J. Math. Kyoto Univ. (JMKYAZ)
24-3 (1984) 521-538

Limit theorems for Lévy processes and Poisson point processes and their applications to Brownian excursions

By

Yuji KASAHARA

(Communicated by Prof. S. Watanabe, February 7, 1983)

0. Introduction

K. Itô's idea of constructing Markov processes from Poisson point processes is a powerful tool in various fields of probability theory and has many applications. One of them is a simple proof of Lévy's downcrossing theorem for Brownian motion: Let $\{B(t);\ t\geqq 0\}$ be a one-dimensional standard Brownian motion starting at 0 and $\phi(t)$ be its local time at $x=0$;

(0.1) $$\phi(t)=\lim_{\varepsilon\to 0}(4\varepsilon)^{-1}\int_0^t 1_{(-\varepsilon,\varepsilon)}(B_s)ds \quad \text{a.s.}.$$

For $\varepsilon>0$, $t\geqq 0$, define

(0.2) $d_\varepsilon(t)=$the number of times that the reflecting Brownian motion $|B(\cdot)|$ crosses down from $x=\varepsilon$ to 0 by time t.

If we apply Itô's idea, we easily see that $d_\varepsilon(\phi^{-1}(t))$ is a Poisson process with intensity $2/\varepsilon$, and from the strong law of large numbers we have the well-known Lévy's down-crossing theorem (see [14]);

(0.3) $$\lim_{\varepsilon\to 0}\varepsilon d_\varepsilon(t)=2\phi(t),\ t\geqq 0, \quad \text{a.s.}.$$

Furthermore, if we apply the CLT (central limit theorem) instead of the law of large numbers, we have

Theorem 0.1. ([7])

$$\tilde{d}_\varepsilon(t)=(\varepsilon d_\varepsilon(t)-2\phi(t))/\sqrt{2\varepsilon}\xrightarrow{\mathscr{D}}\tilde{B}(\phi(t)) \quad as \quad \varepsilon\longrightarrow 0$$

where $\tilde{B}(\cdot)$ is an independent copy of $B(\cdot)$.

Here, $\xrightarrow{\mathscr{D}}$ denotes the weak (i.e. narrow) convergence of the distributions on the function space $D=D([0,\infty))$ (see section 1 for details). Of couse the only thing that needs proof in Theorem 0.1 is the independence of $\tilde{B}(\cdot)$ and $\phi(\cdot)$, and in [7] the author proved it using F. Knight's representation theorem for continuous martingales and did not use Itô's method explicitly. However, the idea of [7] widely depends on a Stroock lemma which gives a continuous martingale approximation to $\tilde{d}_\varepsilon(\cdot)$, and cannot be applied to similar problems for other interesting functionals of Brownian excursions considered by Itô-McKean [6]. In the present article we return to Itô's approach and give another proof of Theorem 0.1 as a special case of a CLT for Poisson point processes. The merit of this approach is that the independence of $\tilde{B}(\cdot)$ and $\phi(\cdot)$ can be understood from the well-known theorem for processes with independent increments (i.e., if $(x(t), y(t))$ is a vector-valued Lévy process, if $x(\cdot)$ is continuous and if $y(\cdot)$ is a jump process then $x(\cdot)$ and $y(\cdot)$ are automatically independent). Using this method we can also prove CLT's for limit theorems for Brownian excursion intervals: Define

(0.4) $\eta_\varepsilon(t)=$ the number of the excursion intervals in $[0, t]$ of length $\geqq \varepsilon$,

(0.5) $\xi_\varepsilon(t)=$ the total length of the excursion intervals in $[0, t]$ of length $<\varepsilon$.

Then it is well known that

$$\lim_{\varepsilon\to 0} \sqrt{\pi\varepsilon/2}\, \eta_\varepsilon(t)=2\phi(t),\ t\geqq 0,\quad \text{a.s.,} \tag{0.6}$$

$$\lim_{\varepsilon\to 0} \sqrt{\pi/2\varepsilon}\, \xi_\varepsilon(t)=2\phi(t),\ t\geqq 0,\quad \text{a.s.,} \tag{0.7}$$

(see page 43 of Itô-McKean [6] and Ikeda-Watanabe [5]). For these two theorems we can prove the following as special cases of a CLT for Poisson point processes.

Theorem 0.2. *As* $\varepsilon\to 0$,

$$\tilde{\eta}_\varepsilon(t)=\{\sqrt{\pi\varepsilon/2}\, \eta_\varepsilon(t)-2\phi(t)\}/(2\pi\varepsilon)^{1/4} \xrightarrow{\mathscr{D}} \tilde{B}(\phi(t))$$

where $\tilde{B}(\phi(t))$ *is the same as before.*

Theorem 0.3. *As* $\varepsilon\to 0$,

$$\tilde{\xi}_\varepsilon(t)=\{\sqrt{\pi/2\varepsilon}\, \xi_\varepsilon(t)-2\phi(t)\}/(2\pi\varepsilon/9)^{1/4} \xrightarrow{\mathscr{D}} \tilde{B}(\phi(t))$$

where $\tilde{B}(\phi(t))$ *is the same as before.*

We now explain the contents of this paper. In section 1 we give a quick review of some fundamental notations and facts of the Skorohod function space D. Basically we shall follow Billingsley [1] and Lindvall [10] but we shall also state some fundamental facts which are well known but, as far as the author knows, have not stated explicitly. In section 2, we prove a CLT for processes of the form $X_\lambda(t)=Z_\lambda(A_\lambda^{-1}(t))$ where $(Z_\lambda(\cdot), A_\lambda(\cdot))$, $\lambda>0$ are vector-valued Lévy processes. (Notice that d_ε, η_ε, ξ_ε are typical examples.) Our theorem itself is an easy consequence of well-known results for Lévy processes. However, it should be emphasized that the independence of $\tilde{B}(\cdot)$ and $\phi(\cdot)$ in Theorems 0.1–0.3 will turn out to depend deeply on the fact that $\tilde{B}(\cdot)$ is continuous. In section 3, we apply the theorem in section 2 to the case where the process $X_\lambda(\cdot)$ in question is based on a Poisson point processes. This theorem will be applied to Brownian excursions in section 4 and Theorems 0.1–

0.3 will also be proved. In section 5, we consider a discrete-time version of the theorem in section 2. Here the processes in question become sums of random number of *i.i.d.* (independent, identically distributed) random variables. This kind of problems has been studied by a number of authors. However, they were concerned only with one-dimensional marginal distributions, and here we shall consider the convergence in the function space for a special but the most interesting case of Kesten's result in [9], which was studied in connection with the occupation time problem for Markov chains. In the last section, we give an extension to multi-dimensional cases, and we shall see that the three Brownian motions $\tilde{B}(\cdot)$'s appearing in Theorems 0.1–0.3 are mutually independent.

1. Preliminaries from Skorohod's function space

For T >0 and $d=1, 2,\dots$, we denote by $D^{(d)}[0, T]=D([0, T]\to R^d)$ the space of all right-continuous R^d-valued functions on $[0, T]$ having left-limits. $D^{(d)}[0, \infty)=D([0, \infty)\to R^d)$ can be defined in a similar way. We endow $D^{(d)}[0, T]$ with Skorohod's J_1-topology. Therefore, $w_n(\cdot)\in D^{(d)}[0, T]$ converges to $w\in D^{(d)}[0, T]$ if and only if there exist strictly increasing, continuous functions $\{\lambda_n(\cdot)\}_{n\geq 1}$ on $[0, T]$ with $\lambda_n(0)=0$, $\lambda_n(T)=T$ such that

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} \{|w_n(\lambda_n(t))-w(t)|+|\lambda_n(t)-t|\}=0. \tag{1.1}$$

It is well known that there exists a metric on $D^{(d)}[0, T]$ compatible with this convergence and that with respect to this metric $D^{(d)}[0, T]$ becomes a complete separable metric space (see Billingsley [1]). The topology of $D^{(d)}[0, \infty)$ (we often denote this space by $D^{(d)}$ for simplicity) is defined as follows. We say that $w_n(\cdot)\in D$ converges to $w(\cdot)\in D^{(d)}$ if and only if there exist strictly increasing continuous functions $\{\lambda_n(\cdot)\}$ on $[0, \infty)$ with $\lambda_n(0)=0$, $\lambda_n(\infty)=\infty$ such that, for every $T>0$, (1.1) holds. This convergence was introduced by C. Stone [11], and Lindvall [10] showed that this convergence is compatible with a metric with which $D^{(d)}$ becomes a complete separable metric space. [10] also proved that w_n converges to w in $D^{(d)}$ if and only if $r_T\circ w_n$ converges $r_T\circ w$ in $D^{(d)}[0, T]$ for every continuity point T of $w(\cdot)$. Here, $r_T\circ w$ denotes the function in $D^{(d)}[0, T]$ which is identical to w on $[0, T]$. (When there is no confusion, we shall simply write w rather than $r_T\circ w$ in the rest of this paper.) Therefore, $\{w_j\}\subset D^{(d)}$ is a convergent series if and only if so it is in $D^{(d)}[0, T_k]$ for every k, where $\{T_k\}_{k=1,2,\dots}$ is some sequence tending to infinity (but depending on $\{w_j\}$). Indeed, if $\{w_j\}$ is a convergent series, then choose $\{T_k\}$ from the continuity points of the limit $w(\cdot)$. Conversely, if $\{w_j\}$ is convergent to $w^{(k)}(\cdot)\in D^{(d)}[0, T_k]$ for every k, then we can find a function $w\in D^{(k)}$ such that $w^{(k)}(t)=w(t)$ on $[0, T_k)$ (but $w^{(d)}(T_k)\neq w(T_k)$ in general unless w is continuous at $t=T_k$). It is not difficult to show that w_j converges to w in $D^{(d)}$, which proves the assertion. This observation leads us to

Lemma 1.1. *A subset $K\subset D^{(d)}$ is relatively compact if there exists $\{T_k\}$ tending to infinity such that K (precisely, $r_{T_k}\circ K$) is relatively compact in $D^{(d)}[0, T_k]$ for* *every* k.

By *relatively compact* we mean that the closure of the set is compact. Since $D^{(d)}$ and $D^{(d)}[0, T]$ are Polish spaces, a set K is relatively compact if and only if every sequence in K contains a convergent subsequence. The reader should notice that the converse of Lemma 1.1 is false. Here is a counter-example;

Example 1.2. Let $\Lambda=[1, \infty)\cup\{\infty\}$ be the one-point compactification of $[1, \infty)$. Define for $\alpha\in[1, \infty)$, $w_\alpha(t)=1$ if $t\in[0, \alpha)$ and $=0$ if $t\in[\alpha, \infty)$, and let $w_\infty(t)=1$ identically. Then clearly $\{w_{\alpha_j}\}$ converges in $D^{(d)}$ to w_α if and only if $\{\alpha_j\}$ converges to α. Therefore, $\{w_\alpha\}_{\alpha\in T}$ is a compact set in $D^{(d)}$. However, for *every* $T>2$, this set (precisely, $\{r_T\circ w_\alpha\}$) is *not* relatively compact in $D^{(d)}[0, T]$ because $\{w_{T-(1/j)}\}_j$ does not contain any convergent series.

Let $X_\lambda(\cdot)$, $\lambda>0$ and $X(\cdot)$ be stochastic processes with sample paths in $D^{(d)}$. By $X_\lambda(\cdot)\xrightarrow{\mathscr{D}}X(\cdot)$, $\lambda\to\infty$ we denote the weak (narrow) convergence of distributions in $D^{(d)}$. We can also define convergence in distribution in $D^{(d)}[0, T]$, and we shall denote it by $X_\lambda(\cdot)\xrightarrow{\mathscr{D}}X(\cdot)$ in $D^{(d)}[0, T]$ as $\lambda\to\infty$. Since $D^{(d)}$ and $D^{(d)}[0, T]$ are Polish spaces, these convergences can be realized by almost everywhere convergences without changing the law of each process (see Skorohod [12] or page 9 of Ikeda-Watanabe [5]). Let T_X consist of those t in $[0, \infty)$ for which $P[X(t)=X(t-)]=1$. The complement of T_X in $[0, \infty)$ is at most countable. We say that $X(\cdot)$ is stochastically continuous if T_X coinsides with $[0, \infty)$. By $X_\lambda(\cdot)\xrightarrow{\text{f.d.}}X(\cdot)$, $\lambda\to\infty$ we mean that the convergence in law of $(X_\lambda(t_1), X_\lambda(t_2),\dots, X_\lambda(t_k))$ to $(X(t_1), X(t_2),\dots, X(t_k))$ for arbitrary $k=1, 2,\dots$ and $\{t_1,\dots, t_k\}$ in T_X. It should be noted that we do not require the convergence of *all* finite-dimensional marginal distributions. Clearly, $X_\lambda\xrightarrow{\mathscr{D}}X$ holds if and only if

(1.2) $$X_\lambda\xrightarrow{\text{f.d.}}X$$

and

(1.3) the laws $\{P_\lambda\}_\lambda$ of $\{X_\lambda\}_\lambda$ form a tight family.

Of course these notations and facts can be translated to $D^{(d)}[0, T]$ in the obvious manner. For the definition of tightness see page 7 of [5]. It is well known that tightness is equivalent to relative-compactness for probabilities on $D^{(d)}$ or $D^{(d)}[0, T]$. We shall use the following lemma repeatedly in this paper implicitly or explicitly.

Lemma 1.3. (i) *If there exists* $\{T_k\}_{k=1}^\infty$ $(\to\infty)$ *such that the laws* $\{P_\lambda\}_\lambda$ *of* $\{X_\lambda\}_\lambda$ *form a tight family on* $D^{(d)}[0, T_k]$, *then it is also tight in* $D^{(d)}=D^{(d)}[0, \infty)$.
(ii) *If there exists* $\{T_k\}_{k=1}^\infty$ $(\to\infty)$ *such that* $X_\lambda\xrightarrow{\mathscr{D}}X$ *on* $D^{(d)}[0, T_k]$ *for every* k, *then* $X_\lambda\xrightarrow{\mathscr{D}}X$ *in* $D^{(d)}$.

Proof. It suffices to prove only (i). By assumption, for every $\varepsilon>0$ we can choose a compactum $K_{\varepsilon,k}$ in $D^{(d)}[0, T_k]$ such that

(1.4) $$\sup_\lambda P_\lambda r_{T_k}^{-1}[K_{\varepsilon,k}]>1-\varepsilon/2^k,$$

where the restriction operator r_T is the same as before. Let $K_\varepsilon=\bigcap_{k=1}^{\infty} r_{T_k}^{-1}K_{\varepsilon,k}$. Then, clearly $P_\lambda[K_\varepsilon]>1-\varepsilon$. It remains to prove that K_ε has compact closure. However, it is proved in Lemma 1.1.

Remarks. (i) Precisely speaking, the assertion of the above lemma should be written as follows: If the laws $\{P_\lambda r_{T_k}^{-1}K_{\varepsilon,k}\}$ form a tight family in the space of all probabilities in $(D^{(d)}[0, T_k], \mathscr{B}_{[0,T_k]})$ then $\{P_\lambda\}$ is tight in the space of all probabilities in $D^{(d)}$. However, the author believes that there will be no confusion.
(ii) The converse of Lemma 1.3 (ii) is true. However, the converse of Lemma 1.3 (i) is false. A counter-example can easily be found in view of Example 1.2. (A relevant result can be found in Corollary in section 4 of [10]. However, P (and hence T_p) in the statement seems to be undefined, and the author could not understand his assertion.)
(iii) The proof of Theorem 3 of [10] also proves Lemma 1.3 (ii).

We next remark on another topology on $D^{(d)}$ (or $D^{(d)}[0, T]$). Since $D^{(d)}$ can be identified with the product space $D^{(1)}\times D^{(1)}\times\cdots\times D^{(1)}$, we can also consider the product topology come from J_1-topology of $D^{(1)}$. For simplicity, we shall call it the product topology. These two topologies are the same if restricted on the space of continuous functions but, in general, the product topology is weaker than the ordinary J_1-topology in $D^{(d)}$. Indeed, $w_j=(w_j{}^{(q)})_{q=1}^d$ converges to $w=(w^{(q)})_q$ in the product topology if and only if $w_j^{(q)}$ converges to $w^{(q)}$ in $D^{(1)}$ for every q, by definition. However, it is known that w_j converges to w in $D^{(d)}$ if and only if (w_j, ξ) converges to (w, ξ) for every $\xi\in R^d$, where $(\cdot, \cdot)$ denotes the usual inner product of R^d (see Appendix (A.26) and (A.28) of Holly-Stroock [4]). However, in the present paper, we consider only the cases where all coodinates except one of the limiting processes are continuous with probability one. In these cases one can easily see that the convergences in these two topologies are equivalent to each other. Therefore we shall not distinguish these two in the rest of this article: When we need to prove tightness of measures, we use the product topology and the conclusion may be stated in $D^{(d)}$-topology. However, this abuse of terms will cause no confusion in the problems we are concerned with.

Finally we explain a notation. Throughout, inverse function (or process) of a nonnegative, nondecreasing function $\phi(t)$, $t\geqq 0$ will always defined by $\phi^{-1}(t)=\inf\{s: \phi(s)>t\}$. Therefore, $\phi^{-1}(t)$, if defined, is assumed to be right-continuous.

2. A limit theorem for Lévy processes

A stochastic process $\{X(t);\ t\geqq 0\}$ with sample paths in $D^{(d)}$ is called a d-dimensional Lévy process if it has independent increments and is stochastically continuous. It is well known that the law of a temporally homogeneous Lévy process is completely determined by its characteristic function $E[\exp i\xi X(1)]$, if $X(0)=0$. Using the independent-increment property, we see that the convergence of all finite-dimensional distributions is equivalent to the convergence of the characteristic functions. Furthermore, Skorohod [13] proved that this also implies the convergence in law in

$D^{(d)}[0, T]$ (and therefore, in $D^{(d)}=D^{(d)}[0, \infty)$).

Theorem 2.1. *Let $(A_\lambda(\cdot), B_\lambda(\cdot))$, $\lambda>0$ be temporally homogeneous 2-dimensional Lévy provess starting at (0, 0) such that $A_\lambda(\cdot)$ is nondecreasing with probability one. Suppose*

$$\lim_{\lambda\to\infty} E[\exp\{i\xi B_\lambda(1)\}]=e^{-\xi^2/2}, \qquad \xi\in R \tag{2.1}$$

and

$$\lim_{\lambda\to\infty} E[\exp\{-sA_\lambda(1)\}]=\exp\left[c-\int_0^\infty (1-e^{-su})n(du)\right], \tag{2.2}$$

for $s>0$, where $n(du)$ is a measure on $(0, \infty)$ such that

$$\int_0^\infty \min\{1, u\}n(du)<\infty \tag{2.3}$$

and

$$c<0 \quad or \quad \int_0^\infty n(du)=\infty. \tag{2.4}$$

Then,

$$B_\lambda(A_\lambda^{-1}(t)) \xrightarrow{\mathscr{D}} B(A^{-1}(t)) \quad as \quad \lambda\longrightarrow\infty \tag{2.5}$$

where $A^{-1}(\cdot)$ is the inverse process of temporally homogeneous Lévy process $A(t)$ with Laplace transform $\exp t\{c-\int_0^\infty (1-e^{-su})n(du)\}$ and $B(\cdot)$ is a standard Brownian motion independent of $A(\cdot)$.

Proof. Let $A(\cdot)$ and $B(\cdot)$ be the processes in (2.5). Observe that (2.1) and (2.2) imply $B_\lambda \xrightarrow{\mathscr{D}} B$ and $A_\lambda \xrightarrow{\mathscr{D}} A$, respectively. Therefore the laws of $(A_\lambda(\cdot), B_\lambda(\cdot))$ form a tight family (apply Lemma 1.3 (i) with $T_k=k$). Clearly we see that any limiting process $(\tilde{A}(\cdot), \tilde{B}(\cdot))$ does not have fixed points of discontinuity and hence is a temporally homogeneous Lévy process and that $\tilde{A}$ and $\tilde{B}$ are identical in law to A and to B, respectively. However, since $\tilde{B}$ is continuous a.s., it is independent of increasing process $\tilde{A}$ (consider the Lévy-Itô decomposition of $\tilde{B}+\tilde{A}$). This proves that

$$(A_\lambda(\cdot), B_\lambda(\cdot)) \xrightarrow{\mathscr{D}} (A(\cdot), B(\cdot)) \quad \text{as} \quad \lambda\longrightarrow\infty. \tag{2.6}$$

By the Skorohod theorem we stated in section 1, we can realize the convergence in (2.6) by an almost everywhere convergence. Since there is no confusion, we assume that $(A_\lambda(\cdot), B_\lambda(\cdot))$ itself converges to $(A(\cdot), B(\cdot))$ a.s. to avoid complicated notations. Since $A(\cdot)$ is strictly increasing by assumption (2.4), $A^{-1}(\cdot)$ is continuous *a.s.*. Now we can choose $\{\tau_\lambda(\cdot)\}_\lambda$ such that $A_\lambda(\tau_\lambda(\cdot))$ and $\tau_\lambda(\cdot)$ converge to $A(\cdot)$ and $\tau(t)=t$ uniformly on every finite intervals *a.s.*. This proves that the inverse process of $A_\lambda(\tau_\lambda)$, which can be defined by the strong law of large numbers, converges to A^{-1} uniformly on every finite intervals. From this it follows that A_λ^{-1} converges to A^{-1}

in $D^{(1)}$. (The reader should notice that if nondecreasing functions converge to a continuous function on a set which is dense in $[0, \infty)$ then the convergence is uniform on every finite interval.) However, since the limit process is continuous, this convergence also implies the uniform convergence on every finite intervals *a.s.*. Now it is easy to see that $B_\lambda(A_\lambda^{-1})$ converges to $B(A^{-1})$ uniformly on every compact interval, which proves the assertion.

The reader should observe that in Theorem 2.1 the independence of $A(\cdot)$ and $B(\cdot)$ widely depends on the assumption that $B(\cdot)$ is a Brownian motion and that the condition (2.4) played an essential roll to prove the weak convergence in $D^{(1)}$-topology instead of that of all finite-dimensional marginal distributions since inverse processes are involved.

We next consider the case where $(A_\lambda(t), B_\lambda(t))$, $\lambda>0$ are of the form $((v(\lambda))^{-1}A(\lambda t), (1/\sqrt{\lambda})Z(\lambda t))$ for a fixed Lévy process $(A(t), Z(t))$ and for a suitable normalization $v(\lambda)$ tending to ∞ as λ goes to ∞. Clearly, (2.1) is satisfied if

$$E[Z(1)]=0 \quad \text{and} \quad E[Z(1)^2]=1. \tag{2.7}$$

(2.2) holds if $A(1)$ belongs to the domain of attraction of one-sided stable law of index α $(0<\alpha<1)$. We can state this condition using the Lévy measure of $A(t)$. We have

$$E[e^{-stA(t)}]=\exp\{-t\int_0^\infty (1-e^{-su})n(du)\}, \quad s>0 \tag{2.8}$$

for a Radon measure $n(du)$ on $(0, \infty)$ such that

$$\int_0^\infty \min\{1, u\}n(du)<\infty. \tag{2.9}$$

It is easy to see that (2.2) holds if and only if

$$\int_{[x,\infty)} n(du)\sim 1/\{\Gamma(1-\alpha)x^\alpha L(x)\} \quad \text{as} \quad x \longrightarrow \infty \tag{2.10}$$

for $0<\alpha<1$ and a slowly varying $L(x)$. Here $a(x)\sim b(x)$ as $x\to\infty$ denotes that the ratio converges to 1 as $x\to\infty$. If (2.10) holds then letting $v(x)$ be the asymptotic inverse of $u(x)=x^\alpha L(x)$ (i.e., $u(v(x))\sim v(u(x))\sim x$ as $x\to\infty$), we have

$$(1/v(\lambda))A(\lambda t) \xrightarrow{\text{f.d.}} A_\alpha(t) \quad \text{as} \quad \lambda \longrightarrow \infty$$

or equivalently,

$$(1/\lambda)A(u(\lambda)t) \xrightarrow{\text{f.d.}} A_\alpha(t) \quad \text{as} \quad \lambda \longrightarrow \infty$$

where $A_\alpha(t)$ is the one-sided stable process such that

$$E[\exp(-sA_\alpha(t))]=e^{-ts^\alpha}.$$

(*cf.* page 446 of Feller [3] replacing $L(x)$ by $1/L(x)$ and ρ by $1-\alpha$). Therefore, by Theorem 2.1, we have

Theorem 2.2. *Let* $(A(t), Z(t))$, $t \geqq 0$ *be temporally homogeneous Lévy process satisfying* (2.7) *and* (2.10). *Then,*

$$(\lambda^{\alpha}L(\lambda))^{-1/2}Z(A^{-1}(\lambda t)) \xrightarrow{\mathscr{D}} B(\ell_{\alpha}(t)) \quad \text{as} \quad \lambda \longrightarrow \infty$$

where ℓ_{α} *is the inverse process of the one-sided stable process with Laplace transform* $e^{-ts^{\alpha}}$ *and* $B(\cdot)$ *is a Brownian motion starting at* 0 *which is independent of* ℓ_{α}

Proof. Observe that the inverse process of $(1/v(\lambda))A(\lambda t)$ is $(1/\lambda)A^{-1}(v(\lambda)t)$. Since $u(v(\lambda)) \sim \lambda$, the assertion is clear from the above argument.

Remark 2.3. From the relationship between the stable law and the Mittag-Leffler function (see page 453 of [3]), it follows that

$$E[\exp\{s\ell_{\alpha}(t)\}] = \textstyle\sum_{k=0}^{\infty}(st^{\alpha})^{k}/\Gamma(1+k\alpha), \ s \in R \quad \text{and that}$$

$$E[\exp\{sB(\ell_{\alpha}(t))\}] = \textstyle\sum_{k=0}^{\infty}(s^{2}t^{\alpha}/2)^{k}/\Gamma(1+k\alpha), \ s \in R \qquad \text{for} \quad t \geqq 0.$$

3. A CLT for Poisson point processes

We refer to the textbook of Ikeda-Watanabe [5] for the definition of Poisson point processes and throughout we shall use the notation and terminology of [5]. Therefore, we shall only explain a few notations which are necessary to state our theorem.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a right-continuous increasing family $(\mathscr{F}_t)_{t \geqq 0}$ of sub-σ-fields of $\mathscr{F}$ each containing all P-null sets. Let $n(dx)$ be a σ-finite measure on a measurable space $(X, \mathscr{F}_X)$. By p we denote a stationary $(\mathscr{F}_t)$-Poisson point processes on $(0, \infty) \times X$ with characteristic measure $n(dx)$. Therefore, the counting measure $N_p(dtdx)$ is a Poisson random measure such that $E[N_p(dt\,dx)] = dt\,n(dx)$. For $U \in \mathscr{F}_X$ such that $n(U) < \infty$ we define a martingale $\tilde{N}_p(t, U)$ by

$$\tilde{N}_p(t, U) = N_p([0, t] \times U) - tn(U). \tag{3.1}$$

We now state our theorem. Let $\{g_{\lambda}(x), \lambda > 0\}$ be measurable functions on X satisfying

$$\lim_{\lambda \to \infty} \int_X g_{\lambda}(x)^2 n(dx) = 1 \tag{3.2}$$

and

$$\lim_{\lambda \to \infty} \int_{\{x: |g_{\lambda}(x)| > \delta\}} g_{\lambda}(x)^2 n(dx) = 0, \tag{3.3}$$

for every $\delta > 0$.
A sufficient condition for (3.2) is

$$\lim_{\lambda \to \infty} \int |g_{\lambda}(x)|^{2+\varepsilon} n(dx) = 0 \qquad \text{for some} \quad \varepsilon > 0. \tag{3.4}$$

Define

$$B_\lambda(t)=\int_0^{t+}\int_X g_\lambda(x)\tilde{N}_p(dt\,dx)\,. \tag{3.5}$$

The right-hand side of (3.5) is the stochastic integral with respect $\tilde{N}_p$ defined in (3.1), and if $g_\lambda(x)$ is integrable with respect to $n(dx)$, (3.5) can be written as

$$B_\lambda(t)=\int_0^{t+}\int_X g_\lambda(x)N_p(dt\,dx)-t\int_X g_\lambda(x)\,n(dx)\,. \tag{3.6}$$

Theorem 3.1. *Suppose* (3.2) *and* (3.3) (*or* (3.4)) *are satisfied. Let* $f(x)\geqq 0$ *be a measurable function on* X. *We further assume*

$$\int_0^\infty \min\{1, f(x)\}n(dx)<\infty \tag{3.7}$$

and

$$n\{x: f(x)>0\}=\infty. \tag{3.8}$$

Define $A(t)=\int_0^{t+}\int_X f(x)N_p(dx\,ds)$. *Then,*

$$B_\lambda(A^{-1}(t)) \xrightarrow{\mathscr{D}} B(A^{-1}(t)) \quad as \quad \lambda \longrightarrow \infty$$

where $B(\cdot)$ *is a standard Brownian motion independent of* $A(\cdot)$.

Proof. Applying Itô's formula (see page 66 of [5]) and taking expectations, we obtain

$$\begin{aligned}
&E[\exp i\xi B_\lambda(t)]\\
&\quad =1+E\Big[\int_0^t\int_X \{\exp i\xi(B_\lambda(s)+g_\lambda(x))-\exp i\xi B_\lambda(s)\\
&\qquad -i\xi \exp i\xi B_\lambda(s)\cdot g_\lambda(x)\}n(dx)dx\Big]\\
&\quad =1+\int_0^t E[\exp i\xi B_\lambda(s)]ds\int_X(\exp i\xi g_\lambda(x)-1-i\xi g_\lambda(x))n(dx)\,.
\end{aligned}$$

Consequently, we have,

$$\begin{aligned}
&E[\exp i\xi B_\lambda(t)]\\
&\quad =\exp\left\{t\int_X(\exp i\xi g_\lambda(x)-1-i\xi g_\lambda(x))n(dx)\right\}.
\end{aligned} \tag{3.9}$$

Observe that from (3.2) and (3.3) it follows that

$$\lim_{\lambda\to\infty}\int_X(\exp i\xi g_\lambda(x)-1-i\xi g_\lambda(x))n(dx)=-(1/2)\xi^2. \tag{3.10}$$

Indeed, by (3.2) it suffices to prove

$$\lim_{\lambda\to\infty}\int_X\Big(\exp i\xi g_\lambda(x)-1-i\xi g_\lambda(x)+\frac{1}{2}\xi^2 g_\lambda(x)^2\Big)n(dx)=0. \tag{3.11}$$

However, for every $\delta>0$, the left-hand side is less than or equal to

$$\limsup_{\lambda\to\infty}\int_{X_\delta}\left(\exp i\xi g_\lambda - 1 - i\xi g_\lambda + \frac{1}{2}\xi^2 g_\lambda^2\right)n(dx)$$

$$+\limsup_{\lambda\to\infty}\int_{X-X_\delta}\left(\exp i\xi g_\lambda - 1 - i\xi g_\lambda + \frac{1}{2}\xi^2 g_\lambda^2\right)n(dx)$$

$$\leqq \limsup_{\lambda\to\infty}\int_{X_\delta}(\xi^3/6)|g_\lambda(x)|^3 n(dx)$$

$$+\limsup_{\lambda\to\infty}\int_{X-X_\delta}\xi^2 g_\lambda(x)^2 n(dx)$$

$$\leqq \limsup_{\lambda\to\infty}(\xi^3/6)\int g_\lambda(x)^2 n(dx)+0$$

$$=\xi^3\delta/6,$$

where $X_\delta=\{x\colon |g_\lambda(x)|\leqq\delta\}$, $\delta>0$.
Since $\delta>0$ is arbitrary, we have (3.11) by letting $\delta\to 0$. Therefore (3.9) combined with (3.10) implies that (2.1) of Theorem 2.1 is satisfied. We can also prove in a similar way that

$$E[e^{-sA(t)}]=\exp\left\{t\int(e^{-sf(x)}-1)n(dx)\right\},\quad s>0.$$

This implies that the Lévy measure of $A(\cdot)$ is $n(f(x)\in du)$. Since $(A(\cdot), B_\lambda(\cdot))$ is a temporally homogeneous Lévy process, our assertion follows from Theorem 2.1.

Similarly, it follows from Theorem 2.2 that

Theorem 3.2. *Let $f(x)\geqq 0$ and $g(x)$ satisfy*

$$\int g(x)^2 n(dx)=1$$

and

$$n\{x\colon f(x)\geqq u\}\sim 1/\{\Gamma(1-\alpha)u^\alpha L(u)\}\quad as\quad u\longrightarrow\infty \tag{3.12}$$

for α $(0<\alpha<1)$ and slowly varying $L(u)$.
Define

$$Z(t)=\int_0^{t+}\int g(x)\tilde{N}_p(ds\,dx)$$

and

$$A(t)=\int_0^{t+}\int f(x)N_p(ds\,dx).$$

Then,

$$(\lambda^\alpha L(\lambda))^{-1/2}Z(A^{-1}(\lambda t))\xrightarrow{\mathscr{D}} B(\ell_\alpha(\cdot))\quad as\quad \lambda\longrightarrow\infty,$$

where $B(\ell_\alpha)$ is the same as in Theorem 2.2.

4. Applications to Brownian excursions

Let us start with construction of a Brownian motion and its functionals $\phi(t)$, $d_\varepsilon(t)$, $\eta_\varepsilon(t)$ and $\xi_\varepsilon(t)$ defined in section 0 using a Poisson point process. As in the previous section, we refer to the textbook of Ikeda-Watanabe [5] for details.

Let $\mathscr{W}^+$ (or $\mathscr{W}^-$) be the totality of all continuous functions w: $[0, \infty)\to R$ such that $w(0)=0$ and there exists $\sigma(w)>0$ such that if $0<t<\sigma(w)$ then $w(t)>0$ (resp. $w(t)<0$), and if $t\geqq\sigma(w)$ then $w(t)=0$. Then it is known that there exists σ-finite measures n^+ and n^- on $\mathscr{W}^+$ and on $\mathscr{W}^-$ such that

$$n^{\pm}(\{w: w(t_1)\in A_1,\dots, w(t_k)\in A_k\})$$
$$=\int_{A_1} K(t_1, x_1)dx_1\int_{A_2} p^0(t_2-t_1, x_1, x_2)dx_2\int_{A_3}\cdots$$
$$\int_{A_k} p^0(t_k-t_{k-1}, x_{k-1}, x_k)dx_k,$$

where $K(t, x)=(2/\pi t^3)^{1/2}|x|\exp(-x^2/2t)$, $t>0$, $x\in R$ and $p^0(t, x, y)=\{\exp\cdot(-(x-y)^2/2t)-\exp(-(x+y)^2/2t)\}/\sqrt{2\pi t}$. Let n be the σ-finite measure on $\mathscr{W}=\mathscr{W}^+\cup\mathscr{W}^-$ such that $n|\mathscr{W}^{\pm}=n^{\pm}$. The reader should notice that $\mathscr{W}$ and w correspond to X and x in the previous section. Define

$$B(t)=\int_0^{\phi(t)+}\int_{\mathscr{W}} w(t-A(s-))N_p(ds\,dw) \tag{4.1}$$

where $\phi(t)$ is the inverse process of

$$A(t)=\int_0^{t+}\int_{\mathscr{W}}\sigma(w)N_p(ds\,dw). \tag{4.2}$$

Here, N_p denotes the counting measure of the Poisson point process p with characteristic measure n. Then $B(t)$, $t\geqq 0$ is a standard Brownian motion having $\phi(t)$ as its local time at $x=0$. (Therefore this notation is not in conflict with $B(\cdot)$ in the previous sections.) We also have

$$n(\{w: \max_{0<t<\sigma(w)}|w(t)|>u\})=2/u \tag{4.3}$$

$$n(\{w: \sigma(w)\geqq u\})=2\sqrt{2/\pi u},\quad u>0 \tag{4.4}$$

and

$$E[e^{-sA(t)}]=\exp(-2\sqrt{2s}),\quad s>0, \tag{4.5}$$

(see Ikeda-Watanabe [5] pp. 123–131). By (4.4) we have

$$\int_{\{\sigma<\varepsilon\}}\sigma(w)n(dw)=\sqrt{8\varepsilon/\pi} \tag{4.6}$$

$$\int_{\{\sigma<\varepsilon\}}\sigma(w)^2 n(dw)=(8\varepsilon^3/9\pi)^{1/2}. \tag{4.7}$$

Now define for $\varepsilon>0$,

(4.8) $$g_{1,\varepsilon}(w)=(\varepsilon/2)^{1/2}1_{\{\max_t|w(t)|\geqq\varepsilon\}}(w)$$

(4.9) $$g_{2,\varepsilon}(w)=(\pi\varepsilon/8)^{1/4}1_{\{\sigma(w)\geqq\varepsilon\}}(w)$$

(4.10) $$g_{3,\varepsilon}(w)=(9\pi/8\varepsilon^3)^{1/4}\sigma(w)1_{\{\sigma(w)<\varepsilon\}}(w).$$

Then we have,

$$\int g_{1,\varepsilon}(w)n(dw)=(2/\varepsilon)^{1/2}=2/(2\varepsilon)^{1/2},$$

$$\int g_{2,\varepsilon}(w)n(dw)=(8/\pi\varepsilon)^{1/4}=2/(2\pi\varepsilon)^{1/4},$$

and

$$\int g_{3,\varepsilon}(w)n(dw)=(72/\pi\varepsilon)^{1/4}=2/(2\pi\varepsilon/9)^{1/4}.$$

By (4.3)–(4.7), we also have that each of $g_{j,\varepsilon}(j=1, 2, 3)$ satisfies the assumptions of Theorem 3.1 as $\varepsilon=1/\lambda\to0$. Thus it follows that

(4.11) $$\begin{aligned}&\int_0^{\phi(t)+}\int_{\mathscr{W}} g_{j,\varepsilon}(w)\tilde{N}_p(ds\,dw)\\ &=\int_0^{\phi(t)+}\int_{\mathscr{W}} g_{j,\varepsilon}(w)N_p(ds\,dw)-\phi(t)\int g_{j,\varepsilon}(w)n(dw)\\ &\xrightarrow{\mathscr{D}} \tilde{B}(\phi(\cdot)) \quad\text{as}\quad \varepsilon\longrightarrow 0,\end{aligned}$$

where $\tilde{B}$ is an independent copy of B.
One the other hand, keeping in mind that $\phi(\cdot)$ is continuous *a.s.*, we easily see

(4.12) $$\begin{aligned}&\int_0^{\phi(t)-}\int_{\mathscr{W}} g_{1,\varepsilon}(w)\tilde{N}_p(ds\,dw)\leqq\tilde{d}_\varepsilon(t)\\ &\leqq\int_0^{\phi(t)+}\int_{\mathscr{W}} g_{1,\varepsilon}(w)\tilde{N}_p(ds\,dw).\end{aligned}$$

where $\tilde{d}_\varepsilon$ is the same as in Theorem 0.1. Applying Theorem 3.1 with $\sigma(w)$ in place of $f(x)$, we see that the right side of (4.12), converges to $\tilde{B}(\phi(t))$. On the other hand the difference of the left and the right side of (4.12) is less than or equal to

$$\sup_{0\leqq s\leqq\phi(T)}|B_\varepsilon(s)-B_\varepsilon(s-)|,\quad t\leqq T,$$

where $B_\varepsilon(t)=\int_{\mathscr{W}} g_{1,\varepsilon}(w)\tilde{N}_p(ds\,dw)$. However, this clearly converges to 0 in distribution because $B_\varepsilon(t)$ converges to a Brownian motion which is continuous. Therefore, we have proved that $\tilde{d}_\varepsilon(\cdot)$ itself converges to $\tilde{B}(\phi(\cdot))$, which completes the proof of Theorem 0.1. Theorems 0.2 and 0.3 can be proved in a similar way from (4.11).

As another example of Theorem 3.1, we next consider the occupation-time problem for Brownian motions. Let $V(x)$, $x\in R$ be a bounded measurable function vanishing outside a compact set. Define

$$g(w)=\int_0^{\sigma(w)} V(w(s))ds.$$

Then it is not difficult to see

$$\bar{V}=\int_{\mathscr{W}} g(w)n(dw)=2\int_{-\infty}^{\infty} V(u)du$$

and

$$(4.13) \qquad \langle V\rangle=\int_{\mathscr{W}} g(w)^2 n(dw)$$

$$=8\int_0^{\infty}\Big(\int_t^{\infty} V(u)du\Big)^2 dt+8\int_{-\infty}^{0}\Big(\int_{-\infty}^{t} V(u)du\Big)^2 dt.$$

Let $A(t)$ be as in (4.2). By (4.4) all assuptions of Theorem 3.2 are satisfied with $L(u)=1/\sqrt{8}.$ Thus we have

$$\lambda^{-1/4}Z(A^{-1}(\lambda t)) \xrightarrow{\mathscr{D}} (\langle V\rangle/\sqrt{8})^{1/2}B(\ell_{1/2}(t)) \quad \text{as} \quad \lambda \longrightarrow \infty.$$

However, $8^{-1/4}B(\ell_{1/2}(\cdot))$ is equivalent in law to $\tilde{B}(\phi(\cdot))$. Therefore, we have the following well-known theorem due to Papanicolaou-Stroock-Varadhan (see page 137 of [5]).

Theorem 4.1.

$$\lambda^{-1/4}\Big[\int_0^t V(B_u)du-2\int_{-\infty}^{\infty} V(x)dx\cdot\phi(t)\Big]$$

$$\xrightarrow{\mathscr{D}} \langle V\rangle^{1/2}\tilde{B}(\phi(t)) \quad as \quad \lambda \longrightarrow \infty$$

where $\tilde{B}(\phi)$ is the same as in Theorem 0.1.

Proof. The only thing to be proven is that the error term converges to 0. However, this can be proved in a similar way as in the proof of Theorem 0.1.

For more general theorems for occupation-times of Markov processes, see [2] and [8].

5. Sums of random number of i.i.d. random variables

Let $\{(X_j, \tau_j); j=1, 2,\dots\}$ be R^2-valued independent, identically distributed random variables. Notice that we assume that X_j (or τ_j) is independent of $\{X_1, \tau_1,\dots, X_{j-1}, \tau_{j-1}\}$ but that X_j is not necessarily independent of τ_j. We further assume

$$(5.1) \qquad E[X_1]=0, \quad E[X_1^2]=1$$

and

$$(5.2) \qquad P(\tau_1\geqq 0)=1.$$

Define $T(t)=T(t, \omega)$, $t\geqq 0$ by

$$T(t)=\begin{cases} k & \text{if } \tau_1+\cdots+\tau_k\leqq t<\tau_1+\cdots+\tau_{k+1} \\ 0 & \text{if } t<\tau_1. \end{cases} \tag{5.3}$$

Thus $T(t)$ is the inverse process of

$$S(t)=\sum_{k\leqq t}\tau_k. \tag{5.4}$$

Recall that τ_1 belongs to the domain of attraction of a stable law of index α $(0<\alpha<1)$ if and only if

$$P(\tau_1>x)\sim 1/\{\Gamma(1-\alpha)x^{\alpha}L(x)\} \quad \text{as} \quad x\longrightarrow\infty \tag{5.5}$$

for slowly varying L, and if (5.5) holds, then

$$S(\lambda t)/v(\lambda) \xrightarrow{\text{f.d.}} A_{\alpha}(t) \quad \text{as} \quad \lambda\longrightarrow\infty, \tag{5.6}$$

where $v(\lambda)$ is the (asymptotic) inverse of $u(\lambda)=\lambda^{\alpha}L(\lambda)$ and $A_{\alpha}(\cdot)$ is the stable process with Laplace transform $e^{-s^{\alpha}t}$ as before. (See page 448 of Feller [3]. The reader should notice that $L(x)$ of [3] plays the roll of the reciprocal of $L(x)$ in (5.5).) Now by an easy modification of the proof of Theorem 2.2, we have

Theorem 5.1. *Suppose* (5.1), (5.2) *and* (5.5) *are satisfied for* $0<\alpha<1$, *and define* $T(t)$ *by* (5.3). *Then*

$$(\lambda^{\alpha}L(\lambda))^{-1/2}\sum_{j=1}^{T(\lambda t)}X_j \xrightarrow{\mathscr{D}} B(l_{\alpha}(t)) \quad \text{as} \quad \lambda\longrightarrow\infty$$

where $B(l_{\alpha}(t))$ *is the same as in Theorem* 2.2.

We next apply this theorem for occupation-time problems of Markov chains. Consider a recurrent, irreducible Markov chain $Y_0, Y_1, Y_2,\ldots$ with a denumerable state space, say $Z=\{0, +1, +2,\ldots\}$, and k-step transition probability $P_{i,j}^{(k)}$. Let $N(0)=0$, $N(j)=\min\{k>N(j-1)\colon Y_k=0\}$, $j=1, 2, 3,\ldots$. Thus $N(k)$ is the time of k^{th} visit to 0. Define $\tau_j=N(j)-N(j-1)$, for $j=1, 2,\ldots$ and $T(t)=\min\{k\colon N(k)>t\}$. Let $V(j)$, $j\in Z$ be a function vanishing outside a finite set and we define the occupation-time process by $\xi(t)=\sum_{j\leqq t}V(Y_j)$. Then $\xi(t)$ is approximately equal to $A(t)=\sum_{j\leqq T(t)}X_j$ where $X_j=\sum_{N(j-1)\leqq k<N(j)-1}V(Y_k)$. It is well known that $\bar{V}=E_0[X_1]$ and $\langle V\rangle=E_0[X_1^2]-\bar{V}^2$ are finite. By the strong Markov property, we see that $\{(\tau_k, X_k);\ k=1, 2,\ldots\}$ are independent and identically distributed. If

$$\sum_{k=0}^{\infty}z^kP_{0,0}^{(k)}\sim(1-z)^{-\alpha}L(1/(1-z)) \quad \text{as} \quad z\longrightarrow 1- \tag{5.7}$$

for some slowly varying $L(\cdot)$, then we have

$$P_0(\tau_1>n)\sim 1/\{\Gamma(1-\alpha)n^{\alpha}L(n)\} \quad \text{as} \quad n\longrightarrow\infty. \tag{5.8}$$

Indeed, since

$$\sum_{k=0}^{\infty}z^kP_0(Y_k=0)=\{(1-z)\sum_{k=0}^{\infty}z^kP_0(\tau_1>k)\}^{-1},$$

(5.7) and (5.8) are equivalent to each other in view of the Tauberian theorem (see page 447 of Feller [3]). Thus we have from Theorem 5.1 the following;

Theolem 5.2. *Suppose* (5.7) *is satisfied. Then*

$$(\lambda^\alpha L(\lambda))^{-1/2}\{\textstyle\sum_{j\leq\lambda t} V(Y_j)-\bar{V}T(\lambda t)\}$$
$$\xrightarrow{\mathscr{D}} \sqrt{\langle V\rangle}B(\ell_\alpha(t)) \quad as \quad \lambda\longrightarrow\infty,$$

where $B(\ell_\alpha(\cdot))$ *is the same as in Theorem* 2.2.

For the proof that the error term is negligible, see section 4. As we mensioned in section 0, this kind of problems has been studied by many authors, and Theorem 5.2 is a functional-version of a theorem in [9] by Kesten. The constants $\bar{f}$ and $\langle f\rangle$ are also given in [9]. Another way to compute these constants is also given in [8]: As we have seen in Remark 2.3, $E[B(\ell_\alpha(1))^2]=1/\Gamma(1+k\alpha)$. Therefore, $\langle f\rangle$ can be obtained by computing the second moment of $\sum_{j\leq t} V(Y_j)$. For details see [8].

As an example, let us consider the simplest random walk $\{Y_j\}$ on $Z=\{0, \pm 1, \pm 2,\dots\}$. Let $V(j)$ be a function on Z vanishing outside a finite set. Then it is well known that $\sum_{k=0}^\infty z^k P_{0,0}^{(k)}=(1-z^2)^{-1/2}$. Therefore, (5.7) is satisfied with $\alpha=1/2$ and $L(t)=1/\sqrt{2}$. By Theorem 5.2, we have, as $\lambda\to\infty$,

$$\lambda^{-1/4}\{\textstyle\sum_{j\leq\lambda t} V(Y_j)-\bar{V}T(\lambda t)\} \xrightarrow{\mathscr{D}} \sqrt{\langle V\rangle}2^{-1/4}B(\ell_{1/2}(t)),$$

where $\bar{V}=2\sum_j V(j)$ and $\langle V\rangle=4\sum_j j\tilde{V}(j)^2+\sum_{i<j}\tilde{V}(i)\tilde{V}(j)-\sum_j\tilde{V}(j)^2=-2\sum_{i,j}|i-j|\tilde{V}(i)\tilde{V}(j)-\sum_j\tilde{V}(j)^2$, ($\tilde{V}(j)=V(0)-\sum_k V(k)$ if $j=0$ and $V(j)$ otherwise). This result is a functional-version of Dobrusin's theorem ([15]). To find the constants $\bar{V}$ and $\langle V\rangle$, compute the first and the second moments using

$$\textstyle\sum_{j=0}^k P_{0,0}^{(j)}\sim\sqrt{2/\pi}\,k^{1/2} \quad as \quad k\longrightarrow\infty$$

and

$$\textstyle\sum_{k=0}^\infty\{P_{i,j}^{(k)}-P_{0,0}^{(k)}\}=-|i-j|,\ i,j\in Z.$$

6. Multi-dimensional case

In section 2 we considered the limiting process of $B_\lambda(A_\lambda^{-1}(t))$ when $(A_\lambda(\cdot), B_\lambda(\cdot))$, $\lambda>0$ are temporally homogeneous Lévy processes such that A_λ and B_λ converge to an increasing process and to a Brownian motion, respectively. However, all ideas we used there can be applied to more general cases where $B_\lambda(\cdot)$, $\lambda>0$ are vector-valued and converge to a Gaussian process.

Theorem 6.1. *Let* $(A_\lambda(\cdot), B_\lambda^{(1)}(\cdot),\dots, B_\lambda^{(d)}(\cdot))$, $\lambda>0$ *be temporally homogeneous* $(d+1)$*-dimensional Lévy process such that* $A_\lambda(\cdot)$ *is nonnegative and nondecreasing with probability one for every* $\lambda>0$. *Assume that there exists a* $d\times d$ *matrix* $Q=(Q_{jk})$ *such that*

$$\lim_{\lambda\to\infty} E[\exp i\textstyle\sum\xi_k B^{(k)}(1)]=\exp-\frac{1}{2}(\xi, Q\xi), \tag{6.1}$$

for every $\xi={}^T(\xi_1,\dots,\xi_d)\in R^d$,
and

(6.2) $$\lim_{\lambda\to\infty} E[\exp -sA_\lambda(1)]=\exp\left[c-\int_0^\infty(1-e^{-su})n(du)\right],\quad s>0$$

where $n(du)$ *is a Radon measure on* $(0,\infty)$ *with*

(6.3) $$\int_0^\infty \min\{1, u\}n(du)<\infty$$

and

(6.4) $$c<0 \quad or \quad \int_0^\infty n(du)=\infty.$$

Then,

(6.5) $$(B_\lambda^{(1)}(A_\lambda^{-1}(t)),\dots, B_\lambda^{(d)}(A_\lambda^{-1}(t)))$$
$$\xrightarrow{\mathscr{D}} (X^{(1)}(A^{-1}(t)),\dots, X^{(d)}(A^{-1}(t)))$$

where $A^{-1}(t)$ *is the same as in Theorem* 2.1 *and* $X=(X^{(1)}),\dots, X^{(d)}$ *is a* R^d*-valued Gaussian process with covariance matrix* $E[X^{(j)}(t)X^{(k)}(s)]=Q_{jk}\min\{t, s\}$ *and is independent of* $A(\cdot)$.

Proof. This theorem can be proved by a slight modification of the proof of Theorem 2.1. The details are omitted.

Theorem 6.2. *Let p be a Poisson point process with characteristic measure* $n(dx)$ *as in section* 3. *Suppose measurable functions* $g_\lambda^{(k)}(x)$ $(k=1, 2,\dots, d)$ *defined on X satisfy*

(6.6) $$\lim_{\lambda\to\infty}\int_X g_\lambda^{(j)}(x)g_\lambda^{(k)}(x)n(dx)=Q_{jk}\ (\in R),\quad 1\leqq j, k\leqq d$$

and

(6.7) $$\lim_{\lambda\to\infty}\int_{\{x:|g_\lambda^{(k)}(x)|>\delta\}} g_\lambda^{(k)}(x)^2 n(dx)=0$$

for every $\delta>0$ *and* $k=1,\dots, d$.
Define

$$Z_\lambda^{(k)}=\int_0^{t+}\int_X g_\lambda^{(k)}(x)\ \tilde{N}_p(ds\, dx),\quad k=1, 2,\dots, d.$$

Then, as $\lambda\to\infty$,

$$(Z_\lambda^{(1)}(A^{-1}(t)),\dots, Z_\lambda^{(d)}(A^{-1}(t)))$$
$$\xrightarrow{\mathscr{D}} (X^{(1)}(A^{-1}(t)),\dots, X^{(d)}(A^{-1}(t)))$$

where $(X^{(1)}(\cdot),\dots, X^{(d)}(\cdot))$ *is a Gaussian process with covariance matrix* $(Q_{jk})\cdot\min\{t, s\}$ *and* $A^{-1}(\cdot)$ *is the same as in Theorem* 3.1 *and* $(X^{(1)}(\cdot),\dots, X^{(d)}(\cdot))$ *is* *independent of* $A(\cdot)$.

Proof. The proof can be carried out in a similar way as in section 3. The only thing we need to prove is that (6.7) implies

$$\lim_{\lambda\to\infty}\int_{\{x:|g_\lambda(x)|>\delta\}} g_\lambda(x)^2 n(dx)=0$$

for every linear combination $g_\lambda(x)$ of $\{g_\lambda^{(k)}(x)\}_k$. However, this can be easily seen by the following lemma.

Lemma 6.3. *Let* $g_\lambda(x)$, $\lambda>0$ *be measurable functions on* X *such that*

$$\limsup_{\lambda\to\infty}\int_X g_\lambda(x)^2 n(dx)<\infty. \tag{6.8}$$

Then,

$$\limsup_{\lambda\to\infty}\int_{\{x:|g_\lambda(x)|>\delta\}} g_\lambda(x)^2 n(dx)=0 \tag{6.9}$$

holds if and only if

$$\lim_{\lambda\to\infty}\int_{E_\lambda} g_\lambda(x)^2 n(dx)=0 \tag{6.10}$$

for arbitrary $\{E_\lambda\}\subset\mathscr{B}_X$ *such that* $\sup_\lambda n(E_\lambda)<\infty$.

Proof. Assume (6.9). Then we have

$$\int_{E_\lambda} g_\lambda(x)^2 u(dx)=\int_{E_\lambda\cap\{|g_\lambda(x)|\leqq\delta\}}+\int_{E_\lambda\cap\{|g_\lambda(x)|>\delta\}}$$
$$\leqq\delta^2 n(E_\lambda)+\int_{\{|g_\lambda(x)|>\delta\}} g_\lambda(x)^2 u(dx),\ \delta>0.$$

Letting $\lambda\to\infty$ and then letting $\delta\to 0+$, we have (6.10). Suppose (6.10) is satisfied. Since (6.8) implies that $n\{x: |g_\lambda(x)|>\delta\}$ is bounded for every $\delta>0$, (6.9) follows immediately from (6.10).

As an example of Theorem 6.2, let us consider $g_{j,\varepsilon}(w)$, $j=1, 2, 3$ defined in (3.9)–(3.11). Then we have

$$\limsup_{\varepsilon\to 0}\left|\int g_{1,\varepsilon}(w)g_{2,\varepsilon}(w)n(dw)\right|$$
$$\leqq\limsup_{\varepsilon\to 0}(\varepsilon/2)^{1/2}(\pi\varepsilon/8)^{1/4}n\{w: \sigma(w)\geqq\varepsilon\}$$
$$=\limsup_{\varepsilon\to 0}(\varepsilon/2)^{1/2}(\pi\varepsilon/8)^{1/4}2(2/\pi\varepsilon)^{1/2}=0.$$

Therefore it is easy to see that (6.6) is satisfied with $(R_{jk})=I^{(3)}$, where $I^{(3)}$ denotes the identity matrix of dimension 3. (6.7) is already seen in section 3. Thus we have,

Theorem 6.4. *Let* $\tilde{d}_\varepsilon$, $\tilde{\eta}_\varepsilon$ *and* $\tilde{\varepsilon}_\varepsilon$ *be as in Theorems* 0.1–0.3. *Then, as* $\varepsilon\downarrow 0$,

$$(\tilde{d}_\varepsilon(\cdot),\ \tilde{\eta}_\varepsilon(\cdot),\ \tilde{\varepsilon}_\varepsilon(\cdot))\xrightarrow{\mathscr{D}}(B_1(\phi(\cdot)),\ B_2(\phi(\cdot)),\ B_3(\phi(\cdot)))$$

where $(B_1(\cdot), B_2(\cdot), B_3(\cdot))$ *is a three-dimensional standard Brownian motion starting at* 0 *and* $\phi(\cdot)$ *is the local time at* 0 *of a one-dimensional standard Brownian motion independent of* $(B_1(\cdot), B_2(\cdot), B_3(\cdot))$.

Added in Proof

In the proof of Theorem 3.1, we proved directly that $B_\lambda(t)$ converges to a Brownian motion. However, this fact can also be proven as a special case of CLT for semimartingales studied by a number of authors (see eg. Liptser-Shiryayev [16]).

INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA

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