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Local solutions to the initial and initial boundary value problem for the Boltzmann equation with an external force I

By

Kiyoshi ASANO

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Since Boltzmann introduced his equation in 1872 to interpret the behavior of gas particles, many authors have studied the existence (and behavior) of solutions of the Boltzmann equation. As for the spatially homogeneous case, the global existence theory of the initial value problem was established at earlier time by Carleman [4], Arkeryd and others. Unfortunately the physical consevative quantities such as total mass and energy (and also entropy) are not useful in the existence theory for the spatially non-homogeneous Boltzmann equation, while they played decisive roles in the study of spatially homogeneous equation. Thus only two types of existence theorems are expected for the initial value problem, the local eixstence theorems for arbitrary initial data (at least in a wide class), and the global existence for sufficiently small (i.e. near an equilibrium) initial data, until a strong idea lightens the whole problem some day.

Grad [9], [10] and [11] introduced the concept of angular cuoff scattering potentials in the calculation of collision integrals, and then established the fundamental frame work for the global existence theory of the initial value problem, after he showed the local existence of solutions for the Boltzmann equation of Maxwell gas ([8] §20).

Since 1974 Ukai [17], [18], Nishida-Imai [14], Shizuta-Asano [16] and Ukai-Asano [20] (and also Asano [1] and [2]) showed the global existence (and asymptotic appraoach to the prescribed equilibrium) of solutions for the initial or initial boundary value problem of the Boltzmann equation with an angular cutoff hard potential. The initial data were always assumed to be sufficiently small in some sense.

Recently Caflisch [3] and Ukai-Asano [19] proved the corresponding results for the initial value problem of the angular cutoff soft potential case. In this case, it is easy to prove the local existence of solutions for large initial data.

All these works were done for the Boltzmann equation without an external force. Only Glikson [6] and [7] considered that equation and showed the unique local existence of solutions to the initial value problem for sufficiently small initial data. While Kaniel-Shinbrot [13] showed the local existence of solutions to the initial boundary value problem of the Boltzmann equation without any external force and with a general (hard or soft) angular cutoff potential. Their initial data might be arbitrary large and the solution exists in a time interval determined by the size of the initial data.

In this paper we show the local existence of solutions to the initial and initial boundary value problem of the Boltzmann equation with an external force and a general (hard or soft) angular cutoff scattering potential. The initial data may be arbitrary large in some function spaces and the solution exists in a time interval corresponding to the size of the initial data. Our method is a combination of those of Glikson and Kaniel-Shinbrot to some extent, but simpler and more natural.

This paper is the first part of the study and concerned with the initial value problem and the initial boundary value problem with the reverse or specular boundary condition at the boundary. In the second part we will study more general boundary conditions including so called Maxwell's boundary condition (i.e. the convex combination of specular and diffuse boundary condition), and show the local existence of solutions.

Finally we note that in our scheme the solution exists in a backword time interval as well as in the forword. (The positivity is not preserved in the backwrod.) This might show that the scheme is not adequate in the study of the global existence.

§ 1. Introduction

Let $f=f(t, x, \xi)$ be the density distribution of interacting gas particles at time $t\geq 0$ and position $x\in\Omega\subset\boldsymbol{R}^n$ with velocity $\xi\in\boldsymbol{R}^n$, $n\geq 3$. The change of f is described by the Boltzman equation:

$$\text{(1.1)}\qquad \begin{aligned} &\frac{\partial f}{\partial t}+\xi\cdot\nabla_x f+a(x)\cdot\nabla_\xi f=Q[f, f],\\ &f|_{t=0}=f_0(x, \xi) \qquad \text{(the initial condition)}, \end{aligned}$$

where $\xi\cdot\nabla_x=\xi_1\dfrac{\partial}{\partial x_1}+\cdots+\xi_n\dfrac{\partial}{\partial x_n}$, $a(x)\cdot\nabla_\xi=a_1(x)\dfrac{\partial}{\partial\xi_1}+\cdots+a_n(x)\dfrac{\partial}{\partial\xi_n}$, $a(x)$ is the external force and $Q[\cdot,\cdot]$ is the collision integral.

If $\Omega=\boldsymbol{R}^n$, the initial value problem (1.1) is well posed in a suitable sense stated later. If Ω is a domain in $\boldsymbol{R}^n$ with a piecewise smooth boundary $S=\partial\Omega$, we need a boundary condition:

$$\text{(1.2)}\qquad f|_{\tilde{S}^+}=C(f|_{\tilde{S}^-}).$$

Here we define with the unit inner normal $n(x)$ at $x\in S$

$$\text{(1.3)}\qquad \begin{aligned} &S^\pm(\xi)=\{x\in S;\ \langle n(x), \xi\rangle\gtrless 0\} \quad \text{and}\\ &\tilde{S}^\pm=\bigcup_{\xi\neq 0} S^\pm(\xi)\times\{\xi\}, \end{aligned}$$

and C is a linear operator from a suitable function space on $\tilde{S}^-$ to a similar one on $\tilde{S}^+$. With a suitable "dissipative condition" on the operator C, the initial boundary value problem (1.1)–(1.2) will be well posed.

We state assumptions.

[A] The external force $a(x)$ is a (piecewise) smooth potential force whose potential $b(x)$ is bounded from below:

$$a(x) = -\nabla_x b(x),\ b(x) \geq 1 \quad \text{and} \quad b(x) \in C^2(\bar{\Omega}).$$

[Q] The scattering cross section $q(v, \theta) \geq 0$ in the collision integral is induced from the inverse power intermolecular force and satisfies the angular cutoff assumption of Grad [9]:

(Q.1) $q_1(V) = \int_{S^{n-1}} q(V, \theta) d\omega$ exists and is continuous in $(0, \infty)$, where $V = |\xi - \eta|$ and $\langle \xi - \eta, \omega \rangle = |\xi - \eta| \cos\theta$.

(Q.2) $$0 \leq q_1(V) \leq C_0(V^\lambda + V^\mu), \quad -n < \lambda \leq \mu \leq 2.$$

For the later use we write the explicit form of $Q[f, f]$:

(1.4) $$Q[f, f](\xi) = \iint_{R^n \times S^{n-1}} q(|\xi - \xi_1|, \theta) \{f(\xi') f(\xi_1') - f(\xi) f(\xi_1)\} d\xi_1 d\omega,$$

$$\xi' = \xi - \langle \xi - \xi_1, \omega \rangle \omega, \quad \langle \xi - \xi_1, \omega \rangle = |\xi - \xi_1| \cos\theta$$

$$\xi_1' = \xi_1 + \langle \xi - \xi_1, \omega \rangle \omega,\ \omega \in S^{n-1} \quad \text{and}$$

$$f(\xi') = f(t, x, \xi'), \qquad f(\xi_1') = f(t, x, \xi_1') \quad \text{etc.}$$

Sometimes we rewrite (1.4) as

(1.4)′ $$\begin{aligned} &Q[f, f] = Q_1[f, f] - v[f]f \\ &v[f](\xi) = \iint q(|\xi - \xi_1|, \theta) f(\xi_1) d\xi_1 d\omega = \int q_1(|\xi - \xi_1|) f(\xi_1) d\xi_1. \end{aligned}$$

We note that v, Q_1 and Q are invariant under the orthogonal transformations in R^n_ξ.

Assuming $\Omega = \boldsymbol{R}^n$, we consider the bicharacteristic equation of (1.1) in $R^n_x \times R^n_\xi$:

(1.5) $$\begin{aligned} &\frac{dx}{dt} = \xi, \qquad \frac{d\xi}{dt} = a(x), \\ &x|_{t=0} = x_0, \qquad \xi|_{t=0} = \xi_0. \end{aligned}$$

We describe the (unique) solution of (1.5) by

(1.6) $$x = X(t, x_0, \xi_0), \qquad \xi = \Xi(t, x_0, \xi_0).$$

If we define the energy function (Hamiltonian) $e(x, \xi)$ [6], [7] by

(1.7) $$e(x, \xi) = b(x) + \frac{1}{2} |\xi|^2,$$

then we can show easily that $e(x, \xi)$ is constant on each bicharacteristic curve, i.e, (1.8) $$e(X(t, x_0, \xi_0), \Xi(t, x_0, \xi_0))=e(x_0, \xi_0).$$

This implies that the solution (1.6) exists globally in time for each $(x_0, \xi_0) \in \boldsymbol{R}^n \times \boldsymbol{R}^n$. If we define a family of mappings $\{T_t;\ t \in R\}$ by

(1.9) $$T_t(x, \xi)=(X(t, x, \xi), \Xi(t, x, \Xi)),$$

then $\{T_t\}$ is a group of (piecewise) C^1-diffeomorphisms in $\boldsymbol{R}^{2n}$. In particular, we have

(1.10) $$T_t T_{-t}=T_0=\text{identity map}.$$

The following lemma is needed later.

Lemma 1.1. *Assume [A]. Let $T>0$ and $R>0$, and define the subset $B(R)=\{(x, \xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n;\ |(x, \xi)|^2=|x|^2+|\xi|^2 \leq R^2\}$ of $\boldsymbol{R}^n \times \boldsymbol{R}^n$. Then, there exists a constant $c=c(T, R)>0$ such that T_t maps the outside of $B(cR)$ into the outside of $B(R)$ for $|t| \leq T$.*

Proof. The continuity of $T_t(x, \xi)$ in (t, x, ξ) implies that T_t maps $B(R)$ into $B(cR)$ for some $c>0$, $|t| \leq T$. This and (1.10) show the lemma.

Now we define function spaces on $\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n$. Let E_α be the set of all measurable (or continuous) functions $f(x, \xi)$ on $\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n$ such that

(1.11) $$|f|_\alpha=\sup_{x, \xi} e^{\alpha e(x, \xi)}|f(x, \xi)|<\infty \quad (\alpha \geq 0).$$

The subspace $\dot{E}_\alpha$ of E_α is defined with the same norm $|\ |_\alpha$ by $\dot{E}_\alpha=\{f(x, \xi) \in E_\alpha;\ \sup_{|x|+|\xi| \geq r} e^{\alpha e(x, \xi)}|f(x, \xi)| \to 0 \text{ as } r \to \infty\}$.

Let $E_{\rho,\gamma,T}$ be the set of all measurable (or continuous) functions $f(t, x, \xi)$ on $[0, T] \times R_x^n \times R_\xi^n$ such that $f(t, \cdot) \in E_{\rho-\gamma t}$, $0 \leq t \leq T$, and

[1.12] $$|f|_{\rho,\gamma,T}=\sup_{0 \leq t \leq T} \sup_{x, \xi} e^{(\rho-\gamma t) e(x, \xi)}|f(t, x, \xi)|<\infty.$$

$\dot{E}_{\rho,\gamma,T}$ is defined similary, i.e,

$$\dot{E}_{\rho,\gamma,T}=\{f(t, x, \xi) \in E_{\rho,\gamma,T};\ f(t, \cdot) \in \dot{E}_{\rho-\gamma t},\ 0 \leq t \leq T\}.$$

Finally we define a group $U_0(t)$ of operators by

(1.13) $$U_0(t) f(x, \xi)=f(T_{-t}(x, \xi))=f(X(-t, x, \xi), \Xi(-t, x, \xi)).$$

$U_0(t)$ is a contraction mapping in E_α, $\dot{E}_\alpha$, $E_{\rho,\gamma,T}$ and $\dot{E}_{\rho,\gamma,T}$, because of Lemma 1.1. Clearly $U_0(t)$ is generated by $A_0=-\xi \cdot \nabla_x-a(x) \cdot \nabla_\xi$. If $\dot{E}_\alpha$ consists of continuous functions only, then $U_0(t)$ is a C_0 semi-group in $\dot{E}_\alpha$. We note that $U_0(t)$ has the following properties:

(1.14) $$\begin{aligned} & U_0(t)(fg)=(U_0(t) f)(U_0(t) g), \\ & |U_0(t) f|=U_0(t)|f|, \\ & U_0(t) 1=1 \end{aligned}$$

$$U_0(t)f \leq U_0(t)g \quad \text{if} \quad f \leq g.$$

The equation (1.1) with $\Omega = \boldsymbol{R}^n$ is almost equivalent to

$$f(t, x, \xi) = U_0(f)f_0 + \int_0^t U_0(t-s)Q[f(s), f(s)]ds. \tag{1.15}$$

In this paper we are concerned with solutions of (1.15) or the corresponding evolutional integral equation (3.6). They are called the mild solutions of (1.1) or (1.1)–(1.2).

§ 2. The initial value problem

We ask for the solution of (1.15) in the space $E_{\rho,\gamma,T}$. Let $f_0 \in E_\rho$, $\rho > 0$, i.e,

$$e^{\rho e(x,\xi)} |f_0(x, \xi)| \leq |f_0|_\rho. \tag{2.1}$$

We put

$$\begin{aligned} &f_0(t, x, \xi) = f_0(x, \xi), \\ &f_{l+1}(t, x, \xi) = U_0(t)f_0 + \int_0^t U_0(t-s)Q[f_l(s), f_l(s)]ds \end{aligned} \tag{2.2}$$

for $l = 0, 1, \ldots$. Using (1.14), we have

$$\begin{aligned} &e^{(\rho-\gamma t)e(x,\xi)} |f_{l+1}(t, x, \xi)| \leq |f_0|_\rho + \\ &+ \int_0^t e^{-\gamma(t-s)e(x,\xi)} U_0(t-s)\{e^{(\rho-\gamma s)e(x,\xi)} |Q[f_l(s), f_l(s)]|\} ds. \end{aligned} \tag{2.3}$$

To estimate the second term on the righthand side, we need

Lemma 2.1. *Assume the condition* $[Q]$ *and let* $f \in E_\alpha$, $\alpha > 0$. *Then we have*

$$\begin{aligned} &|Q_1[f, f]| \leq Q_1[|f|, |f|] \leq \nu_\alpha(\xi)\, e^{-\alpha}\, e^{-\alpha e(x,\xi)} |f|_\alpha^2, \\ &|\nu[f]| \leq \nu[|f|] \leq \nu_\alpha(\xi)\, e^{-\alpha} |f|_\alpha, \end{aligned} \tag{2.4}$$

where

$$\nu_\alpha(\xi) \equiv \int q_1(|\xi - \xi_1|)\, e^{-\alpha|\xi_1|^2/2}\, d\xi_1 \leq C(\alpha)\left(1 + \frac{1}{2}|\xi|^2\right)^{\mu/2}. \tag{2.5}$$

Moreover, $\nu_\alpha(\xi)^{-1}Q_1$ *maps* E_α *(resp.* $\dot{E}_\alpha$*) into* E_α *(resp.* $\dot{E}_\alpha$*) continuously.*

Proof. Using the equality $|\xi'|^2 + |\xi_1'|^2 = |\xi|^2 + |\xi_1|^2$, we have

$$\begin{aligned} |Q_1[f, f](\xi)| &\leq \iint_{\boldsymbol{R}^n \times S^{n-1}} q(|\xi - \xi_1|, \theta) |f(\xi')|\, |f(\xi_1')| d\xi_1 d\omega \\ &\leq e^{-\alpha e(x,\xi)} \iint q(|\xi - \xi_1|, \theta) |f|_\alpha^2\, e^{-\alpha e(x,\xi_1)}\, d\xi_1 d\omega \\ &= e^{-\alpha e(x,\xi)} |f|_\alpha^2 \int q_1(|\xi - \xi_1|) e^{-\alpha e(x,\xi_1)}\, d\xi_1 \\ &= e^{-\alpha e(x,\xi)} |f|_\alpha^2\, e^{-\alpha b(x)}\, \nu_\alpha(\xi). \end{aligned}$$

This proves the first inequality of (2.4). The second is proved in a similary way.

The assumption [Q] implies

$$\nu_\alpha(\xi) \le \int C_0(|\xi-\xi_1|^\lambda + |\xi-\xi_1|^\mu)\, e^{-\alpha|\xi_1|^2/2}\, d\xi_1,$$

from which (2.5) is easily obtained.

Using lemma 2.1, we obtain for $\rho-\gamma s>0$

$$\text{(2.6)}\qquad \begin{aligned} e^{(\rho-\gamma s)e(x,\xi)}\,|Q[f_l(s), f_l(s)]| &\le 2\nu_{\rho-\gamma s}(\xi)\, e^{-(\rho-\gamma s)}\,|f_l(s)|^2_{\rho-\gamma s} \\ &\le 2\nu_{\rho-\gamma t}(\xi)\, e^{-(\rho-\gamma t)}\,|f_l|^2_{\rho,\gamma,t}, \quad 0\le s\le t. \end{aligned}$$

Hence we can estimate the integral of (2.3) as follows:

$$\text{(2.7)}\qquad \begin{aligned} &\int_0^t e^{-\gamma(t-s)e(x,\xi)} U_0(t-s)\{\dots\}\, ds \\ &\le \int_0^t e^{-\gamma(t-s)e(x,\xi)} 2\nu_{\rho-\gamma s}(\Xi(-t+s, x, \xi)) e^{-(\rho-\gamma t)} |f_l|^2_{\rho,\gamma,t} ds \\ &\le \int_0^t e^{-\gamma(t-s)e(x,\xi)}\, 2C(\rho-\gamma t)\left\{b(x) + \frac{1}{2}|\xi|^2\right\}^{\mu/2} e^{-(\rho-\gamma t)}\,|f_l|^2_{\rho,\gamma,t}\, ds \\ &\le 2C(\rho-\gamma t)\, e^{-(\rho-\gamma t)}\,\frac{1}{\gamma}\, e(x,\xi)^{\mu/2-1}|f_l|^2_{l,\gamma,t}, \end{aligned}$$

since we have by (2.5) and (1.8)

$$\begin{aligned} \nu_\alpha(\xi(-s, x, \xi)) &\le C(\alpha)\left\{1 + \frac{1}{2}|\Xi(-s, x, \xi)|^2\right\}^{\mu/2} \\ &\le C(\alpha)\left\{b(X(-s, x, \xi)) + \frac{1}{2}|\Xi(-s, x, \xi)|^2\right\}^{\mu/2} \\ &= C(\alpha)e(-s, x, \xi) = C(\alpha)e(x, \xi). \end{aligned}$$

From (2.3) and (2.7) we obtain for $0\le t\le T=\rho/2\gamma$

$$\text{(2.8)}\qquad |f_{l+1}|_{\rho,\gamma,t} \le |f_0|_\rho + 2C\left(\frac{\rho}{2}\right) e^{-\rho/2}\,\frac{1}{\gamma}\,|f_l|^2_{\rho,\gamma,t}.$$

If we choose $\gamma>0$ satisfying the inequality

$$\text{(2.9)}\qquad D \equiv 1 - 8C\left(\frac{\rho}{2}\right) e^{-\rho/2}\,\frac{1}{\gamma}\,|f_0|\rho > 0,$$

then the following algebraic equation has two positive roots $0<X_0<X_1$:

$$\text{(2.10)}\qquad 2C\left(\frac{\rho}{2}\right) e^{-\rho/2}\,\frac{1}{\gamma}\, X^2 - X + |f_0|_\rho = 0.$$

Clearly we have

$$|f_0| < X_0 = \frac{2|f_0|_\rho}{1+\sqrt{D}} < 2|f_0|_\rho, \tag{2.11}$$

$$4C\left(\frac{\rho}{2}\right)e^{-\rho/2}\frac{1}{\gamma}X_0 < 1,$$

which shows that (2.8) implies

$$|f_l|_{\rho,\gamma,T} \le X_0 < 2|f_0|_\rho. \tag{2.12}$$

From (2.2) we have

$$f_{l+1}(t) - f_l(t) = \int_0^t U_0(t-s)\{Q[f_l(s), f_l(s)] - Q[f_{l-1}(s), f_{l-1}(s)]\}ds. \tag{2.13}$$

A similar calculation as above shows

$$|f_{l+1} - f_l|_{\rho,\gamma,T} \le 2C\left(\frac{\rho}{2}\right)e^{-\rho/2}\frac{1}{\gamma}|f_l + f_{l-1}|_{\rho,\gamma,T}|f_l - f_{l-1}|_{\rho,\gamma,T} \tag{2.14}$$

$$\le 4C\left(\frac{\rho}{2}\right)e^{-\rho/2}\frac{1}{\gamma}X_0|f_l - f_{l-1}|_{\rho,\gamma,T}.$$

Thus (2.11) implies that $\{f_l\}$ is convergent in $E_{\rho,\gamma,T}$. The limit $f(t)$ satisfies in $E_{\rho,\gamma,T}$ the equality

$$f(t) = U_0(t-s)f_0 + \int_0^t U_0(t-s)Q[f(s), f(s)]ds. \tag{2.15}$$

Note 1. For a given $\rho > 0$, γ and T are calculated by

$$\gamma = \gamma(\rho)|f_0|_\rho, \tag{2.16}$$

$$T = \frac{\rho}{2\gamma} = \frac{\rho}{2\gamma(\rho)|f_0|_\rho} = \frac{T(\rho)}{|f_0|_\rho}. \tag{2.17}$$

Note 2. If $\mu \le 0$ in the assumption [Q] (This occurs in the angular cutoff soft potential case. See Grad [9]), we have

$$|f_{l+1}(t)|_\rho \le |f_0|_\rho + 2C(\rho)e^{-\rho}t|f_l(t)|^2 \tag{2.18}$$

instead of (2.8), since $\nu_\rho(\xi) \le C(\rho)$ for $\mu \le 0$. Thus we can show that the solution $f(t)$ is in $E_{\rho,\gamma,T}$.

Note 3. If $f_0 \in \dot{E}_\rho$, then all $f_l(t)$ is in $\dot{E}_{\rho,\gamma,T}$. Hence the solution $f(t)$ is in $\dot{E}_{\rho,\gamma,t}$.

If $f_0 \in \dot{E}_\rho$ is continuous in (x, ξ), then $f_l(t, x, \xi)$ is continuous in (t, x, ξ) for $l = 0, 1, \dots$. Thus the solution $f(t, x, \xi)$ of (2.15) is also continuous in (t, x, ξ). Moreover $f(t, x, \xi)$ satisfies

$$\left[\frac{\partial}{\partial t} + \xi \cdot \nabla_x + a(x) \cdot \nabla_\xi\right] f(t, x, \xi) = Q[f(t, x, \cdot), f(t, x, \cdot)](\xi). \tag{2.19}$$

To prove the uniqueness of the solution of (2.15), we assume that $f(t)$ and $g(t)$

be the mild solutions of (2.15) in $E_{\rho,\gamma,T}$ with the initial data f_0 and g_0. We choose $\gamma_1 \geq \gamma$ so that the condition (2.9) holds for $|f_0|_\rho$ and $|g_0|_\rho$. If we take $T_1 = \rho/2\gamma_1$ according to (2.17), then we can show easily

$$(2.20) \qquad |f|_{\rho,\gamma_1,T_1}, \quad |g|_{\rho,\gamma_1,T_1} \leq X_0,$$

where X_0 satisfies the second inequality of (2.11) with γ replaced by γ_1, as the root of the corresponding quadratic equation. In a similar way as used to show (2.14), we obtain

$$(2.21) \qquad |f-g|_{\rho,\gamma_1,T_1} \leq |f_0-g_0|_\rho + 4C\left(\frac{\rho}{2}\right)e^{-\rho/2}\frac{1}{\gamma_1}X_0|f-g|_{\rho,\gamma_1,T_1}.$$

This proves the uniqueness in $[0, T_1]$. Iterating this argument, we can prove the uniqueness in $[0, T]$.

Now we state theorems.

Theorem 2.2. *Assume [A] and [Q], and let $f_0 \in E_\rho(\rho>0)$. Then the initial value problem (1.1) has a unique mild solution $f(t)$ in $E_{\rho,\gamma,T}$, where γ and T are determined by (2.16) and (2.17). If $f_0 \in \dot{E}_\rho$, then $f(t) \in \dot{E}_{\rho,\gamma,T}$. If $f_0 \in \dot{E}_\rho$ and is continuous in (x, ξ), then $f(t)$ is also continuous in (t, x, ξ) and satisfies (2.19).*

Theorem 2.3. *Assume [A] and [Q]. Let $f_0 \in E_\rho$ $(\rho>0)$ and $f_0(x, \xi) \geq 0$ a.e. Then the mild solution $f(t)$ of (1.1) is also non-negative a.e.*

To prove Theorem 2.3 we need a slightly different iteration scheme. First, we consider the linear equation

$$(2.22) \qquad \begin{aligned} &\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + a(x) \cdot \nabla_\xi f + v(t, x, \xi)f = 0, \\ &f|_{t=\tau} = f_0(x, \xi), \quad (x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n. \end{aligned}$$

The above equation is very easily solved. We define a "group of operators" $U(t, \tau; v)$ to solve (2.22):

$$(2.23) \qquad \begin{aligned} U(t, \tau; v)f_0(x, \xi) &= e^{-\int_\tau^t v(s, T_{-t+s}(x,\xi))ds} f_0(T_{-t+\tau}(x, \xi)) \\ &= \exp\left\{-\int_\tau^t U_0(t-s)v(s)ds\right\}U_0(t-\tau)f_0. \end{aligned}$$

If $v(t, x, \xi) \geq 0$, then $U(t, \tau; v)$ is a contraction in E_ρ and $\dot{E}_\rho$ for $t \geq \tau$. $U(t, \tau; v)$ is generated by $A(t) = -\xi \cdot \nabla_x - a(x) \cdot \nabla_\xi - v(t)$ and has the following properties

$$(2.24) \qquad \begin{aligned} &U(t, \tau; v)U(\tau, s; v) = U(t, s; v), \quad t \geq \tau \geq s, \\ &U(t, t; v) = \text{identity map}, \\ &|U(t, \tau; v)f| = U(t, \tau; v)|f|, \\ &U(t, \tau; v)f \leq U(t, \tau; v)g \quad \text{if} \quad f \leq g. \end{aligned}$$

We define $\{f_l(t)\}$ by

$$(2.25)\quad \begin{aligned} &f_0(t)=f_0(x,\xi)\geq 0,\\ &\frac{\partial f_{l+1}}{\partial t}+(\xi\cdot\nabla_x+a(x)\cdot\nabla_\xi)f_{l+1}=Q_1[f_l,f_l]-\nu[f_l]f_{l+1},\ f_{l+1}|_{t=0}=f_0(x,\xi).\end{aligned}$$

Then $f_{l+1}(t)$ is solved by

$$(2.26)\quad \begin{aligned} f_{l+1}(t)=&\,U(t,0;\nu[f_l])f_0\\ &+\int_0^t U(t,s;\nu[f_l])Q_1[f_l(s),f_l(s)]ds.\end{aligned}$$

Hence $f_{l+1}(t)\geq 0$, if $f_l(t)\geq 0$.

By a similar calculation as used to show (2.8) we obtain

$$(2.27)\quad |f_{l+1}|_{\rho,\gamma,t}\leq|f_0|_\rho+C\left(\frac{\rho}{2}\right)e^{-\rho/2}\frac{1}{\gamma}|f_l|^2_{\rho,\gamma,t},\quad 0\leq t\leq T=\frac{\rho}{2\gamma}.$$

If we take $\gamma>0$ and $X_0>0$ as in (2.9) and (2.10), we have the same estimates (2.11) and (2.12) for the new $f_l(t)$ defined by (2.25) or (2.26).

From (2.25) we obtain another representation of $f_l(t)$ similar to (2.2), i.e.,

$$(2.28)\quad f_{l+1}(t)=U_0(t)f_0+\int_0^t U_0(t-s)\{Q_1[f_l(s),f_l(s)]-\nu[f_l(s)]f_{l+1}(s)\}ds.$$

The same calculation used to derive (2.14) shows

$$(2.29)\quad |f_{l+1}-f_l|_{\rho,\gamma,T}\leq 2C\left(\frac{\rho}{2}\right)e^{-\rho/2}\frac{1}{\gamma}X_0\{|f_l-f_{l-1}|_{\rho,\gamma,T}+|f_{l+1}-f_l|_{\rho,\gamma,T}\}.$$

Taking the second inequality of (2.11) into account, we have

$$(2.30)\quad |f_{l+1}-f_l|_{\rho,\gamma,T}\leq 4C\left(\frac{\rho}{2}\right)e^{-\rho/2}\frac{1}{\gamma}X_0|f_l-f_{l-1}|_{\rho,\gamma,T}.$$

Thus the new $f_l(t)$ is also convergent in $E_{\rho,\gamma,T}$, and the limit $f(t)$ is the solution of (2.15) or the following

$$(2.31)\quad f(t)=U(t,0;\nu[f])f_0+\int_0^t U(t,s;\nu[f])Q_1[f(s),f(s)]ds.$$

The coincidence of the solutions of (1.1), (2.15) and (2.31) (and also those of (2.25), (2.26) and (2.28)) is proved easily by the integration theory. This completes the proof of Theorem 2.2.

§3. Initial boundary value problem - Reverse and specular reflection cases

In the study of the initial boundary value problem (1.1)–(1.2), we need the bicharacteristic curve starting from each point $(x,\xi)\in\Omega\times\boldsymbol{R}^n$ under some reflection law at the wall $S=\partial\Omega$. In this paper we study only two typical cases, that is, the reverse reflection and the specular reflection, which are described as below:

(3.1) $f(x, \xi)=f(x, -\xi), \quad (x, \xi)\in \tilde{S}^+$ (reverse refl.),

(3.2) $f(x, \xi)=f(x, \xi-2\langle n(x), \xi\rangle n(x)), \quad (x, \xi)\in \tilde{S}^+$ (specular refl.).

1. Reverse reflection boundary condition

First we state the basic assumptions on the spatial domain Ω.

$[\Omega]_R$ Ω is a domain in $\boldsymbol{R}^n$ with a piecewise smooth boundary $S=\partial\Omega$ which consists of at most a countable number of $(n-1)$-dimensional C^1 surfaces $S_1,\dots, S_{l_0}$ $(1\le l_0\le\infty)$, which are closed in $\boldsymbol{R}^n$.

The bicharacteristic equation of (1.1)–(3.1) is described as below:

$$\frac{dx}{dt}=\xi, \quad \frac{d\xi}{dt}=a(x),$$

(3.3)

$$x|_{t=0}=x_0, \quad \xi|_{t=0}=\xi_0, \quad (x_0, \xi_0)\in\Omega\times\boldsymbol{R}^n,$$

with the reverse reflection law

$$\xi(t\pm 0)=-\xi(t\mp 0)$$

(3.4)

$$\text{at } x(t+0)=x(t-0)=x(t)\in S.$$

We can construct easily the solution of (3.3)–(3.4) in the time interval $(-\infty, \infty)$. In fact, $(x(t), \xi(t))$ given by (1.6) is the solution of (3.3) in $\Omega\times\boldsymbol{R}^n$ in a maximal time interval $(-t^+(x_0, \xi_0), t^-(x_0, \xi_0))$. If $t^+(x_0, \xi_0)$ (resp. $^-t(x_0, \xi_0)$) is finite, we have

(3.5) $$X(-t^+(x_0, \xi_0), x_0, \xi_0)\in S$$

$$(\text{resp. } X(t^-(x_0, \xi_0), x_0, \xi_0)\in S).$$

If we extend $(x(t), \xi(t))$ by

$$x(t\mp t^\pm(x_0, \xi_0))=X(-t\mp t^\pm(x_0, \xi_0), x_0, \xi_0), \quad t\lessgtr 0$$

$$\xi(t\mp t^\pm(x_0, \xi_0))=-\Xi(-t\mp t^\pm(x_0, \xi_0), x_0, \xi_0), \quad t\lessgtr 0,$$

then $(x(t), \xi(t))$ satisfies the first two equation of (3.3) in $(-2t^+(x_0, \xi_0)-t^-(x_0, \xi_0), -t^+(x_0, \xi_0))$ or $(t^-(x_0, \xi_0), 2t^-(x_0, \xi_0)+t^+(x_0, \xi_0))$ with the reflection law (3.4). Repeating this procedure if necessary, we have the solution of (3.3)–(3.4) in the whole interval $(-\infty, \infty)$, which is also denoted without confusion by

$$x(t)=X(t, x_0, \xi_0),$$

(3.5)

$$\xi(t)=\Xi(t, x_0, \xi_0).$$

If we define T_t by (1.9) with (X, Ξ) constructed as above, then $\{T_t\}$ is a group of piecewise C^0- (and also C^1-) diffeomorphisms in $\Omega\times\boldsymbol{R}^n$, and enjoys the properties stated in (1.8) and Lemma 1.1.

We define the function spaces E_ρ and $\dot{E}_\rho$ on $\Omega\times\boldsymbol{R}^n$, and $E_{\rho,\gamma,T}$ and $\dot{E}_{\rho,\gamma,T}$ on $[0, T]\times\Omega\times\boldsymbol{R}^n$ as in §1. Then the operator $U_0(t)$ defined from the new $\{T_t\}$ by (1.13) is a contraction mapping in E_ρ and $\dot{E}_\rho$, and enjoys all the properties in (1.14).

Moreover, if f satisfies the boundary condition (3.1), then $U_0(t)f$ also does. Thus we can rewrite (1.1)–(1.2) (Here (1.2) means the reverse reflection boundary condition (3.1).) as in §1, which results to

$$f(t)=U_0(t)f_0+\int_0^t U_0(t-s)Q[f(s), f(s)]ds. \tag{3.6}$$

Clearly $Q[f(s), f(s)]$ satisfies the condition (3.1), if $f(s)$ does. Hence all the arguments in §2 are applicable to study (3.6). Thus we obtain the corresponding results to Theorem 2.2 and 2.3 with same constants. We note that in the successive approximation formula (2.2) or (2.26) $f_{l+1}(t)$ satisfies (3.1), if $f_l(t)$ does.

Theorem 3.1. *Assume* $[A]$, $[Q]$ *and* $[\Omega]_R$. *Let* $f_0\in E_\rho$ $(\rho>0)$. *Then the initial boundary value problem* (1.1)–(3.1) *has a unique mild solution* $f(t)$ *in* $E_{\rho,\gamma,T}$ *with the same* γ *and* T *as in Theorem* 2.2. *If* $f_0\in \dot{E}_\rho$, *then* $f(t)\in \dot{E}_{\rho,\gamma,T}$. *If* $f_0\in \dot{E}_\rho$, *is piecewise continuous on* $\bar{\Omega}\times \boldsymbol{R}^n$ *with discontinuities only on and not along the bicharacteristics defined by* (3.3)–(3.4), *and satisfies the reverse reflection law* (3.1), *then* $f(t, x, \xi)$ *is also piecewise continuous with discontinuities on and not along the bicharacteristics and satisfies* (2.19) *and* (3.1). *Moreover, if* $f_0(x, \xi)\geq 0$ *a.e. in* $\Omega\times \boldsymbol{R}^n$, *then* $f(t, x, \xi)\geq 0$ *a.e.*

2. Specular reflection boundary condition.

The bicharacteristic curve of (1.1)–(3.2) is given by (3.3) with the specular reflection law

$$\begin{aligned}&\xi(t\pm 0)=\xi(t\mp 0)-2\langle n(x(t)), \xi(t\mp 0)\rangle n(x(t)),\\ &\text{at } x(t+0)=x(t-0)=x(t)\in S.\end{aligned} \tag{3.7}$$

The construction of the solution of (3.3)–(3.7) is rather difficult, and carried over under the more restrictive assumption on Ω.

$[\Omega]_s$ Ω is a domain in $\boldsymbol{R}^n$ with the boundary $S=\partial\Omega$ which consists of a countable number of $(n-1)$ dimensional surfaces $S_1,\dots, S_{l_0}$ $(1\leq l_0<\infty)$, which are closed, regular and disjoint with each other. The curvature of S_i's is nuiformly bounded.

If we construct the solution of (3.3)–(3.7) in the time interval $(-\infty, 0]$ for each initial data $(x_0, \xi_0)\in\Omega\times \boldsymbol{R}^n$, then we have the existence theorem for the initial boundary value problem (1.1)–(3.2), combining the arguments used to show Theorem 2.2, 2.3 and 3.1. Therefore we focus our efforts to solve (3.3)–(3.7).

For any initial $(x_0, \xi_0)\in\Omega\times \boldsymbol{R}^n$, $(x(t), \xi(t))$ given by (1.6) is the unique solution of (3.3) in a maximal time interval $(-t^+(x_0, \xi_0), t^-(x_0, \xi_0))$, while $x(t)$ remains in Ω. If $t^+(x_0, \xi_0)<\infty$ (resp. $t^-(x_0, \xi_0)<\infty$), $x(t)$ reaches S, as $t\to -t^+(x_0, \xi_0)$ (resp. $t\to t^-(x_0, \xi_0)$). In other words

$$\begin{aligned}&X(-t^+(x_0, \xi_0), x_0, \xi_0)\in \bar{S}^+(\Xi(-t^+(x_0, \xi_0), x_0, \xi_0)\\ &(\text{resp. } X(t^-(x_0, \xi_0), x_0, \xi_0)\in \bar{S}^-(\Xi(t^-(x_0, \xi_0), x_0, \xi_0)\,.\end{aligned} \tag{3.8}$$

We put

$$X(-t^+(x_0, \xi_0), x_0, \xi_0)=X^+(x_0, \xi_0), \tag{3.9}$$
$$\Xi(-t^+(x_0, \xi_0), x_0, \xi_0)=\Xi^+(x_0, \xi_0).$$

Similarly we define $X^-(x_0, \xi_0)$ and $\Xi^-(x_0, \xi_0)$. Furthermore we put

$$C(x, \xi)=\xi-2\langle n(x), \xi\rangle n(x). \tag{3.10}$$

Roughly speaking, two cases occur at $(X^+(x_0, \xi_0), \Xi^+(x_0, \xi_0))$, that is,

(i) there exists a bicharacteristic curve $(x_1(t), \xi_1(t))$ starting from $(X^+(x_0, \xi_0), C(X^+(x_0, \xi_0), \Xi^+(x_0, \xi_0)))$ which remains in $\Omega\times \boldsymbol{R}^n$ for a time,

(ii) there exists no such a solution.

If $\langle n(X^+(x_0, \xi_0)), \Xi^+(x_0, \xi_0)\rangle>0$, then the case (i) occurs. In this case we can extend $(x(t), \xi(t))$ into $\Omega\times \boldsymbol{R}^n$ by the specular reflection law of (3.7). We have only to put

$$x(-t-t^+(x_0, \xi_0))=x_1(t), \tag{3.11}$$
$$\xi(-t-t^+(x_0, \xi_0))=\xi_1(t)$$

for $0\leq t<t^-(X^+(x_0, \xi_0)), C(X^+(x_0, \xi_0), \Xi^+(x_0, \xi_0)))$. When the extended $(x(t), \xi(t))$ reaches to $S\times \boldsymbol{R}^n$ and the case (i) occurs again, we repeat the same procedure.

If the case (ii) occurs at $(X^+(x_0, \xi_0), \Xi^+(x_0, \xi_0))$, we have to extend $(x(t), \xi(t))$ in the tangent bundle $T(S)$ of S. In this case we have

$$\langle n(X^+(x_0, \xi_0)), \Xi^+(x_0, \xi_0)\rangle=0. \tag{3.12}$$
$$\langle n(X^+(x_0, \xi_0)), C(X^+(x_0, \xi_0), \Xi^+(x_0, \xi_0))\rangle=0.$$

Denote by $\bar{a}(x)$ the orthogonal projection of $a(x)$ onto the tangent space $T_x(S)$ of S at $x\in S$. Then there holds

$$a(x)=\bar{a}(x)+a_1(x)n(x), \tag{3.13}$$
$$a_1(x)=\langle a(x), n(x)\rangle.$$

Let $\kappa(x)$ be the curvature form on S. A smooth curve $x=x(t)$ lies on S, if and only if the following equation holds

$$\frac{dx}{dt}=\xi(t),$$

$$\frac{d\xi}{dt}\equiv\langle\kappa(x)\xi, \xi\rangle n(x) \quad (\mathrm{mod}\ T_{x(t)}).$$

For the initial $(X_0, \Xi_0)\in T_{X_0}(S)$, we can solve the following "bicharacteristic equation"

$$\frac{dx}{dt}=\xi, \tag{3.14}$$
$$\frac{d\xi}{dt}=\langle\kappa(x)\xi, \xi\rangle n(x)+\bar{a}(x),$$

(3.15) $$x|_{t=0}=X_0, \quad \xi|_{t=0}=\Xi_0.$$

The solution $(X(t), \Xi(t))$ (or more precisely $(X(t, X_0, \Xi_0), \Xi(t, x_0, \Xi_0))$ of (3.14)–(3.15) exists uniquely for a time and satisfies the following equality:

$$\frac{d}{dt}\langle n(X(t)), \Xi(t)\rangle=0,$$

$$\frac{d}{dt}\left\{b(X(t))+\frac{1}{2}|\Xi(t)|^2\right\}=0.$$

Thus $e(X(t), \Xi(t))=e(X_0, \Xi_0)$ and $(X(t), \Xi(t))$ exists globaly in time. We can extend the solution of (3.3)–(3.7) by putting for $t\leq 0$

(3.16)
$$\begin{aligned} x(-t-t^+(x_0, \xi_0))&=X(t, X^+(x_0, \xi_0), C(X^+(x_0, \xi_0), \Xi^+(x_0, \xi_0))),\\ \xi(-t-t^+(x_0, \xi_0))&=\Xi(t, X^+(x_0, \xi_0), C(X^+(x_0, \xi_0), \Xi^+(x_0, \xi_0))). \end{aligned}$$

in the case (ii). Another choise of extension is possible under some condition. For example, we can pull the bicharacteristic curve into $\Omega\times \boldsymbol{R}^n$ if the following condition holds for $(X_0, \Xi_0)\in T_{X_0}(S)$,

(3.17) $\langle\kappa(X_0)\Xi_0, \Xi_0\rangle<a_1(X_0)$.

Under this condition we have only to solve the usual bicharacteristic equation (3.3) with initial (X_0, Ξ_0). Thus at any point of the "bicharacteristic curve" lying in $T(S)$ where (3.17) holds, we can pull back the curve into $\Omega\times \boldsymbol{R}^n$. However these changes of extensions give no influence to the specular reflection law (3.7).

Thus we have the solution $(x(t), \xi(t))$ of (3.3)–(3.7) in $(-\infty, 0]$. Similarly we can obtain the solution in $[0, \infty)$. Again we denote the solution of (3.3)–(3.7) by $(X(t, x_0, \xi_0), \Xi(t, x_0, \xi_0))$. Then we define T_t by (1.9) and $U_0(t)$ by (1.13) with the new bicharacteristics defined above. Clearly (1.8) and other properties (e.g. Lemma 1.1) enjoyed by the old one hold for this case. Exceptionally (1.10) hold for almost every point of $\bar{\Omega}\times \boldsymbol{R}^n$ and not for every point. But the proof of Lemma 1.1 is available, because $T_t(x, \xi)$ is piecewise continuous in (t, x, ξ). Especially $U_0(t)f$ satisfies (3.2), if f_0 does.

Thus all the arguments used for the reverse reflection case are applicable for this specular reflection case, and we have the following

Theorem 3.2. *Assume* $[A]$, $[Q]$ *and* $[\Omega]_s$. *Let* $f_0\in E_\rho$ $(\rho>0)$. *Then the initial boundary value problem* (1.1)–(3.2) *has a unique mild solution* $f(t)$ *in* $E_{\rho,\gamma,T}$ *with the same* γ *and* T *as in Theorem* 2.2. *If* $f_0\in E_\rho$, *then* $f(t)\in E_{\rho,\gamma,T}$. *If* $f_0\in \dot{E}_\rho$, *is piecewise continuous on* $\bar{\Omega}\times \boldsymbol{R}^n$ *with discontinuities only on and not along the bicharacteristics defined by* (3.3)–(3.7), *and satisfies the specular reflection law* (3.2), *then* $f(t, x, \xi)$ *is also piecewise continuous with discontinuities on and not along the bicharacteristics and satisfies* (2.19) *and* (3.2). *Moreover, if* $f_0(x, \xi)\geq 0$ *a.e. in* $\Omega\times \boldsymbol{R}^n$, *then* $f(t, x, \xi)\geq 0$ *a.e.*

INSTITUTE OF MATHEMATICS
YOSHIDA COLLEGE
KYOTO UNIVERSITY

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