition 3.1. $\boldsymbol{H}(G)$ denotes the totality of closed normal subgroups N of G such that G/N is finite dimensional (cf. Lemma 1.14), and $\boldsymbol{H}_1(G)$ the subset of $\boldsymbol{H}(G)$ consisting of all $N\in\boldsymbol{H}(G)$ such that G/N is separable. (Needless to say, $\boldsymbol{H}_0(G)\subseteq\boldsymbol{H}_1(G)$.)

Note that if G is finite dimensional, $\boldsymbol{H}(G)$ consists of all closed normal subgroups of G, containing in particular the subgroup $\{e\}$.

Lemma 3.1. *Let N be a closed normal subgroup of G. The following three are equivalent*: (a) $N\in\boldsymbol{H}(G)$; (b) $c(N)\in\boldsymbol{H}(G)$; (c) $c(N_0)\subseteq N$ *for some* $N_0\in\boldsymbol{H}_0(G)$.

Proof. Evidently $c(N)$ is, as well as N, a closed normal subgroup of G, and $G/N\cong(G/c(N))/(N/c(N))$ (as topological groups). Here $N/c(N)$ is totally disconnected. Hence, by Lemma 1.8, (iii) and Corollary to Lemma 1.10, $\Lambda(G/N)\cong\Lambda(G/c(N))$. This proves that (a) and (b) are equivalent. Next, if we assume (c), then $c(N_0)\in\boldsymbol{H}(G)$ and $G/N\cong(G/c(N_0))/(N/c(N_0))$. Hence $\Lambda(G/N)$ is finite dimensional. That is, (a) holds. Conversely, assume (a). Then, by Lemma 1.14, there exists a closed normal subgroup N_0 of G containing N such that N_0/N is totally disconnected and $(G/N)/(N_0/N)$ $(\cong G/N_0)$ is Lie or finite. Hence $c(N_0)\subseteq N$ and $N_0\in\boldsymbol{H}_0(G)$. That is, (c) holds. q.e.d.

A topological group is said to be locally connected if the connected open sets containing its unity e form a neighbourhood base at e. A finite dimensional compact Hausdorff group is locally connected if and only if it is a Lie or finite group ([5], §47). The direct product of a family of compact Hausdorff groups G_λ is locally connected if and only if each G_λ is locally connected and all except at most a finite number of G_λ's are connected.

Lemma 3.2. *The following three are equivalent*: (a) $\boldsymbol{H}(G)=\boldsymbol{H}_0(G)$; (b) $c(N)\in\boldsymbol{H}_0(G)$ *for every* $N\in\boldsymbol{H}_0(G)$; (c) G *is locally connected.*

Proof. Since $c(N)\in\boldsymbol{H}(G)$ for $N\in\boldsymbol{H}(G)$ (Lemma 3.1), (a) implies (b). Next assume (b), and take any neighbourhood U of e. Then we can choose a neighbourhood V of e and a connected $N\in\boldsymbol{H}_0(G)$ so that $VN\subseteq U$. Since G/N is Lie or finite, there exists a connected open neighbourhood W' of $\pi_N(e)$ contained in $\pi_N(V)$. Then $W=\pi_N^{-1}(W')$ is a connected open neighbourhood of e contained in U. Indeed, we have only to check its connectedness. Suppose contrary. Then there exists open sets O_1 and O_2 in G such that $(W\cap O_1)\cup(W\cap O_2)=W$ and $(W\cap O_1)\cap(W\cap O_2)=\emptyset$. Since N is connected, each coset xN of N contained in W meets only one of O_1 and O_2. Hence, for $O_i'=\pi_N(O_i)$ $(i=1, 2)$, $(W'\cap O_1')\cup(W'\cap O_2')=W'$ and $(W'\cap O_1')\cap(W'\subset O_2')=\emptyset$, which contradicts the connectedness of W'. Thus we have shown that (b) implies (c). Finally assume (c). Then, for $N\in\boldsymbol{H}(G)$, G/N is finite-dimensional and locally connected, i.e., a Lie or finite group. Hence (a). q.e.d.

Lemma 3.3. *Suppose a subset $\mathscr{B}$ of $\mathscr{E}_1(G)$ satisfies the condition*

$$\sup_{x\in G\, f\in\mathscr{B}} |d_\alpha^{(r)} f(x)| < \infty$$

for each $\alpha \in R(G)$. Then there exists $N \in \boldsymbol{H}(G)$ such that $\mathscr{B} \subseteq \mathscr{E}_1(G, N)$. The same holds if the left derivative is employed instead of the right one.

Proof. We can assume with no loss of generality that $\mathscr{B}$ consists only of real valued functions. Then, for each $f \in \mathscr{B}$ and $x \in G$, the map φ_{fx}: $\alpha \mapsto d_\alpha^{(r)} f(x)$ of $R(G)$ into $\boldsymbol{R}$ belongs to the dual space $R(G)'$ (Lemma 2.8). By assumption, the set $\{\varphi_{fx}; f \in \mathscr{B}, x \in G\}$ is weakly bounded, and so, equicontinuous because of the barrelledness of $R(G)$. Therefore there exists a neighbourhood $\mathscr{V}$ of 0 in $\Lambda(G)$ such that

$$(3.1) \qquad |d_\alpha^{(r)} f(x)| < 1 \quad \text{for} \quad f \in \mathscr{B}, x \in G \quad \text{and} \quad \alpha \in h_G^{-1}(\mathscr{V}).$$

Take $N \in \boldsymbol{H}_0(G)$ so that $\Lambda_N(G) \subseteq \mathscr{V}$ (Lemma 1.11, (iii)). Since $\Lambda_N(G)$ is linear and $h_G^{-1}(\Lambda_N(G)) = R(N)$, (3.1) demands that

$$d_\alpha^{(r)} f(x) = 0 \quad \text{for} \quad f \in \mathscr{B}, x \in G \quad \text{and} \quad \alpha \in R(N).$$

Hence, for each $f \in \mathscr{B}$, $x \in G$ and $\alpha \in R(N)$, the function $f(x\alpha(t))$ of real variable t is constant. Since the union of orbits of all $\alpha \in R(N)$ is dense in $c(N)$ (Lemma 1.15), it follows that $f(xy) = f(x)$ for $f \in \mathscr{B}$, $x \in G$ and $y \in c(N)$, i.e., that $\mathscr{B} \subseteq \mathscr{E}_1(G, c(N))$. Since $c(N) \in \boldsymbol{H}(G)$ (Lemma 3.1), this completes the proof. q.e.d.

Theorem 3.1. *Each function f in $\mathscr{E}_1(G)$ belongs to $\mathscr{E}_1(G, N)$ for some $N \in \boldsymbol{H}_1(G)$.*

Proof. Applying Lemma 3.3 to the singleton $\{f\}$ of $\mathscr{E}_1(G)$, we see that $f \in \mathscr{E}_1(G, N_1)$ for some $N_1 \in \boldsymbol{H}(G)$. Let $\hat{f}$ be the Fourier transform of f, i.e.,

$$\hat{f}(\sigma) = \int_G f(x) U_x(\sigma)^* d_G x \quad (\sigma \in \hat{G}).$$

Put $\Delta = \{\sigma \in \hat{G}; \hat{f}(\sigma) \neq 0\}$ and $N_2 = A(G, \Delta)$. Then, for all $y \in N_2$ and $\sigma \in \hat{G}$, $\hat{f_y}(\sigma) = U_y(\sigma)\hat{f}(\sigma) = \hat{f}(\sigma)$. So $f_y = f$ ($y \in N_2$), because both sides have the same Fourier coefficients. Thus $f \in \mathscr{E}_1(G, N_2)$. Besides, since Δ is countable, so is $[\Delta]$ $(=(G/N_2)^\wedge)$. That is, G/N_2 is separable. Now put $N = N_1 N_2$. Then, obviously, $f \in \mathscr{E}_1(G, N)$. And, since $G/N \cong (G/N_i)/(N/N_i)$ $(i = 1, 2)$, G/N is finite-dimensional and separable, i.e., $N \in \boldsymbol{H}_1(G)$. q.e.d.

Corollary. *For each $n = \infty, 1, 2, \ldots$, the space $\mathscr{E}_n(G)$ coincides with the union of all $\mathscr{E}_n(G, N)$'s with $N \in \boldsymbol{H}_1(G)$.*

By Theorem 3.1 the study of continuously differentiable functions on compact groups is reduced to that of those on separable finite-dimensional compact groups. In Paragraph 3.3 we shall show that $\mathscr{E}_\infty(G)$ coincides with $\mathscr{D}(G)$ only when G is locally connected. Therefore, in the above Corollary, we can not replace $\boldsymbol{H}_1(G)$ by $\boldsymbol{H}_0(G)$ unless G is locally connected. A continuous function on a connected compact Lie group belongs to the C^∞-class if and only if its Fourier transform decreases rapidly ([7]). Hence, if the compact group G is connected, each $\mathscr{E}_\infty(G, N)$ with $N\in \boldsymbol{H}_0(G)$ consists of all functions on G expressed as

$$f(x)=\sum_{\sigma\in A(\hat{G},N)} d_\sigma^{-1}\operatorname{Tr}(U_x(\sigma)C(\sigma)) \quad (x\in G)$$

with a rapidly decreasing $(C(\sigma))_{\sigma\in A(\hat{G},N)}\in\Sigma(G/N)$. But, for a non locally connected G, even if it is connected, the present author does not know yet how the functions in $\mathscr{E}_\infty(G)\smallsetminus\mathscr{D}(G)$ might be characterized.

Definition 3.2. For $f\in\mathscr{E}_1(G)$, N_f denotes the largest of all closed normal subgroups N of G such that $f\in\mathscr{E}_1(G, N)$. (The largest member really exists. It can be defined as the closed subgroup of G generated by the union of all such N's.)

The following are easy to check through Theorem 3.1 and the isomorphism theorem used in its proof: (i) for any $f\in\mathscr{E}_1(G)$, $N_f\in\boldsymbol{H}_1(G)$; (ii) if $f\in\mathscr{D}(G)$, then $N_f\in\boldsymbol{H}_0(G)$. Conversely, for a supplement to Theorem 3.1, we have the following

Proposition 3.1. *For any $N\in\boldsymbol{H}_1(G)$, there exists a function $f\in\mathscr{E}_\infty(G)$ such that $N_f=N$.*

Proof. As Example 2 in Paragraph 2.2 shows, each $\mathscr{E}_\infty(G/N)$ with $N\in\boldsymbol{H}_1(G)$ contains a function g such that $N_g=\{\pi_N(e)\}$. Put $f=g\circ\pi_N$. Then $f\in\mathscr{E}_\infty(G)$ and $N_f=N$. q.e.d.

Remark. Each of $\boldsymbol{H}(G)$ and $\boldsymbol{H}_1(G)$ is closed under finite intersections and so, lower directed under inclusion. Indeed, since $\boldsymbol{H}_0(G)$ has this property, we see by Lemma 3.1 that so does $\boldsymbol{H}(G)$. As for $\boldsymbol{H}_1(G)$, we reason as follows. Take any $N_1, N_2\in\boldsymbol{H}_1(G)$. Put $N=N_1\cap N_2$, $\Omega_j=A(\hat{G}, N_j)$ $(j=1, 2)$ and $\Delta=\Omega_1\cup\Omega_2$. Then $N\subseteq A(G, \Delta)\subseteq N$, and hence, $A(\hat{G}, N)=[\Delta]$. Since $[\Delta]$ is countable as well as Ω_1 and Ω_2, this shows that G/N is separable.

3.2. Inductive limit topology for the spaces $\mathscr{E}_n(G)$.

Definition 3.3. For each pair (p, q) of non-negative integers and for any $\alpha_1,\dots,\alpha_p$, $\beta_1,\dots,\beta_q\in R(G)$, define the seminorm $P^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}$ on $\mathscr{E}_{p+q}(G)$ as

$$P^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}(f)=\sup_{x\in G}|d^{(r)}_{\alpha_1}\cdots d^{(r)}_{\alpha_p}d^{(l)}_{\beta_1}\cdots d^{(l)}_{\beta_q}f(x)| \quad (f\in\mathscr{E}_{p+q}(G)).$$

In case $p=0$ or $q=0$, this coincides with one of $P_{\alpha_1\cdots\alpha_p}$, $P^{\beta_1\cdots\beta_q}$ and P_0 defined in Definition 2.6. For $n=\infty$, 1, 2,..., let $\mathfrak{F}^{(r)}_n$ and $\mathfrak{F}^{(l)}_n$ be the same as in Definition 2.6, and put

$$\mathfrak{F}_n=\{P^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p};\ 0\leq p+q<n+1,\ \alpha_1,\dots,\alpha_p,\ \beta_1,\dots,\beta_q\in R(G)\}.$$

The topologies for each $\mathscr{E}_n(G)$ defined by $\mathfrak{F}^{(r)}_n$, $\mathfrak{F}^{(l)}_n$ and $\mathfrak{F}_n$ are denoted by $\boldsymbol{\tau}_r$, $\boldsymbol{\tau}_l$ and $\boldsymbol{\tau}_+$ respectively, without regard to n. The space $\mathscr{E}_n(G)$ equipped with $\boldsymbol{\tau}_r$, $\boldsymbol{\tau}_l$

and τ_+ is denoted by $\mathscr{E}_n^{(r)}(G)$, $\mathscr{E}_n^{(l)}(G)$ and $\mathscr{E}_n^{(+)}(G)$ respectively. (This definition of the notations $\mathscr{E}_n^{(r)}(G)$ and $\mathscr{E}_n^{(l)}(G)$ agrees nicely with Definition 2.6.)

Plainly, τ_+ is finer than each of τ_r and τ_l. It is easy to see that all spaces $\mathscr{E}_n^{(r)}(G)$, $\mathscr{E}_n^{(l)}(G)$ and $E_n^{(+)}(G)$ are complete. For $f\in\mathscr{E}_n(G)$ ($n=\infty$, 1, 2,...) and $\alpha_1,\dots,\alpha_k\in R(G)$ $(0\le k<n+1)$, we have $P^{\alpha_1\cdots\alpha_k}(\check{f})=P_{\alpha_1\cdots\alpha_k}(f)$ by Lemma 2.1, (ii). Hence the map $f\mapsto\check{f}$ is a topological linear isomorphism of $\mathscr{E}_n^{(r)}(G)$ onto $\mathscr{E}_n^{(l)}(G)$.

Lemma 3.4. *Suppose $R(G)$ has a countable linear base. Then, for each $\mathscr{E}_n(G)$ ($n=\infty$, 1, 2,...), the topologies τ_r, τ_l and τ_+ coincide with one another, and make it a Fréchet space.*

Proof. Let $\mathfrak{B}$ be a countable linear base of $R(G)$, and $\mathfrak{F}_n'$ the subset of $\mathfrak{F}_n$ consisting of all $P_{\alpha_1\cdots\alpha_p}^{\beta_1\cdots\beta_q}$'s such that $0\le p+q<n+1$ and $\alpha_1,\dots,\alpha_p$, $\beta_1,\dots,\beta_q\in\mathfrak{B}$. Then $\mathfrak{F}_n'$ is countable. Moreover, since the maps $\alpha\mapsto d_\alpha^{(r)}f$ and $\alpha\mapsto d_\alpha^{(l)}f$ of $R(G)$ into $\mathscr{E}_0(G)$ are R-linear for $f\in\mathscr{E}_1(G)$ (Lemma 2.6), $\mathfrak{F}_n'$ defines obviously the topology τ_+ for $\mathscr{E}_n(G)$ ($n=\infty$, 1, 2,...). Hence each $\mathscr{E}_n^{(+)}(G)$ is a Fréchet space. Similarly, we see that either of $\mathscr{E}_n^{(r)}(G)$ and $\mathscr{E}_n^{(l)}(G)$ is a Fréchet space. Therefore, the continuous identity maps $\mathscr{E}_n^{(+)}(G)\ni f\mapsto f\in\mathscr{E}_n^{(r)}(G)$ and $\mathscr{E}_n^{(+)}(G)\ni f\mapsto f\in\mathscr{E}_n^{(l)}(G)$ are both open by the open mapping theorem. Hence the lemma. q.e.d.

Lemma 3.5. *Let N be a closed normal subgroup of G. For each $n=\infty$, 1, 2,..., the algebra isomorphsim (Corollary to Lemma 2.2)*

$$\mathscr{E}_n(G/N)\ni g\longmapsto g\circ\pi_N\in\mathscr{E}_n(G,N)$$

is also a homeomorphism relative to each of τ_r, τ_l and τ_+.

Proof. From Lemma 2.2 we have, for $g\in\mathscr{E}_n(G)$ and $\alpha_1,\dots,\alpha_p$, $\beta_1,\dots,\beta_q\in R(G)$ $(0\le p+q<n+1)$,

$$P_{\alpha_1\cdots\alpha_p}^{\beta_1\cdots\beta_q}(g\circ\pi_N)=P_{\bar{\alpha}_1\cdots\bar{\alpha}_p}^{\bar{\beta}_1\cdots\bar{\beta}_q}(g),$$

where $\bar{\alpha}_i=\bar{\pi}_N(\alpha_i)$ and $\bar{\beta}_j=\bar{\pi}_N(\beta_j)$ $(i=1,\dots,p,\ j=1,\dots,q)$. Since the map $\bar{\pi}_N$ carries $R(G)$ onto $R(G/N)$, this proves the assertion. q.e.d.

Lemma 3.6. *The topologies τ_r, τ_l and τ_+ induce the same topology on each $\mathscr{E}_n(G,N)$, for $N\in\boldsymbol{H}(G)$ and $n=\infty$, 1, 2,..., and make it a Fréchet space.*

Proof. Clear from Lemmas 3.4 and 3.5. q.e.d.

Now we define an inductive limit topology for the spaces $\mathscr{E}_n(G)$, which is rather more proper than τ_+.

Definition 3.4. For any closed normal subgroup N of G, $\mathscr{E}_n^{(+)}(G,N)$ denotes the space $\mathscr{E}_n(G,N)$ equipped with the relativized τ_+. The family $\{\mathscr{E}_n^{(+)}(G,N);\ N\in\boldsymbol{H}(G)\}$ is upper directed since $\boldsymbol{H}(G)$ is lower directed, and its union is $\mathscr{E}_n(G)$ (Theorem 3.1). Hence each $\mathscr{E}_n(G)$ ($n=\infty$, 1, 2,...) can be topologized so as to be the inductive limit of this family. This topology is denoted by τ_* without regard to n, and the space $\mathscr{E}_n(G)$ equipped with it by $\mathscr{E}_n^{(*)}(G)$. For any closed normal subgroup N of G, $\mathscr{E}_n^{(*)}(G, N)$ denotes the space $\mathscr{E}_n(G, N)$ equipped with the relativized $\boldsymbol{\tau}_*$.

$\boldsymbol{\tau}_*$ is, by definition, the finest locally convex topology for $\mathscr{E}_n(G)$ which coincides with $\boldsymbol{\tau}_+$ on each $\mathscr{E}_n(G, N)$ with $N \in \boldsymbol{H}(G)$. As an inductive limit of Fréchet spaces, each $\mathscr{E}_n^{(*)}(G)$ is barrelled and bornological. For any closed normal subgroup N of G, $\mathscr{E}_n(G, N)$ is closed in $\mathscr{E}_n^{(+)}(G)$, and so, in $\mathscr{E}_n^{(*)}(G)$. If G is finite dimensional, $\boldsymbol{\tau}_*$ coincides with $\boldsymbol{\tau}_+$.

Remark. Each $\mathscr{E}_n(G)$ $(n=\infty, 1, 2,\dots)$ can also be topologized so as to be the inductive limit of $\{\mathscr{E}_n^{(+)}(G, N);\ N \in \boldsymbol{H}_1(G)\}$. Denote this topology by $\boldsymbol{\tau}_0$. Let us show that $\boldsymbol{\tau}_0$ coincides with $\boldsymbol{\tau}_*$ for each n. By definition, $\boldsymbol{\tau}_0$ is obviously finer than $\boldsymbol{\tau}_*$. To prove the converse, it suffices to show that the identity map of each $\mathscr{E}_n^{(+)}(G, H)$ with $N \in \boldsymbol{H}(G)$ into $\mathscr{E}_n^{(0)}(G)$, the space $\mathscr{E}_n(G)$ equipped with $\boldsymbol{\tau}_0$, is continuous. Let f_j $(j=1, 2,\dots)$ be any sequence in $\mathscr{E}_n^{(+)}(G, N)$ converging to 0. Put $\Omega_j = A(\hat{G}, N_{f_j})$, $\Delta = \bigcup_{j=1}^{\infty} \Omega_j$ and $N_0 = \bigcap_{j=1}^{\infty} N_{f_j}$. Then $N_0 = A(G, \Delta)$, and so, $A(\hat{G}, N_0) = [\Delta]$. Here, since each Ω_j is countable, so is $[\Delta]$. Hence G/N_0 is separable. On the other hand, $N_0 \in \boldsymbol{H}(G)$ since $N_0 \supseteq N$. Thus $N_0 \in \boldsymbol{H}_1(G)$. Therefore $\boldsymbol{\tau}_0$ coincides with $\boldsymbol{\tau}_+$ on $\mathscr{E}_n(G, N_0)$, and so, $f_j \to 0$ in $\mathscr{E}_n^{(0)}(G)$ because $f_j \to 0$ in $\mathscr{E}_n^{(+)}(G, N_0)$. This shows that the map in question is continuous.

Theorem 3.2. *Let $\mathscr{B}$ be a subset of $\mathscr{E}_n(G)$ $(n=\infty, 1, 2,\dots)$. The following five statements are equivalent*: (a) *$\mathscr{B}$ is $\boldsymbol{\tau}_*$-bounded*; (b) *$\mathscr{B}$ is $\boldsymbol{\tau}_+$-bounded*; (c) *$\mathscr{B}$ is $\boldsymbol{\tau}_r$-bounded*; (d) *$\mathscr{B}$ is $\boldsymbol{\tau}_l$-bounded*; (e) *$\mathscr{B}$ is a bounded subset of $E_n^{(+)}(G, N)$ for some $N \in \boldsymbol{H}(G)$.*

Proof. The implication '(a)⇒(b)⇒(c) and (d)' is obvious. By Lemmas 3.3 and 3.6, each of (c) and (d) implies (e). Since $\boldsymbol{\tau}_*$ coincides with $\boldsymbol{\tau}_+$ on each $\mathscr{E}_n(G, N)$ with $N \in \boldsymbol{H}(G)$, (e) implies (a). q.e.d.

Proposition 3.3 in the next paragraph will show that, in the above theorem, $\boldsymbol{H}(G)$ in (e) can not in general be replaced by $\boldsymbol{H}_1(G)$.

Theorem 3.3. *Each space $\mathscr{E}_n^{(*)}(G)$ $(n=\infty, 1, 2,\dots)$ is complete.*

This theorem is verified by following the way of proof of Theorem 1 in [2]. Let us sketch the proof for our case. For a closed normal subgroup N of G, we define a $\boldsymbol{\tau}_*$-continuous projection ρ_N of any $\mathscr{E}_n(G)$ $(n=\infty, 1, 2,\dots)$ onto $\mathscr{E}_n(G, N)$ as

$$\rho_N(f)(x) = \int_N f(xy)d_N y \quad (f \in \mathscr{E}_n(G),\ x \in G),$$

where d_N denotes the Haar measure on N such that $\int_N d_N = 1$. We have $\rho_{N_1}\rho_{N_2} = \rho_{N_2}\rho_{N_1} = \rho_{N_2}$ if $N_1 \subseteq N_2$.

Lemma 3.7. *The set $\{\rho_N;\ N \in \mathfrak{N}(G)\}$ is equicontinuous on each $\mathscr{E}_n^{(*)}(G)$, where $\mathfrak{N}(G)$ denotes the totality of closed normal subgroups of G.*

Proof. For each $f \in \mathscr{E}_n(G)$, $\{\rho_N(f);\ N \in \mathfrak{N}(G)\}$ is bounded in $\mathscr{E}_n^{(+)}(G)$ and so, equivalently, in $\mathscr{E}_n^{(*)}(G)$. That is, $\{\rho_N;\ N \in \mathfrak{N}(G)\}$ is bounded relative to the simple topology for the space of continuous linear operators on $\mathscr{E}_n^{(*)}(G)$. Since $\mathscr{E}_n^{(*)}(G)$ is barrelled, the equicontinuity follows. q.e.d.

Lemma 3.8. *Let N be a closed normal subgroup of G. The algebraic linear isomorphism $g \mapsto g \circ \pi_N$ of $\mathscr{E}_n^{(*)}(G/N)$ onto $\mathscr{E}_n^{(*)}(G, N)$ is also topological.*

Proof. Denote this map by ψ and put $\boldsymbol{H} = \{N' \in \boldsymbol{H}(G);\ N \subseteq N'\}$. We see that $\boldsymbol{H}(G/H) = \{N'/N;\ N' \in \boldsymbol{H}\}$ and $\mathscr{E}_n(G, N) = \cup\, \mathscr{E}_n(G, N')$ $(N' \in \boldsymbol{H})$. By Lemma 3.5, each $\mathscr{E}_n^{(+)}(G/N, N'/N)$ $(N' \in \boldsymbol{H})$ is topologically isomorphic with $\mathscr{E}_n^{(+)}(G, N')$ under ψ. Hence, in particular, ψ is $\boldsymbol{\tau}_*$-continuous. On the other hand, ψ^{-1} can be extended to the map $\psi^{-1} \circ \rho_N$ of $\mathscr{E}_n(G)$ onto $\mathscr{E}_n(G/N)$. If $N_0 \in \boldsymbol{H}(G)$, then $NN_0 \in \boldsymbol{H}$, and $\psi^{-1} \circ \rho_N$ maps $\mathscr{E}_n^{(+)}(G, N_0)$ into $\mathscr{E}_n^{(+)}(G/N, NN_0/N)$ continuously. Hence $\psi^{-1} \circ \rho_N$ is $\boldsymbol{\tau}_*$-continuous, and so is ψ^{-1}. q.e.d.

Lemma 3.9. *Let $\boldsymbol{H}$ be a directed subfamily of $\boldsymbol{H}(G)$ such that the intersection of its members is $\{e\}$. Then for any $N \in \boldsymbol{H}_0(G)$, there exists $N_1 \in \boldsymbol{H}$ such that $N_1 \subseteq N$.*

Proof. The proof is the same as for Lemma 1 in [2]. q.e.d.

Proof of Theorem 3.3. Set $\mathscr{E} = \mathscr{E}_n^{(*)}(G)$. Let $\mathscr{E}'$ denote the dual space of $\mathscr{E}$, and $\mathscr{E}^{\#}$ the set of all linear forms of $\mathscr{E}'$ that are $\sigma(\mathscr{E}', \mathscr{E}')$-continuous on each equicontinuous subset of $\mathscr{E}'$. The proof depends on Grothendieck's completeness theorem.

(I) For $N \in \mathfrak{N}(G)$, put $\mathscr{E}_N = \rho_N(\mathscr{E})$, $\mathscr{K}_N = \ker \rho_N$, $\mathscr{E}'_N = \rho'_N(\mathscr{E}')$ and $\mathscr{K}'_N = \ker \rho'_N$, where ρ'_N denotes the adjoint of ρ_N, and $\mathscr{E}_N$ is endowed with the relativized $\boldsymbol{\tau}_*$. Then, (i) $\mathscr{E} = \mathscr{E}_N + \mathscr{K}_N$, $\mathscr{E}' = \mathscr{E}'_N + \mathscr{K}'_N$ (direct sums); (ii) $\mathscr{E}'_N$ can be canonically viewed as the dual space of $\mathscr{E}_N$; (iii) if $N_1 \subseteq N_2$, then $\mathscr{E}'_{N_1} \supseteq \mathscr{E}'_{N_2}$ and $\mathscr{K}'_{N_1} \subseteq \mathscr{K}'_{N_2}$.

(II) For $T \in \mathscr{E}'$, put $X_T = \{\rho'_N(T);\ N \in \boldsymbol{H}(G)\}$. Its $\sigma(\mathscr{E}', \mathscr{E})$-closure $\overline{X}_T$ is equicontinuous on $\mathscr{E}$ and contains T. Here we use Lemma 3.7 and the fact that the net $\rho'_N(T)$ $(N \in \boldsymbol{H}(G))$ on $\boldsymbol{H}(G)$ converges to T in $\sigma(\mathscr{E}', \mathscr{E})$.

(III) Put $X = \cup\{\mathscr{E}'_N;\ N \in \boldsymbol{H}(G)\}$. Then, for each $u \in \mathscr{E}^{\#}$, there exists an $N_u \in \boldsymbol{H}(G)$ such that $u(T) = 0$ for $T \in X \cap \mathscr{K}'_{N_u}$. Indeed, if otherwise, we can choose $N_j \in \boldsymbol{H}(G)$ and $T_j \in \mathscr{E}'$ $(j = 1, 2, \ldots)$ so that $N_j \supseteq N_{j+1}$, $T_j \in \mathscr{E}'_{N_{j+1}} \cap \mathscr{K}'_{N_j}$ and $u(T_j) = 1$. Here, by (iii) of (I) and Lemma 3.1, we can assume that each N_j is connected. Put $N_\infty = \bigcap_{j=1}^{\infty} N_j$. Then $N_j/N_\infty \in \boldsymbol{H}(G/N_\infty)$ and $\bigcap_{j=1}^{\infty} N_j/N_\infty = \{\pi_{N_\infty}(e)\}$. Since each N_j/N_∞ is connected, this shows by Lemmas 3.9 and 3.1 that $\{N_j/N_\infty;\ j = 1, 2, \ldots\}$ is cofinal in $\boldsymbol{H}(G/N_\infty)$. Therefore, by Lemma 3.8, $\mathscr{E}_{N_\infty}$ is an (LF)-space and so complete. Hence there exists an $f \in \mathscr{E}_{N_\infty}$ such that $u(T) = T(f)$ for $T \in \mathscr{E}'_{N_\infty}$. Since $\mathscr{E}_{N_\infty} = \bigcup_{j=1}^{\infty} \mathscr{E}_{N_j}$, and so, $f \in \mathscr{E}_{N_{j_0}}$ for some j_0, it follows that $u(T_{j_0}) = T_{j_0}(f) = 0$. This is a contradiction.

(IV) Suppose $T \in \mathscr{K}'_{N_u}$. Then we see easily that $X_T \subseteq X \cap \mathscr{K}'_{N_u}$. Hence $u = 0$ on X_T and so, by (II), $u(T) = 0$. On the other hand, since $\mathscr{E}_{N_u}$ is complete, there exists an $f_u \in \mathscr{E}_{N_u}$ such that $u(T) = T(f_u)$ for $T \in \mathscr{E}'_{N_u}$. Then $u(T) = T(f_u)$ for all

$T\in\mathscr{E}'$. This completes the proof. q.e.d.

3.3. Results from the structure theorem for finite dimensional compact groups.

Lemma 3.10. *Let K be a compact (i.e., closed) subset of G and O a neighbourhood of K. Then there exists a function $f\in\mathscr{D}(G)$ such that $f(x)\equiv 1$ on K and, $\equiv 0$ on $G\smallsetminus O$.*

Proof. See Proposition 2 in [2]. q.e.d.

Lemma 3.11. *If G is totally disconnected, the space $\mathscr{T}(G)$ of trigonometric polynomials on G coincides with $\mathscr{D}(G)$.*

Proof. For $N\in \boldsymbol{H}_0(G)$, the group G/N is finite, and so, the set $(G/N)^\wedge$ is also finite. Hence every $g\in\mathscr{E}_0(G/N)$ $(=\mathscr{E}_\infty(G/N))$ permits the Fourier expansion with finite terms. Hence the lemma. q.e.d.

Lemma 3.12. *Suppose G is finite dimensional. Then, for each totally disconnected $N\in\boldsymbol{H}_0(G)$, there exists a topological linear isomorphism $g\mapsto\tilde{g}$ of $\mathscr{E}_0(N)$ onto a closed linear subspace of $\mathscr{E}_\infty^{(+)}(G)$ having the following two properties:* (a) *$\tilde{g}=g$ on N for all $g\in\mathscr{E}_0(N)$, and* (b) *$g\in\mathscr{D}(N)\Leftrightarrow\tilde{g}\in\mathscr{D}(G)$.*

Proof. It suffices to show that there exists a continuous linear map $g\mapsto\tilde{g}$ of $\mathscr{E}_0(N)$ into $\mathscr{E}_\infty^{(+)}(G)$ fulfilling (a) and (b). By the structure theorem for finite dimensional compact groups ([5], §47), there exists a subset L of G fulfiilling the following four conditions: (i) $e\in L$; (ii) L is homeomorphic under π_N with an open neighbourhood of $\pi_N(e)$; (iii) $U=LN$ is an open neighbourhood of e; (iv) if we put $U'=\pi_N(L)\times N$ and $y'=\pi_N(y)$ for $y\in L$, then the map ψ: $U'\ni(y',z)\mapsto yz\in U$ is a local isomorphsim of $(G/N)\times N$ onto G. Now take open sets V and W in G so that $N\subseteq V$, $\overline{V}\subseteq W$ and $\overline{W}\subseteq U$, and $\theta\in\mathscr{D}(G)$ so that $\theta(x)\equiv 1$ on V and $\theta(x)\equiv 0$ on $G\smallsetminus W$ (Lemma 3.10). For $g\in\mathscr{E}_0(N)$, denote by g' the trivial extension of g to U', and regard it as a function on U thourght ψ. Then g' is continuous on U, and for any $\alpha\in R(G)$ and $x\in U$, $d_\alpha^{(r)}g'(x)$ exists and $=0$. Define

$$\tilde{g}(x)=\theta(x)g'(x)\quad(x\in G).$$

Then, obviously, $\tilde{g}\in\mathscr{E}_0(G)$ and $\tilde{g}=g$ on N. Moreover, for any $\alpha_1,\dots,\alpha_p\in R(G)$ $(p=1,2,\dots)$, we have

$$d_{\alpha_1}^{(r)}\cdots d_{\alpha_p}^{(r)}\tilde{g}(x)=(d_{\alpha_1}^{(r)}\cdots d_{\alpha_p}^{(r)}\theta)(x)g'(x)\qquad(x\in G).$$

Hence $\tilde{g}\in\mathscr{E}_\infty(G)$. Obviously from this equality and construction of $\tilde{g}$, the map $g\mapsto\tilde{g}$ is linear and continuous. Now assume that $g\in\mathscr{D}(N)$. Since $\mathscr{D}(N)=\mathscr{T}(N)$ (Lemma 3.11), g is expressed as $\sum_i c_i{}^{(\tau_i)}$ (finite sum), where $c_i\in\boldsymbol{C}$, $\tau_i\in\hat{N}$, and each $v^{(\tau_i)}$ is a matrix element of τ_i. Recall that each τ_i is an irreducible component of the restriction $\sigma_i|_N$ of some $\sigma_i\in\hat{G}$ ([3], (27.48)). Put $N'=\bigcap_i A(G,\{\sigma_i\})$ $(\in\boldsymbol{H}_0(G))$. Then $g\in\mathscr{E}_\infty(N, N'\cap N)$. Hence g' is constant on each coset of $N'\cap N$ contained in U. Therefore, if we take $N_0\in\boldsymbol{H}_0(G)$ so that $\theta\in\mathscr{E}_\infty(G,N_0)$, then $\tilde{g}\in\mathscr{E}_\infty(G, N_0\cap N'\cap N)$.

Since N_0, N', $N \in \boldsymbol{H}_0(G)$, this shows that $\tilde{g} \in \mathscr{D}(G)$. Conversely assume that $\tilde{g} \in \mathscr{D}(G)$, i.e., $\tilde{g} \in \mathscr{E}_\infty(G, N'')$ for some $N'' \in \boldsymbol{H}_0(G)$. Then $g = \tilde{g}|_N \in \mathscr{E}_\infty(N, N \cap N'')$ and $N/(N \cap N'') \cong NN''/N''$. Here $N/(N \cap N'')$ is totally disconnected and NN''/N'' is Lie or finite. Therefore these isomorphic groups are finite. Hence $N \cap N'' \in \boldsymbol{H}_0(N)$, and so, $g \in \mathscr{D}(N)$. q.e.d.

Proposition 3.2. *If G is not locally connected, then $\mathscr{E}_\infty(G) \neq \mathscr{D}(G)$.*

Proof. (I) First suppose G is totally disconnected. Since G is not finite by assumption, neither is $\hat{G}$. So we can choose an infinite sequence σ_i $(i=1, 2,\dots)$ in $\hat{G}$ and a convergent series $\sum_{i=1}^{\infty} c_i$ of positive numbers. Define $f(x) = \sum_{i=1}^{\infty} c_i d_i^{-1} \chi_i(x)$ $(x \in G)$, where χ_i denotes the character of σ_i. Then $f \in \mathscr{E}_0(G)$ $(=\mathscr{E}_\infty(G))$ but $f \notin \mathscr{T}(G)$ $(=\mathscr{D}(G)$ by Lemma 3.11).

(II) Next suppose G is finite dimensional. Then we can choose a totally disconnected $N \in \boldsymbol{H}_0(G)$ so that G is locally isomorphic with $(G/N) \times N$ ([5], §47). Since G is not locally connected, neither is N. Therefore, by (I), there exists a function $g \in \mathscr{E}_0(N) \smallsetminus \mathscr{D}(N)$. By Lemma 3.12, g can be extended to a function in $\mathscr{E}_\infty(G) \smallsetminus \mathscr{D}(G)$.

(III) Finally we treat the general case. By assumption and by Lemma 3.2, there exists $N \in \boldsymbol{H}(G) \smallsetminus \boldsymbol{H}_0(G)$. And, by (II), there exists a function $g \in \mathscr{E}_\infty(G/N) \smallsetminus \mathscr{D}(G/N)$. Put $f = g \circ \pi_N$ $(\in \mathscr{E}_\infty(G))$. Then $f \notin \mathscr{D}(G)$. Indeed, if we assume contrary, $N_f \in \boldsymbol{H}_0(G)$. Since $N_f \supseteq N$, $(G/N)/(N_f/N) \cong G/N_f$ and $N_f/N = N_g$, this implies that $N_g \in \boldsymbol{H}_0(G/N)$. But this is absurd, since $g \notin \mathscr{D}(G/N)$. q.e.d.

From Lemma 3.2, Theorem 3.1 and Proposition 3.2 we have the following

Corollary. *The following four conditions are equivalent*:
(a) *G is locally connected*; (b) $\boldsymbol{H}(G) = \boldsymbol{H}_0(G)$;
(c) $\boldsymbol{H}_1(G) = \boldsymbol{H}_0(G)$; (d) $\mathscr{E}_\infty(G) = \mathscr{D}(G)$.

Proposition 3.3. *If $\boldsymbol{H}(G) \neq \boldsymbol{H}_1(G)$, then there exists a bounded subset of $\mathscr{E}_\infty^{(+)}(G)$ included in $\mathscr{D}(G)$ but not in any $\mathscr{E}_\infty(G, N)$ with $N \in \boldsymbol{H}_1(G)$.*

Proof. (I) First suppose G is finite dimensional. The assumption $\boldsymbol{H}(G) \neq \boldsymbol{H}_1(G)$ means in this case that G is not separable. Choose a totally disconnected $N \in \boldsymbol{H}_0(G)$ so that G is locally isomorphic with $(G/N) \times N$. Then N is non-separable as well as G. Let $\mathscr{B}$ be the set of the coordinate functions of all continuous, irreducible, unitary representations of N, and $\tilde{\mathscr{B}}$ the image of $\mathscr{B}$ under the map in Lemma 3.12. Then, since $\mathscr{B}$ is bounded in $\mathscr{E}_0(N)$ and included in $\mathscr{D}(N)$, $\tilde{\mathscr{B}}$ is bounded in $\mathscr{E}_\infty^{(+)}(G)$ and included in $\mathscr{D}(G)$. We show that $\tilde{\mathscr{B}}$ is not included in any $\mathscr{E}_\infty(G, N)$ with $N \in \boldsymbol{H}_1(G)$. Suppose contrary, i.e., that $\mathscr{B} \subseteq \mathscr{E}_\infty(G, N_1)$ for some $N_1 \in \boldsymbol{H}_1(G)$. Then $\mathscr{B} \subseteq \mathscr{E}_\infty(N, N \cap N_1)$. This together with definition of $\mathscr{B}$ demands that $N \cap N_1 = \{e\}$. Hence $N = N/(N \cap N_1) \cong NN_1/N_1$. Therefore N is separable as well as G/N_1. But this is absurd.

(II) Next let G be arbitrary. By assumption there exists an $N \in \boldsymbol{H}(G) \smallsetminus \boldsymbol{H}_1(G)$. Obviously, $\boldsymbol{H}_0(G/N) = \{N'/N;\ N \subseteq N' \in \boldsymbol{H}_0(G)\}$ and $\boldsymbol{H}_1(G/N) = \{N'/N;\ N \subseteq N' \in$

$\boldsymbol{H}_1(G)\}$. By (I) there exists a bounded subset $\mathscr{B}$ of $\mathscr{E}_\infty^{(+)}(G/N)$ included in $\mathscr{D}(G/N)$ but not in any $\mathscr{E}_\infty(G/N, N'/N)$ with $N'/N \in \boldsymbol{H}_1(G/N)$. Put $\mathscr{B}'=\{g\circ\pi_N;\ g\in\mathscr{B}\}$. Then it is easy to see that $\mathscr{B}'$ is a bounded subset of $\mathscr{E}_\infty^{(+)}(G)$ included in $\mathscr{D}(G)$ but not in any $\mathscr{E}_\infty(G, N')$ with $N'\in\boldsymbol{H}_1(G)$. q.e.d.

Each $\mathscr{E}_\infty^{(+)}(G, N)$ with $N\in\boldsymbol{H}_0(G)$ is, evidently, a Montel space. While, in case G is not locally connected, we have the following

Proposition 3.4. *Each $\mathscr{E}_\infty^{(+)}(G, N)$ with N in $\boldsymbol{H}(G)\setminus\boldsymbol{H}_0(G)$ includes a closed, bounded and non-compact subset.*

Proof. (I) First suppose G is finite dimensional and not locally connected. Then there exists a totally disconnected and non locally connected $N\in\boldsymbol{H}_0(G)$. Set $\boldsymbol{H}_0(N)=\{K_\lambda;\ \lambda\in\Lambda\}$. Since each K_λ is open in N, its indicating function f_λ is continuous on N. The set $\{f_\lambda;\ \lambda\in\Lambda\}$ is bounded in $\mathscr{E}_0(N)$ but not equicontinuous at e, because $\cap\{K_\lambda;\ \lambda\in\Lambda\}=\{e\}$ and N is not discrete. Hence the closure $\mathscr{B}$ of $\{f_\lambda;\ \lambda\in\Lambda\}$ in $\mathscr{E}_0(N)$ is bounded in $\mathscr{E}_0(N)$ but not compact. Now let $\tilde{\mathscr{B}}$ be the image of $\mathscr{B}$ under the map in Lemma 3.12. Then $\tilde{\mathscr{B}}$ is closed and bounded in $\mathscr{E}_\infty^{(+)}(G)$ but not compact.

(II) Let G be arbitrary and $N\in\boldsymbol{H}(G)\setminus\boldsymbol{H}_0(G)$. By (I), $\mathscr{E}_\infty^{(+)}(G/N)$ includes a closed, bounded and non compact subset. Since $\mathscr{E}_\infty^{(+)}(G/N)$ is topologically isomorphic with $\mathscr{E}_\infty^{(+)}(G, N)$, this completes the proof. q.e.d.

Proposition 3.4 shows that if G is not locally connected, the space $\mathscr{E}_\infty^{(*)}(G)$ is not Montel, and so not nuclear because it is complete and barrelled. As for a locally connected G, $\mathscr{E}_\infty^{(*)}(G)$ is nuclear if and only if G is separable ([2], p. 53, Remark).

3.4. For a closed normal subgroup N of G, put

$$\mathscr{T}(G, N)=\mathscr{T}(G)\cap\mathscr{E}_\infty(G, N).$$

This is a subalgebra of $\mathscr{T}(G)$ isomorphic with $\mathscr{T}(G/N)$ under the map $\mathscr{T}(G/N)\ni g\mapsto g\circ\pi_N$ (cf. [3], (28.72), (*k*)).

Proposition 3.5. *For each $N\in\boldsymbol{H}(G)$ and $n=\infty, 1, 2,\dots$, $\mathscr{T}(G, N)$ is dense in $\mathscr{E}_n^{(+)}(G, N)$.*

Proof. In view of Lemmas 3.4 and 3.5, it suffices to show, for any G and n, that $\mathscr{T}(G)$ is dense in $\mathscr{E}_n^{(r)}(G)$. Take any $f\in\mathscr{E}_n(G)$, $\varepsilon>0$ and any finite number of operators $D_1,\dots, D_k$ each of which has the form $d_{\alpha_1}^{(r)}\cdots d_{\alpha_p}^{(r)}$ $(0\le p<n+1)$. By Lemma 2.5, we can choose a regular function θ on G so that

$$\sup_{\substack{x\in G\\ j=1\cdots,k}} |D_j(\theta*f)(x)-D_jf(x)|<\varepsilon. \tag{3.2}$$

Since θ is regular, the functions $G\times G\ni(y, x)\mapsto D_j(\theta_y)(x)$ $(j=1,\dots, k)$ are continuous. Hence

$$D_j(\theta * f)(x) = \int_G D_j(\theta_{y^{-1}})(x) f(y) d_G y. \tag{3.3}$$

Next put $M = \sup \{|D_j(\theta_y)(x)|;\ x,\ y \in G,\ j = 1, \ldots, k\}$. By the Peter-Weyl theorem, there exists a $g \in \mathscr{T}(G)$ such that $\sup_{y \in G} |g(y) - f(y)| < \varepsilon/M$. Then, by (3.3),

$$\begin{aligned} &\sup_{\substack{x \in G \\ j=1,\ldots,k}} |D_j(\theta * g)(x) - D_j(\theta * f)(x)| \\ &= \sup_{\substack{x \in G \\ j=1,\ldots,k}} \left| \int_G D_j(\theta_{y^{-1}})(x)(g(y) - f(y)) d_G y \right| \\ &< M \times \frac{\varepsilon}{M} = \varepsilon. \end{aligned} \tag{3.4}$$

From (3.2) and (3.4), we have

$$\sup_{\substack{x \in G \\ j=1,\cdots,k}} |D_j(\theta * g)(x) - D_j f(x)| < 2\varepsilon.$$

Here $\theta * g$ belongs to $\mathscr{T}(G)$ as well as g. This completes the proof.

q.e.d.

The above proposition implies *ipso fact* that $\mathscr{T}(G)$ is dense in $\mathscr{E}_n^{(*)}(G)$ ($n = \infty$, 1, 2,...). It generalizes the classical Weierstrass approximation theorem for the tori to any compact groups.

§4. Enveloping algebra

Most statements in this section are made in regard to the right differentiation. But the parallel statements for the left differentiation also hold.

4.1. Derivations associated to one-parameter subgroups. Let $\mathscr{E}$ be an algebra over $\boldsymbol{C}$. A linear map d on $\mathscr{E}$ is called a derivation if $d(fg) = (df)g + f(dg)$ holds for $f,\ g \in \mathscr{E}$. The totality of derivations on $\mathscr{E}$ forms a complex Lie algebra under the usual linear operations and commutator product.

Definition 4.1. $R(G)^c$ denotes the complexification of the Lie algebra $R(G)$. For each $\alpha + i\beta \in R(G)^c$ ($\alpha,\ \beta \in R(G)$, $i = \sqrt{-1}$), $d^{(r)}_{\alpha+i\beta}$ (resp. $d^{(l)}_{\alpha+i\beta}$) is defined to be the linear map

$$f \longmapsto d^{(r)}_\alpha f + i d^{(r)}_\beta f$$

$$(\text{resp.}\ f \longmapsto d^{(l)}_\alpha f + i d^{(l)}_\beta f)$$

of $\mathscr{E}_1(G)$ into $\mathscr{E}_0(G)$. Each $d^{(r)}_{\alpha+i\beta}$ (resp. $d^{(l)}_{\alpha+i\beta}$), restricted to $\mathscr{E}_\infty(G)$, is a derivation on the algebra $\mathscr{E}_\infty(G)$, which we call the right (resp. left) derivation associated to $\alpha + i\beta \in R(G)^c$.

All the derivations $d^{(r)}_{\alpha+i\beta}$ and $d^{(l)}_{\alpha+i\beta}$ are $\boldsymbol{\tau}_*$-continuous. Actually the operators

$d^{(r)}_{\alpha+i\beta}$ and $d^{(l)}_{\alpha+i\beta}$ belong to $L(\mathscr{E}_1^{(*)}(G), \mathscr{E}_0(G))$ and, if restricted to $\mathscr{E}_n(G)$ ($n=\infty, 2, 3,\dots$), belong to $L(\mathscr{E}_n^{(*)}(G), \mathscr{E}_{n-1}^{(*)}(G))$. Moreover, for $f \in \mathscr{E}_1(G)$ and $x_0 \in G$, we have

$$\text{(left invariancy)} \qquad d^{(r)}_{\alpha+i\beta}({}_{x_0}f) = {}_{x_0}(d^{(r)}_{\alpha+i\beta}f),$$

$$\text{(right invariancy)} \qquad d^{(l)}_{\alpha+i\beta}(f_{x_0}) = (d^{(l)}_{\alpha+i\beta}f)_{x_0}.$$

Our first task in this section is to prove

Theorem 4.1. *The map $\alpha+i\beta \mapsto d^{(r)}_{\alpha+i\beta}$, where $d^{(r)}_{\alpha+i\beta}$ is taken as restricted to $\mathscr{E}_\infty(G)$, is a Lie algebra isomorphism of $R(G)^c$ into the Lie algebra of derivations on $\mathscr{E}_\infty(G)$. The image of this map coincides with the totality of left invariant and τ_*-continuous derivations on $\mathscr{E}_\infty(G)$.*

Lemma 4.1. (i) *The map $\alpha+i\beta \mapsto d^{(r)}_{\alpha+i\beta}$ is linear from $R(G)^c$ into $L(\mathscr{E}_1^{(*)}(G), \mathscr{E}_0(G))$.*

(ii) *If $d^{(r)}_{\alpha+i\beta}f=0$ for every $f \in \mathscr{T}(G)$, then $\alpha+i\beta=0$ (i.e. $\alpha=\beta=0$).*

(iii) *For $\alpha+i\beta$, $\alpha'+i\beta' \in R(G)^c$,*

$$d^{(r)}_{[\alpha+i\beta, \alpha'+i\beta']} = [d^{(r)}_{\alpha+i\beta}, d^{(r)}_{\alpha'+i\beta'}]$$

holds on $\mathscr{E}_2(G)$, where the right hand side designates the usual commutator product of operators.

Proof. (i) is obvious from Lemma 2.6. (ii). Put $h_G(\alpha)=H_\alpha$ and $h_G(\beta)=H_\beta$. Then, by assumption, $U(\alpha)(H_\alpha(\sigma)+iH_\beta(\sigma))=0$ for every $\sigma \in \hat{G}$. Hence $H_\alpha(\sigma)+iH_\beta(\sigma)=0$ ($\sigma \in \hat{G}$). Since $H_\alpha(\sigma)$ and $H_\beta(\sigma)$ are skew-hermite, this shows that $H_\alpha(\sigma)=H_\beta(\sigma)=0$ ($\sigma \in \hat{G}$). Hence $H_\alpha=H_\beta=0$, i.e. $\alpha=\beta=0$. (iii). In view of (i), it suffices to show that $d^{(r)}_{[\gamma,\gamma']}=[d^{(r)}_\gamma, d^{(r)}_{\gamma'}]$ holds on $\mathscr{E}_2(G)$ for γ, $\gamma' \in R(G)$. Put $h_G(\gamma)=H_\gamma$ and $h_G(\gamma')=H_{\gamma'}$. For $\sigma \in \hat{G}$ we have

$$\begin{aligned} d^{(r)}_{[\gamma,\gamma']}U(\sigma) &= U(\sigma)[H_\gamma(\sigma), H_{\gamma'}(\sigma)] \\ &= U(\sigma)(H_\gamma(\sigma)H_{\gamma'}(\sigma)-H_{\gamma'}(\sigma)H_\gamma(\sigma)) = [d^{(r)}_\gamma, d^{(r)}_{\gamma'}]U(\sigma). \end{aligned}$$

Hence $d^{(r)}_{[\gamma,\gamma']}$ and $[d^{(r)}_\gamma, d^{(r)}_{\gamma'}]$ coincide on $\mathscr{T}(G)$. Since $d^{(r)}_{[\gamma,\gamma']}$ and $[d^{(r)}_\gamma, d^{(r)}_{\gamma'}]$, considered as mappings of $\mathscr{E}_2^{(*)}(G)$ into $\mathscr{E}_0(G)$, are continuous, and $\mathscr{T}(G)$ is dense in $\mathscr{E}_2^{(*)}(G)$, this shows that they coincide on $\mathscr{E}_2(G)$. q.e.d.

The first assertion of Theorem 4.1 follows immediately from Lemma 4.1. Therefore it remains only to prove that each left invariant and τ_*-continuous derivation on $\mathscr{E}_\infty(G)$ has the form $d^{(r)}_{\alpha+i\beta}$ for some α, $\beta \in R(G)$.

Lemma 4.2. *Let C^G be the complex vector space of all C-valued functions on G and d a linear map of $\mathscr{T}(G)$ into C^G. The following three are equivalent:*

(a) *$d=d^{(r)}_{\alpha+i\beta}|_{T(G)}$ for some α, $\beta \in R(G)$;*

(b) *for f, $g \in \mathscr{T}(G)$ and $x_0 \in G$,*

$$d(fg)=(df)g+f(dg) \quad \text{and} \quad d({}_{x_0}f)={}_{x_0}(df);$$

(c) *for f, $g \in \mathscr{T}(G)$ and $x_0 \in G$,*

$$d(fg)(e)=(df)(e)g(e)+f(e)(dg)(e) \quad \text{and}$$

$$d({}_{x_0}f)(e)={}_{x_0}(df)(e).$$

Proof. The implications (a)⇒(b)⇒(c) are obvious. We show that (c) implies (a). Evidently (c) is equivalent to the following: (c′) for σ, $\sigma' \in \hat{G}$ and $x_0 \in G$,

$$(4.1) \qquad d(U(\sigma)\otimes U(\sigma'))(e)$$

$$=(dU(\sigma))(e)\otimes U_e(\sigma')+U_e(\sigma)\otimes(dU(\sigma'))(e) \quad \text{and}$$

$$(4.2) \qquad d(U_{x_0^{-1}}(\sigma)U(\sigma))(e)=(dU(\sigma))(x_0^{-1}).$$

Here note that $U_e(\sigma)=I(\sigma)$ and $U_e(\sigma')=I(\sigma')$. Now suppose (c′) holds. Put $T(\sigma)=(dU(\sigma))(e)$ for $\sigma \in \hat{G}$ and define $T=(T(\sigma))_{\sigma\in\hat{G}}$ ($\in \Sigma(G)$). Then T satisfies the condition (C2) in Definition 1.2. Indeed, if an irreducible decomposition of $U(\sigma)\otimes U(\sigma')$ for σ, $\sigma' \in \hat{G}$ is given by

$$U(\sigma)\otimes U(\sigma')=V^{-1}(U(\sigma_1)\oplus\cdots\oplus U(\sigma_m))V$$

(V a unitary matrix), then, by (4.1),

$$V^{-1}(T(\sigma_1)\oplus\cdots\oplus T(\sigma_m))V=T(\sigma)\otimes I(\sigma')+I(\sigma)\otimes T(\sigma').$$

Now put $H_1=\frac{1}{2}(T-T^*)$ and $H_2=-\frac{i}{2}(T+T^*)$. These matrix fields are hermite and satisfy (C2), because so does T^* as well as T. That is, H_1, $H_2 \in \Lambda(G)$. So put $\alpha_j=h_G^{-1}(H_j)$ ($j=1, 2$). Then , by (4.2),

$$(dU(\sigma))(x_0)=d(U_{x_0}(\sigma)U(\sigma))(e)=U_{x_0}(\sigma)T(\sigma)$$

$$=U_{x_0}(\sigma)(H_1(\sigma)+iH_2(\sigma))=(d^{(r)}_{\alpha_1+i\alpha_2}U(\sigma))(x_0) \quad (\sigma\in\hat{G},\ x_0\in G).$$

This shows that (a) holds. q.e.d.

Corollary. *Let d be a continuous linear map of $\mathscr{E}_n^{(*)}(G)$ ($n=\infty$, 1, 2,...) into $\boldsymbol{C}^G$ equipped with the pointwise convergence topology. Then the following three are equivalent*:

(a) $d=d^{(r)}_{\alpha+i\beta}|_{\mathscr{E}_n(G)}$ *for some* α, $\beta \in R(G)$;

(b) *for* f, $g \in \mathscr{E}_n(G)$ *and* $x_0 \in G$,

$$d(fg)=(df)g+f(dg) \quad \text{and} \quad d({}_{x_0}f)={}_{x_0}(df).$$

(c) *for* f, $g \in \mathscr{T}(G)$ *and* $x_0 \in G$,

$$d(fg)(e)=(df)(e)g(e)+f(e)(dg)(e) \quad \text{and}$$

$$d({}_{x_0}f)(e)={}_{x_0}(df)(e).$$

Proof. The implications (a)⇒(b)⇒(c) are obvious. If (c) is assumed, there exists, by Lemma 4.2, α, $\beta \in R(G)$ such that $df=d^{(r)}_{\alpha+i\beta}f$ for $f \in \mathscr{T}(G)$. Then, since d and $d^{(r)}_{\alpha+i\beta}|_{\mathscr{E}_n(G)}$ are continuous from $\mathscr{E}_n^{(*)}(G)$ into $\boldsymbol{C}^G$, and $\mathscr{T}(G)$ is dense in $\mathscr{E}_n^{(*)}(G)$, this equality holds for $f\in\mathscr{E}_n(G)$. Hence (a). q.e.d.

It is clear from the above corollary that each left invariant and $\boldsymbol{\tau}_*$-continuous derivation on $\mathscr{E}_\infty(G)$ has the form $d^{(r)}_{\alpha+i\beta}$ for some α, $\beta\in R(G)$. Thus Theorem 4.1 has been proved.

4.2. Invariant differential operators.

Definition 4.2. $\boldsymbol{D}(G)$ denotes the set of all linear operators on $\mathscr{E}_\infty(G)$ that are generated by the derivations $d^{(r)}_{\alpha+i\beta}$, $d^{(l)}_{\alpha+i\beta}$ (α, $\beta\in R(G)$) and $\mathscr{E}_\infty(G)$, where each member of $\mathscr{E}_\infty(G)$ is taken as a multiplication operator. That is, $\boldsymbol{D}(G)$ consists of the operators on $\mathscr{E}_\infty(G)$ expressed as

$$D=\sum a^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}d^{(r)}_{\alpha_1}\cdots d^{(r)}_{\alpha_p}d^{(l)}_{\beta_1}\cdots d^{(l)}_{\beta_q}\qquad\text{(finite sum)},$$

where $\alpha_1,\dots,\alpha_p$, $\beta_1,\dots,\beta_p\in R(G)$, $a^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_q}\in\mathscr{E}_\infty(G)$, and if $p=q=0$, $d^{(r)}_{\alpha_1}\cdots d^{(r)}_{\alpha_p}d^{(l)}_{\beta_1}\cdots d^{(l)}_{\beta_q}$ means 1, the identity operator on $\mathscr{E}_\infty(G)$. Each member of $\boldsymbol{D}(G)$ is called a differential operator on G. A differential operator D on G is said to be left (resp. right) invariant if $D({}_{x_0}f)={}_{x_0}(Df)$ (resp. $D(f_{x_0})=(Df)_{x_0}$) holds for $f\in\mathscr{E}_\infty(G)$ and $x_0\in G$. $\boldsymbol{D}_l(G)$ (resp. $\boldsymbol{D}_r(G)$) denotes the set of all left (resp. right) invariant differential operators.

A differential operator is $\boldsymbol{\tau}_*$-continuous, and so, determined by the restriction to $\mathscr{T}(G)$. Obviously $\boldsymbol{D}(G)$ is an algebra over $\boldsymbol{C}$ with 1 as its identity element, and also a left module over $\mathscr{E}_\infty(G)$. Each of $\boldsymbol{D}_l(G)$ and $\boldsymbol{D}_r(G)$ is a subalgebra of $\boldsymbol{D}(G)$ containing 1.

Lemma 4.3. *A differential operator D on G is left* (resp. *right*) *invariant if and only if it can be expressed as*

$$D=\sum c_{\alpha_1\cdots\alpha_p}d^{(r)}_{\alpha_1}\cdots d^{(r)}_{\alpha_p}$$

$$\text{(resp.}\quad D=\sum c^{\alpha_1\cdots\alpha_p}\,d^{(l)}_{\alpha_1}\cdots\cdots d^{(l)}_{\alpha_p})\qquad\text{(finite sum)},$$

where $\alpha_1,\dots,\alpha_p\in R(G)$, $c_{\alpha_1\cdots\alpha_p}$ (resp. $c^{\alpha_1\cdots\alpha_p}$)$\in\boldsymbol{C}$, *and* $d^{(r)}_{\alpha_1}\cdots d^{(r)}_{\alpha_p}$ (resp. $d^{(l)}_{\alpha_1}\cdots d^{(l)}_{\alpha_p}$) *means the identity operator* 1 *if* $p=0$.

Proof. We shall prove for left invariance. A differential operator D has the form

$$D=\sum a^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}\,D^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}\qquad\text{(finite sum)},$$

where $a^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}\in\mathscr{E}_\infty(G)$ and $D^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}=d^{(r)}_{\alpha_1}\cdots d^{(r)}_{\alpha_p}d^{(l)}_{\beta_1}\cdots d^{(l)}_{\beta_q}$. By Lemma 2.11, $d^{(l)}_\beta f(e)=-d^{(r)}_\beta(e)$ for $f\in\mathscr{E}_\infty(G)$ and $\beta\in R(G)$. Hence

$$\begin{aligned}D^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}f(e)&=-D^{\beta_2\cdots\beta_q}_{\beta_1\alpha_1\cdots\alpha_p}f(e)\\&=\cdots\cdots\quad=(-1)^q D_{\beta_q\cdots\beta_1\alpha_1\cdots\alpha_p}f(e).\end{aligned}$$

Therefore, if D is left invariant,

$$(Df)(x)=D({}_{x^{-1}}f)(e)$$

$$=\sum a^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}(e)\,(-1)^q(D_{\beta_q\cdots\beta_1\alpha_1\cdots\alpha_p}({}_{x^{-1}}f))\,(e)$$

$$=\sum(-1)^q a^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}(e)(D_{\beta_q\cdots\beta_1\alpha_1\cdots\alpha_p}f)\,(x)\quad(x\in G).$$

This proves the 'only if' part of the lemma. The 'if' part is obvious. q.e.d.

Let ρ be the algebra automorphism $f\mapsto\check{f}$ of $\mathscr{E}_\infty(G)$. For each $D\in \boldsymbol{D}(G)$, define $\check{D}=\rho D\rho^{-1}$. If D has the form

$$D=\sum a^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}\,d^{(r)}_{\alpha_1}\cdots d^{(r)}_{\alpha_p}\,d^{(l)}_{\beta_1}\cdots d^{(l)}_{\beta_q}\qquad\text{(finite sum)},$$

where $a^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}\in\mathscr{E}_\infty(G)$, then, by Lemma 2.1, (ii),

$$\check{D}=\sum(a^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p})^\vee\,d^{(l)}_{\alpha_1}\cdots d^{(l)}_{\alpha_p}\,d^{(r)}_{\beta_1}\cdots d^{(r)}_{\beta_q}.$$

Therefore the transformation $D\mapsto\check{D}$ is an automorphism of the algebra $\boldsymbol{D}(G)$, and the subalgebras $\boldsymbol{D}_l(G)$ and $\boldsymbol{D}_r(G)$ are carried onto each other under this transformation.

Definition 4.3. $\boldsymbol{U}(G)$ denotes the universal enveloping algebra of the Lie algebra $R(G)^c$. $\boldsymbol{C}$ and $R(G)^c$ are identified with their canonical images in $\boldsymbol{U}(G)$.

Each element of $U(G)$ is expressed as

$$\sum c_{\alpha_1\cdots\alpha_p}\alpha_1\ldots\alpha_p\quad\text{(finite sum)},$$

where $\alpha_1,\ldots,\alpha_p\in R(G)$, $c_{\alpha_1\cdots\alpha_p}\in\boldsymbol{C}$, and the product $\alpha_1\cdots\alpha_p$ means the identity element 1 if $p=0$.

Theorem 4.2. *The map $\alpha+i\beta\mapsto d^{(r)}_{\alpha+i\beta}$ of $R(G)^c$ into $\boldsymbol{D}_l(G)$ extends uniquely to an algebra isomorphsim of $\boldsymbol{U}(G)$ onto $\boldsymbol{D}_l(G)$ mapping the identity of $\boldsymbol{U}(G)$ to that of $\boldsymbol{D}_l(G)$.*

This theorem is well known if G is a Lie group, and makes clear the structure of the algebra $\boldsymbol{D}_l(G)$. By Theorem 4.1, the map $\alpha+i\beta\mapsto d^{(r)}_{\alpha+i\beta}$ is a Lie algebra homomorphism of $R(G)^c$ into $\boldsymbol{D}_l(G)$, regarded as Lie algebra under the commutator product. Hence this map extends uniquely to an algebra homomorphism of $\boldsymbol{U}(G)$ into $\boldsymbol{D}_l(G)$ mapping the identity of $\boldsymbol{U}(G)$ to that of $\boldsymbol{D}_l(G)$. Denote this extension by ψ. Then, by Lemma 4.3, ψ is obviously surjective. Therefore, for the proof of Theorem 4.2, it remains only to show that ψ is injective.

Lemma 4.4. *For any finite number of linearly independent members $H_1,\ldots,H_m$ of $A(G)$, there exists a finite subset Δ of $\hat{G}$ such that the matrix fields $H_1,\ldots,H_m$, restricted to Δ, are linearly independent.*

Proof. This can be proved by finite induction. Assume that, for some $k<m$, there has been chosen a finite subset Δ_k so that $H_1,\ldots,H_k$, restricted to Δ_k, are linearly independent. This assumption obviously holds for $k=1$. Now if $H_1,\ldots,H_k,H_{k+1}$, restricted to Δ_k, are linearly independent, we define $\Delta_{k+1}=\Delta_k$. Otherwise, there exist uniquely determined real numbers $c_1,\ldots,c_k$ such that

$$H_{k+1}(\sigma)=c_1H_1(\sigma)+\cdots+c_kH_k(\sigma) \qquad \text{for all} \quad \sigma\in\Delta_k.$$

But, since $H_1,\dots, H_k, H_{k+1}$ are linearly independent on $\hat{G}$, there exists $\sigma'\in\hat{G}$ such that

$$H_{k+1}(\sigma')\neq c_1H_1(\sigma')+\cdots+c_kH_k(\sigma').$$

Put $\Delta_{k+1}=\Delta_k\cup\{\sigma'\}$. Then $H_1,\dots, H_k, H_{k+1}$ are linearly independent on it. This completes the induction. q. e. d.

Proof of Theorem 4.2. *We prove the injectivity of* ψ.

(I) Take a linear base $\mathfrak{B}$ of $R(G)^c$ consisting of elements in $R(G)$. Introduce a total ordering in $\mathfrak{B}$, and denote by $\mathfrak{B}'$ the set of all finite increasing sequences of elements of $\mathfrak{B}$, containing the void sequence. Then $\{\alpha\beta\cdots\gamma;\ (\alpha, \beta,\dots, \gamma)\in\mathfrak{B}'\}$ is a linear base for $\boldsymbol{U}(G)$. Therefore, for the injectivity of ψ, it suffices to show that $\{d_\alpha^{(r)}d_\beta^{(r)}\cdots d_\gamma^{(r)};\ (\alpha, \beta,\dots, \gamma)\in\mathfrak{B}'\}$ is linearly independent in $\boldsymbol{D}_l(G)$. So our task is to prove the linear independency of $\{d_\alpha^{(r)}d_\beta^{(r)}\cdots d_\gamma^{(r)};\ (\alpha, \beta,\dots, \gamma)\in S\}$ for any finite subset $S\subseteq\mathfrak{B}'$.

(II) Let F be the set of all members of $R(G)$ appearing in the sequences in S. Put $F'=h_G(F)$. Then, since F' is a finite and linearly independent subset of $\Lambda(G)$, there exists a finite subset $\Delta\subseteq\hat{G}$ such that the members of F', restricted to it, are linearly independent (Lemma 4.4). Set $N=A(G, \Delta)$. Then, since $A(G, N)=[\Delta]$, the map r_N (see Definition 1.7) carries F' onto a linearly independent subset of $\Lambda(G/N)$ in a one-to-one way. That is, $\bar{\pi}_N$ carries F onto a linearly independent subset of $R(G/N)$ in a one-to-one way (Lemma 1.9, (ii)). Hence $\{\bar{\alpha}\bar{\beta}\cdots\bar{\gamma};\ (\alpha, \beta,\dots, \gamma)\in S\}$, where $\bar{\alpha}=\bar{\pi}_N(\alpha)$, $\bar{\beta}=\bar{\pi}_N(\beta),\dots, \bar{\gamma}=\bar{\pi}_N(\gamma)$, is linearly independent in $U(G/N)$. Since $N\in\boldsymbol{H}_0(G)$ (Lemma 1.11, (i)), i.e., G/N is Lie or finite, this implies that $\{d_{\bar{\alpha}}^{(r)}d_{\bar{\beta}}^{(r)}\cdots d_{\bar{\gamma}}^{(r)};\ (\alpha, \beta,\dots, \gamma)\in S\}$ is linearly independent in $\boldsymbol{D}_l(G/N)$. Under the isomorphism $g\mapsto g\circ\pi_N$ of $\mathscr{E}_\infty(G/N)$ onto $\mathscr{E}_\infty(G, N)$, each $d_{\bar{\alpha}}^{(r)}d_{\bar{\beta}}^{(r)}\cdots d_{\bar{\gamma}}^{(r)}$ is transformed to $d_\alpha^{(r)}d_\beta^{(r)}\cdots d_\gamma^{(r)}|_{\mathscr{E}_\infty(G,N)}$ (cf. (2.1)). Hence, as a matter of course, $\{d_\alpha^{(r)}d_\beta^{(r)}\cdots d_\gamma^{(r)};\ (\alpha, \beta,\dots, \gamma)\in S\}$ is linearly independent in $\boldsymbol{D}_l(G)$. q. e. d.

4.3. The center of $\boldsymbol{U}(G)$. Let $\Lambda(G)^c$ denote the set of all members of $\Sigma(G)$ satisfying the condition (C2) in Definition 1.2. Then it is easy to see that, under the linear operations and the commutator product in $\Sigma(G)$, $\Lambda(G)^c$ is a complex Lie algebra isomorphic with the complexification of $\Lambda(G)$. The Lie algebra isomorphism h_G of $R(G)$ onto $\Lambda(G)$ extends uniquely to that of $R(G)^c$ onto $\Lambda(G)^c$, also denoted by h_G.

Definition 4.4. For $x\in G$, let $\mathrm{Ad}(x)$ denote the map

$$\alpha+i\beta\longmapsto x\alpha x^{-1}+ix\beta x^{-1} \quad (\alpha, \beta\in R(G))$$

on $R(G)^c$, or equivalently in virtue of h_G, the map

$$H_1+iH_2\longmapsto U_x(H_1+iH_2)U_{x^{-1}} \quad (H_1, H_2\in\Lambda(G))$$

on $\Lambda(G)^c$. Then $\mathrm{Ad}(x)$ is an automorphism of the Lie algebra $R(G)^c$, and extends uniquely to an algebra automorphism of $\boldsymbol{U}(G)$ mapping 1 to 1, which is again denoted by Ad (x).[3]

Lemma 4.5. *Let the algebra $\boldsymbol{U}(G)$ be realized by $\boldsymbol{D}_l(G)$ through the isomorphism established in Theorem 4.2. Then*

$$\mathrm{Ad}\,(x)D = R_x \circ D \circ R_{x^{-1}} \quad (x \in G,\ D \in \boldsymbol{D}_l(G)),$$

where R_x denotes the right translation $f \mapsto f_x$ on $\mathscr{E}_\infty(G)$.

Proof. For $x \in G$, $\alpha \in R(G)$ and $f \in \mathscr{E}_\infty(G)$,

$$(\mathrm{Ad}(x) d_\alpha^{(r)}) f = d_{x\alpha x^{-1}}^{(r)} f$$

$$= \frac{d}{dt} f(\cdot x\alpha(t)x^{-1})|_{t=0} = (R_x \circ d_\alpha^{(r)} \circ R_{x^{-1}}) f.$$

Since $\boldsymbol{D}_l(G)$ is generated by 1 and $d_\alpha^{(r)}$ ($\alpha \in R(G)$), this proves the lemma. q.e.d.

For $\gamma \in R(G)^c$, define the map ad (γ) on $\boldsymbol{U}(G)$ as

$$\mathrm{ad}\,(\gamma) D = [\gamma, D] = \gamma D - D\gamma \quad (D \in \boldsymbol{U}(G)).$$

ad (γ) is a derivation on the algebra $\boldsymbol{U}(G)$. For each positive integer m, put

$$\boldsymbol{U}_m(G) = \omega(\boldsymbol{C} + \textstyle\sum_{k=1}^{m} \otimes^k R(G)^c),$$

where ω denotes the canonical homomorphism of the tensor algebra $\boldsymbol{C} + \sum_{k=1}^{\infty} \otimes^k R(G)^c$ onto $\boldsymbol{U}(G)$. Each $\boldsymbol{U}_m(G)$ is a linear subspace of $\boldsymbol{U}(G)$ stable under every ad (γ) ($\gamma \in R(G)^c$). Put $\mathrm{ad}_m(\gamma) = \mathrm{ad}\,(\gamma)|_{\boldsymbol{U}_m(G)}$. Now let us suppose G is finite dimensional. Then so is each $\boldsymbol{U}_m(G)$. Therefore we can equip it with the usual finite dimensional vector space topology, and define the map $\exp \mathrm{ad}_m(\gamma)$ on it by the Taylor series $\sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}_m(\gamma)^k$. If $m < m'$, then $\exp \mathrm{ad}_{m'}(\gamma)$ extends $\exp \mathrm{ad}_m(\gamma)$. Therefore, for each $\gamma \in R(G)^c$, the union of the maps $\exp \mathrm{ad}_m(\gamma)$ ($m = 1, 2, \ldots$) defines a linear map on $\boldsymbol{U}(G)$, which we denote by exp ad (γ).

The next lemma is verified in the same argument as for Lie groups with simple modifications.

Lemma 4.6. *Assume that G is finite dimensional.*

(i) $\mathrm{Ad}\,(\alpha(t)) = \exp \mathrm{ad}\,(t\alpha)$ $(\alpha \in R(G),\ t \in \boldsymbol{R})$.

(ii) *For $\alpha \in R(G)$ and $D \in \boldsymbol{U}(G)$,*

$$\mathrm{ad}\,(\alpha) D = 0 \Longleftrightarrow \mathrm{Ad}\,(\alpha(t)) D = D \text{ for all } t \in \boldsymbol{R}.$$

(iii) *Let $\boldsymbol{Z}(G)$ denote the center of $\boldsymbol{U}(G)$. Then, for $D \in \boldsymbol{U}(G)$,*

$$D \in Z(G) \Longleftrightarrow \mathrm{Ad}\,(x) D = D \text{ for all } x \in c(G).$$

---

(3) The linear space $A(G)^c$ is locally convex, complete and Hausdorff relative to the coordinatewise convergence topology, and the map Ad: $G \ni x \mapsto \mathrm{Ad}\,(x)$ gives a representation of G on $A(G)^c$ such that $G \times A(G)^c \ni (x, H) \mapsto \mathrm{Ad}\,(x)H \in A(G)^c$ is continuous.

Proof. (i). For any $D_1, D_2 \in \boldsymbol{U}(G)$, choose a positive integer m so large that $D_1, D_2, D_1D_2 \in \boldsymbol{U}_m(G)$. Then, since $\mathrm{ad}\,(t\alpha)$ is a derivation on $\boldsymbol{U}(G)$,

$$(\exp \mathrm{ad}(t\alpha))\ (D_1D_2) = \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{ad}_m(t\alpha)^k(D_1D_2))$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i+j=k} \frac{k!}{i!j!} (\mathrm{ad}_m(t\alpha)^i D_1)\ (\mathrm{ad}_m(t\alpha)^j D_2)$$

$$= \left(\sum_{i=0}^{\infty} \frac{1}{i!} (\mathrm{ad}_m(t\alpha)^i D_1)\right)\left(\sum_{j=0}^{\infty} \frac{1}{j!} (\mathrm{ad}_m(t\alpha)^j D_2)\right)$$

$$= ((\exp \mathrm{ad}(t\alpha))(D_1)((\exp \mathrm{ad}(t\alpha))D_2).$$

Hence $\exp \mathrm{ad}(t\alpha)$ is an endomorphism of the algebra $\boldsymbol{U}(G)$. Now take any $\beta \in R(G)$. The elements

$$\mathrm{Ad}\,(\alpha(t))\beta = \alpha(t)\beta\alpha(-t)$$

and

$$(\exp \mathrm{ad}(t\alpha))\beta = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\mathrm{ad}_1\,(\alpha)^k\beta)$$

of $R(G)$ correspond under h_G with

$$(U_{\alpha(t)}(\sigma)H_\beta(\sigma)U_{\alpha(-t)}(\sigma))_{\sigma\in\hat{G}}$$

and

$$\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \mathrm{ad}\,(H_\alpha(\sigma))^k H_\beta(\sigma)\right)_{\sigma\in\hat{G}}$$

respectively, where $(H_\alpha(\sigma))_{\sigma\in\hat{G}} = h_G(\alpha)$, $(H_\beta(\sigma))_{\sigma\in\hat{G}} = h_G(\beta)$, and $\mathrm{ad}\,(H_\alpha(\sigma))$ denotes the map $M \mapsto [H_\alpha(\sigma), M] = H_\alpha(\sigma)M - MH_\alpha(\sigma)$ on $\mathfrak{M}(d_\sigma, \boldsymbol{C})$. For each $\sigma \in \hat{G}$, the σ-th coordinate of each of these matrix fields satisfies the equation

$$\frac{d}{dt} M(t) = H_\alpha(\sigma)M(t) - M(t)H_\alpha(\sigma) \quad (M(t) \in \mathfrak{M}(d_\sigma, \boldsymbol{C}))$$

with the initial condition $M(0) = H_\beta(\sigma)$. Hence these two matrix fields coincide with each other for all $t \in \boldsymbol{R}$. Thus, for each $t \in \boldsymbol{R}$, the endomorphisms $\mathrm{Ad}\,(\alpha(t))$ and $\exp \mathrm{ad}(t\alpha)$ of $\boldsymbol{U}(G)$ coincide on $R(G)$. Besides, from definition, each of them fixes the identity 1 of $\boldsymbol{U}(G)$. Therefore they coincide on $\boldsymbol{U}(G)$.

(ii). Choose m so that $D \in \boldsymbol{U}_m(G)$. Then, by (i),

$$\mathrm{Ad}\,(\alpha(t))D = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\mathrm{ad}_m\,(\alpha)^k D) \quad (t \in \boldsymbol{R}).$$

Hence the implication $\Rightarrow$. The reverse one follows from

$$\frac{d}{dt} \mathrm{Ad}\,(\alpha(t))D|_{t=0} = \mathrm{ad}_m(\alpha)D.$$

(iii). D belongs to $\boldsymbol{Z}(G)$ if and only if it commutes with every $\alpha \in R(G)$. By (ii) this is the case if and only if $\mathrm{Ad}\,(\alpha(t))D = D$ for all $\alpha \in R(G)$ and $t \in \boldsymbol{R}$. Hence the implication $\Leftarrow$. Now take m so that $D \in \boldsymbol{U}_m(G)$. To prove the reverse implication, it suffices to show that the map $x \mapsto \mathrm{Ad}\,(x)D$ of G into $\boldsymbol{U}_m(G)$ is continuous, because $\cup\, \alpha(R)$ $(\alpha \in R(G))$ is dense in $c(G)$. Take a linear base $\{\alpha_1, \dots, \alpha_n\}$ of $R(G)$. Then D is expressed as

$$\sum_{0 \le k_1 + \cdots + k_n \le m} c_{k_1 \cdots k_n} \alpha_1^{k_1} \cdots \alpha_n^{k_n} \quad (c_{k_1 \cdots k_n} \in \boldsymbol{C}).$$

Hence

$$\mathrm{Ad}\,(x)D = \sum_{0 \le k_1 + \cdots + k_n \le m} c_{k_1 \cdots k_n}\, (\mathrm{Ad}\,(x)\alpha_1)^{k_1} \cdots (\mathrm{Ad}\,(x)\alpha_n)^{k_n}.$$

The continuity in question is clear from this together with Lemma 2.10. q.e.d.

The assertion (iii) of the above lemma holds without finite dimensionality of G. That is, we have

Theorem 4.3. *Let $\boldsymbol{Z}(G)$ denote the center of the algebra $\boldsymbol{U}(G)$, and $c(G)$ the connected component of e in G. Then*

$$\boldsymbol{Z}(G) = \{D \in \boldsymbol{U}(G);\ \mathrm{Ad}\,(x)D = D \text{ for all } x \in c(G)\}.$$

Proof. Realize $\boldsymbol{U}(G)$ by $\boldsymbol{D}_l(G)$ through the isomorphism in Theorem 4.2. For any closed normal subgroup N of G, the Lie algebra homomorphism $\bar{\pi}_N \colon \alpha \mapsto \pi_N \circ \alpha$ of $R(G)$ onto $R(G/N)$ extends uniquely to an algebra homomorphism of $\boldsymbol{D}_l(G)$ onto $\boldsymbol{D}_l(G/N)$, mapping 1 to 1. And, evidently,

$$\bar{\pi}_N(\mathrm{Ad}\,(x)D) = \mathrm{Ad}\,(\pi_N(x))(\bar{\pi}_N(D)) \quad (x \in G,\ D \in \boldsymbol{D}_l(G)). \tag{4.3}$$

Now assume that a $D \in \boldsymbol{D}_l(G)$ is fixed by $\mathrm{Ad}\,(x)$ for all $x \in c(G)$. Take any $D' \in \boldsymbol{D}_l(G)$ and $f \in \mathscr{E}_\infty(G)$, and choose $N \in \boldsymbol{H}(G)$ and $g \in \mathscr{E}_\infty(G/N)$ so that $f = g \circ \pi_N$. By (4.3) we have,

$$\mathrm{Ad}\,(\pi_N(x))(\bar{\pi}_N(D)) = \bar{\pi}_N(\mathrm{Ad}\,(x)D) = \bar{\pi}_N(D) \quad (x \in c(G)).$$

Since G/N is finite dimensional and $\pi_N(c(G)) = c(G/N)$, this shows by (iii) of Lemma 4.6 that $\bar{\pi}_N(D) \in \boldsymbol{Z}(G/N)$. Hence

$$(DD')f = (\bar{\pi}_N(D)\bar{\pi}_N(D')g) \circ \pi_N = (\bar{\pi}_N(D')\bar{\pi}_N(D)g) \circ \pi_N = (D'D)f.$$

Since f and D' are arbitrary, this shows that $D \in \boldsymbol{Z}(G)$. Conversely assume that $D \in \boldsymbol{Z}(G)$. Let f, N and g be the same as above. Then $\bar{\pi}_N(D) \in \boldsymbol{Z}(G/N)$ and so, again by (4.3) and (iii) of Lemma 4.6,

$$\begin{aligned}(\mathrm{Ad}\,(x)D)f &= (\bar{\pi}_N(\mathrm{Ad}\,(x)D)g) \circ \pi_N \\ &= ((\mathrm{Ad}(\pi_N(x))\bar{\pi}_N(D))g) \circ \pi_N = (\bar{\pi}_N(D)g) \circ \pi_N = Df \quad (x \in c(G)).\end{aligned}$$

Hence $\mathrm{Ad}\,(x)D = D$ $(x \in c(G))$. This completes the proof. q.e.d.

Corollary. *If $\boldsymbol{U}(G)$ is realized by $\boldsymbol{D}_l(G)$, then*

$$\boldsymbol{Z}(G)=\{D\in\boldsymbol{D}_l(G);\ D\circ R_x=R_x\circ D\ \text{for all}\ x\in c(G)\}.$$

In particular, $\boldsymbol{Z}(G)$ includes $\boldsymbol{D}_l(G)\cap\boldsymbol{D}_r(G)$, and coincides with it if G is connected.

Proof. Obvious from Lemma 4.5 together with

$$\boldsymbol{D}_l(G)\cap\boldsymbol{D}_r(G)=\{D\in\boldsymbol{D}_l(G);\ D\circ R_x=R_x\circ D\ (x\in G)\}.$$ q. e. d.

Department of Mathematics
College of Science and Technology
Nihon University

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Added in proof. After our manuscript had been finished, the author learned the papers: (a) H. Boseck and G. Czichowski, Grundfunktionen und verallgemeinerte Funktionen auf topologischen Gruppen I, Math. Nachr., 58 (1973), 215–240; (b) K. P. Rudolph, Michal-Bastiani differentiation on topological groups, Abh. Akad. Wiss. DDR, Abt. Math. Naturwiss. Tech., 1979 (1980), 161–166. In (a) our Theorem 2.1 is obtained for any LC groups but under a stronger assumption on continuous differentiability of functions. (b) announces our Lemma 2.9, the key to Theorem 1.1, also for any LC groups without proof. Recently, Prof. K. Sakai kindly informd us that these results were given in full treatment in a book by the above three authors: Analysis on topological groups - General Lie theory, Teubner-Texte zur Math., Band 37, Leibzig, 1981. Unlike our construction, their method uses only the inverse limit technique together with computational observation on one-parameter subgroups.