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On an isomorphism of the algebra of pseudo-differential operators

By

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I. Let M be a compact connected smooth manifold without boundary, and denote by $L^m(M)$ the space of pseudo-differential operators of order $m \in \boldsymbol{Z}$. We assume that the total symbol of $P \in L^m(M)$ (in each local coordinate) has an asymptotic expansion in homogeneous functions of integer order. Also put $L^\infty(M) = \bigcup_{m \in Z} L^m(M)$, then $L^\infty(M)$ is an algebra over $\boldsymbol{C}$, and $L^{-\infty}(M) = \bigcap_{m \in Z} L^m(M)$ is a two sided ideal consisting of all operators with smooth kernel (for non-compact manifolds see Remark 1 below, and for the definition of pseudo-differential operators see [3] and also for more details see [4] and [5]).

We denote by T^*M the cotangent bundle of M and by T_0^*M the complement of the zero section in T^*M, and also by S^*M the cotangent sphere bundle of M.

Let $\alpha: L^\infty(M) \cong L^\infty(N)$ be an order-preserving algebra isomorphism, i.e., $\alpha(L^m(M)) = L^m(N)$, for all $m \in \boldsymbol{Z}$, then in [1] Duistermaat — Singer has shown the

Theorem A. *If $H^1(S^*M, \boldsymbol{C}) = 0$, then α is equal to a conjugation by an invertible elliptic Fourier integral operator $A: C^\infty(M) \cong C^\infty(N)$, that is, $\alpha(P) = A \circ P \circ A^{-1}$ for all $P \in L^\infty(M)$. Here $H^1(\cdot, \boldsymbol{C})$ is the first de Rham cohomology group with coefficients in $\boldsymbol{C}$.*

The canonical relation of this operator A is defined by a homogeneous symplectomorphism $C: T_0^*M \cong T_0^*N$. If C is defined over all T^*M, then C is the lifting of a diffeomorphism $\mathscr{F}: M \cong N$ (see [6, p. 34]), and the Fourier integral operator A in Theorem A is equal to $\mathscr{F}^*$ up to an invertible elliptic pseudo-differential operator.

On the other hand, in [2] Pursell — Shanks has shown the

Theorem B. *Let $i: X(M) \cong X(N)$ be an isomorphism between Lie algebras of smooth vector fields on the manifolds M and N. Then the isomorphism i is of the form $i = \mathrm{d}F$, that is, $i(X) = (F^{-1})^* \circ X \circ F^*$, $X \in X(M)$, where $F: M \cong N$ is a diffeomorphism.*

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Immediately from Theorem B we see that the isomorphism $i = dF: X(M) \simeq X(N)$ can be extended to an order-preserving algebra isomorphism $i: L^\infty(M \simeq L^\infty(N)$, $i(P) = (F^{-1})^* \circ P \circ F^*$.

Based on these results, we show in this note the following

Theorem 1. *Let* $\alpha: L^\infty(M) \simeq L^\infty(M)$ *be an algebra isomorphism such that*
(i) $\alpha(L^{-\infty}(M)) = L^{-\infty}(M)$, *and*
(ii) $\alpha(X(M)) \subset X(M)$.
Then there exists a unique diffeomorphism $F: M \simeq M$ *such that* $\alpha(P) = (F^{-1})^* \circ P \circ F^*$, *for all* $P \in L^\infty(M)$.

Consequently we have at once

Corollary 1. *Under the same assumption as in Theorem* 1, α *is order-preserving.*

Also as a corollary of Theorem 1 together with Theorem B we have

Corollary 2. *Let* $\alpha: L^\infty(M) \simeq L^\infty(M)$ *be an algebra isomorphism such that*
(i) $\alpha(L^{-\infty}(M)) \subset L^{-\infty}(M)$,
(ii) $\alpha(X(M)) = X(M)$.
Then α *is of the same form as in Theorem* 1.

Remark 1. In Theorems A and B the manifolds need not be compact. If M is not compact, pseudo-differential operators must be restricted to the class of P's to each of which corresponds a kernel distribution K_P with the following property: if $Pu = \int_M K_P(x, y)u(y)\,dy$, then the projection $(x, y) \mapsto x$ restricted to the support of K_P is proper. Of course differential operators always satisfy this condition.

II. Before proving Theorem 1 we give an outline of a proof of the following Proposition 1.

Proposition 1. *Let* $i: L^{-\infty}(M) \simeq L^{-\infty}(M)$ *be an isomorphism of the algebra of the operators with smooth kernel. Then there exists a topological linear automorphism* $A: C^\infty(M) \simeq C^\infty(M)$ *such that* i *is the conjugation by* A. *The operator* A *is unique up to constant multiples.*

In [1] a more general result is proved, and the proof below is done along the same line as in [1]. Here we give it for the sake of the self-containedness of this note.

There are five steps for the proof.

Step 1. First, we fix a smooth positive measure ω_0 on M. For elements $u, v \in C^\infty(M)$ we denote by $u \otimes v$ an operator $u \otimes v: C^\infty(M) \to C^\infty(M)$, $u \otimes v(f) = \left(\int_M f \cdot v\omega_0\right) \cdot u$, and we define a pairing $\langle u, v \rangle$ by $\langle u, v \rangle = \int_M u \cdot v\omega_0$. For $S, T \in L^{-\infty}(M)$ put $B(S, T) = \{S \circ P \circ T;\ P \in L^{-\infty}(M)\}$, then we have
(i) $i(B(S, T)) = B(i(S), i(T))$,
(ii) for $u, v, \tilde{u}, \tilde{v} \in C^\infty(M)$ $\dim B(\tilde{u} \otimes v, u \otimes \tilde{v}) \leq 1$.

From (i) and (ii) we have

(iii) for any elements $u, v \in C^{\infty}(M)$ there exist $\varphi, \psi \in C^{\infty}(M)$ such that $i(u \otimes v) = \varphi \otimes \psi$.

Step 2. Fix $u_0, v_0 \in C^{\infty}(M)$ such that $\langle u_0, v_0 \rangle \neq 0$, and write $i(u_0 \otimes v_0) = \varphi_0 \otimes \psi_0$. Then we have $\langle u_0, v_0 \rangle = \langle \varphi_0, \psi_0 \rangle$ and there exist operators A and B: $C^{\infty}(M) \to C^{\infty}(M)$ such that

(i) $i(u \otimes v_0) = Au \otimes \psi_0$ for all $u \in C^{\infty}(M)$,

(ii) $i(u_0 \otimes v) = \varphi_0 \otimes Bv$ for all $v \in C^{\infty}(M)$.

Consequently we have $i(u \otimes v) = Au \otimes Bv$, because

$$u \otimes v_0 \circ u_0 \otimes v = \langle u_0, v_0 \rangle u \otimes v$$

and so

$$i(u \otimes v_0) \circ i(u_0 \otimes v) = Au \otimes \psi_0 \circ \varphi_0 \otimes Bv = \langle \varphi_0, \psi_0 \rangle Au \otimes Bv = \langle u_0, v_0 \rangle i(u \otimes v).$$

Step 3. Since i is an isomorphism we see that the operators A and B must be automorphisms. Also it holds $\langle Au, v \rangle = \langle u, B^{-1} \rangle$ for all $u, v \in \in C^{\infty}(M)$.
In fact, $\langle A\tilde{u}, v \rangle Au \otimes B\tilde{v} = Au \otimes v \circ A\tilde{u} \otimes B\tilde{v} = i(u \otimes B^{-1}v) \circ i(\tilde{u} \otimes \tilde{v}) = \langle \tilde{u}, B^{-1}v \rangle i(u \otimes \tilde{v}) = \langle \tilde{u}, B^{-1}v \rangle Au \otimes B\tilde{v}$.

This relation implies in a standard manner the continuity of the operators A and B, owing to the closed graph theorem for the Fréchet space $C^{\infty}(M)$ with C^{∞}-topology.

Step 4. For all $P \in L^{-\infty}(M)$ we have
$A(Pu) \otimes Bv = i(Pu \otimes v) = i(P \circ u \otimes v) = i(P) \circ Au \otimes Bv = (i(P)Au) \otimes Bv$. Hence $A \circ P = i(P) \circ A$.

Step 5. Uniqueness of A up to constant multiples. If $A \circ P \circ A^{-1} = P$ for all $P \in L^{-\infty}(M)$, where A: $C^{\infty}(M) \simeq C^{\infty}(M)$ is an automorphism, then we have
$(A \circ u \otimes v)(f) = (u \otimes v \circ A)(f)$ for every u, v, and $f \in C^{\infty}(M)$. Hence putting $u \equiv 1$ we see that $A(1) =$ constant function $(= c_1)$ and $c_1 \langle f, v \rangle = \langle Af, v \rangle$. Therefore $A(f) = c_1 f$ for every $f \in C^{\infty}(M)$.

III. Proof of Theorem 1

1. By the same way as in the proof of Proposition 1 (Step 4) we see that there exists an automorphism A: $C^{\infty}(M) \simeq C^{\infty}(M)$ such that $\alpha(P) = A \circ P \circ A^{-1}$ for all $P \in L^{\infty}(M)$. If P is a vector field, then by the assumption (ii) we have $A \circ P \circ A^{-1}(1) = 0$, which means that $P(A^{-1}1) = 0$ for every vector field P. Hence we see that $A^{-1}(1) =$ constant function. So $A(1)$ is also a constant function. Hence we can put $A(1) \equiv 1$.

2. For an element $f \in C^{\infty}(M)$ we denote by $M_f \in L^0(M)$ the operator $M_f(g) = f \cdot g$. Let $X_1, \ldots, X_l$ be vector fields on M such that at each point $x \in M$, the tangent space $T_x M$ is spanned by $X_{1,x}, \ldots, X_{l,x}$, then the differential operator $\sum_{i=1}^{l} X_i^2$ is elliptic. By the assumption (i) the operator $\alpha(\sum_{i=1}^{l} X_i^2)$ is also elliptic. Because an operator $P \in L^m(M)$ is elliptic, if and only if, there exists an $Q \in L^{-m}(M)$ such that $P \circ Q - Id \in L^{-\infty}(M)$, where Id is the identity operator. Therefore $\{\alpha(X_i)\}_{i=1}^{l}$ also spans the tangent space $T_x M$ at each point $x \in M$.

3. For $P\in L^m(M)$ we denote by $\sigma_P(x,\theta)$ the total symbol of P with respect to a local coordinate system (x,θ). For $f\in C^\infty(M)$ and $X\in X(M)$ we have

$$\sigma_{\alpha(fX)}(x,\theta)\sim\sum_{\gamma}(iD_\theta)^\gamma(\sigma_{\alpha(M_f)}(x,\theta))\cdot D_x^\gamma(\sigma_{\alpha(X)}(x,\theta))/\gamma!,$$

where $D_x^\gamma=(1/i)^{|\gamma|}\dfrac{\partial^{|\gamma|}}{\partial x^\gamma}$, $\gamma=(\gamma_1,\dots,\gamma_n)$ and $|\gamma|=\sum\gamma_i$. This is the asymptotic expansion formula for the total symbol of the composition of two operators $\alpha(M_f)$ and $\alpha(X)$. By the assumption (ii) the total symbol of $\alpha(fX)$ is of the form:

$$\sigma_{\alpha(fX)}(x,\theta)=\sum_{k=1}^{n}a_k(f;x)\theta_k.$$

Also the total symbol of $\alpha(M_f)$ has an asymptotic expansion:

$$\sigma_{\alpha(M_f)}(x,\theta)\sim\sum_{k=-\infty}^{k_0}\sigma_k(f;x,\theta),$$

with $\sigma_k(f;x,\theta)$ homogeneous of degree k. From these we have

$$\sigma_{k_0}(f;x,\theta)\sum_{n=1}^{k}a_k(1;x)\theta_k=\sum_{k=1}^{n}a_k(f;x)\theta_k,\quad\text{and also}$$

we see that k_0 must be zero (see Step 2, above). Hence $\alpha(M_f)\in L^0(M)$ for all $f\in C^\infty(M)$. Simultaneously we have

$$\sigma_0(f;x,\theta)=\sigma_0(f;x)\in C^\infty(M),$$

that is, the prinicpal symbol of the operator $\alpha(M_f)$ is the lifting of a function on M. Inductively we have

$$\sigma_k(f;x,\theta)=0\qquad\text{for}\quad k<0,$$

by the above asymptotic expansions. Consequently we can conclude that

$$\alpha(M_f)-M_{\phi(f)}=R_f\in L^{-\infty}(M),$$

where we put $\phi(f)(x)=\sigma_0(f;x)$.

Also we have

$R_f\circ\alpha(X)=0$ for all vector fields X. This follows from the equality:

$$R_f\circ\alpha(X)=\alpha(fX)-M_{\phi(f)}\circ\alpha(X).$$

Because the right hand side is a first order differential operator and the left is in $L^{-\infty}(M)$. So both sides must be zero.

The map ϕ: $C^\infty(M)\to C^\infty(M)$, $f\mapsto\phi(f)$ satisfies

(i) $\phi(f\cdot g)=\phi(f)\cdot\phi(g)$,

(ii) $\phi(1)=1$.

4. From the formula $X\circ M_f-M_f\circ X=M_{X(f)}$ $(X\in X(M))$ we have

$$\alpha(X)(\phi(f))=\phi(X(f)).$$

We can choose such $f_i \in C^\infty(M)$ and $X_i \in X(M)$, $i=1,\dots, s$, that $\sum_{i=1}^{s} X_i(f_i) \equiv 1$ (see Lemma 1 below). Hence from the formula

$$A \circ M_f \circ A^{-1} = \alpha(M_f) = M_{\phi(f)} + R_f,$$

we have

$$A(f) = \phi(f) + R_f(1) = \phi(f) + R_f(\sum \alpha(X_i)(\phi(f_i))) = \phi(f).$$

Finally this implies that there exists a diffeomorphism $F\colon M \stackrel{\sim}{\to} M$ such that $A = F^*$.

Remark 2. The correspondence between α and F is as follows:

(i) From the above proof we have $\alpha(M_f) = M_{\phi(f)}$, $f \in C^\infty(M)$.

(ii) Since every maximal ideal I in $C^\infty(M)$ is of the form $I = I_x = \{f; f(x)=0\}$, $F(x) = y$ if and only if $\phi(I_x) = I_y$.

Lemma 1. *Let $I = (-1, 1)$ be an open interval.* (i) *Let φ, σ and $\rho \in C_0^\infty(I^n)$ be such that*

$\operatorname{supp}[\varphi] \subset \{x \in I^n;\ \sigma(x) = 1\}$ *and*

$\operatorname{supp}[\sigma] \subset \{x \in I^n;\ \rho(x) = 1\}$.

Then we have

$$\sigma(x) \frac{\partial}{\partial x_1} \left(\rho(x) \cdot \int_{-1}^{x_1} \varphi(t, x_2, \dots, x_n) dt\right) \equiv \varphi(x).$$

(ii) *Let $\{\varphi_i\}_i$ be a partition of unity on a paracompact manifold M. Here we assume that each φ_i has its support in a coordinate neighborhood U_i diffeomorphic to I^n ($n = \dim M$). Let σ_i, ρ_i and φ_i ($=\varphi$) be as in* (i), *and put*

$$X_i = \sigma_i(x) \frac{\partial}{\partial x_1} \quad \text{and} \quad f_i(x) = \rho_i(x) \int_{-1}^{x_1} \varphi_i(t, x') dt.$$

Then we have $\sum_i X_i(f_i) = \sum \varphi_i \equiv 1$.

Proof. (i) follows by a straightforward caluculation:

$$\sigma(x) \left(\frac{\partial}{\partial x_1} \rho\right) \cdot \int_{-1}^{x_1} \varphi(t, x') dt + \sigma(x) \cdot \rho(x) \cdot \varphi(x) \equiv \varphi(x).$$

(ii) is also easily shown by noticing that X_i's and f_i's can be seen as globally defined vector fields and functions on M respectively.

IV. Proof of Corollary 2. It is enough to show that $\alpha(L^{-\infty}(M)) = L^{-\infty}(M)$. Assume that there exists a $P_0 \in L^m(M)$ such that $P_0 \notin L^{-\infty}(M)$ and $\alpha(P_0) \in L^{-\infty}(M)$, then we see that P_0 is not an elliptic operator by the same way as the step 2 in the proof of Theorem 1.

Let Q_0 be an elliptic operator of order $1-m$. By composing P_0 and Q_0 we can assume from the beginning that the above operator P_0 is in $L^1(M)$ and not in $L^0(M)$.

From the assumption for P_0 we see that the characteristic set $Ch(P_0) = \{\sigma_1(P_0) = 0\} \neq \emptyset$ and $Ch(P_0) \subsetneq T_0^*M$, where $\sigma_1(P_0)$ is the principal symbol of P_0.

Let X_i, $i=1,\dots,t$, be vector fields on M such that the operator

$$P_0^*\circ P_0+\sum_{i=1}^{t} X_i^2$$

is elliptic, but not elliptic $\sum_{i=1}^{t} X_i^2$ itself. Here $P_0^*\in L^1(M)$ is an adjoint operator of P_0 with respect to a suitable inner product in $C^\infty(M)$. Let $F: X \rightleftarrows X$ be the diffeomorphism mentioned in Theorem B such that $\alpha=\mathrm{d}F$, i.e., $\alpha(X)=(F^{-1})^*\circ X\circ F^*$ on $X(M)$. Then we have

$$\alpha(\sum X_i^2)+\alpha(P_0^*)\circ\alpha(P_0)$$

is elliptic and $\alpha(P_0^*\circ P_0)\in L^{-\infty}(M)$. Hence

$$\alpha(\sum_i X_i^2)=\sum_i \mathrm{d}F(X_i)\circ\mathrm{d}F(X_i)$$

is already elliptic, which contradicts that the operator $\sum X_i^2$ is not elliptic. Therefore there exist no such P_0, that is, $\alpha(L^{-\infty}(M))=L^{-\infty}(M)$.

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