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Scattering and spectral theory for the linear Boltzmann operator

Dedicated to Professor Sigeru MIZOHATA on his sixtieth birthday

By

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1. Introduction

In this paper we shall study the scattering and spectral theory for the linear Boltzmann operator. Consider the linear Boltzmann equation

$$(1.1) \qquad \frac{d}{dt} n(x, v, t) = -v \cdot grad_x n(x, v, t) + \int k(x, v', v) n(x, v', t) dv' - \sigma_a(x, v) n(x, v, t)$$
$$(x \in \boldsymbol{R}^d,\ v \in \boldsymbol{R}^d,\ t \in \boldsymbol{R}),$$

which describes a beam of non-self-interacting neutrons. A positive function $n(x, v, t)$ represents the neutron density at time t with position x and velocity v. The first term on the right-hand side of (1.1) describes the free classical motion of neutrons. The second term represents neutrons produced at a point (x, v) in phase space due to processes such as scattering and fission. Later we need

$$\sigma_p(x, v) = \int_{\boldsymbol{R}^d} k(x, v, v') dv'$$

which is the total rate of production at a point (x, v). The last term on the right-hand side of (1.1) represents the loss of neutrons from a point (x, v) in phase space due to scattering or to absorption.

We work in the Banach space $L^1(\boldsymbol{R}^{2d}_{x,v})$, because $L^1(\boldsymbol{R}^{2d}_{x,v})$ is a natural space for the linear Boltzmann operator. As we shall see later, under certain assumptions the linear Boltzmann operator

$$(Bn)(x, v) = v \cdot grad_x n(x, v) - \int k(x, v', v) n(x, v') dv' + \sigma_a(x, v) n(x, v)$$

generates a strongly continuous group $\{e^{-tB} \mid -\infty < t < \infty\}$. It is known that the dynamical operator e^{-tB} is positivity preserving (i.e., it leaves invariant $L^1_+(\boldsymbol{R}^{2d}_{x,v})$, the cone of positive functions in $L^1(\boldsymbol{R}^{2d}_{x,v})$) for $t\geqslant 0$, and that in general the interacting dynamics is only one-sided (see Simon [10]). On the other hand, the collision-free linear Boltzmann operator

$$B_0 = v\cdot grad_x$$

generates a strongly continuous group $\{e^{-tB_0}\mid -\infty<t<\infty\}$ where the dynamical operator e^{-tB_0} is positivity preserving for $t\in\boldsymbol{R}$. Thus the free dynamics is two-sided. The basic objects of the scattering theory are:

$$W_- = \operatorname*{s-lim}_{t\to-\infty} e^{tB}e^{-tB_0} \tag{1.2}$$

$$\tilde{W}_+ = \operatorname*{s-lim}_{t\to+\infty} e^{tB_0}e^{-tB} \tag{1.3}$$

and the scattering operator $S=\tilde{W}_+W_-$. Notice that the free dynamics e^{-tB_0} occurs in (1.2) and (1.3) for $t\leqslant 0$, but that e^{-tB} occurs only for $t\geqslant 0$.

In the first part of the paper, we investigate the range of the inverse wave operator $\tilde{W}_+$. More specifically, we prove that the range of $\tilde{W}_+$ is dense in $L^1(\boldsymbol{R}^{2d}_{x,v})$. We follow the line of Enss [2] which was exploited to show the asymptotic completeness for quantum mechanical potential scattering, but there are many differences in detail.

First, our work is carried out in the Banach space L^1 whereas Enss's work was performed in the Hilbert space L^2.

Secondly, we do not use any asymptotic equality of B and B_0. Since we treat the inverse wave operator, we need only to control e^{-tB_0}. In the Enss analysis, as Simon [9] pointed out, it is important to prove a certain asymptotic equality of an interacting and a free Hamiltonian.

Finally, so-called decomposition operators used to describe the Enss decomposition principle are merely multiplication operators in our case, while they are pseudo-differential operators in the case of quantum mechanical potential scattering. The proof of the Enss decomposition principle for the Boltzmann case is easier and more elementary than for the Schrödinger case. Indeed, we do not require the method of stationary phase (compare Simon [9], Perry [5], Davies [1]).

In the second part of the paper, we analyze the spectral properties of B_0 and B. We show that the spectrum of B_0 consists only of the residual spectrum and coincides with the imaginary axis. Moreover, using our result on $\tilde{W}_+$, we can show that the spectrum of B includes that of B_0. We emphasize that the wave operator method is useful for the spectral analysis of certain operators in Banach space as well as self-adjoint operators in Hilbert space.

The plan of the paper is as follows. Section 2 contains the main theorems. In Section 3, we establish the Enss decomposition principle. In Section 4, we show that the range of $\tilde{W}_+$ is dense in $L^1(\boldsymbol{R}^{2d}_{x,v})$. The spectral properties of B_0 and B are discussed in Section 6 after we prove some abstract theorems in Section 5.

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2. Main theorems

We define the collision-free linear Boltzmann operator B_0 to be the closure of the operator defined on $C_0^\infty(\boldsymbol{R}_{x,v}^{2d})$ by

$$(B_0 n)(x, v) = (v \cdot grad_x n)(x, v).$$

It is known that $-B_0$ is the infinitesimal generator of a strongly continuous group of positivity preserving isometries on $L^1(\boldsymbol{R}_{x,v}^{2d})$ and that

$$[e^{-tB_0}n](x, v) = n(x - tv, v). \tag{2.1}$$

(See Hejtmanek [3], Reed-Simon [8], Simon [10].)

Throughout the paper we shall assume that the pair (k, σ_a) is *regular*, i.e.,

(i) *$k(x, v', v)$ is a nonnegative measurable function on $\boldsymbol{R}^{3d}$ and $\sigma_a(x, v)$ is a nonnegative measurable function on $\boldsymbol{R}^{2d}$;*

(ii) *For each (x, v'), $k(x, v', \cdot)$ is in $L^1(\boldsymbol{R}_v^d)$;*

(iii) *$\sigma_a(x, v)$ and $\sigma_p(x, v) = \int k(x, v, v')dv'$ are essentially bounded functions on $\boldsymbol{R}^{2d}$;*

(iv) *There is a compact set D in $\boldsymbol{R}_x^d$ so that $k(x, v', v)$ and $\sigma_a(x, v)$ vanish if $x \notin D$.*

The linear collision operator A is a sum of two operators:

$$(A_1 n)(x, v) = -\int k(x, v', v) n(x, v') dv'$$

$$(A_2 n)(x, v) = \sigma_a(x, v) n(x, v).$$

As is easily seen, A_1 and A_2, and hence A, are bounded operators with norms $\|\sigma_p\|_\infty$ and $\|\sigma_a\|_\infty$, respectively. Here $\|\cdot\|_\infty$ denotes the norm in $L^\infty(\boldsymbol{R}_{x,v}^{2d})$.

Now, define the linear Boltzmann operator B by

$$B = B_0 + A.$$

In order to show that $-B$ generates a strongly continuous group, we need the following

Proposition 2.1. *Let $-T$ generate a contraction group on a Banach space X. If A is a bounded operator on X, then $-(T+A)$ generates a strongly continuous group with*

$$\|e^{-t(T+A)}\| \leqslant e^{|t| \|A\|}$$

for t real.

The proof of Proposition 2.1 (in a more general set-up) is given in Appendix.

From Proposition 2.1 it follows that $-B$ generates a strongly continuous group with

$$\|e^{-tB}\| \leqslant \exp[|t|(\|\sigma_p\|_\infty + \|\sigma_a\|_\infty)] \tag{2.2}$$

for t real. (An estimate sharper than (2.2) can be found in Reed-Simon [8].) Simon [10] showed that e^{-tB} is positivity preserving for t positive.

The existence of W_- and $\tilde{W}_+$ was investigated by Hejtmanek [3], Simon [10], and Voigt [11]. In this paper, we shall not examine the existence of W_- and $\tilde{W}_+$.

Before stating the main theorems, we introduce some notation. Let T be a linear operator in a Banach space. Then $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$ respectively represent the spectrum, the point spectrum, the continuous spectrum and the residual spectrum of T. For these definitions, see Yosida [12, p. 209].

One of our main results now reads:

Theorem 2.2. *Let (k, σ_a) be a regular pair. Assume that the inverse wave operator $\tilde{W}_+$ exists. Then* Ran $(\tilde{W}_+)$ *is a dense subspace of $L^1(\boldsymbol{R}^{2d}_{x,v})$.*

We shall determine the spectrum of B_0 completely.

Theorem 2.3. *The spectrum $\sigma(B_0)$ consists of the imaginary axis and coincides with the residual spectrum $\sigma_r(B_0)$*

From Theorem 2.3 it is clear that $\sigma_p(B_0)$ and $\sigma_c(B_0)$ are empty. Regarding the spectrum of B, we have

Theorem 2.4. *Let (k, σ_a) be a regular pair and assume that $\tilde{W}_+$ exists. Then the spectrum $\sigma(B)$ is included in the strip*

$$\{\lambda \in \boldsymbol{C} \mid 0 \leqslant \operatorname{Re}\lambda \leqslant \|\sigma_p\|_\infty + \|\sigma_a\|_\infty\}.$$

Moreover, the residual spectrum $\sigma_r(B)$ includes the imaginary axis.

3. The Enss decomposition principle

In this section we establish the Enss decomposition principle. Before stating it, we introduce some notation which will be employed in the sequel without further reference. For any interval $I \subset \boldsymbol{R}$, we define

$$\Omega(I) = \{\xi \in \boldsymbol{R}^d \mid |\xi| \in I\}.$$

When it is convenient, we also use the notation $\Omega(I)_x$, $\Omega(I)_v$. The multiplication operator by the characteristic function of $\Omega(I)_x \times \boldsymbol{R}^d_v$ will be denoted by $F(|x| \in I)$.

Theorem 3.1. *Let (k, σ_a) be a regular pair, and let M be a positive number with $D \subset \Omega([0, M])_x$. Let $0 < a < b < +\infty$. Then there exist three families, $\{D_r^\pm\}_{r>0}$ and $\{D_r^0\}_{r>0}$, of positivity preserving bounded operators in $L^1(\boldsymbol{R}^{2d}_{x,v})$ with the following properties*:

(i) *For every $r > 0$ and every $n \in L^1(\boldsymbol{R}^{2d}_{x,v})$ with* supp $n \subset \boldsymbol{R}^d_x \times \Omega([a, b])_v$

$$(D_r^+ + D_r^- + D_r^0)n = n,$$

(*here* supp n *denotes the support of* n).

(ii) *For every* $r>2M$

$$e^{-tB}D_r^{\pm}=e^{-tB_0}D_r^{\pm},\ t\gtrless 0$$

(iii) *For every* $r>0$

$$F\left(|x|\leqslant \frac{r}{2}\right)e^{-tB_0}D_r^{\pm}=0, \qquad t\gtrless 0$$

and

$$F\left(|x|\leqslant \frac{a}{4}|t|\right)e^{-tB_0}D_r^{\pm}=0, \qquad t\gtrless 0.$$

(iv) *For every* $r>0$, t *real and every* $n\in L^1(\boldsymbol{R}^{2d}_{x,v})$

$$\operatorname{supp}\left[e^{tB_0}D_r^{\pm}e^{-tB_0}n\right]\subset \operatorname{supp} n.$$

(v) *For every* $r>0$

$$\operatorname*{s-lim}_{t\to\pm\infty} D_r^0 e^{-tB_0}=0.$$

Basically, D_r^+n (respectively, D_r^-n) is the part of n outside the ball $\Omega([0, r])_x$ and with velocities pointing outwards (respectively, inwards).

Theorem 3.1 has an important corollary.

Corollary. (i) *If* W_- *exists, then* $(W_- - I)D_r^-=0$ *for every* $r>2M$.
(ii) *If* $\tilde{W}_+$ *exists, then* $(\tilde{W}_+ - I)D_r^+=0$ *for every* $r>2M$.

Remark. As mentioned in the introduction, e^{-tB} is in general positivity preserving only for t positive. But it follows from Theorem 3.1 that for every $r>2M$, $e^{-tB}D_r^-$ is positivity preserving for t negative. This may be physically reasonable.

To prove Theorem 3.1, we need a lemma.

Lemma 3.2. *Let* $0<a<b<+\infty$, *and let* $0<\tau<\sqrt{3}/2$. *Then there exist two functions* $g_{\pm}\in C^\infty((\boldsymbol{R}^d_x\setminus\{0\})\times\boldsymbol{R}^d_v)$ *with the following properties*:

(i) $0\leqslant g_{\pm}(x, v)\leqslant 1$ *for all* (x, v).

(ii) $g_+(x, v)+g_-(x, v)=1$ *if* $(x, v)\in(\boldsymbol{R}^d\setminus\{0\})\times\Omega([a, b])$.

(iii) $\operatorname{supp} g_{\pm}\subset\{(x, v)\in(\boldsymbol{R}^d\setminus\{0\})\times\boldsymbol{R}^d \mid v\in\Omega([a/2, 2b]), \cos\theta(x, v)\gtrless\mp\tau\}$.

Here $\theta(x, v)$ *denotes the angle between* x *and* v.

Proof. Let ψ be a function in $C_0^\infty(\boldsymbol{R})$ which is 0 off $(a/2, 2b)$, 1 on $[a, b]$ with $0\leqslant\psi\leqslant 1$. Let η_+ be a function in $C^\infty([-1, 1])$ which is 0 on $[-1, -\tau]$, 1 on $[\tau, 1]$ with $0\leqslant\eta_+\leqslant 1$. Set $\eta_-=1-\eta_+$. Now, define

$$g_{\pm}(x, v)=\psi(|v|)\eta_{\pm}(\cos\theta(x, v)).$$

Since

$$\cos\theta(x, v)=|x|^{-1}|v|^{-1}\textstyle\sum_{j=1}^d x_j v_j,$$

it follows that $g_\pm$ are in $C^\infty((\boldsymbol{R}_x^d \setminus \{0\}) \times \boldsymbol{R}_v^d)$. It is then obvious that $g_\pm$ have properties (i), (ii) and (iii) of the lemma. Q. E. D.

Proof of Theorem 3.1. First choose τ so that $0<\tau<\sqrt{3}/2$. It follows from Lemma 3.2 that there exist two functions $g_\pm$ in $C^\infty((\boldsymbol{R}_x^d \setminus \{0\}) \times \boldsymbol{R}_v^d)$ with properties (i), (ii) and (iii) of Lemma 3.2. Pick a function φ in $C^\infty(\boldsymbol{R})$ which is 0 on $(-\infty, 1]$, 1 on $[2, \infty)$ with $0 \leqslant \varphi \leqslant 1$. For every $r>0$, set

$$\varphi_r(x) = \varphi(|x|/r).$$

We define operators $D_r^\pm$ and D_r^0 by

$$(D_r^\pm n)(x, v) = g_\pm(x, v)\varphi_r(x)n(x, v)$$

$$(D_r^0 n)(x, v) = (1-\varphi_r(x))n(x, v).$$

It is then obvious that $D_r^\pm$ and D_r^0 are positivity preserving and bounded. Since $D_r^\pm$ are multiplication operators, it follows from the expression (2.1) that (iv) of the theorem holds. Moreover, it follows from (ii) of Lemma 3.2 that (i) of the theorem holds.

Now, we shall prove (iii) of the theorem for D_r^+. (The proof for D_r^- is similar.) It suffices to show that for every $t \geqslant 0$, $r>0$ and every $n \in L^1(\boldsymbol{R}_{x,v}^{2d})$

$$\text{supp}\,[e^{-tB_0} D_r^+ n] \subset \Omega\left(\left[\max\left\{\frac{r}{2}, \frac{a}{4}t\right\}, \infty\right)\right)_x \times \boldsymbol{R}_v^d. \tag{3.1}$$

Let (x, v) be in $\text{supp}\,[e^{-tB_0} D_r^+ n]$. Writing

$$[e^{-tB_0} D_r^+ n](x, v)$$

$$= g_+(x-tv, v)\varphi_r(x-tv)n(x-tv, v)$$

and using (iii) of Lemma 3.2, we see that

$$v\cdot(x-tv) \geqslant -\frac{\sqrt{3}}{2}|v|\,|x-tv|,$$

and thus

$$2tv\cdot(x-tv) \geqslant -\frac{1}{2}\varepsilon^2 t^2|v|^2 - \frac{3}{2}\varepsilon^{-2}|x-tv|^2 \tag{3.2}$$

for every $t>0$ and every $\varepsilon>0$. Combining (3.2) and the identity

$$|x|^2 = t^2|v|^2 + 2tv\cdot(x-tv) + |x-tv|^2,$$

we get

$$|x|^2 \geqslant \left(1-\frac{1}{2}\varepsilon^2\right)t^2|v|^2 + \left(1-\frac{3}{2}\varepsilon^{-2}\right)|x-tv|^2$$

for every $t \geqslant 0$ and every $\varepsilon>0$. Taking $\varepsilon=\sqrt{2}$ and noting that $x-tv$ is in $\text{supp}\,\varphi_r$, we have

$$|x|^2 \geqslant \frac{1}{4} r^2. \tag{3.3}$$

Similarly, taking $\varepsilon=\sqrt{3/2}$ and noting that $(x-tv, v)$ is in supp g_+, we have

$$|x|^2 \geqslant \frac{1}{4}\left(\frac{1}{2} at\right)^2 \tag{3.4}$$

for $t \geqslant 0$. Thus, (3.3) and (3.4) yield (3.1).

Next we prove (ii) of the theorem for D_r^+. (The proof for D_r^- is similar.) It suffices to show that for all n in $C_0^\infty(\boldsymbol{R}_{x,v}^{2d})$

$$D_r^+ n = e^{tB} e^{-tB_0} D_r^+ n \tag{3.5}$$

holds for $r > 2M$ and $t \geqslant 0$. Differentiating the right-hand side of (3.5) with respect to t, we have

$$\frac{d}{dt}(e^{tB} e^{-tB_0} D_r^+ n) = e^{tB}(A_1 + A_2) e^{-tB_0} D_r^+ n \tag{3.6}$$

for n in $C_0^\infty(\boldsymbol{R}_{x,v}^{2d})$. By (3.1) we have

$$\operatorname{supp}[e^{-tB_0} D_r^+ n] \subset \Omega((M, \infty))_x \times \boldsymbol{R}_v^d \tag{3.7}$$

for $t \geqslant 0$ and $r > 2M$. Since, by the hypothesis of the theorem, $D \subset \Omega([0, M])_x$, it follows from (3.7) that the right-hand side of (3.6) vanishes for $t \geqslant 0$ and $r > 2M$ (recall the definition of the regular pair). This implies (3.5).

All that remains is property (v) of the theorem. It suffices to show that

$$D_r^0 e^{-tB_0} n \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \pm\infty$$

for every n in $C_0^\infty(\boldsymbol{R}_{x,v}^{2d})$ with

$$\operatorname{supp} n \cap \{(x, v) \mid v = 0\} = \emptyset.$$

This follows from the fact that for such a data

$$\operatorname{supp}[e^{-tB_0} n] \subset \Omega([2r, \infty))_x \times \boldsymbol{R}_v^d$$

for large $|t|$. Q. E. D.

4. Proof of Theorem 2.2.

In this section, we prove Theorem 2.2 with the aid of Theorem 3.1. We begin with the intertwining property.

Proposition 4.1. *Let (k, σ_a) be a regular pair and assume that W_- and $\tilde{W}_+$ exist. Then the following relations*

$$e^{-tB} W_- = W_- e^{-tB_0},$$

$$e^{-tB_0} \tilde{W}_+ = \tilde{W}_+ e^{-tB}$$

hold for all $t\in \boldsymbol{R}$.

The proof is the same as in the case of strongly continuous unitary groups generated by self-adjoint operators. See Kato [4, p. 532].

We now turn to the

Proof of Theorem 2.2. Suppose the contrary. Then we can find a non-trinial function n in $C_0^\infty(\boldsymbol{R}^{2d}_{x,v})$ with

$$\operatorname{supp} n\cap\{(x, v)\mid v=0\}=\emptyset$$

and $n\notin \mathrm{Cl}\,\mathrm{Ran}\,(\tilde{W}_+)$ (Cl=closure). By the Hahn-Banach theorem, there is an $f\in L^\infty(\boldsymbol{R}^{2d}_{x,v})$ (the adjoint space of $L^1(\boldsymbol{R}^{2d}_{x,v})$) such that $(f, n)=1$ and

$$(f, m)=0 \tag{4.1}$$

for all $m\in \mathrm{Cl}\,\mathrm{Ran}\,(\tilde{W}_+)$. Choose positive numbers R, a and b so that

$$\operatorname{supp} n\subset\Omega([0, R])_x\times\Omega([a, b])_v.$$

Let $\{D_r^\pm\}$ and $\{D_r^0\}$ be bounded operators having all the properties specified in Theorem 3.1. Noting

$$\operatorname{supp}[e^{-tB_0}n]\subset \boldsymbol{R}^d_x\times\Omega([a, b])_v$$

and using (i) of Theorem 3.1, we have

$$e^{-tB_0}n=D_r^+e^{-tB_0}n+D_r^-e^{-tB_0}n+D_r^0e^{-tB_0}n.$$

We now write

$$(f, n)=((e^{tB_0})^*f, e^{-tB_0}n)=\mathrm{I}+\mathrm{II}+\mathrm{III},$$

where

$$\mathrm{I}=((e^{tB_0})^*f, D_r^+e^{-tB_0}n),$$

$$\mathrm{II}=((e^{tB_0})^*f, D_r^-e^{-tB_0}n),$$

$$\mathrm{III}=((e^{tB_0})^*f, D_r^0e^{-tB_0}n).$$

By the corollary to Theorem 3.1 and Proposition 4.1, we see that for $r>2M$

$$\mathrm{I}=(f, \tilde{W}_+e^{tB}D_r^+e^{-tB_0}n)$$

which equals zero by (4.1). Since, by (iii) of Theorem 3.1,

$$F\left(|x|\leqslant\frac{r}{2}\right)e^{tB_0}D_r^-=0$$

for $t\geqslant 0$, we see that

$$(f, F\left(|x|\leqslant\frac{r}{2}\right)e^{tB_0}D_r^-e^{-tB_0}n)=0 \tag{4.2}$$

for $t \geqslant 0$. Since, by (iv) of Theorem 3.1,

$$\operatorname{supp}\left[e^{tB_0}D_r^- e^{-tB_0}n\right] \subset \Omega([0, R])_x \times \Omega([a, b])_v,$$

we have

$$(f, F\left(|x| > \frac{r}{2}\right) e^{tB_0} D_r^- e^{-tB_0}n) = 0 \tag{4.3}$$

for $r > 2R$. Hence, combining (4.2) and (4.3), we see that $\mathrm{II} = 0$ for $t \geqslant 0$ and $r > 2R$. Writing

$$\mathrm{III} \leqslant \|f\|_\infty \|D_r^0 e^{-tB_0} n\|_1,$$

it follows from (v) of Theorem 3.1 that III goes to zero as $t \to +\infty$. We have thus shown that $(f, n) = 0$. Since $(f, n) = 1$, this is impossible. Q.E.D.

5. Abstract theorems

In this section we give a few theorems in Banach space which will be used in the proofs of the next section. We are mainly interested in the residual spectrum of the infinitesimal generator of a group.

Let X be a Banach space. By the adjoint space X^* of X we mean the set of all bounded anti-linear forms on X. Let T be a densely defined linear operator in X. The adjoint operator T^* of T is defined in the following way: $D(T^*)$ consists of all $g \in X^*$ such that there exists an $f \in X^*$ with the property

$$(g, Tu) = (f, u) \qquad \text{for all} \quad u \in D(T).$$

T^* is defined by setting $T^*g = f$. (For this definition see Kato [4, p. 167].)

We start from the following

Proposition 5.1. *Let X be a Banach space, and let T be a densely defined linear operator in X. Then*

(i) *if λ is in $\sigma_r(T)$, then $\bar{\lambda}$ is in $\sigma_p(T^*)$;*

(ii) *if $\bar{\lambda}$ is in $\sigma_p(T^*)$, then λ is in either $\sigma_p(T)$ or $\sigma_r(T)$.*

The proof is similar to that of Proposition, p. 194 of Reed-Simon [6] and is omitted here.

Theorem 5.2. *Let $-T$ be the generator of a strongly continuous semigroup on a Banach space X. Suppose that λ is in $\sigma_p(T^*)$ and that f is a corresponding eigenvector of T^*. Then*

$$(f, e^{-tT}u) = e^{-t\lambda}(f, u) \tag{5.1}$$

for all $u \in X$ and all $t \geqslant 0$.

Proof. Since $D(T)$ is dense in X, it suffices to prove (5.1) for $u \in D(T)$. We thus suppose that $u \in D(T)$. Set

$$\varphi(t)=(f,\ e^{-t(T-\bar{\lambda})}u).$$

Then φ is continuously differentiable, and

$$\varphi'(t)=-((T^*-\lambda)f,\ e^{-t(T-\bar{\lambda})}u).$$

By the hypotheses of the theorem, $\varphi'(t)=0$ for $t\geqslant 0$. Hence we obtain

$$(f,\ e^{-t(T-\bar{\lambda})}u)=(f,\ u)$$

which proves (5.1). Q.E.D.

Theorem 5.2 has a converse if $-T$ is the generator of a group.

Theorem 5.3. *Let $-T$ be the generator of a strongly continuous group on a Banach space X, and let λ be a complex number. Suppose that there are a non-trivial form f in X^* and an open interval I such that*

$$(f,\ e^{-tT}u)=e^{-t\lambda}(f,\ u) \tag{5.2}$$

for all $u \in D(T)$ and all $t \in I$. Then λ is in $\sigma_p(T^*)$.

Proof. Differentiate the both sides of (5.2) with respect to t. Then by (5.2)

$$(f,\ Te^{-tT}u)=(\lambda f,\ e^{-tT}u) \tag{5.3}$$

for all $u \in D(T)$ and all $t \in I$. Since $\{e^{-tT}\}$ is a group, e^{-tT} takes $D(T)$ onto $D(T)$. From this fact and (5.3), it follows that $(f,\ Tu)=(\lambda f,\ u)$ for all $u \in D(T)$. This means that $\lambda \in \sigma_p(T^*)$. Q.E.D.

6. Spectra of B_0 and B

We now turn to the proofs of Theorems 2.3 and 2.4. As mentioned in the introduction, arguments used in the proof of Theorem 2.4 show that scattering theory is a useful tool in spectral analysis of certain operators in Banach space.

We shall denote by $i\boldsymbol{R}$ the imaginary axis.

Proof of Theorem 2.3. Since $\|e^{-tB_0}\|=1$ for all $t \in \boldsymbol{R}$, it follows from the Hille-Yosida theorem (see Reed-Simon [7, p. 238]) that $\sigma(B_0)$ is included in the imaginary axis. For μ real, define f_μ in $L^\infty(\boldsymbol{R}^{2d}_{x,v})$ by

$$f_\mu(x,\ v)=\exp\{i\mu x\cdot v/|v|^2\}.$$

Then one can easily check that $f_\mu \in D(B_0^*)$ and that

$$B_0^* f_\mu=-i\mu f_\mu. \tag{6.1}$$

By Proposition 5.1 (ii), $i\mu$ is in either $\sigma_p(B_0)$ or $\sigma_r(B_0)$.

We shall show that $\sigma_p(B_0)$ is empty. To this end, suppose that there are a pure imaginary number $i\xi$ ($\xi \in \boldsymbol{R}$) and $n \in L^1(\boldsymbol{R}^{2d}_{x,v})$ such that $B_0 n=i\xi n$. It suffices to show that $n=0$. We have

$$e^{-tB_0}n=e^{-it\xi}n. \tag{6.2}$$

Let K be a given compact set in $\boldsymbol{R}_x^d\times(\boldsymbol{R}_v^d\setminus\{0\})$. From (6.2), it follows that

$$\|e^{-tB_0}n\|_{1,K}=\|n\|_{1,K} \tag{6.3}$$

where $\|n\|_{1,K}=\int_K |n(x, v)|dxdv$. Using the expression (2.1) for e^{-tB_0} and noting the fact that the velocity of every particle in K is bounded away from zero, one can see that

$$\|e^{-tB_0}n\|_{1,K}\longrightarrow 0 \tag{6.4}$$

as $t\to\pm\infty$. It follows from (6.3) and (6.4) that $\|n\|_{1,K}=0$. Since K was arbitrary, it follows that $n=0$. Hence $\sigma_p(B_0)$ is empty. Summing up, we have shown

$$\sigma(B_0)\subset i\boldsymbol{R}\subset\sigma_r(B_0)$$

which proves the theorem. Q.E.D.

Proof of Theorem 2.4. Since, by the assumption, $\tilde{W}_+$ exists, it follows upon application of the principle of uniform boundedness (see Reed-Simon [6, p. 81]) that $\|e^{tB_0}e^{-tB}\|\leqslant C$ for all $t\geqslant 0$, where C is a constant. We get

$$\|e^{-tB}\|\leqslant\|e^{-tB_0}\|\,\|e^{tB_0}\,e^{-tB}\|\leqslant C$$

for all $t\geqslant 0$. Applying the Hille-Yoside-Phillips theorem (see Reed-Simon [7, p. 247]), we have

$$\{\lambda\in\boldsymbol{C}\mid \operatorname{Re}\lambda<0\}\subset\rho(B). \tag{6.5}$$

We now recall the estimate

$$\|e^{tB}\|\leqslant\exp\{t(\|\sigma_p\|_\infty+\|\sigma_a\|_\infty)\},\quad t\geqslant 0$$

(see (2.2)). Applying once more the Hille-Yosida-Phillips theorem, we get

$$\{\lambda\in\boldsymbol{C}\mid \operatorname{Re}\lambda>\|\sigma_p\|_\infty+\|\sigma_a\|_\infty\}\subset\rho(B). \tag{6.6}$$

From (6.5) and (6.6), the first statement of the theorem follows.

To prove the second, we first claim that

$$\sigma_p(B)\cap i\boldsymbol{R}=\emptyset. \tag{6.7}$$

Indeed, given $i\xi$ ($\xi\in\boldsymbol{R}$), let $n\in L^1(\boldsymbol{R}_{x,v}^{2d})$ satisfy the equation $Bn=i\xi n$. Then we have

$$e^{-tB}n=e^{-it\xi}n.$$

Let K be a given compact set in $\boldsymbol{R}_x^d\times(\boldsymbol{R}_v^d\setminus\{0\})$. Then, as in the proof of Theorem 2.3, we can show that

$$\|e^{-tB_0}\tilde{W}_+n\|_{1,K}\longrightarrow 0 \tag{6.8}$$

as $t\to+\infty$. Writing

$$\begin{aligned}\|n\|_{1,K}&=\|e^{-it\xi}n\|_{1,K}\\&\leqslant\|e^{-tB_0}\tilde{W}_+n\|_{1,K}\\&\quad+\|e^{-tB_0}\tilde{W}_+n-e^{-tB}n\|_{1,K}\end{aligned}$$

and using (6.8) and the fact that

$$\|e^{-tB_0}\tilde{W}_+n-e^{-tB}n\|_1\longrightarrow 0$$

as $t\to+\infty$, we see that $\|n\|_{1,K}=0$. Since K was arbitrary, we conclude that $n=0$. These arguments show that any pure imaginary number cannot be an eigenvalue of B. Thus we have shown (6.7).

Next, we shall show that

$$\sigma_r(B)\supset i\boldsymbol{R} \tag{6.9}$$

which completes the proof of the theorem. Since, by Theorem 2.2, $\operatorname{Ran}(\tilde{W}_+)$ is dense in $L^1(\boldsymbol{R}^{2d}_{x,v})$, $(\tilde{W}_+)^*$ is one-to-one. For μ real, let f_μ be as given in the proof of Theorem 2.3. Then it follows from (6.1) and Theorem 5.2 that

$$(f_\mu,\ e^{-tB_0}n)=e^{it\mu}(f_\mu,\ n) \tag{6.10}$$

for all $n\in L^1(\boldsymbol{R}^{2d}_{x,v})$ and all $t\in\boldsymbol{R}$. Replace n in the left-hand side of (6.10) by $\tilde{W}_+n$ and use Proposition 4.1. Then

$$(f_\mu,\ e^{-tB_0}\tilde{W}_+n)=((\tilde{W}_+)^*f_\mu,\ e^{-tB}n) \tag{6.11}$$

for all $n\in L^1(\boldsymbol{R}^{2d}_{x,v})$ and all $t\in\boldsymbol{R}$. Replace n in (6.10) once more by $\tilde{W}_+n$. Then

$$(f_\mu,\ e^{-tB_0}\tilde{W}_+n)=e^{it\mu}((\tilde{W}_+)^*f_\mu,\ n) \tag{6.12}$$

for all $n\in L^1(\boldsymbol{R}^{2d}_{x,v})$ and all $t\in\boldsymbol{R}$. Combining (6.11) and (6.12), we get

$$((\tilde{W}_+)^*f_\mu,\ e^{-tB}n)=e^{it\mu}((\tilde{W}_+)^*f_\mu,\ n)$$

for all $n\in L^1(\boldsymbol{R}^{2d}_{x,v})$ and all $t\in\boldsymbol{R}$. Since $(\tilde{W}_+)^*$ is one-to-one, $(\tilde{W}_+)^*f_\mu$ is non-trivial. By Theorem 5.3, $-i\mu$ is in $\sigma_p(B^*)$. It follows from Proposition 5.1 that $i\mu$ is in either $\sigma_p(B)$ or $\sigma_r(B)$. Noting (6.7), we see that $i\mu$ is in $\sigma_r(B)$. Hence we have proved (6.9). Q.E.D.

Appendix

In connection with Proposition 2.1 we establish the following more general result.

Proposition. *Let $-T$ generate a strongly continuous group on a Banach space X with*

$$\|e^{-tT}\| \leqslant Me^{|t|\beta}, \quad t \in \boldsymbol{R},$$

where M and β are constants. If A is a bounded operator in X, then $-(T+A)$ generates a strongly continuous group with

$$\|e^{-t(T+A)}\| \leqslant Me^{|t|(\beta+M\|A\|)}, \quad t \in \boldsymbol{R}.$$

Proof. Since $-T$ generates a semigroup $\{e^{-tT}\}_{t>0}$ with $\|e^{-tT}\| \leqslant Me^{t\beta}$, it follows from Theorem 2.1, p. 497 of Kato [4] that $-T-A$ generates a semigroup $\{U_t\}_{t>0}$ with

$$\|U_t\| \leqslant Me^{t(\beta+M\|A\|)}$$

for t positive. Similarly, since T generates a semigroup $\{e^{tT}\}_{t>0}$ with $\|e^{tT}\| \leqslant Me^{t\beta}$, $T+A$ generates a semigroup $\{V_t\}_{t>0}$ with

$$\|V_t\| \leqslant Me^{t(\beta+M\|A\|)}$$

for t positive. To prove the proposition, it suffices to show that

(A.1) $$U_tV_t = 1, \qquad t \geqslant 0$$

(A.2) $$V_tU_t = 1, \qquad t \geqslant 0.$$

Let $u \in D(T+A)$. Since $D(T+A) = D(T) = D(-T-A)$, we have

$$V_tu \in D(-T-A).$$

Differentiating U_tV_tu with respect to t, we get

$$\frac{d}{dt}(U_tV_tu) = U_t(-T-A+T+A)V_tu = 0.$$

Thus

(A.3) $$U_tV_tu = u, \quad t \geqslant 0$$

for all $u \in D(T)$. Since $D(T)$ is dense in X, and since U_tV_t is bounded, (A.3) implies (A.1). The proof of (A.2) is similar. Q. E. D.

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