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# Another limit theorem for slowly incrasing occupation times

By

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## 0. Introduction

S. Kotani and the author [6] proved two limit theorems for occupation times of two-dimensional Brownian motion and in [5] we generalized one of them for a class of Markov processes. This article is its continuation and we will prove a generalization of the other theorem.

## 1. Main theorems

Let $B(t)=(B_1(t), B_2(t))$ be a two-dimensional standard Brownian motion starting at (0, 0) and $f(x)$, $x \in \boldsymbol{R}^2$ be a bounded measurable function vanishing outside a compact set. Define

$$\ell(t)=\lim_{\varepsilon \to 0} (4\varepsilon)^{-1} \int_0^t 1_{(-\varepsilon,\varepsilon)}(B_1(s))ds$$

$$\sigma_t=\inf\{u;\ B_1(u)=t\}$$

Then [6] proved the following two theorems:

**Theorem A.**

$$(1/\lambda)\int_0^{e^{2\lambda t}} f(B(s))ds \xrightarrow{\text{f.d.}} \bar{f}\,\ell(\sigma_t) \quad \text{as } \lambda \longrightarrow \infty$$

*where* $\bar{f}=(1/\pi)\int f(y)dy$.

**Theorem B.** *If, in addition,* $\bar{f}=0$, *then*

$$(1/\sqrt{\lambda})\int_0^{e^{2\lambda t}} f(B(s))ds \xrightarrow{\text{f.d.}} C\ B_2(\ell(\sigma_t)) \quad \text{as } \lambda \longrightarrow \infty$$

*where* $C^2=-(2/\pi^2)\iint \log|x-y|f(x)f(y)dxdy$.

Here, $\xrightarrow{\text{f.d.}}$ denotes the weak (*or* narrow) convergence of all finite-dimensional marginal distributions. (In [6], we proved $M_1$-convergence. However, we will not discuss it here.) We now generalize these theorems as follows.

Let $S$ be a locally compact Hausdorff space and $(X_t)_{t\geqq 0}$ be a consevative, time-homogeneous strong Markov process with right-continuous paths in $S$. Further, we assume that the transition function $p(t, x, dy)$ is absolutely continuous with respect to a Radon measure $\mu(dx)$ and that the density $p(t, x, y)$ has the following decomposition:

$$p(t, x, y)=p(t)+q(t, x, y), \quad t\geqq 1, \tag{1.1}$$

where $p(t)$ $(\geqq 0)$ and $q(t, x, y)$ are measurable in $(t, x, y)$ and

$$L(t)=\int_1^t p(u)du \ (\uparrow \infty \text{ as } t\to\infty) \text{ varies slowly at } \infty, \tag{1.2}$$

(1.3) for every compact set $K$,

$$\lim_{N\to\infty}\int_1^N dt\int_K |q(t, x, y)|\mu(dy)$$

converges uniformly for $x$ on compacts. (We define $q(t, x, y)=p(t, x, y)$ if $t\in(0, 1)$ for convenience.)

A typical example is the case of two-dimensional Brownian motion. If we put $\mu(dx)=(1/\pi)dx$, then $p(t, x, y)=(1/2t)\times\exp\{-|x-y|^2/2t\}$. Therefore, (1.1)–(1.3) are satisfied with $p(t)=1/(2t)$ and $L(t)=(1/2)\log t$. Indeed, we have

$$\begin{aligned}&\int_K |q(t, x, y)|\mu(dy)\\ &=\int_K (1/2t)(1-\exp\{-|x-y|^2/2t\})\,(1/\pi)\,dy\\ &\leqq(4\pi t^2)^{-1}\int_K |x-y|^2dy, \quad t\geqq 1\end{aligned}$$

Therefore, (1.3) is clear.

Our main theorems are as follows:

**Theorem 1.** *Assume* (1.1)–(1.3). *Let* $n(t)$ $(\geqq 0)$ *be a non-decreasing function such that* $L(n(t))/t\to 1$ *as* $t\to\infty$. *Let* $f(x)$ *be a bounded measurable function vanishing outside a compact set. Then*

$$\frac{1}{\lambda}\int_0^{n(\lambda t)} f(X_s)ds \xrightarrow{\text{f.d.}} \bar{f}\,\ell(\sigma_t) \quad \text{as} \quad \lambda\longrightarrow\infty$$

*where* $\bar{f}=\int f(x)\mu(dx)$, *and* $\ell(\sigma_t)$ *is the same as before.*

**Theorem 2.** *If, in addition,* $\bar{f}=0$, *then*

$$(1/\sqrt{\lambda})\int_0^{n(\lambda t)} f(X_s)ds \xrightarrow{\text{f.d.}} \sqrt{2\langle f\rangle}B_2(\ell(\sigma_t)), \quad \lambda\longrightarrow\infty$$

*where* $\langle f\rangle=\int_0^\infty dt\left\{\iint f(x)f(y)p(t,x,y)\mu(dx)\mu(dy)\right\}$.

**Remark 1.1.** If we take the Laplace transforms of (1.1), then (1.2) and (1.3) imply

$$\int_0^\infty e^{-st}p(t,x,y)dt=L(1/s)+u(x,y)+o(1)\quad\text{as}\quad s\downarrow 0 \tag{1.4}$$

where $u(x,y)=\int_0^1 p(t,x,y)dt+\int_1^\infty q(t,x,y)dt$.

Therefore, as a special case $\alpha=0$ of the result of [4], we see that

$$v(t)^{-1}\int_0^t f(X_s)ds$$

converges to a bilateral exponential distribution or to an exponential distribution according as $\bar{f}$ vanishes or not, where $v(t)=\sqrt{L(t)}$ if $\bar{f}=0$ and $=L(t)$ otherwise. So the assertions of Theorems 1 and 2 are already proven for one-dimensional marginal distributions. To see the assumption (A) of [4], notice that if $\bar{f}=0$, $G_sf(x)$ converges to $g(x)=\int f(y)u(x,y)\mu(dy)$ uniformly on compacts as $s\downarrow 0$. Thus $G_sf(x)$ is uniformly bounded on compacts. However, since $f(x)$ has compact support, it is not difficult to see that $G_sf(x)$ is uniformly bounded on the whole space as a consequence of the strong Markov property.

**Remark 1.2.** In the definition of $\langle f\rangle$, it should be noted that

$$\int_0^\infty dt\left|\iint f(x)f(y)p(t,x,y)\mu(dx)\mu(dy)\right|<\infty$$

although $\int_0^\infty p(t,x,y)dt$ diverges because of the recurrence. Indeed, for $t\geqq 1$, keeping $\bar{f}=0$ in mind, we see that

$$\begin{aligned}&\iint f(x)f(y)p(t,x,y)\mu(dx)\mu(dy)\\&=(\bar{f})^2p(t)+\iint f(x)f(y)q(t,x,y)\mu(dx)\mu(dy)\\&=\iint f(x)f(y)q(t,x,y)\mu(dx)\mu(dy).\end{aligned}$$

Since $f(x)$ had compact support, this function belongs to $L^1(dt)$ by assumption (1.3). It should also be noticed that

$$\langle f\rangle=\lim_{s\downarrow 0}\int_0^\infty e^{-st}dt\left\{\iint f(x)f(y)p(t,x,y)\mu(dx)\mu(dy)\right\}.$$

Therefore, using the notation of (1.4) we have another expression:

$$\langle f\rangle=\iint f(x)f(y)u(x,y)\mu(dx)\mu(dy). \tag{1.5}$$

**Remark 1.3.** Formally, Theorem A (or B) is included in Theorem 1 (or 2, respectively). Indeed, as we have seen before Theorem 1, two-dimensional Brownian motions satisfy (1.1)–(1.3) with $\mu(dy)=(1/\pi)dy$ and $L(t)=(1/2)\log t\,(n(t)=e^{2t})$. Clearly, $\bar{f}$ in Theorem A is compatible with that in Theorem 1. To see that $C^2$ in Theorem B equals $2\langle f\rangle$, recall the well-known formula;

$$\int_0^\infty (e^{-ax}-e^{-bx})/x\,dx=\log(b/a), \quad a,\ b>0.$$

Therefore,

$$\begin{aligned}2\langle f\rangle &=\pi^{-2}\int_0^\infty dt\left\{\iint f(x)f(y)(1/t)\exp(-|x-y|^2/2t)dxdy\right\}\\ &=\pi^{-2}\int_0^\infty dt\left\{\iint f(x)f(y)(1/t)\exp(-t|x-y|^2/2)dxdy\right\}\\ &=\pi^{-2}\int_0^\infty dt\iint f(x)f(y)\{\exp(-t|x-y|^2/2)-e^{-t/2}\}/t\,dxdy\\ &=\pi^{-2}\iint f(x)f(y)\log(1/|x-y|^2)\,dxdy\\ &=C^2.\end{aligned}$$

Thus, formally, Theorems A and B are special cases of Theorems 1 and 2. However, we shall use Theorem B to prove Theorem 2 (see Lemma 3.4), and therefore Theorem 2 does not materially contain Theorem B. Nonetheless this inconvenience can be removed if one note that Lemma 3.4 can be proven without using Theorem B but with tedious calculus.

**Remark 1.4.** If $f(x)$ is nonnegative, then Theorem 1 is already proven in [5] as we mentioned in the previous section. In the general case where $f(x)$ may take negative values, observe that $f(x)$ can be expressed as a sum of two functions $f_1(x)$ and $f_2(x)$: $f_1(x)$ is nonnegative or nonpositive and $\bar{f}_2=0$. Thus Theorem 1 follows from the result of [5] combined with Theorem 2. Therefore we need only to prove Theorem 2.

## 2. The case of Cauchy process

In this section we consider the case of one-dimensional Cauchy process as an example. The transition function is given by

$$p(t,x,dy)=t/\{(x-y)^2+t^2\}\,(1/\pi)\,dy, \quad x,\ y\in R.$$

In this case, $p(t,x,y)=t/\{(x-y)^2+t^2\}$ and $\mu(dx)=(1/\pi)dy$ satisfy the assumptions of Theorems A and B with $L(t)=\log t$ and $n(t)=e^t$. Indeed to see (1.3) holds, note that for every compact set $K$,

$$\int_K |q(t,x,y)|\,(1/\pi)dy$$

$$= \int_K \{(1/t) - t/\{(x-y)^2 + t^2\}\} (1/\pi) dy$$

$$\leqq t^{-3} \int_K (x-y)^2 (1/\pi) dy.$$

Thus (1.3) is satisfied and therefore we have,

**Theorem 3.** *Let $\{X_t\}$ be the Cauchy process stated above and $f(x)$ be a bounded measurable function vanishing outside a compact set.*
*Then,*

( i) $(1/\lambda) \int_0^{e^{\lambda t}} f(X_s) ds \xrightarrow{\text{f.d.}} \bar{f}\, \ell(\sigma_t)$ as $\lambda \longrightarrow \infty$

*where* $\bar{f} = (1/\pi) \int f(x) dx$, *and*
(ii) *if* $\bar{f} = 0$,

$$(1/\sqrt{\lambda}) \int_0^{e^{\lambda t}} f(X_s) ds \xrightarrow{\text{f.d.}} \sqrt{2\langle f \rangle}\, B_2(\ell(\sigma_t)) \quad \text{as} \quad \lambda \longrightarrow \infty$$

*where* $\langle f \rangle = -(1/\pi^2) \iint \log |x-y| f(x) f(y) dx\, dy$.

*Proof.* The only thing remaining to be proven is

$$(1.6) \qquad (1/\pi^2) \int_0^\infty dt \iint f(x) f(y) t / \{(x-y)^2 + t^2\} dx dy$$

$$= -(1/\pi^2) \iint \log |x-y| f(x) f(y) dx dy.$$

However, since

$$\int_0^T t/\{(x-y)^2 + t^2\} dt = (1/2) \log \{(x-y)^2/T^2 + 1\} + \log T - \log |x-y|$$

we have that the left-hand side of (1.6) equals

$$\pi^{-2} \lim_{T \to \infty} \iint \{(1/2) \log \{(x-y)^2/T^2 + 1\} - \log |x-y|\} f(x)\, f(y) dx dy.$$

thanks to $\bar{f} = 0$. But this equals the right-hand side of (1.6).

## 3. Proof of Theorem 2

As we mentioned in Remark 1.4, we will only prove Theorem 2, so from now on we will assume all assumptions in Theorem 2, and for simplicity we will assume that $\langle f \rangle = 1$. Let

$$(3.1) \qquad A_\lambda(t) = (1/\sqrt{\lambda}) \int_0^{n(\lambda t)} f(X_\xi) d\xi, \quad \lambda > 0$$

and for $s_i > 0$, define $\phi^{(2k)}$ $(k = 1, 2, \ldots)$ as follows by induction.

$$(3.2)\qquad \phi^{(2)}(s_1, s_2; t)=\int_t^\infty e^{-(s_1+s_2)\xi}\,d\xi+e^{-(s_1+s_2)t}\,t$$

$$(3.3)\qquad \phi^{(2k)}(s_1,\dots, s_{2k}; t) = \int_t^\infty e^{-(s_1+s_2)\xi}\phi^{(2k-2)}(s_3,\dots, s_{2k}; \xi)\,d\xi + te^{-(s_1+s_2)t}\phi^{(2k-2)}(s_3, \dots, s_{2k}; t).$$

Then, we have

**Proposition 3.1.** *For every $x\in S$, and $k=1, 2,\dots,$*

$$(3.4)\qquad \lim_{\lambda\to\infty} \mathscr{L}E_x[A_\lambda(t_1)A_\lambda(t_2)\cdots A_\lambda(t_k)](s_1,\dots, s_k) = \begin{cases} \sum_\pi \phi^{(k)}(s_{\pi(1)},\dots, s_{\pi(k)}; 0) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.}\end{cases}$$

Here $\mathscr{L}F$ denotes the Laplace transform of $F$:

$$(3.5)\qquad \mathscr{L}F(t_1, t_2,\dots, t_k)(s_1, s_2,\dots, s_k) = s_1 s_2\cdots s_k\int_0^\infty\int_0^\infty \exp(-\sum_1^k s_i t_i)F(t_1,\dots, t_k)\,dt_1\cdots dt_k.$$

We will postpone the proof of Proposition 3.1 until section 4. Our next step is to obtain the convergence of $E_x[A_\lambda(t_1)\cdots A_\lambda(t_k)]$ itself from (3.4). It should be noticed that if $f(x)$ is non-negative, this follows immediately from (3.4) by the well-known continuity theorem for Laplace transforms (cf. [2] page 431). However, in our case, $f(x)$ is not non-negative and we need the following auxiliary result.

**Lemma 3.2.** *For every $x\in S$, and $k=1, 2,\dots$*
(i) *$\{E_x[A_\lambda(t_1)\cdots A_\lambda(t_k)]\}_{\lambda>1}$ is equi-continuous.*
(ii) *There exists $C>0$ such that*

$$\sup_{\lambda>1} |E_x[A_\lambda(t_1)\cdots A_\lambda(t_k)]| \leqq C\prod_{i=1}^k (1+t_i)$$

*Proof.* As we mentioned in Remark 1.1, we can apply the results of [4]. Define

$$g(x)=\int_0^\infty dt\int p(t, x, y)f(y)\mu(dy).$$

By (1.3) $g(x)$ is locally bounded, and it is not difficult to see that $g(x)$ is bounded on $S$ by the strong Markov property. We can also prove that

$$(3.6)\qquad M_t=g(X_t)+\int_0^t f(X_s)ds$$

is a martingale. In [4] we have proved that for $k=1, 2,\dots,$

$$(3.7)\qquad E_x[M_t^{2k}]/L(t)^k \longrightarrow (2k)!,$$

$$E_x\left[\left(\int_0^t f(X_s)ds\right)^{2k}\right]/L(t)^k \longrightarrow (2k)!, \quad \text{as} \quad t \longrightarrow \infty \tag{3.8}$$

Since $L(n(\lambda t))/\lambda \to t$ as $\lambda \to \infty$ for every $t \geqq 0$, (3.7) can be written as

$$E_x[M_{n(\lambda t)}^{2k}]/\lambda^k \longrightarrow (2k)!\ t^k \quad \text{as} \quad \lambda \longrightarrow \infty. \tag{3.9}$$

If a sequence of non-decreasing functions defined on $[0, \infty)$ converges to a continuous function, then the convergence is uniform on compacts. Therefore, (3.9) holds uniformly for $t$ on compacts. Keeping in mind that $g(x)$ in (3.6) is bounded, we have from (3.9) that

$$E_x[A_\lambda(t)^{2k}] \longrightarrow (2k)!\ t^k \quad \text{as} \quad \lambda \longrightarrow \infty \tag{3.10}$$

uniformly for $t$ on compacts.
As a special case of $k=1$, we have

$$(1/\lambda)E_x[(M_{n(\lambda t)} - M_{n(\lambda s)})^2] = (1/\lambda)E_x[M_{n(\lambda t)}{}^2] - (1/\lambda)E_x[M_{n(\lambda s)}{}^2]$$

converges uniformly for $(s, t)$ on compacts in $[0, \infty)^2$. Thus we have

$$\lim_{\delta\to 0} \limsup_{\lambda\to\infty} \sup_{\substack{|t-s|<\delta \\ 0\leqq t,s\leqq T}} (1/\lambda)E_x[(M_{n(\lambda t)} - M_{n(\lambda s)})^2] = 0, \quad T>0. \tag{3.11}$$

By the definition of $M_t$ (see (3.6)), we obtain from (3.11) that

$$\lim_{\delta\to 0} \limsup_{\lambda\to\infty} \sup_{\substack{|t-s|<\delta \\ 0\leqq t,s\leqq T}} (1/\lambda)E_x[(A_\lambda(t) - A_\lambda(s))^2] = 0, \quad T>0. \tag{3.12}$$

Now it is easy to prove (i) by combining (3.10) and (3.12) with Hölder's inequality. (ii) is clear by (3.10).

**Lemma 3.3.** *For every* $x \in S$ *and* $k = 1, 2,$

$$\Phi(t_1, \dots, t_k) = \lim_{\lambda\to\infty} E_x[A_\lambda(t_1)\cdots A_\lambda(t_k)], \quad t_i \geqq 0,$$

*exists and its Laplace transform equals the right-hand side of* (3.4).

*Proof.* By Ascoli-Arzelà's theorem, we can choose $\lambda_1 < \lambda_2 < \cdots \to \infty$ such that $E_x[A_\lambda(t_1)\cdots A_\lambda(t_k)]$ converges on $[0, \infty)^k$. By Proposition 1.1 and Lemma 3.2 (ii), we see that the Laplace transform of the limit function equals the right-hand side of (3.4). By the uniqueness of Laplace transforms, we have the assertion using a standard argument.

**Lemma 3.4.** *Let* $Z(t) = \sqrt{2}\, B_2(\ell(\sigma_t))$ *be the process as before and* $\Phi(t_1, \dots, t_k)$ *be the same as in Lemma* 3.3. *Then,*

$$\Phi(t_1, \dots, t_k) = E[Z(t_1)\cdots Z(t_k)], \quad t_i \geqq 0. \tag{3.13}$$

*Proof.* Since the Laplace transform of $\Phi$ equals the right-hand side of (3.4), $\Phi$ does not depend on the choice of $X_t$ and $f(x)$. Therefore, it suffices to show (3.13) for suitably chosen $X_t$ and $f(x)$. Let $X_t = B_t$ and $f(x)$ be as in Theorem B. Then as we have seen before, all our assumptions are satisfied. (Recall that $C^2 = 2\langle f \rangle$.)

Keeping Lemma 3.2 (ii) in mind (cf [2] page 251), we see from Theorem B,

$$\lim_{\lambda\to\infty} E_x[A_\lambda(t_1)\cdots A_\lambda(t_k)] = E[Z(t_1)\cdots Z(t_k)].$$

(We assumed that $\langle f\rangle = 1$.) Therefore, by Lemma 3.3, we have the assertion from the uniqueness of the limit.

Now it is easy to complete the proof of Theorem 2. Observe that by Lemmas 3.3 and 3.4 we have

$$\lim_{\lambda\to\infty} E_x[A_\lambda(t_1)\cdots A_\lambda(t_k)] = E[Z(t_1)\cdots Z(t_k)]$$

for arbitrary $k = 1, 2, \dots$ and $t_i \geqq 0$. Since repetition is allowed, we obtain

$$\lim_{\lambda\to\infty} E_x[A_\lambda(t_1)^{k_1}\cdots A_\lambda(t_j)^{k_j}] = E[Z(t_1)^{k_1}\cdots Z(t_j)^{k_j}]$$

for $t_i \geqq 0$, $k_i = 1, 2, \dots$ and $j = 1, 2, \dots$.

By Lemma 3.2 (ii), this proves our assertion thanks to the well-known Carleman test (see page 227 of [2]). For details see section 4 of [5].

**Remarks 3.5.** (i) This moment method is due to Darling-Kac [3] and Bingham [1].

(ii) Combining Lemma 3.4 with (3.2) and (3.4) we have

$$\mathscr{L}E[Z(t_1)Z(t_2)](s_1, s_2) = \mathscr{L}\Phi(t_1, t_2)(s_1, s_2) = 2/(s_1 + s_2).$$

Reversing the Laplace transform, we see

$$E[Z(t_1)Z(t_2)] = 2\min(t_1, t_2).$$

This fact can be confirmed by a direct computation: If $t_1 \leqq t_2$,

$$\begin{aligned} E[Z(t_1)Z(t_2)] &= E[2B_2(\ell(\sigma_{t_1}))B_2(\ell(\sigma_{t_2}))] \\ &= 2E[\min\{\ell(\sigma_{t_1}), \ell(\sigma_{t_2})\}] \\ &= 2E[\ell(\sigma_{t_1})] \\ &= 2t_1 \end{aligned}$$

because $\ell(\sigma_t)$ is an exponential random variable with expectation $t$.

## 4. Proof of Proposition 3.1

In this section we will prove Proposition 3.1. For simplicity, we assume again that $\langle f\rangle = 1$. Now observe that, by integration by parts,

$$\begin{aligned} (4.1)\qquad & E_x[A_\lambda(t_1)\cdots A_\lambda(t_k)](s_1, \dots, s_k) \\ &= \int_0^\infty\cdots\int_0^\infty \exp\left(-\sum_1^k s_i t_i\right) E_x\left[\prod_1^k f(X_{n(\lambda t_i)})\right] \prod_1^k d_{t_i} n(\lambda t_i). \end{aligned}$$

Therefore, taking the symmetry in mind, we see that it suffices to show

$$\lim_{\lambda\to\infty}(1/\sqrt{\lambda})^k \int\cdots\int_{0<t_1<\cdots<t_k} \exp(-\sum s_i t_i) E_x[\prod f(X_{n(\lambda t_i)})] \prod d_{t_i} n(\lambda t_i) \tag{4.2}$$
$$= \phi^{(k)}(s_1, s_2, \dots, s_k; 0).$$

In order to make use of induction, we introduce a new parameter $t$, and prove the following proposition. Of course, (4.2) can be obtained by putting $t=0$.

**Proposition 4.1.** *Define* $\phi_\lambda = \phi_\lambda^{(k)}$, $k=1, 2, \dots$, *by*

$$\phi_\lambda(s_1, \dots, s_k; x, t) \tag{4.3}$$
$$= \int\cdots\int_{t<t_1<\cdots<t_k} \exp(-\sum s_i t_i) E_x[\prod f(X_{n(\lambda; t_i, t)})] \prod d_{t_i} n(\lambda t_i)$$

*where* $n(\lambda; t_i, t) = n(\lambda t_i) - n(\lambda t) + n(0)$. *Then,*

$$\lim_{\lambda\to\infty} \phi_\lambda(s_1, \dots, s_k; x, t) = \phi^{(k)}(s_1, \dots, s_k; t).$$

To simplify notations, we assume that $n(0)=0$ and define $Q_\lambda^s$ as follows.

**Definition 4.2.** For bounded measurable function $\Phi(x, t)$, $x \in S$, $t \geqq 0$, we define for $s>0$ and $\lambda>0$,

$$Q_\lambda^s[\Phi](x, t) = \int_t^\infty e^{-s\xi} T_{n(\lambda\xi)-n(\lambda t)} \Phi(\cdot, \xi)(x)\, d_\xi n(\lambda\xi),$$

where $T_t\psi(\cdot)(x) = E_x[\psi(X_t)]$.

It should be noticed that by Markov property we have

$$\phi_\lambda^{(k)}(s_1, \dots, s_k; x, t) \tag{4.4}$$
$$= (1/\sqrt{\lambda}) Q_\lambda^{s_1}[f(x)\phi_\lambda^{(k-1)}(s_2, \dots, s_k; x, t)](x, t).$$

**Lemma 4.3.** *Let $K$ be a compact set in $S$ and $\phi(t)$, $t \geqq 0$ be a bounded measurable function. Then*

$$\sup_{x\in K} |Q_\lambda^s[f(x)\phi(t)](x, t)| \leqq C_K e^{-st} \cdot \sup_{\xi \geqq 0} |\phi(\xi)|$$

*where $C_K$ is a positive number which does not depend on $\lambda$, $t$ or $\phi(t)$.*

*Proof.* Since $\bar{f}=0$ by assumption, we have

$$T_t f(x) = \int q(t, x, y) f(y) \mu(dy).$$

Therefore,

$$|Q^s[f(x)\phi(t)](x, t)|$$
$$= \left| \int_t^\infty e^{-s\xi}\phi(\xi) d\xi \int q(n(\lambda\xi) - n(\lambda t), x, y) f(y) \mu(dy) \right|$$

$$\leqq \sup_{\xi} |\phi(\xi)| \int_t^{\infty} e^{-s\xi}\, d_{\xi} n(\lambda\xi) \int |q(n(\lambda\xi)-n(\lambda t),\, x,\, y) f(y)| \mu(dy)$$

$$= \sup_{\xi} |\phi(\xi)| s \int_t^{\infty} e^{-s\xi}\, d\xi \int_0^{n(\lambda\xi)-n(\lambda t)} du \int |q(u,\, x,\, y) f(y)| \mu(dy)$$

$$\leqq \sup_{\xi} |\phi(\xi)|\ e^{-st} \int_0^{\infty} du \int |q(u,\, x,\, y) f(y)| \mu(dy)$$

However, by assumption (1.3),

$$C_K = \sup_{x \in K} \int_0^{\infty} du \int |q(u,\, x,\, y) f(y)| \mu(dy) < \infty.$$

**Lemma 4.4.** *For every* $T>0$,

$$\lim_{\lambda\to\infty} \sup_{t \geqq 0}\ |L(T+n(\lambda t))/\lambda - t|\, e^{-st} = 0.$$

*Proof.* The assertion can be proved by the assumption that $L(t)$ varies slowly. The details are easy and hence omitted.

**Lemma 4.5.** *For every compact set* $K$,

(i) $\displaystyle \sup_{\lambda>1} \sup_{t\geqq 0}\ (1/\lambda) \int_t^{\infty} e^{-s\xi}\, p(n(\lambda\xi)-n(\lambda t)) d_{\xi} n(\lambda\xi) < \infty$

(ii) $\displaystyle \sup_{\lambda>1} \sup_{x\in K} \sup_{t\geqq 0} \int_t^{\infty} e^{-s\xi} d_{\xi} n(\lambda\xi) \int |q(n(\lambda\xi)-n(\lambda t),\, x,\, y) f(y)| \mu(dy) < \infty.$

*Proof.* (i) By integration by parts, we see that the left-hand side equals

$$\sup_{\lambda>1} \sup_{t\geqq 0}\ s \int_t^{\infty} e^{-s\xi}\, d\xi\, (1/\lambda) \int_0^{n(\lambda\xi)-n(\lambda t)} p(u) du.$$

However, since $(1/\lambda) \int_0^{n(\lambda)} p(u)\, du$ converges to 1 by assumption, it is easy to see that $(1/\lambda) \int_0^{n(\lambda\xi)-n(\lambda t)} p(u)\, du$ is dominated by $C(1+\xi)$ for suitably chosen $C>0$. Therefore,

$$\sup_{\lambda>1} \sup_{t\geqq 0}\ (1/\lambda) \int_t^{\infty} e^{-s\xi} p(n(\lambda\xi)-n(\lambda t)) d_{\xi} n(\lambda\xi)$$

$$\leqq \sup_{t\geqq 0}\ C\, s \int_t^{\infty} e^{-s\xi} (1+\xi) d\xi < \infty.$$

(ii) As in (i), we have,

$$\int_t^{\infty} e^{-s\xi}\, d_{\xi} n(\lambda\xi) \int |q(n(\lambda\xi)-n(\lambda t),\, x,\, y) f(y)| \mu(dy)$$

$$= s \int_t^{\infty} e^{-s\xi}\, d\xi \int_0^{n(\lambda\xi)-n(\lambda t)} du \int |f(y) q(u,\, x,\, y)| \mu(dy).$$

which is clearly dominated by

$$s\int_0^\infty e^{-s\xi}\,d\xi\int_0^\infty du\int |f(y)q(u,x,y)|\mu(dy).$$

By assumption (1.3), it is easy to see that this is bounded on compacts.

**Lemma 4.6.** *For every compact set $K(\subset S)$, there exists a positive constant $C$ such that*

$$\sup_{x\in K}\sup_{\lambda>1}|(1/\lambda)Q_\lambda^{s_1}[f(x)Q_\lambda^{s_2}[f(x)\Psi(x,t)]](x,t)|$$
$$\leqq C\,e^{-s_1t/2}\sup\{|\Psi(x,t)|;\,t\geqq 0, f(x)\neq 0\},$$

*for every bounded measurable function $\Psi(x,t)$, $x\in S$, $t\geqq 0$.*

*Proof.* It is harmless to assume that $K$ includes the support of $f(x)$. Define

$$r_\lambda(t)=\int_t^\infty e^{-s_2\xi}\,p(n(\lambda\xi)-n(\lambda t))\int f(y)\Psi(y\xi)\mu(dy)d_\xi n(\lambda\xi)$$

and

$$u_\lambda(x,t)=\int_t^\infty e^{-s_2\xi}d_\xi n(\lambda\xi)\int q(n(\lambda\xi)-n(\lambda t),x,y)f(y)\Psi(y,\xi)\mu(dy).$$

Then, of course we have,

$$Q_\lambda^{s_2}[f(x)\Psi(x,t)]=r_\lambda(t)+u_\lambda(x,t).$$

By Lemma 4.5, we also see that there exist constants $C_1$ and $C_2$ such that

(4.5) $\qquad (1/\lambda)|r_\lambda(t)|\leqq C_1\sup\{|\Psi(x,t)|;\,t\geqq 0,\,x\in K\},\ \lambda>1,\ t\geqq 0,$

and

(4.6) $\qquad |u_\lambda(x,t)|\leqq C_2\sup\{|\Psi(x,t)|;\,t\geqq 0,\,x\in K\},\ \lambda>1,\ t\geqq 0,\ x\in K.$

On the other hand, by Lemma 4.3,

$$\begin{aligned}&(1/\lambda)Q_\lambda^{s_1}[fQ_\lambda^{s_2}[f\Psi]]\,(x,t)\\&\leqq|Q_\lambda^{s_1}[f(x)(1/\lambda)r_\lambda(t)]|+|(1/\lambda)Q_\lambda^{s_1}[f(x)u_\lambda(x,t)]|\\&\leqq C_KC_1\,e^{-s_1t}\sup_{t\geqq 0}|(1/\lambda)r_\lambda(t)|\\&\quad+\sup\{|u_\lambda(x,t)|;f(x)\neq 0,\,t\geqq 0\}\,(1/\lambda)Q_\lambda^{s_1/2}[|f(x)|]\,e^{-s_1t/2}.\end{aligned}$$

By Lemma 4.5, it is easy to see that $(1/\lambda)Q_\lambda^{s_1/2}[|f|](x,t)$ is dominated by a constant (and that it converges to $\int|f|\mu(dx)\times(s_1/2)\int_t^\infty e^{-s_1\xi/2}\,\xi\,d\xi$). This combined with (4.5) and (4.6) implies our assertion.

**Lemma 4.7.** *Let $\phi(t)$, $t\geqq 0$, be a bounded continuous function and $h(x)$, $x\in S$ be a bounded measurable function vanishing outside a compact set. Then, for every compact set $K(\subset S)$,*

(i) $\lim_{\lambda\to\infty} Q_\lambda^{s}[\phi(t)f(x)](x,t)=e^{-st}\,\phi(t)g(x)$

*the convergence being uniform for* $(t, x)\in[0, \infty)\times K$, *where* $g(x)=\int f(y)q(t, x, y)\mu(dy)$.

(ii) $\lim_{\lambda\to\infty} (1/\lambda)Q_\lambda^s[\phi(t)h(x)](x, t)=\bar{h}(\int_t^\infty e^{-s\xi}\phi(\xi)d\xi+t\,e^{-st}\phi(t))$,
*the convergence being uniform for* $(t, x)\in[0, \infty)\times K$.

*Proof.* Obseve that $Q_\lambda^s[\phi(t)f(x)]=Q_\lambda^{s/2}[e^{-st/2}\phi(t)f(x)]$. Therefore, without loss of generality, we can assume that $\phi(t)$ vanishes at infinity. Further, by Lemmas 4.3 and 4.5, we can apply standard approximation arguments, and it is easy to see that it suffices to show the assertion for smooth functions. So we assume that $\phi(t)$ has continuous derivatives vanishing at infinity. By integration by parts, we obtain that

$$\begin{aligned}&Q_\lambda^s[\phi(t)f(x)](t, x)\\&=\int_t^\infty e^{-s\xi}\{s\phi(\xi)-\phi'(\xi)\}d\xi\int_0^{n(\lambda\xi)-n(\lambda t)}T_uf(x)du\\&=g(x)\int_t^\infty e^{-s\xi}\{s\phi(\xi)-\phi'(\xi)\}d\xi+\varepsilon(\lambda, x, t)\end{aligned}$$

where

$$\begin{aligned}\varepsilon(\lambda, x, t)&=\int_t^\infty e^{-s\xi}\{s\phi(\xi)-\phi'(\xi)\}d\xi\int_{n(\lambda\xi)-n(\lambda t)}^\infty T_uf(x)du\\&=\int_t^\infty e^{-s\xi}\{s\phi(\xi)-\phi'(\xi)\}d\xi\int_{n(\lambda\xi)-n(\lambda t)}^\infty du\int q(n, x, y)f(y)\mu(dy).\end{aligned}$$

However, by (1.3), we have that for every $\eta>0$ there exists $T>0$ such that

$$\sup_{x\in K}\int_T^\infty du\int|q(u, x, y)f(y)|\mu(dy)<\eta.$$

Therefore,

$$\begin{aligned}&|\varepsilon(\lambda, x, t)|\leqq\eta\sup_{\xi\geqq 0}|s\phi(\xi)-\phi'(\xi)|(1/s)\,e^{-st}+\\&\quad\left|\int_t^{L(T+n(\lambda t))/\lambda}e^{-s\xi}\{s\phi(\xi)-\phi'(\xi)\}d\xi\int_{n(\lambda\xi)-n(\lambda t)}^\infty du\int q(u, x, y)f(y)\mu(dy)\right|\\&\leqq[\{(1/\lambda)L(T+n(\lambda t))-t\}\int_0^\infty du\int|q(u, x, y)f(y)|\mu(dy)\\&\qquad+(\eta/s)]\,e^{-st}\sup_{\xi\geqq 0}|s\phi(\xi)-\phi'(\xi)|.\end{aligned}$$

Combining this with Lemma 4.4, we have

$$\limsup_{\lambda\to\infty}|\varepsilon(\lambda, x, t)|\leqq\eta\,e^{-st}(1/s)\sup_{\xi\geqq 0}|s\phi(\xi)-\phi'(\xi)|.$$

Thus we obtain (i) letting $\eta\to 0$. (ii) is proved in Proposition 5.3 of [5].

**Lemma 4.8.** *Let* $\phi(t)$, $t\geqq 0$, *be a bounded continuous function. Then*

$$(4.7)\qquad \lim_{\lambda\to\infty}(1/\lambda)Q_\lambda^{s_1}[f(x)(Q_\lambda^{s_2}[f(x)\phi(t)])](x,\ t)$$

$$=\int_t^\infty e^{-(s_1+s_2)\xi}\,\phi(\xi)d\xi+t\,e^{-(s_1+s_2)t}\,\phi(t)$$

*the convergence being uniform for* $(t,\ x)\in[0,\ \infty)\times K$ *for every compact set* $K$.

*Proof.* As in the proof of the previous lemma, we can assume that $\phi(t)$ has continuous derivatives which vanish at infinity (cf. Lemma 4.6). By Lemma 4.7, we see that the left-hand side of (4.12) equals

$$\lim_{\lambda\to\infty}(1/\lambda)Q_\lambda^{s_1}[f(x)\,e^{-s_2t}\,\phi(t)g(x)]\,.$$

Using Lemma 4.7 again, this equals

$$\int f(x)g(x)\mu(dx)\left\{\int_t^\infty e^{-(s_1+s_2)\xi}\,\phi(\xi)d\xi+t\,e^{-(s_1+s_2)\xi}\,\phi(t)\right\}.$$

This proves the assertion.

We are now ready to prove Proposition 4.1. In Lemma 4.7 (i), set $\phi(t)=1$. Then we have

$$\lim_{\lambda\to\infty}Q_\lambda^s[f](x,\ t)=e^{-st}\,g(x)\,.$$

Thus we obtain that

$$(4.8)\qquad \phi_\lambda^{(1)}(s;\ x,\ t)=(1/\sqrt{\lambda})Q_\lambda^s[f](x,\ t)\longrightarrow 0\quad\text{as}\quad\lambda\longrightarrow\infty.$$

Similarly, by Lemma 4.8, we see

$$(4.9)\qquad \phi_\lambda^{(2)}(s_1,\ s_2;\ x,\ t)=(1/\lambda)Q^{s_1}[fQ^{s_2}[f]](x,\ t)\longrightarrow\phi^{(2)}(s_1,\ s_2;\ t)$$

We will prove Proposition 4.1 by induction. Assume that

$$(4.10)\qquad \lim_{\lambda\to\infty}\phi_\lambda^{(2k-1)}(s_1,\dots,\ ;\ x,\ t)=0$$

and

$$(4.11)\qquad \lim_{\lambda\to\infty}\phi_\lambda^{(2k)}(s_1,\dots,\ ;\ x,\ t)=\phi^{(2k)}(s_1,\dots,\ ;\ t)\,.$$

Since $\phi_\lambda^{(2k+1)}(s_1,\dots,\ x,\ t)=(1/\lambda)Q_\lambda^{s_1}[fQ_\lambda^{s_2}f\phi_\lambda^{(2k-1)}]$, we see that $\phi_\lambda^{(2k+1)}$ converges to 0 by (4.10) and Lemma 4.6. Here, it should be noticed that if $\phi_\lambda^{(2k-1)}$ converges to 0 uniformly on compacts, then $f(x)\phi_\lambda^{(2k-1)}$ converges uniformly on $S$ because $f(x)$ has compact support. Similarly, we have from Lemma 4.7 (i),

$$\lim_{\lambda\to\infty}\ \phi_\lambda^{(2k+2)}(s_1,\dots,\ x,\ t)$$

$$=\lim_{\lambda\to\infty}\ (1/\lambda)Q_\lambda^{s_1}[fQ_\lambda^{s_2}f\phi_\lambda^{(2k)}]$$

$$=\lim_{\lambda\to\infty}\ (1/\lambda)Q_\lambda^{s_1}[f\,e^{-s_2t}\,\phi^{(2k)}g]$$

However, by Lemma 4.8, this equals

$$\int_t^\infty e^{-(s_1+s_2)\xi}\phi^{(2k)}d\xi + te^{-(s_1+s_2)t}\phi(2k)$$

By definition (3.3), this is equal to $\phi^{(2k+2)}$, which completes the proof of Proposition 4.1. Therefore, as we mentioned at the beginning of this section, Proposition 3.1 is proved.

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