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Singular Cauchy problems for second order partial differential operators with non-involutory characteristics

By

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§0. Introduction

We denote by (x, y) the variables of C^{n+1}, where $x\in C$ and $y=(y_1, y')\in C\times C^{n-1}$, and by (ξ, η) the dual variables of (x, y), where $\eta=(\eta_1, \eta')$. In this paper, we consider partial differential operators of second order written in the following form:

$$P(x, y, D_x, D_y)=\sum_{i+|\alpha|\leqslant 2} x^{\kappa(i,\alpha)}a_{i\alpha}(x, y)D_x^iD_y^\alpha.$$

Here $D_x=\partial/\partial x$, $D_y=\partial/\partial y$, and $\kappa(i, \alpha)$, $i+|\alpha|\leqslant 2$, are integers defined by

$$\kappa(i, \alpha)=\begin{cases} q|\alpha| & \text{if } \ i+|\alpha|=2,\\ q' & \text{if } \ i=0,\ |\alpha|=1,\\ 0 & \text{otherwise.}\end{cases}$$

q and q' are integers satisfying $0\leqslant q'\leqslant q-2$. Furthermore, $a_{i\alpha}(x, y)$, $i+|\alpha|\leqslant 2$, are holomorphic at the origin and $a_{2,0}=1$.

Remark. If $q'=q-1$, the above operators are said to satisfy Levi condition. Several authers considered singular Cauchy problems for operators which satisfy Levi condition. In this case, the solutions have at most poles along the chracteristic hypersurfaces issuing from the singularities of the Cauchy data provided the latter have at most poles (See Nakane [5], Takasaki [7] and Urabe [9]). Perhaps we can also treat this case, but in this paper we only consider operators which do not satisfy Levi condition.

We assume that the equation

$$\sum_{i+|\alpha|=2} x^{q|\alpha|}a_{i\alpha}(x, y)\xi^i\eta^\alpha=0$$

has two roots $\xi=x^q\lambda_i(x, y, \eta)$, $i=1, 2$, where $\lambda_i(x, y, \eta)$ are holomorphic at $x=0$, $y=0$, $\eta=(1,\dots, 0)$, homogeneous of degree 1 with respect to η, and satisfy

$$\lambda_1(x, y, \eta) \neq \lambda_2(x, y, \eta) \quad \text{at} \quad x=0,\ y=0,\ \eta=(1, 0, \dots, 0).$$

We consider the following problems:

$$(1) \qquad \begin{cases} Pu(x, y)=0 \\ D_x^i u(0, y)=\mathring{u}_i(y) \qquad i=0, 1. \end{cases}$$

Here $\mathring{u}_i(y)$, $i=0, 1$, are multi-valued holomorphic functions defined on $\{y \in \boldsymbol{C}^n;\ |y_j|<R,\ y_1=0\}$ with some $R>0$, which satisfy

$$|\mathring{u}_i(y)| \leqslant C \exp\{C|y_1|^{-(q-1-q')/(q+1)}\}$$

with some $C>0$ there.

To state the main theorem, we define $\varphi_i(x, y)$, $i=1, 2$, by

$$\begin{cases} \partial_x \varphi_i(x, y) - x^q \lambda_i(x, y, \nabla_y \varphi_i(x, y))=0 \\ \varphi_i(0, y)=y_1, \end{cases}$$

and $\psi_i(x, y')$, $i=1, 2$, by

$$\varphi_i(x, y)=0 \quad \text{if and only if} \quad y_1=\psi_i(x, y').$$

We have the following

Theorem. *Let $\varepsilon>0$ be small enough, and θ an arbitrary real number. We define $\omega_{\varepsilon,\theta}=\omega'_{\varepsilon,\theta} \cup \omega''_{\varepsilon,\theta}$ by*

$$\omega'_{\varepsilon,\theta}=\Big\{(x, y) \in \boldsymbol{C} \times \boldsymbol{C}^n;\ |x|<\varepsilon,\ |y_j|<\varepsilon,\ j=1,\dots, n,$$
$$|\arg(y_1-\psi_i(x, y'))-\theta|<\frac{\pi}{2}+\varepsilon,\ i=1, 2\Big\}$$

$$\omega''_{\varepsilon,\theta}=\Big\{(x, y) \in \boldsymbol{C} \times \boldsymbol{C}^n;\ |x|<\varepsilon,\ |y_j|<\varepsilon,\ j=1,\dots, n,$$
$$|\arg(y_1-\psi_i(x, y'))-\theta-\pi|<\frac{\pi}{2}+\varepsilon,\ i=1, 2\Big\}.$$

Then there exists a unique solution $u(x, y)$ of (1) *which is holomorphic on $\omega_{\varepsilon,\theta}$ which satisfies*

$$|u(x, y)| \leqslant C \sum_{i=1,2} \exp\{C|\varphi_i(x, y)|^{-(q-1-q')/(q+1)}\}$$

with some $C>0$ on $\omega_{\varepsilon,\theta}$.

Remark. Let us fix $(x, y') \in \boldsymbol{C} \times \boldsymbol{C}^{n-1}$ arbitrarily. Let us define θ by $\theta=\arg\{\psi_1(x, y')-\psi_2(x, y')\}+\frac{\pi}{2}$. Then it is easy to see that $\omega_{\varepsilon,\theta}$ is a domain in the universal covering space of $\omega_\varepsilon=\{(x, y) \in \boldsymbol{C} \times \boldsymbol{C}^n;\ |x|<\varepsilon,\ |y_j|<\varepsilon,\ j=1,\dots, n,\ \varphi_i(x, y) \neq 0,\ i=1, 2\}$ whose projection to ω_ε is ω_ε itself. However, we cannot construct the solution on an arbitrary domain in the universal covering space of ω_ε.

Remark. Moreover, we can give the concrete representation of the solution using some class of operators. The details are given in §2. We can also prove that the solution is single-valued on $\{(x, y) \in \boldsymbol{C} \times \boldsymbol{C}^n;\ |\psi_i(x, y')| < |y_1|,\ i=1, 2\}$.

§1. Preliminaries

In this section, we define a class of operators which act on some (quotient) space of holomorphic functions. Our operator class is very much like that of holomorphic microlocal operators which act on the space of holomorphic microfunctions (for holomorphic microlocal operators, see Sato, Kawai and Kashiwara [6], Kataoka [4], and Aoki [1]). Aoki defined the notion of the symbol of a holomorphic microlocal operator, and this notion is very useful for our study. However, though we need to know the relation between an operator and its symbol in a concrete manner, he gave this relation in a rather abstract manner. The purpose of this section is to give this relation in a concrete manner. However, we have not obtained a result which is interesting of itself, and our result is no more than a provisional one which is only enough for our later use. Thus we only give the sketch of our result, and omit the proof. The idea is due to Aoki [1].

In this section, we fix a real number θ arbitrarily, and define ρ by $\rho = (q-1-q')/(2q-q')$. Let R, r and ε be real numbers which satisfy $0 < \varepsilon \ll r \ll R < 1$. We denote by $\mathscr{S}^{R,r}$ the set of holomorphic functions $a(y, \eta)$ defined on

$$(2) \qquad \{(y, \eta) \in \boldsymbol{C}^n \times \boldsymbol{C}^n;\ |y_j| < R,\ j=1,\dots, n,\ |\eta_j| < R|\eta_1|,\ j=2,\dots, n,$$
$$|\eta_1| > R^{-1},\ |\arg(e^{\sqrt{-1}\theta}\eta_1)| < R,\}$$

which satisfy

$$|a(y, \eta)| \leqslant C \exp\{C|\eta_1|^\rho + r \sum_{j=2}^{n} |\eta_j|\}$$

with some $C > 0$ on (2). Assume that $f(y)$ is holomorphic on $\{|y_j| < R,\ j=1,\dots, n\}$ and satisfies $|f(y)| < \varepsilon$ there. We define $\mathscr{S}_f^{R,r}$ by $\mathscr{S}_f^{R,r} = \{\exp\{f(y)\eta_1\} a(y, \eta);\ a(y, \eta) \in \mathscr{S}^{R,r}\}$. If $b(y, \eta) = \exp\{f(y)\eta_1\} b(y, \eta) \in \mathscr{S}_f^{R,r}$ we define $\check{b}(y, z-y)$ by

$$\check{b}(y, z-y) = \frac{1}{2\pi\sqrt{-1}} \sum_{\boldsymbol{Z}_+^{n-1}} \int_{\left(\frac{2e^2(|\alpha|+1)}{rRe^{\sqrt{-1}\theta}}\right)}^{\infty} e^{-(z_1 - y_1 - f(y))\eta_1} a_\alpha(y, \eta_1)\, d\eta_1$$
$$\times \Theta_\alpha(-z'-y'),$$

where $\boldsymbol{Z}_+ = \{0, 1, 2,\dots\}$, $a_\alpha(y, \eta_1) = \frac{1}{\alpha!}[\partial_{\eta'}^\alpha a(y, \eta)]_{\eta'=0}$, and

$$\Theta_\alpha(y') = \prod_{j=2}^{n} \frac{\alpha_j!}{2\pi\sqrt{-1}} (-y_j)^{-\alpha_j - 1}.$$

Then we can prove that $\check{b}(y, z-y)$ is holomorphic on

$$(3) \qquad \{(y, z) \in \boldsymbol{C}^n \times \boldsymbol{C}^n;\ |z_1 - y_1 - f(y)| < rR/2e^2,$$
$$|\arg\{e^{\sqrt{-1}\theta}(z_1 - y_1 - f(y))\}| < (\pi + R)/2,$$
$$|z_j - y_j| > 2r,\ j=2,\dots, n,\ |y_j| < R,\ j=1,\dots, n\}$$

and satisfies

$$(4)\qquad |\check{b}(y, z-y)| < C\exp\{C|z_1-y_1-f(y)|^{-\rho/(1-\rho)}\}\prod_{j=2}^{n}|z_j-y_j|^{-1}$$

with some $C>0$ on (3). We denote by $S_f^{R,r}$ the set of holomorphic functions satisfying (4) on (3). We remark that $-\frac{\rho}{1-\rho}=\frac{q-1-q'}{q+1}$.

Let us denote by $\tilde{\mathcal{O}}_f^R$ the set of functions $u(y)$ holomorphic on

$$(5)\qquad \Big\{y\in C^n;\ |y_1+f(y)|<R,\ |\arg(e^{-\sqrt{-1}\theta}(y_1+f(y))-\pi|<\frac{\pi}{2}+\frac{R}{4},\ |y_j|<R,\ j=2,\dots,n\Big\}$$

which satisfy

$$|u(y)|<C\exp\{C|y_1+f(y)|^{-\rho/(1-\rho)}\}$$

with some $C>0$ on (5). We denote by $\mathcal{O}_f^R$ the set of holomorphic functions $u(y)\in\tilde{\mathcal{O}}_f^R$ which can be continued to

$$\{y\in C^n;\ |y_1+f(y)|<R_1,\ |y_j|<R,\ j=2,\dots,n\}$$

with some $R_1\gg\varepsilon$. If $f=0$, we write $\tilde{\mathcal{O}}^R$ (resp. $\mathcal{O}^R$) instead of $\tilde{\mathcal{O}}_f^R$ (resp. $\mathcal{O}_f^R$).

Now let us define the action of $h(y, z-y)\in S_f^{R,r}$ from $\tilde{\mathcal{O}}^R/\mathcal{O}^R$ to $\tilde{\mathcal{O}}_f^{R-2r}/\mathcal{O}_f^{R-2r}$. We define $s_1, s_2\in C$ by

$$s_1=\frac{Rr}{3e^2}\exp\left\{\sqrt{-1}\left(\theta-\frac{\pi}{2}-\frac{R}{3}\right)\right\},\quad s_2=\frac{Rr}{3e^2}\exp\left\{\sqrt{-1}\left(\theta+\frac{\pi}{2}+\frac{R}{3}\right)\right\}.$$

If $u(y)\in\tilde{\mathcal{O}}^R$ (resp. $\mathcal{O}^R$), we define $v(y)$ by

$$v(y)=\int_{\gamma'}\int_{\gamma_1}h(y, z-y)u(z)d(z_1-y_1-f(y))d(z'-y').$$

Here $\gamma_1\subset C$ is a path connecting s_1 and s_2, $\gamma'=\gamma_2\times\cdots\times\gamma_n$ where $\gamma_j=\{z_j-y_j\in C;\ |z_j-y_j|=2r\}$, $j=2,\dots,n$. It is easy to see that $v(y)\in\tilde{\mathcal{O}}_f^{R-2r}$ (resp. $\mathcal{O}_f^{R-2r}$). Thus we obtain a map from $\tilde{\mathcal{O}}^R/\mathcal{O}^R$ to $\tilde{\mathcal{O}}_f^{R-2r}/\mathcal{O}_f^{R-2r}$. We denote this operator class from $\tilde{\mathcal{O}}^R/\mathcal{O}^R$ to $\tilde{\mathcal{O}}_f^{R-2r}/\mathcal{O}_f^{R-2r}$ by $Op_f(R, r)$. If $h(y, z-y)=\check{b}(y, z-y)$, $b(y, \eta)\in\mathcal{S}_f^{R,r}$, we denote the above $v(y)$ considered as an element of $\tilde{\mathcal{O}}_f^{R-2r}/\mathcal{O}_f^{R-2r}$ by $b(y, D_y)u(y)$. We call the function $b(y, \eta)$ the symbol of the operator $b(y, D_y)$, and denote the symbol of an operator $b(y, D_y)$ by $\sigma(b)(y, \eta)$.

Now we give some elementary results of this operator class $Op_f(R, r)$. Let $b(y, \eta)\in\mathcal{S}_f^{R,r}$ and $b_j(y, \eta)$, $j\in Z_+$, a sequence of functions holomorphic on (2) which satisfy

$$|b_j(y, \eta)|<C(R_1)^{-j}(r^j+|\eta_1|^{-j}j!)\exp(nr|\eta_1|)\qquad j\in Z_+$$

with some $C, R_1>0$, $r\ll R_1$ on (2). We write $b(y, \eta)\sim\sum_{j=0}^{\infty}b_j(y, \eta)$ if

$$|b(y, \eta)-\sum_{j=0}^{J}b_j(y, \eta)|\leqslant C(R_1)^{-J}(r^J+|\eta_1|^{-J}J!)\exp(nr|\eta_1|)\qquad J\in Z_+$$

on (2). Now we have the following.

Proposition 1. *Let us assume that* $b(y, D_y) \in Op_f(R, r)$ *is defined by the symbol* $\sigma(b)(y, \eta) \in \mathscr{S}_f^{R,r}$. *If* $\sigma(b)(y, \eta) \sim 0$, *we have* $b(y, D_y)=0$.

Now let us consider the relation between an operator and its symbol. Assume that $h(y, z-y) \in S_f^{R,r}$ and define $\hat{h}(y, \eta)$ by

$$\hat{h}(y, \eta)=\int_\gamma e^{(z-y)\eta} h(y, z-y) d(z-y), \quad \gamma=\gamma_1 \times \gamma'.$$

It is easy to see that $\hat{h}(y, \eta) \in \mathscr{S}_f^{R/4,2r}$. Therefore we can define an operator $h'(y, D_y) \in Op_f(R/4, 2r)$ by $\sigma(h')(y, \eta)=\hat{h}(y, \eta)$. We can prove the following

Proposition 2. *The action of* $h(y, z-y) \in S_f^{R,r} \subset S_f^{R/4,2r}$ *from* $\tilde{\mathcal{O}}^{R/4}/\mathcal{O}^{R/4}$ *to* $\tilde{\mathcal{O}}_f^{R/4-2r}/\mathcal{O}_f^{R/4-2r}$ *coincides with that of* $h'(y, Dy)$.

This means that we can determine an operator from its symbol and vice versa, though we must replace R (resp. r) by $R/4$ (resp. $2r$) in the latter case.

Now let $f'(y)$ be holomorphic on $\{|y_j|<R, j=1,\dots, n\}$ and satisfy $|f'(y)|<\varepsilon$ there. Assume that $b(y, D_y) \in Op_f(R, r)$ (resp. $b'(y, D_y) \in Op_{f'}(R, r)$) is defined by its symbol $\sigma(b)(y, \eta) \in \mathscr{S}_f^{R,r}$ (resp. $\sigma(b')(y, \eta) \in \mathscr{S}_{f'}^{R,r}$). Then we can define the composite operator $c(y, D_y)=b(y, D_y)b'(y, D_y)$ in some sense, and we can prove the following

Proposition 3. *Let us define* $g(y)$ *by* $g(y)=f(y)+f'(y_1+f(y), y'))$. *Then* $c(y, D_y)$ *is an element of* $Op_g(R/4, 2r)$ *which is defined by its symbol* $\sigma(c)(y, \eta) \in \mathscr{S}_g^{R/4,2r}$. *Furthermore, we have*

$$\sigma(c)(y, \eta) \sim \sum_{j=0}^{\infty}\Big(\sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\eta^\alpha \sigma(b)(y, \eta) \partial_y^\alpha \sigma(b')(y, \eta)\Big).$$

§2. Reduction of the problem to *Main Lemma*

In this section, we reduce the singular Cauchy problem (1) to *Main Lemma* which we shall state. Let us define $\lambda_i(x, y, D_y)$, $i=1, 2$, by $\sigma(\lambda_i)(x, y, \eta)=\lambda_i(x, y, \eta)$. Then there exists an operator $\mu(x, y, \eta)$ such that $\sigma(\mu)(x, y, \eta)$ is defined on $\{(x, y, \eta) \in \boldsymbol{C} \times \boldsymbol{C}^n \times \boldsymbol{C}^n; |x|<R, (y, \eta) \text{ satisfies the condition (2)}\}$, $|\sigma(\mu)(x, y, \eta)|< C|\eta_1|$ with some $C>0$ there, and

$$\begin{aligned} P(x, y, D_x, D_y)=&\{D_x-x^q\lambda_2(x, y, D_y)\}\{D_x-x^q\lambda_1(x, y, D_y)\} \\ &+a_{1,0}(x, y)\{D_x-x^q\lambda_1(x, y, D_y)\} \\ &+x^{q'}\mu(x, y, D_y)+a_{0,0}(x, y). \end{aligned}$$

Thus if we neglect a function which is holomorphic on some neighborhood of the origin, the singular Cauchy problem (1) is equivalent to

$$(6)\qquad \begin{cases} \{D_x - x^q \Lambda(x, y, D_y) + A(x, y, D_y)\}\vec{u}(x, y) = 0 \\ \vec{u}(0, y) = \begin{pmatrix} \mathring{u}_0(y) \\ \mathring{u}_1(y) \end{pmatrix} \end{cases}$$

Here $\Lambda(x, y, D_y)$ and $A(x, y, D_y)$ are 2×2 matrices of operators defined by

$$\Lambda(x, y, D_y) = \begin{pmatrix} \lambda_1(x, y, D_y) & \\ & \lambda_2(x, y, D_y) \end{pmatrix}$$

and

$$A(x, y, D_y) = \begin{pmatrix} 0 & -1 \\ x^{q'}\mu(x, y, D_y) + a_{0,0}(x, y) & a_{1,0}(x, y) \end{pmatrix}.$$

Furthermore, $\vec{u}(x, y)$ is a vector defined by

$$\vec{u}(x, y) = \begin{pmatrix} u(x, y) \\ \{D_x - x^q\lambda_1(x, y, D_y)\}u(x, y) \end{pmatrix}.$$

In §3–§6, we shall prove the following

Main Lemma. *We define* $\theta_0 \in \boldsymbol{R}$ *by*

$$\theta_0 = -\arg\{[\lambda_2(x, y, \eta) - \lambda_1(x, y, \eta)]_{x=0, y=0, \eta=(1,0,\dots,0)}\},$$

and $\theta_l \in \boldsymbol{R}$, $l \in \boldsymbol{Z}$, *by* $\theta_l = \theta_0 + \pi l$. *We assume that* R, r, $\varepsilon > 0$ *are small enough and that* $\varepsilon \ll r \ll R$. *We define* $\Omega_{\theta,l} = \Omega^1_{\theta,l} \cup \Omega^2_\theta$, $\theta \in \boldsymbol{R}$, $l \in \boldsymbol{Z}$, *by*

$$\Omega^1_{\theta,l} = \left\{(x, y, \eta) \in \boldsymbol{C} \times \boldsymbol{C}^n \times \boldsymbol{C}^n;\ \frac{|\eta_1|^{-1/(2q-q')}}{\varepsilon \sin\frac{\pi}{12}} < |x| < \varepsilon \sin\frac{\pi}{12},\right.$$
$$|(q+1)\arg x - (\theta_l + \theta) - \pi/2| < \frac{3}{4}\pi,$$
$$\left.(y, \eta) \text{ satisfies the condition } (2)\right\},$$

$$\Omega^2_{\theta,l} = \left\{(x, y, \eta) \in \boldsymbol{C} \times \boldsymbol{C}^n \times \boldsymbol{C}^n;\ \frac{2|\eta_1|^{-1/(2q-q')}}{\varepsilon \sin\frac{\pi}{12}} > |x|,\right.$$
$$\left.(y, \eta) \text{ satisfies the condition } (2)\right\}.$$

1° *For any* $\theta \in \boldsymbol{R}$ *and* $l \in \boldsymbol{Z}$ *there exist* 2×2 *matrices* $U^{\pm}(x, y, \eta) = U^{\pm}(\theta, l, x, y, \eta)$ *holomorphic on* $\Omega_{\theta,l}$ *such that*

$$(7)\qquad |U^{\pm}(x, y, \eta)| \leqslant C\exp\{C|\eta_1|^{(q-1-q')/(2q-q')}\}$$

with some $C>0$ *on* $\Omega_{\theta,l}$. *Thus if we fix a point* $x \in \{0\} \cup \left\{x \in \boldsymbol{C};\ |x| < \varepsilon \sin\frac{\pi}{12},\ |(q+1)\arg x - (\theta_l + \theta) - \pi/2| < \frac{3}{4}\pi\right\}$ *arbitrarily, we can define* $U^{\pm}(x, y, D_y) = U^{\pm}(\theta,$

l, x, y, D_y) by $\sigma(U^{\pm})(x, y, \eta)=U^{\pm}(x, y, \eta)$. These are 2×2 matrices whose entries belong to $Op(R, r)$ for any x as above fixed. We have

$$\{D_x - x^q A(x, y, D_y) + A(x, y, D_y)\}U^+(x, y, D_y) = U^+(x, y, D_y)\{D_x - x^q A(x, y, D_y)\} \tag{8}$$

and

$$U^{\pm}(x, y, D_y)U^{\mp}(x, y, D_y) = I_2. \tag{9}$$

2° *For any $\theta\in \boldsymbol{R}$, there exists a 2×2 matrix $E(x, y, \eta)=E(\theta, x, y, \eta)$ such that $E(x, y, \eta)=E_0(x, y, \eta)E_1(x, y, \eta)$, $E_1(x, y, \eta)=E_1(\theta, x, y, \eta)$. Here*

$$E_0(x, y, \eta) = \exp\begin{pmatrix} (\varphi_1(x, y)-y_1)\eta_1 & \\ & (\varphi_2(x, y)-y_1)\eta_1 \end{pmatrix}$$

and $E_1(x, y, \eta)$ is a 2×2 matrix holomorphic on

$$\{(x, y, \eta)\in \boldsymbol{C}\times \boldsymbol{C}^n\times \boldsymbol{C}^n;\ |x|<\varepsilon,\ (y, \eta) \text{ satisfies the condition } (2)\} \tag{10}$$

and

$$|E_1(x, y, \eta)| < C\exp\{\varepsilon \sum_{j=2}^{n} |\eta_j|\} \tag{11}$$

with some $C>0$ on (10). If we fix a point x with $|x|<\varepsilon$, we can define a 2×2 matrix $E(x, y, D_y)$ by $\sigma(E)(x, y, \eta)=E(x, y, \eta)$. This is a 2×2 matrix whose entries belong either to $Op_{\varphi_1(x,y)-y_1}(R, r)$ or to $Op_{\varphi_2(x,y)-y_1}(R, r)$. This matrix satisfies

$$\{D_x - x^q A(x, y, D_y)\}E(x, y, D_y) = E(x, y, D_y)D_x, \quad E(O, y, D_y) = I_2. \tag{12}$$

Remark. Let $M=(M_{(\mu,\nu)})_{1<\mu,\nu<m}$ be a $m\times m$ matrix. We define $|M|$ by $|M|=m(\max_{1<\mu,\nu<m} |M_{\mu,\nu)}|)$.

Admitting *Main Lemma* for a moment, let us construct the solution of (6) on $\omega'_{\varepsilon,\theta}$. We define $\tilde{v}(x, y)=\tilde{v}(\theta, l, x, y)$ by

$$\tilde{v}(x, y) = U^+(x, y, D_y)E(x, y, D_y)U^-(0, y, D_y)\begin{pmatrix} \mathring{u}_0(y) \\ \mathring{u}_1(y) \end{pmatrix}.$$

Since $U^+(x, y, D_y)E(x, y, D_y)U^-(O, y, D_y)$ is a 2×2 matrix whose entries belong to $\sum_{j=1,2} Op_{\varphi_j(x,y)-y_1}(R/16, 4r)$, each element of $\tilde{v}(x, y)$ belongs to $\sum_{j=1,2} \tilde{\mathcal{O}}^{R/16-8r}_{\varphi_j(x,y)-y_1}/\mathcal{O}^{R/16-8r}_{\varphi_j(x,y)-y_1}$ provided x satisfies

$$|x|<\varepsilon\sin\frac{\pi}{12}, \quad \left|(q+1)\arg x-(\theta_l+\theta)-\frac{\pi}{2}\right|<\frac{3}{4}\pi. \tag{13}$$

It is easy to see that

$$\{D_x - x^q A(x, y, D_y) + A(x, y, D_y)\}\tilde{v}(x, y) = 0.$$

Furthermore, since $(U^+EU^-)(0, y, D_y)=1$, it follows that $\tilde{v}((x, y)$ can be represented as the quotient class of some vector $\tilde{v}'(x, y)$ whose elements belong to $\sum_{j=1,2} \tilde{\mathcal{O}}^{R/16-8r}_{\varphi_j(x,y)-y_1}$ such that

$$\lim_{x\to 0} v'(x, y) \equiv \begin{pmatrix} \mathring{u}_0(y) \\ \mathring{u}_1(y) \end{pmatrix} \qquad \text{modulo } \mathcal{O}^{R/16-8r}.$$

Now let us define $v(x, y)$ to be the first element of the vector $\tilde{v}'(x, y)$. It is an element of $\sum_{j=1,2} \tilde{\mathcal{O}}^{R/16-8r}_{\varphi_j(x,y)-y_1}$ provided x satisfies (13), and holomorphic in x there. It is easy to see that

$$\begin{cases} Pv(x, y) = -f(x, y) \\ \lim_{x\to 0} D_x^i v(x, y) = \mathring{u}_i(y) - \mathring{v}_i(y) \qquad i=0, 1, \end{cases}$$

where $f(x, y)$ (resp. $\mathring{v}_i(y)$) is some function which is holomorphic on $\{(x, y); x$ satisfies (13), $|y_j| < R', j=1,\dots, n\}$ (resp. $\{|y_j| < R', j=1,\dots, n\}$) with some $R' \gg \varepsilon$, and bounded there. Now let us consider the following problem:

$$(14) \qquad Pw(x, y) = f(x, y), \ \lim_{x\to 0} D_x^i w(x, y) = \mathring{v}_i(y), \ i=0, 1.$$

Since $f(x, y)$ and $\mathring{v}_i(y)$, $i=0, 1$, are holomorphic on y on a neighborhood of the origin, we can solve (14) easily. In fact, let us define $w^{(j)}(x,y)$, $j\in \boldsymbol{Z}_+$, inductively by

$$w^{(0)}(x, y) = \int_0^x\int_0^x f(x, y)dxdx + \mathring{v}_0(y) + x\mathring{v}_i(y)$$

$$w^{(j)}(x, y) = -\int_0^x\int_0^x \{P(x, y, \partial_x, \partial_y) - \partial_x^2\} w^{(j-1)}(x, y)dxdx \qquad j\geq 1.$$

It is easy to see that $w^{(j)}(x, y)$ are holomorphic on $\{(x, y)\in \boldsymbol{C}\times \boldsymbol{C}^n;\ x$ satisfies the condition (13), $|y_j|<\varepsilon$, $j=1, 2,\dots, n\}$, and that the series $w(x, y) = \sum_{j=0}^{\infty} w^{(j)}(x, y)$ converges there. $w(x, y)$ is the solution of (14). Let us define $u(x, y)$ by $u(x, y) = v(x, y) + w(x, y)$. Since $|\arg \varphi_i(x, y) - \arg\{y_1 - \psi_i(x, y')\}| \ll 1$, $i=1, 2$, and $0<\varepsilon\ll R$, it is holomorphic on

$$(15) \qquad \left\{(x, y)\in \omega'_{\varepsilon \sin \frac{\pi}{12}, \theta};\ |(q+1)\arg x - (\theta_l+\theta) - \pi/2| < \frac{3}{4}\pi\right\},$$

and satisfies $Pu(x, y)=0$ on (15). Since $\lim_{x\to 0}\partial_x^i u(x, y) = \mathring{u}_i(y)$, $i=0, 1$, and the hypersurface $\{(x, y)\in \boldsymbol{C}\times\boldsymbol{C}^n;\ x=0\}$ is non-characteristic with respect to $P(x, y, \partial_x, \partial_y)$, it follows that $u(x, y)$ can be continued to a neighborhood of $\{(x, y)\in \boldsymbol{C}\times \boldsymbol{C}^n;\ x=0,\ y_1\neq 0,\ |y_j|<\varepsilon \sin\frac{\pi}{12},\ j=1, 2,\dots, n\}$, and that $\partial_x^i u(0, y) = \mathring{u}_i(y)$, $i=0, 1$. Thus we have solved (1) on (15). It is easy to see that any $x\in \boldsymbol{C}\backslash 0$ satisfies $|(q+1)\arg x - (\theta_l+\theta) - \pi/2| < \frac{3}{4}\pi$ with some $l\in \boldsymbol{Z}$. Let us denote by $u(\theta, l, x, y)$ the above solution on (15) to emphasize θ and l. We define $u(\theta, x, y)$ holomorphic on $\omega'_{\varepsilon \sin \frac{\pi}{12}, \theta}$ by

$$u(\theta, x, y)=u(\theta, l, x, y) \qquad \text{if} \quad |(q+1)\arg x-(\theta_l+\theta)-\pi/2|<\frac{3}{4}\pi.$$

$u(\theta, x, y)$ is well defined because of the theorem of Cauchy and Kowalewski. Thus we have constructed the solution on $\omega'_{\varepsilon\sin\frac{\pi}{12},\theta}$. We can construct the solution on $\omega''_{\varepsilon\sin\frac{\pi}{12},\theta}$ just in the same way. In fact, we only have to replace θ by $\theta+\pi$ in the above argument. These two solutions coincide on $\omega'_{\varepsilon\sin\frac{\pi}{12},\theta}\cap\omega''_{\varepsilon\sin\frac{\pi}{12},\theta}$ because of the theorem of Cauchy and Kowalewski. It is single-valued on $\{|\psi_i(x, y')|<|y_1|, i=1, 2\}$ because of the theorem of Cauchy and Kowalewski. Finally, we remark the fact that the solution is represented in terms of the operators mentioned in *Main Lemma*.

Thus we have reduced the problem to *Main Lemma*. We shall prove it in §3–§6. Here we give the plan of its proof. In §3–§5, we shall construct $U^{\pm}(x, y, \eta)$. Inspired by the theory about ordinary differential equations containing large parameters due to Iwano and Shibuya [3], we divide its construction into two parts: In §3 and §4, we shall construct $U^{\pm}(x, y, \eta)$ on $\Omega^1_{\theta,l}$ using the classical WKB method. Then we shall study this matrix on $\Omega^2_{\theta,l}$. In §5 we shall construct $E(x, y, \eta)$.

Remark. Boutet de Monvel [2] constructed parametrices for such operators as our P under certain conditions. He treated the case of $q=1$ and $q'=0$ in our notation. Though this case is excluded in our paper, his argument seems to be very much like ours. He defined a symbol class $\mathscr{H}^m$, $m\in \boldsymbol{Z}$, by $\mathscr{H}^m=\{a(x, y, \eta)\in C^\infty(\boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^n);\ |a(x, y, \eta)|<C_N|\eta|^m(|x|^{2q-q'}|\eta|+1)^{-N}$ with some C_N, $N=0, 1, 2,\dots\}$ with $q=1$, $q'=0$. At first he constructed the parametrices neglecting those operators whose symbols belong to $\mathscr{H}^m$, and then modified such error terms. We also consider the problem at first on $\Omega^1_{\theta,l}$ where $|x|^{2q-q'}|\eta|\gg 1$, and then complete the analysic considering the problem on $\Omega^2_{\theta,l}$.

§3. Construction of $U^{\pm}(x, y, \eta)$ on $\Omega^1_{\theta,l}$ (I)

The purpose of this and the next sections is to prove the following

Proposition 4. *Let ε, $R>0$ satisfy $0<\varepsilon\ll R$, and K, $k\in\boldsymbol{Z}_+$ satisfy $k\leqslant K$. There exist 2×2 matrices $U^{\pm,K,k}(x, y, \eta)$ holomorphic on $\Omega^1_{\theta,l}$ such that*

$$(16)^{K,k} \qquad |\partial_y^\beta\partial_\eta^\gamma U^{\pm,K,k}|\leqslant CR_1^{-2K-|\beta+\gamma|}(\varepsilon/R_1)^k|\eta_1|^{-K+k-|\gamma|}(K+k+|\beta+\gamma|)!$$
$$\times\exp\{|\eta_1|^{(q-1-q')/(2q-q')}\}$$

with some C, $R_1>0$ on $\Omega^1_{\theta,l}$ for K, $k\in\boldsymbol{Z}_+$. Here R_1 satisfies $R_1\gg\varepsilon$ (We may choose ε as small as we like, and R_1 is some constant which does not depend on ε). Furthermore, defining $U^{\pm,K}(x, y, \eta)$, $K\in\boldsymbol{Z}_+$, by $U^{\pm,K}=\sum_{k=0}^{K}U^{\pm,K,k}$, we have

$$(17)^{+,K} \qquad \partial_x U^{+,K}-\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\{\partial_\eta^\alpha\sigma(x^q\Lambda-A)\partial_y^\alpha U^{+,J}-\partial_\eta^\alpha U^{+,J}\partial_y^\alpha\sigma(x^q\Lambda)\}=0,$$

$$(17)^{-,K} \qquad \partial_x U^{-,K}-\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\{\partial_\eta^\alpha\sigma(x^q\Lambda)\partial_y^\alpha U^{-,J}-\partial_\eta^\alpha U^{-,J}\partial_y^\alpha\sigma(x^q\Lambda-A)\}=0,$$

and

$$(18)^K \qquad \sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha U^{\pm,I}\partial_y^\alpha U^{\mp,J}=\delta_{K,0}I_2.$$

Though we can construct $U^{\pm}(x, y, \eta)$ on $\Omega_{\theta,l}^1$ using *Proposition 4* (See *Proposition 8*), at first let us prove *Proposition 4*. We consider $(17)^{\pm,K}$, $K\in \boldsymbol{Z}_+$, as ordinary differential equations with respect to x containing a large parameter η_1. In this section we transform these equations to equivalent ones which are more easy to investigate.

Let us consider 2×2 matrices $S^{\pm,K}(x, y, \eta)$, $K\in \boldsymbol{Z}_+$, defined by

$$S^{\pm,K}(x, y, \eta)=\delta_{K,0}I_2\pm\begin{pmatrix} 0 & 0 \\ s^K(x, y, \eta) & 0\end{pmatrix}.$$

Here $s^K(x, y, \eta)$, $K\in \boldsymbol{Z}_+$, are defined inductively by

$$s^K(x, y, \eta)=\begin{cases} \dfrac{x^{q'-q}\sigma(\mu)(x, y, \eta)+x^{-q}a_{0,0}(x, y)}{\lambda_2(x, y, \eta)-\lambda_1(x, y, \eta)} & K=0,\\[2ex] -\dfrac{\sum_{J+|\alpha|=K, J\neq K}\frac{1}{\alpha!}\{\partial_\eta^\alpha\lambda_2\partial_y^\alpha s^J-\partial_\eta^\alpha s^J\partial_y^\alpha\lambda_1\}}{\lambda_2-\lambda_1} & K\geqslant 1.\end{cases}$$

These functions are holomorphic on $\tilde{\Omega}_\theta^1=\{(x, y, \eta)\in \boldsymbol{C}\times \boldsymbol{C}^n\times \boldsymbol{C}^n;\ \varepsilon^{-1}|\eta_1|^{-1/(2q-q')}<|x|<\varepsilon,\ (y, \eta)$ satisfies the condition (2)$\}$, and

$$(19) \qquad |\partial_y^\beta\partial_\eta^\gamma s^K(x, y, \eta)|\leqslant a|x|^{-q+q'}R^{-K-|\beta+\gamma|}|\eta_1|^{-K-|\eta|}(K+|\beta+\gamma|)!$$

with some $a>0$ for $K\in \boldsymbol{Z}_+$, α, $\beta\in \boldsymbol{Z}_+^n$ on $\tilde{\Omega}_\theta^1$ provided $R>0$ is small enough.

It is easy to see that

$$(20) \qquad \sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha S^{\pm,I}\partial_y^\alpha S^{\mp,J}=\delta_{K,0}I_2.$$

Now let us consider the following transformations:

$$(21)^+ \qquad \tilde{U}^{+,K}=\sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha S^{+,I}\partial_y^\alpha U^{+,J}$$

$$(21)^- \qquad \tilde{U}^{-,K}=\sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha U^{-,J}\partial_y^\alpha S^{-,I}.$$

Then we have the following

Lemma 1. 1) *We have*

$$(22)^+ \qquad U^{+,K}=\sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha S^{-,I}\partial_y^\alpha \tilde{U}^{+,J}$$

$$(22)^- \qquad U^{-,K}=\sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha \tilde{U}^{-,J}\partial_y^\alpha S^{+,I}.$$

2) *The equations* $(17)^{\pm,K}$, $K \in \mathbf{Z}_+$, *are equivalent to the following equations* $(23)^{\pm,K}$ *on* $\tilde{\Omega}^1_\theta$:

$$(23)^{+,K} \qquad \partial_x \tilde{U}^{+,K} - \sum_{J+|\alpha|=K} \frac{1}{\alpha!}\{\partial_\eta^\alpha \sigma(x^q \Lambda)\partial_y^\alpha \tilde{U}^{+,J} - \partial_\eta^\alpha \tilde{U}^{+,J}\partial_y^\alpha \sigma(x^q \Lambda)\}$$

$$+ \sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha \tilde{A}^I \partial_y^\alpha \tilde{U}^{+,J} = 0$$

$$(23)^{-,K} \qquad \partial_x \tilde{U}^{-,K} - \sum_{J+|\alpha|=K} \frac{1}{\alpha!}\{\partial_\eta^\alpha \sigma(x^q \Lambda)\partial_y^\alpha \tilde{U}^{-,J} - \partial_\eta^\alpha \tilde{U}^{-,J}\partial_y^\alpha \sigma(x^q \Lambda)\}$$

$$- \sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha \tilde{U}^{-,J} \partial_y^\alpha \tilde{A}^I = 0.$$

Here $\tilde{A}^I(x, y, \eta)$, $I \in \mathbf{Z}_+$, *are defined by*

$$(24) \qquad \tilde{A}^I = \delta_{K,0}\begin{pmatrix} 0 & -1 \\ 0 & a_{1,0} \end{pmatrix}$$

$$+ \begin{pmatrix} -s^K & 0 \\ \partial_x s^K + a_{1,0}s^K + \sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha s^I \partial_y^\alpha s^J & s^K \end{pmatrix}.$$

Proof. 1) We can prove $(22)^{\pm}$ using (20) and $(21)^{\pm}$ easily.

2) If we substitute $(22)^{\pm}$ into $(17)^{\pm,K}$, we obtain $(23)^{\pm,K}$, where $\tilde{A}^I(x, y, \eta)$, $I \in \mathbf{Z}_+$, are given by

$$\tilde{A}^I = \sum_{J+K+|\alpha|=I} \frac{1}{\alpha!}\partial_\eta^\alpha S^{+,J}\partial_y^\alpha \partial_x S^{-,K}$$

$$- \sum_{J+K+|\alpha+\beta|=I} \frac{1}{\alpha!\beta!}\partial_\eta^\alpha\{\partial_\eta^\beta S^{+,J}\partial_y^\beta \sigma(x^q\Lambda - A)\}\partial_y^\alpha S^{-,K}$$

$$= \delta_{I,0}\sigma(-x^q\Lambda + A) + \begin{pmatrix} -s^I & 0 \\ *^I & s^I \end{pmatrix}.$$

Here $*^I$, $I \in \mathbf{Z}_+$, are given by

$$*^I = x^q \sum_{J+|\alpha|=I} \frac{1}{\alpha!}\{\partial_\eta^\alpha s^J \partial_y^\alpha \lambda_1 - \partial_\eta^\alpha \lambda_2 \partial_y^\alpha s^J\} + a_{1,0} + \partial_x s^I$$

$$+ \sum_{J+K+|\alpha|=I} \frac{1}{\alpha!}\partial_\eta^\alpha s^J \partial_y^\alpha s^K.$$

Substituting the definition of $s^I(x, y, \eta)$ into the right-hand sides of these equations, we obtain (24). Conversely, we can obtain $(17)^{\pm,K}$, $K \in \mathbf{Z}_+$ from $(21)^{\pm,K}$ and $(23)^{\pm,K}$, $K \in \mathbf{Z}_+$ in a similar way. Q.E.D.

Now let us consider the following transformations:

$$\tilde{\tilde{U}}^{+,K}(x, y, \eta) = \begin{pmatrix} 1 & \\ & x^{q-q'} \end{pmatrix}\tilde{U}^{+,K}, \quad \tilde{\tilde{U}}^{-,K}(x, y, \eta) = \tilde{U}^{-,K}\begin{pmatrix} 1 & \\ & x^{-q+q'} \end{pmatrix}.$$

In the same way as *Lemma 1*, we obtain the following

Lemma 2. $(23)^{\pm,K}$, $K\in \boldsymbol{Z}_+$, *are equivalent to the following* $(25)^{\pm,K}$, $K\in \boldsymbol{Z}_+$:

$$(25)^{+,K}\qquad \partial_x\tilde{\tilde{U}}^{+,K}-\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\{\partial_\eta^\alpha\sigma(x^q\Lambda)\partial_y^\alpha\tilde{\tilde{U}}^{+,J}-\partial_\eta^\alpha\tilde{\tilde{U}}^{+,J}\partial_y^\alpha\sigma(x^q\Lambda)\}$$

$$+\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha\tilde{\tilde{A}}^I\partial_y^\alpha\tilde{\tilde{U}}^{+,J}=0$$

$$(25)^{-,K}\qquad \partial_x\tilde{\tilde{U}}^{-,K}-\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\{\partial_\eta^\alpha\sigma(x^q\Lambda)\partial_y^\alpha\tilde{\tilde{U}}^{-,J}-\partial_\eta^\alpha\tilde{\tilde{U}}^{-,J}\partial_y^\alpha\sigma(x^q\Lambda)\}$$

$$-\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha\tilde{\tilde{U}}^{-,J}\partial_y^\alpha\tilde{\tilde{A}}^I=0.$$

Here $\tilde{\tilde{A}}^I(x,y,\eta)$, $I\in \boldsymbol{Z}_+$, *are given by*

$$\tilde{\tilde{A}}^I=\delta_{I,0}\begin{pmatrix}0 & -x^{-q+q'}\\ 0 & a_{1,0}-(q-q')x^{-1}\end{pmatrix}$$

$$+\begin{pmatrix}-s^I & 0\\ x^{q-q'}\left\{\partial_x s^I+a_{1,0}s^I+\sum_{J+K+|\alpha|=I}\frac{1}{\alpha!}\partial_\eta^\alpha s^J\partial_y^\alpha s^K\right\} & s^I\end{pmatrix}.$$

Corollary. *From* (19), *it follows that*

$$(26)\qquad |\partial_y^\beta\partial_\eta^\gamma\tilde{\tilde{A}}^I|\leqslant a|x|^{-q+q'}R^{-I-|\beta+\gamma|}|\eta_1|^{-I-|\gamma|}(I+|\beta+\gamma|)!,$$

with some $a>0$ *for* $I\in \boldsymbol{Z}_+$, $\beta,\gamma\in \boldsymbol{Z}_+^n$ *on* $\tilde{\Omega}_\theta^1$ *provided* $R>0$ *is small enough.*

We transform the equations $(25)^{\pm,K}$, $K\in \boldsymbol{Z}_+$, once more. For this purpose, let us prepare the following

Lemma 3. *There exist* 2×2 *matrices* $\Phi^j(x,y,\eta)$, $\bar{B}^{0,j}(x,y,\eta)$, $0\leqslant j\leqslant q-q'$, *and* $B'(x,y,\eta)$, *which are holomorphic on* $\tilde{\Omega}_\theta^1$ *such that*

1) Φ^0 *is an invertible matrix.* Φ^j, $0\leqslant j\leqslant q-q'$, *and* B' *satisfy*

$$|\Phi^j|\leqslant a|x|^j,\quad |(\Phi^0)^{-1}|\leqslant a,\quad |B'|\leqslant a$$

with some $a>0$ *on* $\tilde{\Omega}_\theta^1$.

2) $\bar{B}^{0,j}$, $0\leqslant j\leqslant q-q'$, *are diagonal matrices satisfying*

$$|\bar{B}^{0,j}|\leqslant a|x|^{-q+q'+j}$$

with some $a>0$ *on* $\tilde{\Omega}_\theta^1$.

3) *Defining* $\Phi(x,y,\eta)$ (*resp.* $\bar{B}^0(x,y,\eta)$) *by*

$$\Phi=\Phi^0\Phi',\ \Phi'=I_2+\sum_{j=1}^{q-q'}\Phi^j\ (\text{resp. } \bar{B}^0=\sum_{j=0}^{q-q'}\bar{B}^{0,j}),$$

we have

$$(27)\qquad \Phi^{-1}\{\sigma(x^q\Lambda)-\tilde{\tilde{A}}^0\}\Phi-\Phi^{-1}\partial_x\Phi=\sigma(x^q\Lambda)-\bar{B}^0+B'.$$

Proof. (27) can be rewritten as

$$\{\sigma(x^q\Lambda-\tilde{\tilde{A}}^0\}\Phi^0\Phi'-\Phi^0\Phi'\{\sigma(x^q\Lambda)-\bar{B}^0\}=(\partial_x\Phi^0)\Phi'+\Phi^0\partial_x\Phi'+\Phi^0\Phi'B'. \tag{28}$$

At first, we choose $\Phi^0(x, y, \eta)$ to be the non-singular matrix which transforms $\sigma(x^q\Lambda)-\tilde{\tilde{A}}^0$ into the diagonal matrix $\sigma(x^q\Lambda)-\bar{B}^{0,0}$, i.e., $(\Phi^0)^{-1}\{\sigma(x^q\Lambda)-\tilde{\tilde{A}}^0\}\Phi^0=\sigma(x^q\Lambda)-\bar{B}^{0,0}$. Remark that

$$\frac{1}{x^q\eta_1}\{\sigma(x^q\Lambda)-\tilde{\tilde{A}}^0\}=\begin{pmatrix}\eta_1^{-1}\lambda_1 & \\ & \eta_1^{-1}\lambda_2\end{pmatrix}+\frac{1}{x^q\eta_1}\tilde{\tilde{A}}^0.$$

Since

$$|\eta_1^{-1}\lambda_1(x, y, \eta)-\eta_1^{-1}\lambda_2(x, y, \eta)|>0$$

and

$$\left|\frac{1}{x^q\eta_1}\tilde{\tilde{A}}^0(x, y\ \eta)\right|\leqslant a|x|^{-2q+q'}|\eta_1|^{-1}\leqslant a\varepsilon^{2q-q'}\ll 1$$

on $\tilde{\Omega}^1_\theta$, such a transformation exists and it follows that

$$|\Phi^0|,\quad |(\Phi^0)^{-1}|\leqslant a,\quad |\bar{B}^{0,0}|\leqslant a|x|^{-q+q'}$$

with some $a>0$ on $\tilde{\Omega}^1_\theta$. Defining Φ^0 and $\bar{B}^{0,0}$ in this way (28) can be rewritten as

$$\{\sigma(x^q\Lambda)-\bar{B}^{0,0}\}\Phi'-\Phi'\{\sigma(x^q\Lambda)-\bar{B}^0\}=(\Phi^0)^{-1}\partial_x\Phi^0\Phi'+\partial_x\Phi'-\Phi'B'.$$

Setting each (μ, ν) element of both sides equal, we obtain

$$\begin{aligned}\{x^q(\lambda_\mu-\lambda_\nu)-\bar{B}^{0,0}_{\mu,\mu}+\bar{B}^{0,0}_{\nu,\nu}\}\sum_{j=1}^{q-q'}\Phi^j_{(\mu,\nu)}+\sum_{j=1}^{q-q'}\bar{B}^{0,j}_{(\mu,\nu)}\\
=-\sum_{j=1}^{q-q'}\{\sum_{k=1}^{j-1}\Phi^k_{(\mu,\nu)}\bar{B}^{0,j-k}_{\nu,\nu}-(\Phi^0)^{-1}\partial_x\Phi^0\Phi^j)_{(\mu,\nu)}+\partial_x\Phi^j_{(\mu,\nu)}\}\\
-((\Phi^0)^{-1}\partial_x\Phi^0)_{(\mu,\nu)}+\Phi'B')_{(\mu,\nu)},\qquad 1\leqslant\mu,\ \nu\leqslant 2.\end{aligned}$$

We define $\Phi^1_{(\mu,\nu)}$ and $\bar{B}^{0,1}_{(\mu,\nu)}$, $1\leqslant\mu,\ \nu\leqslant 2$, by

$$\Phi^1_{(\mu,\nu)}=\begin{cases}0 & \mu=\nu\\ ((\Phi^0)^{-1}\partial_x\Phi^0)_{(\mu,\nu)}/\{x^q(\lambda_\mu-\lambda_\nu)-\bar{B}^{0,0}_{(\mu,\mu)}+\bar{B}^{0,0}_{(\nu,\nu)}\} & \mu\neq\nu\end{cases}$$

and

$$\bar{B}^{0,1}_{(\mu,\nu)}=\begin{cases}-((\Phi^0)^{-1}\partial_x\Phi^0)_{(\mu,\nu)} & \mu=\nu\\ 0 & \mu\neq\nu.\end{cases}$$

Let us define $\Phi^j_{(\mu,\nu)}$ and $\bar{B}^{0,j}_{(\mu,\nu)}$, $1\leqslant\mu,\ \nu\leqslant 2$, $j=2, 3,\dots, q-q'$ by induction on j as follows:

$$\Phi^j_{(\mu,\nu)}=\begin{cases}0 & \mu=\nu\\ \dfrac{-\sum_{k=1}^{j-1}\{\Phi^k_{(\mu,\nu)}\bar{B}^{0,j-k}_{(\nu,\nu)}\}+((\Phi^0)^{-1}\partial_x\Phi^0\Phi^{j-1})_{(\mu,\nu)}-\partial_x\Phi^{j-1}_{(\mu,\nu)}}{x^q(\lambda_\mu-\lambda_\nu)-\bar{B}^{0,0}_{(\mu,\mu)}+\bar{B}^{0,0}_{(\nu,\nu)}} & \mu\neq\nu\end{cases}$$

$$\bar{B}^{0,j}_{(\mu,\nu)}=\begin{cases} -\sum_{k=1}^{j-1}\{\Phi^k_{(\mu,\nu)}\bar{B}^{0,j-k}_{(\nu,\nu)}\}+((\Phi^0)^{-1}\partial_x\Phi^0\Phi^{j-1})_{(\mu,\nu)}-\partial_x\Phi^{j-1}_{(\mu,\nu)} & \mu=\nu \\ 0 & \mu\neq\nu. \end{cases}$$

Furthermore, let us define $B'(x, y, \eta)$ by

$$B'=\Phi^{-1}\{\sigma(x^q\Lambda)-\tilde{\tilde{A}}^0\}\Phi-\Phi^{-1}\partial_x\Phi-\sigma(x^q\Lambda)+\bar{B}^0.$$

It is easy to see that 1), 2) and 3) are valid. Q.E.D.

Let us assume that we have chosen $R>0$ small enough. Then it is easy to see that $\Phi(x, y, \eta)$ is an invertible matrix on $\tilde{\Omega}^1_\theta$ and that

$$|\partial_y^\beta\partial_\eta^\gamma\Phi|,\quad |\partial_y^\beta\partial_\eta^\gamma(\Phi)^{-1}|\leqslant aR^{-|\beta+\gamma|}|\eta_1|^{-|\gamma|}|\beta+\gamma|!$$

$$|\partial_y^\beta\partial_\eta^\gamma\bar{B}^0|\leqslant a|x|^{-q+q'}R^{-|\beta+\gamma|}|\eta_1|^{-|\gamma|}|\beta+\gamma|!$$

with some $a>0$ on $\tilde{\Omega}^1_\theta$ for $\beta, \gamma\in \boldsymbol{Z}^n_+$. We can easily construct 2×2 matrices $\Psi^K(x, y, \eta)$, $K\in \boldsymbol{Z}_+$, holomorphic on $\tilde{\Omega}^1_\theta$ such that

$$\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha\Psi^J\partial_y^\alpha\Phi=\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha\Phi\partial_y^\alpha\Psi^J=\delta_{K,0}I_2.$$

It is also easy to see that if we have chosen $R>0$ small enough, we have

$$|\partial_y^\beta\partial_\eta^\gamma\Psi^K|\leqslant aR^{-K-|\beta+\gamma|}|\eta_1|^{-K-|\gamma|}(K+|\beta+\gamma|)!$$

with some $a>0$ on $\tilde{\Omega}^1_\theta$ for $K\in \boldsymbol{Z}_+$, $\beta, \gamma\in \boldsymbol{Z}^n_+$.

Now let us consider the following transformation:

$$V^{+,K}=\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha\Psi^I\partial_y^\alpha\tilde{\tilde{U}}^{+,J}$$

$$V^{-,K}=\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha\tilde{\tilde{U}}^{-,J}\partial_y^\alpha\Phi.$$

As before, we can prove the following

Lemma 4. 1) *We have*

$$\tilde{\tilde{U}}^{+,K}=\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha\Phi\partial_y^\alpha V^{+,J}$$

$$\tilde{\tilde{U}}^{-,K}=\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha V^{-,J}\partial_y^\alpha\Psi^I.$$

2) *The equations* $(25)^{\pm,K}$, $K\in \boldsymbol{Z}_+$, *are equivalent to the following equations* $(29)^{\pm,K}$, $K\in \boldsymbol{Z}_+$:

$(29)^{+,K}$
$$\partial_x V^{+,K}-\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha(x^q\Lambda^I-B^I)\partial_y^\alpha V^{+,J}$$
$$+\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha V^{+,J}\partial_y^\alpha(x^q\Lambda^0)=0$$

$$(29)^{-,K} \qquad \partial_x V^{-,K} - \sum_{J+|\alpha|=K} \frac{1}{\alpha!}\partial_y^\alpha(x^q\Lambda^0)\partial_y^\alpha V^{-,J}$$

$$+ \sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha V^{-,J}\partial_y^\alpha(x^q\Lambda^I - B^I) = 0.$$

Here we have defined $\Lambda^K(x, y, \eta)$, $B^K(x, y, \eta)$, $K \in \boldsymbol{Z}_+$, *by*

$$\Lambda^0 = \sigma(\Lambda),\ B^0 = x^q\sigma(\Lambda) - \tilde{\tilde{A}}^0 - \Psi^0\{\sigma(x^q\Lambda) - \tilde{\tilde{A}}^0\}\Phi - \Psi^0\partial_x\Phi,$$

and

$$\Lambda^K = \sum_{J+|\alpha+\beta|=K} \frac{1}{\alpha!\beta!}\partial_\eta^\alpha(\partial_\eta^\beta\Psi^J\varrho_y^\beta\sigma(\Lambda))\partial_y^\alpha\Phi$$

$$B^K = \sum_{I+J+|\alpha+\beta|=K} \frac{1}{\alpha!\beta!}\partial_\eta^\alpha(\partial_\eta^\beta\Psi^I\partial_y^\beta\tilde{\tilde{A}}^J)\partial_y^\alpha\Phi$$

$$- \sum_{J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha\Psi^J\partial_y^\alpha\partial_x\Phi$$

if $K \geqslant 1$.

Corollary. *If we have chosen* $R>0$ *small enough, we have*

$$(31) \qquad |\partial_y^\beta\partial_\eta^\gamma\Lambda^K| \leqslant aR^{-K-|\beta+\gamma|}|\eta_1|^{1-K-|\gamma|}(K+|\beta+\gamma|)!$$

$$(32) \qquad |\partial_y^\beta\partial_\eta^\gamma B^K| \leqslant a|x|^{-q+q'}R^{-K-|\beta+\gamma|}|\eta_1|^{-K-|\gamma|}(K+|\beta+\gamma|)!$$

with some $a>0$ *on* $\tilde{\Omega}_\theta^1$ *for* $K \in \boldsymbol{Z}_+$, $\beta, \gamma \in \boldsymbol{Z}_+^n$. *Furthermore, we have* $B^0 = \bar{B}^0 + B'$ *and*

$$(33) \qquad \begin{cases} |\partial_y^\beta\partial_\eta^\gamma\bar{B}^0| \leqslant a|x|^{-q+q'}R^{-|\beta+\gamma|}|\eta_1|^{-|\gamma|}|\beta+\gamma|! \\ |\partial_y^\beta\partial_\eta^\gamma B'| \leqslant aR^{-|\beta+\gamma|}|\eta_1|^{-|\gamma|}|\beta+\gamma|! \end{cases}$$

with some $a>0$ *on* $\tilde{\Omega}_\theta^1$ *for* $\beta, \gamma \in \boldsymbol{Z}_+^n$. *We remind the reader that* Λ^0 *and* $\bar{B}^0$ *are diagonal matrices.*

§4. Construction of $U^\pm(x, y, \eta)$ on $\Omega_{\theta,l}^1$ (II).

In this section, we solve the the equations $(29)^{\pm,K}$, $K \in \boldsymbol{Z}_+$, on $\Omega_{\theta,l}^1$, and using these solutions, we construct the matrices $U^\pm(x, y, \eta)$ mentioned in *Main Lemma*, on $\Omega_{\theta,l}^1$. We solve these equations in two steps. At first, we consider the following equations $(34)^{\pm,K}$, $K \in \boldsymbol{Z}_+$:

$$(34)^{+,K} \qquad \partial_x W^{+,K} - (x^q\Lambda^0 - B^0)W^{+,K} + W^{+,K}(x^q\Lambda^0 - \bar{B}^0) = F^{+,K}$$

$$(34)^{-,K} \qquad \partial_x W^{-,K} - (x^q\Lambda^0 - \bar{B}^0)W^{-,K} + W^{-,K}(x^q\Lambda^0 - B^0) = F^{-,K},$$

where $F^{\pm,K} = F^{\pm,K}(x, y, \eta)$, $K \in \boldsymbol{Z}_+$, are given by

$$(35)^{+,K} \qquad F^{+,K} = \sum_{\substack{I+J+|\alpha|=K \\ J\neq K}} \frac{1}{\alpha!}\partial_\eta^\alpha(x^q\Lambda^I - B^I)\partial_y^\alpha W^{+,J}$$

$$-\sum_{\substack{J+|\alpha|=K\\ J\neq K}}\frac{1}{\alpha!}\partial_\eta^\alpha W^{+,J}\partial_y^\alpha(x^q\Lambda^0-\bar{B}^0)$$

$(35)^{-,K}$
$$F^{-,K}=\sum_{\substack{J+|\alpha|=K\\ I\neq K}}\frac{1}{\alpha!}\partial_\eta^\alpha(x^q\Lambda^0-\bar{B}^0)\partial_y^\alpha W^{-,J}$$
$$-\sum_{\substack{I+J+|\alpha|=K\\ J\neq K}}\frac{1}{\alpha!}\partial_\eta^\alpha W^{-,J}\partial_y^\alpha(x^q\Lambda^I-B^I).$$

Then, we consider the following equations $(36)^{\pm,K}$, $K\in \boldsymbol{Z}_+$:

$(36)^{+,K}$
$$\partial_x X^{+,K}+\bar{B}^0X^{+,K}=G^{+,K}$$

$(36)^{-,K}$
$$\partial_x X^{-,K}-X^{-,K}\bar{B}^0=G^{-,K},$$

where $G^{\pm,K}=G^{\pm,K}(x, y, \eta)$, $K\in \boldsymbol{Z}_+$, are given by

$(37)^{+,K}$
$$G^{+,K}=\sum_{\substack{J+|\alpha|=K\\ J\neq K}}\frac{1}{\alpha!}\{\partial_\eta^\alpha(x^q\Lambda^0-\bar{B}^0)\partial_y^\alpha X^{+,J}-\partial_\eta^\alpha X^{+,J}\partial_y^\alpha(x^q\Lambda^0)\}$$

$(37)^{-,K}$
$$G^{-,K}=\sum_{\substack{J+|\alpha|=K\\ J\neq K}}\frac{1}{\alpha!}\{\partial_\eta^\alpha(x^q\Lambda^0)\partial_y^\alpha X^{-,J}-\partial_\eta^\alpha X^{-,J}\partial_y^\alpha(x^q\Lambda^0-\bar{B}^0)\}.$$

Remark. If $W^{\pm,K}$ (resp. $X^{\pm,K}$), $K\in \boldsymbol{Z}_+$, are solutions of $(34)^{\pm,K}$ (resp. $(36)^{\pm,K}$), $K\in \boldsymbol{Z}_+$, then $V^{\pm,K}$, $K\in \boldsymbol{Z}_+$, defined by

$(38)^{+,K}$
$$V^{+,K}=\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha W^{+,I}\partial_y^\alpha X^{+,J}$$

(resp.

$(38)^{-,K}$
$$V^{-,K}=\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha X^{-,J}\partial_y^\alpha W^{-,I})$$

are solutions of $(29)^{\pm,K}$, $K\in \boldsymbol{Z}_+$.

From now on we assume that the integer l which defines the domain $\Omega_{\theta,l}^1$ is even. Let us define $\mathring{x}_\nu^\pm=\mathring{x}_\nu^\pm(\theta, l)$, $\nu=1, 2$, by

$$\mathring{x}_1^+=\mathring{x}_2^-=\varepsilon\cdot\exp\left\{\frac{-1}{q+1}(\theta_l+\theta)\right\},\quad \mathring{x}_2^+=\mathring{x}_1^-=\varepsilon\cdot\exp\left\{\frac{-1}{q+1}(\theta_l+\theta+\pi)\right\}$$

(If l is odd, we define $\mathring{x}_\nu^\pm$, $\nu=1, 2$, by

$$\mathring{x}_1^+=\mathring{x}_2^-=\varepsilon\cdot\exp\left\{\frac{-1}{q+1}(\theta_l+\theta)\right\},\quad \mathring{x}_2^+=\mathring{x}_1^-=\varepsilon\cdot\exp\left\{\frac{-1}{q+1}(\theta_l+\theta-\pi)\right\}.$$

Then the following arguments are also valid for this case).

Now we solve the equations $(34)^{+,K}$, $K\in \boldsymbol{Z}_+$, by successive approximation. Let K, $k\in \boldsymbol{Z}_+$, and consider the following initial value problems:

$(34)^{+,K,k}$
$$\begin{cases}\partial_x W^{+,K,k}-(x^q\Lambda^0-B^0)W^{+,K,k}+W^{+,K,k}(x^q\Lambda^0-\bar{B}^0)=F^{+,K,k}\\ W_{(\mu,\nu)}^{+,K,k}(\mathring{x}_{\nu,y,\eta}^+)=\delta_{K,0}\delta_{k,0}\delta_{\mu,\nu},\end{cases}$$

where

$$F^{+,K,k}=\begin{cases} 0 & k=0 \\ \displaystyle\sum_{\substack{I+J+|\alpha|=K\\ J\neq K}} \frac{1}{\alpha!}\partial_\eta^\alpha(x^q A^I - B^I)\partial_y^\alpha W^{+,J,k-1} & \\ \displaystyle\quad - \sum_{\substack{J+|\alpha|=K\\ J\neq K}} \frac{1}{\alpha!}\partial_\eta^\alpha W^{+,J,k-1}\partial_y^\alpha(x^q A^0 - \bar{B}^0) & k\geqslant 1. \end{cases}$$

Now we prepare the following

Lemma 5. *Let* $(x, y, \eta)\in\Omega^1_{\theta,l}$. *Then there exist two piecewise smooth paths* $\gamma_\nu(x)\subset \boldsymbol{C}$, $\nu=1, 2$, *such that*

1° $\gamma_\nu(x)$ *is a path which connects* $\mathring{x}_\nu^+$ *and* x. *If* $t\in\gamma_\nu(x)$, *then we have* $(t, y, \eta)\subset\tilde{\Omega}^1_\theta$.

2° *We denote by* $d_\nu(t)$ *the length along* $\gamma_\nu(x)$ *from* $\mathring{x}_\nu^+$ *to* $t\in\gamma_\nu(x)$. *If* s, $t\in\gamma_\nu(x)$ *and* $d_\nu(s)\geqslant d_\nu(t)$, *then* $|\arg\{(-1)^\nu(s^{q+1}-t^{q+1})\}-\theta_l-\theta|\leqslant\frac{\pi}{3}$.

3° *If* $t\in\gamma_\nu(x)$, *then we have* $|d_\nu(x)|<a|x|$, $|x|<a|t|$ *with some* $a>0$.

Proof. Let us assume that $\nu=1$. If x satisfies

$$|(q+1)\arg x-(\theta_l+\theta)|\leqslant\frac{\pi}{2},$$

we may take as $\gamma_1(x)$ the segment between $\mathring{x}_1^+$ and x. Assume that

$$\frac{\pi}{2}<(q+1)\arg x-(\theta_l+\theta)<\frac{5}{4}\pi.$$

We define $x_1'\in\boldsymbol{C}$ by

$$x_1'=\exp\left(\frac{\theta_l+\theta}{q+1}+\frac{\pi}{2}\right)\left\{\mathrm{Im}\left(\exp\left(-\frac{\theta_l+\theta}{q+1}\right)x\right)-\left(\tan\frac{\pi}{6}\right)\mathrm{Re}\left(-\frac{\theta_l+\theta}{q+1}\right)x\right)\right\}.$$

It is enough to take as $\gamma_1(x)$ the union of the segment between $\mathring{x}_1^+$ and x_1', and the segment between x_1' and x. In the same way we can construct $\gamma_1(x)$ in the case

$$-\frac{5}{4}\pi<(q+1)\arg x-(\theta_l+\theta)<\frac{\pi}{2}.$$

Thus we can construct $\gamma_1(x)$ if

$$|(q+1)\arg x-(\theta_l+\theta)|<\frac{5}{4}\pi.$$

In the same way we can construct $\gamma_2(x)$ if

$$|(q+1)\arg x-(\theta_l+\theta)-\pi|<\frac{5}{4}\pi. \qquad \text{Q.E.D.}$$

The following proposition is one of the most essential parts in this paper:

Proposition 5. *Let* $(x, y, \eta)\in\Omega^1_{\theta,l}$, *and let* $\gamma_\nu(x)$ *be as in Lemma 5.* There exist solutions $W^{+,K,k}(x, y, \eta)$, K, $k\in\boldsymbol{Z}_+$, *of* $(34)^{+,K,k}$ *such that*

$$(39)^{K,k} \qquad |\partial_y^\beta \partial_\eta^\gamma W^{+,K,k}_{(\mu,\nu)}(t, y, \eta)| \leqslant C R_1^{-2K-k-|\beta+\gamma|} |u_1|^{-K+k-|\gamma|} (d_\nu(t))^k (K+|\beta+\gamma|)!$$

with some C, $R_1>0$ for K, $k \in \boldsymbol{Z}_+$, β, $\gamma \in \boldsymbol{Z}_+^n$, $1 \leqslant \mu$, $\nu \leqslant 2$, if $t \in \gamma_\nu(x)$. Furthermore $W^{+,K,k}=0$ if $K<k$, and $W^{+,0,0}$ is an invertible matrix.

Proof. We use the classical WKB method in the theory of ordinary differential equations. We define $e_{\mu\nu}(x, y, \eta)$, $1 \leqslant \mu$, $\nu \leqslant 2$, by

$$e_{\mu\nu}(x, y, \eta) = \exp\left\{\int_0^x (x^q \lambda_\mu(x, y, \eta) - x^q \lambda_\nu(x, y, \eta)) dx\right\}.$$

Here we prepare the following

Lemma 6. *If (y, η) satisfies the condition (2) and $s, t \in \boldsymbol{C}$ satisfy $|\arg(s^{q+1}-t^{q+1})-\theta_l-\theta| < \frac{\pi}{3}$, then we have*

$$(40) \qquad |e_{2,1}(t, y, \eta)/e_{2,1}(s, y, \eta)| \leqslant \exp(-R|t^{q+1}-s^{q+1}|\,|\eta_1|)$$

provided $R>0$ is small enough, and $|s|$, $|t| < \varepsilon \ll R$.

Proof. Consider the Taylor expansion of $\lambda_\mu(x, y, \eta)$ with respect to x at $x=0$: $\lambda_\mu(x, y, \eta) = \sum_{k=0}^\infty x^k \lambda_{\mu,k}(y, \eta)$. We have $|\lambda_{\mu,k}(y, \eta)| \leqslant a^{k+1}|\eta_1|$ with some $a>0$. Now we have

$$\begin{aligned}
&\int_s^t \{x^q \lambda_2(x, y, \eta) - x^q \lambda_1(x, y, \eta)\} dx \\
&= \frac{1}{q+1}(t^{q+1}-s^{q+1})(\lambda_{2,0}(y, \eta) - \lambda_{1,0}(y, \eta)) \\
&\quad + \sum_{k=1}^\infty \frac{1}{k+q+1}(t^{k+q+1}-s^{k+q+1})(\lambda_{2,k}(y, \eta) - \lambda_{1,k}(y, \eta)) \\
&= \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)}.
\end{aligned}$$

Here

$$\text{(i)} = \frac{1}{q+1}\eta_1(t^{q+1}-s^{q+1})[\lambda_{2,0}(y, \eta) - \lambda_{1,0}(y, \eta)]_{y=0, \eta=(1,0,\dots,0)},$$

$$\text{(ii)} = \frac{1}{q+1}(t^{q+1}-s^{q+1})\{\lambda_{2,0}(y, \eta) - \eta_1[\lambda_{2,0}(y, \eta)]_{y=0, \eta=(1,0,\dots,0)}\},$$

$$\text{(iii)} = \frac{1}{q+1}(s^{q+1}-t^{q+1})\{\lambda_{1,0}(y, \eta) - \eta_1[\lambda_{1,0}(y, \eta)]_{y=0, \eta=(1,0,\dots,0)}\},$$

and

$$\text{(iv)} = \sum_{k=1}^\infty \frac{1}{k+q+1}(t^{k+q+1}-s^{k+q+1})(\lambda_{2,k}(y, \eta) - \lambda_{1,k}(y, \eta)).$$

Since

$$\left(2l+\frac{1}{2}\right)\pi+\frac{\pi}{12}<\text{the argument of (i)}<\left(2l+\frac{3}{2}\right)\pi-\frac{\pi}{12},$$

it follows that

$$\text{the real part of (i)}\leqslant -\left(\sin\frac{\pi}{12}\right)|t^{q+1}-s^{q+1}|\,|\eta_1|.$$

It is easy to see that if $R>0$ is small enough, we have

$$|(\mathrm{ii})|,\ |(\mathrm{iii})|\leqslant R|t^{q+1}-s^{q+1}|\,|\eta_1|.$$

To estimate $|(\mathrm{iv})|$, we remark that if $|s|$, $|t|<\varepsilon$,

$$\begin{aligned}\left|\frac{1}{k+q+1}(t^{k+q+1}-s^{k+q+1})\right| &= \left|\int_s^t x^{k+q}dx\right| \\ &= \left|\int_{s^{q+1}}^{t^{q+1}}\frac{1}{q+1}(x^{q+1})^{k/(q+1)}d(x^{q+1})\right| \\ &\leqslant \frac{\varepsilon^k}{q+1}|t^{q+1}-s^{q+1}|.\end{aligned}$$

Thus we obtain

$$\begin{aligned}|(\mathrm{iv})| &\leqslant \frac{2a}{q+1}\sum_{k=1}^{\infty}(a\varepsilon)^k|t^{q+1}-s^{q+1}|\,|\eta_1| \\ &= \frac{2a^2\varepsilon}{(q+1)(1-a\varepsilon)}|t^{q+1}-s^{q+1}|.\end{aligned}$$

Q.E.D.

Proof of Proposition 5 (continued). Let us consider the 0th approximation $(34)^{+,K,0}$, $K\in \boldsymbol{Z}_+$, of $(34)^{+,K}$. We solve $(34)^{+,K,0}$, $K\in\boldsymbol{Z}_+$, themselves also by successive approximation:

$$W^{+,K,0}=\sum_{j=0}^{\infty}W^{+,K,0,j},$$

where $W^{+,K,0,j}$, $K, j\in\boldsymbol{Z}_+$, are the solutions of

$$(34)^{+,K,0,j}\qquad \begin{cases}\partial_x W^{+,K,0,j}-(x^qA^0-\bar{B}^0)W^{+,K,0,j}+W^{+,K,0,j}(x^qA^0-\bar{B}^0) \\ \qquad =\begin{cases}0 & j=0\\ -B'W^{+,K,0,j-1} & j\geqslant 1\end{cases} \\ W^{+,K,0,j}_{(\mu,\nu)}(\mathring{x}^+,y,\eta)=\delta_{K,0}\delta_{j,0}\delta_{\mu,\nu}.\end{cases}$$

It is trivial that $W^{+,K,0,j}=0$ if $K\geqslant 1$. Thus we consider the case $K=0$. We can solve $(34)^{+,0,0,j}$ by induction on $j=0, 1, 2,\dots$. We may assume that the diagonal matrix $\bar{B}^0(x, y, \eta)$ is written in the form

$$\bar{B}^0(x,y,\eta)=\begin{pmatrix} b_1(x,y,\eta) & \\ & b_2(x,y,\eta)\end{pmatrix},$$

and that

$$|\partial_y^\beta \partial_\eta^\gamma b(x, y, \eta)| \leqslant a|x|^{-q+q'} R^{-|\beta+\gamma|} |\eta_1|^{-|\gamma|} |\beta+\gamma|!$$

with some $a>0$ for $\beta, \gamma \in Z_+^n$ on $\tilde{\Omega}_\theta^1$, provided we have chosen $R>0$ small enough. We define $\bar{e}_{\mu\nu}(x, y, \eta)$, $1 \leqslant \mu, \nu \leqslant 2$, by

$$\bar{e}_{\mu\nu}(x, y, \eta) = e_{\mu\nu}(x, y, \eta) \exp\left\{\int_\varepsilon^x (b_\mu(x, y, \eta) - b_\nu(x, y, \eta) dx\right\}.$$

From *Lemma 6*, it follows that under the condition of *Lemma 6*, we have

(41) $$|\partial_y^\beta \partial_\eta^\gamma (\bar{e}_{2,1}(t, y, \eta)/\bar{e}_{2,1}(s, y, \eta))| \leqslant aR^{-|\beta+\gamma|} |\eta_1|^{-|\gamma|} |\beta+\gamma|!.$$

Let us consider $(34)^{+,0,0,j}$. Assume that $(x, y, \eta) \in \Omega_{\theta,l}^1$, the paths $\gamma_\nu^+(x)$, $\nu=1, 2$, are as in *Lemma 5*, and that $t \in \gamma_\nu(x)$. Then $W_{(\mu,\nu)}^{+,0,0,j}(t, y, \eta)$ are given by

$$W_{(\mu,\nu)}^{+,0,0,j}(t, y, \eta) = \begin{cases} \delta_{\mu\nu} & j=0 \\ -\int_{x_\nu^+}^t \frac{\bar{e}_{\mu\nu}(t, y, \eta)}{\bar{e}_{\mu\nu}(s, y, \eta)} \sum_{\kappa=1,2} \{B'_{(\mu,\kappa)}(s, y, \eta) \\ \qquad \times W_{(\kappa,\nu)}^{+,0,0j-1}(s, y, \eta)\} ds & j \geqslant 1. \end{cases}$$

In the above integrand, we have

$$\bar{e}_{\mu\nu}(t) = \bar{e}_{\mu\nu}(s) = 1$$

if $\mu=\nu$. Furthermore, if $(\mu, \nu)=(2, 1)$, we have (41), and if $(\mu, \nu)=(1, 2)$ we have

(41)′ $$|\partial_y^\beta \partial_\eta^\gamma (\bar{e}_{1,2}(t)/\bar{e}_{1,2}(s))| = |\partial_y^\beta \partial_\eta^\gamma (\bar{e}_{2,1}(s)/\bar{e}_{2,1}(t))| \leqslant aR^{-|\beta+\gamma|} |\eta_1||^{-|\gamma|} \beta+\gamma|!$$

because of the definition of $\gamma_\nu(x)$, $\nu=1, 2$. Using (41) and (41)′, we can prove easily that

(42) $$|\partial_y^\beta \partial_\eta^\gamma W_{(\mu,\nu)}^{+,0,0,j}(x, y, \eta))| \leqslant \left(\frac{2^{4n-3} a^2 d_\nu(x)}{1-4R_1/R}\right)^j R_1^{-|\beta+\gamma|} |\eta_1|^{-|\gamma|} |\beta+\gamma|!$$

with $R_1 < R/8$ for $\beta, \gamma \in Z_+^n$ on $\Omega_{\theta,l}^1$, by induction on j. Since $|d_\nu(x)| < a|x| < a\varepsilon$, it follows that

$$|\partial_y^\beta \partial_\eta^\gamma W_{(\mu,\nu)}^{+,0,0}(x, y, \eta)| = |\partial_y^\beta \partial_\eta^\gamma \sum_{j=0}^\infty W_{(\mu,\nu)}^{+,0,0,j}| \leqslant \left(1 - \frac{2^{4n-3} a^3 \varepsilon}{1-4R_1/R}\right)^{-1} R_1^{-|\beta+\gamma|} |\eta_1|^{-|\gamma|} |\beta+\gamma|!$$

with $R_1 < R/8$ for $\beta, \gamma \in Z_+^n$ on $\Omega_{\theta,l}^1$ provided ε is small enough. Furthermore, since $W^{+,0,0,0} = I_2$ and

$$|\sum_{j=1}^\infty W^{+,0,0,j}| \ll 1, \qquad \text{if } \varepsilon \text{ is small enough,}$$

it follows that $W^{+,0,0}$ is invertible. Thus we have proved *Proposition 5* in the case $k=0$.

Now let us assume that $k \geqslant 1$ and that $(39)^{K,k'}$, are valid if $k' \leqslant k-1$, and that $W^{+,K,k'}=0$ if $K<k' \leqslant k-1$. Then we can prove that if $(x, y, \eta) \in \Omega^1_{\theta,l}$ and $t \in \gamma_\nu(x)$, we have $F^{+,K,k}=0$ if $K<k$ and

$$(43)\quad |\partial^\beta_y \partial^\gamma_\eta F^{+,K,k}_{(\mu,\nu)}(t, y, \eta)| \leqslant \frac{2^{3n}aR_1}{(1-8R_1/R)^3} C R_1^{-2K-k-|\beta+\gamma|} |\eta_1|^{-K+k-|\gamma|}(K+|\beta+\gamma|)! \frac{(d_\nu(t))^{k-1}}{(k-1)!}.$$

Let us solve the kth approximation $(34)^{+,K,k}$ of $(34)^{+,K}$. We solve $(34)^{+,K,k}$, K, $k \in \boldsymbol{Z}_+$ themselves also by successive approximation:

$$W^{+,K,k} = \sum_{j=0}^{\infty} W^{+,K,k,j},$$

where $W^{+,K,k,j}$, $K, k, j \in \boldsymbol{Z}_+$, are the solutions of

$$\begin{cases} \partial_x W^{+,K,k,j} - (x^q A^0 - \bar{B}^0) W^{+,K,k,j} + W^{+,K,k,j}(x^q A^0 - \bar{B}^0) \\ \qquad = \begin{cases} F^{+,K,k} & j=0 \\ -B' W^{+,K,k,j-1} & j \geqslant 1 \end{cases} \\ W^{+,K,k,j}_{(\mu,\nu)}(\mathring{x}^+_\nu, y, \eta) = 0. \end{cases}$$

It is easy to see that $W^{+,K,k,j}=0$ if $K<k$. As before, we have

$$W^{+,K,k,j}_{(\mu,\nu)}(t, y, \eta) = \begin{cases} \displaystyle\int_{\mathring{x}^+}^t \frac{\bar{e}_{\mu\nu}(t)}{\bar{e}_{\mu\nu}(s)} F^{+,K,k}_{(\mu,\nu)}(s, y, \eta) ds & j=0 \\ \displaystyle -\int_{\mathring{x}^+}^t \frac{\bar{e}_{\mu\nu}(t)}{\bar{e}_{\mu\nu}(s)} \sum_{\kappa=1,2} B'_{(\mu,\kappa)}(s, y, \eta) W^{+,K,k,j-1}_{(\kappa,\nu)}(s, y, \eta) ds & j \geqslant 1. \end{cases}$$

Thus we can prove that if $(x, y, \eta) \in \Omega^1_{\theta,l}$ and $t \in \gamma_\nu(x)$, then we have

$$|\partial^\beta_y \partial^\gamma_\eta W^{+,K,k,j}_{(\mu,\nu)}(t, y, \eta)| \leqslant \left(\frac{2^{4n-3}a^2}{1-4R_1/R}\right)^j \frac{2^{3na}aR_1}{(1-8R_1/R)^3} C R_1^{-2K-k-|\beta+\gamma|} |\eta_1|^{-K+k-|\gamma|} \times (K+|\beta+\gamma|)! \frac{(d_\nu(t))^k}{k!}.$$

Thus we obtain $(39)^{K,k}$, provided that R_1, $\varepsilon>0$ are small enough. Q.E.D.

Thus we have constructed the solutions of $(34)^{+,K,k}$ which satisfy $(39)^{K,k}$. Now let us consider the equations $(34)^{-,K,k}$. For this purpose we prepare the following

Lemma 7. *Let $W^{+,K,k}(x, y, \eta)$ be as above. Then there exist 2×2 matrices $W^{-,K,k}(x, y, \eta)$, K, $k \in \boldsymbol{Z}_+$, holomorphic on $\Omega^1_{\theta,l}$ such that*

$$(44)\qquad \sum_{\substack{I+J+|\alpha|=K\\ i+j=k}} \frac{1}{\alpha!}\partial_\eta^\alpha W^{+,I,i}\partial_y^\alpha W^{-,J,j}=\delta_{K,0}\delta_{k,0}I_2,$$

$$(45)\qquad \sum_{\substack{I+J+|\alpha|=K\\ i+j=k}} \frac{1}{\alpha!}\partial_\eta^\alpha W^{-,I,i}\partial_y^\alpha W^{+,J,j}=\delta_{K,0}\delta_{k,0}I_2.$$

Furthermore, we have $W^{\pm,K,k}=0$ *if* $K<k$, *and*

$$(46)^{\pm,K,k}\qquad |\partial_y^\beta\partial_\eta^\gamma W^{\pm,K,k}(x, y, \eta)|\leqslant CR_1^{-2K-|\beta+\gamma|}(R_1^{-1}\varepsilon)^k|\eta_1|^{-K+k-|\gamma|} \times (K+|\beta+\gamma|)!/k!$$

with some C, $R_1>0$ *on* $\Omega_{\theta,l}^1$ *for* K, $k\in \boldsymbol{Z}_+$, β, $\gamma\in \boldsymbol{Z}_+^n$. *We may choose* $\varepsilon>0$ *as small as we like and* C, $R_1>0$ *are some constants which do not depend on* ε.

Proof. In $(39)^{K,k}$, let us take as t the end point x of $\gamma_\nu(x)$. Since $d_\nu(x)<a|x|<a\varepsilon$ with some $a>0$ independent of ε, we obtain $(46)^{+,K,k}$ directly from $(39)^{K,k}$. Let us construct $W^{-,K,k}$. We define

$$W^{-,K,k}=\begin{cases} (W^{+,K,})^{k-1} & (K, k)=(0, 0)\\ -(W^{+,0,0})^{-1}\displaystyle\sum_{\substack{I+J+|\alpha|=K\\ i+j=k\\ (J,j)\neq(K,k)}} \frac{1}{\alpha!}\partial_\eta^\alpha W^{+,I,i}\partial_y^\alpha W^{-,J,j} & (K, k)\neq(0, 0).\end{cases}$$

Then (44) follows directly. We can also prove (45) and $(46)^{-,K,k}$ easily (We may have to take another constants C and R_1 in $(46)^{-,K,k}$). It is also easy to see that $W^{-,K,k}=0$ if $K<k$. Q. E. D.

From $(46)^{\pm,K,k}$ it follows that $W^{\pm,K}=\sum_{k=0}^{K} W^{\pm,K,k}$, $K\in \boldsymbol{Z}_+$, converge on $\Omega_{\theta,l}^1$. Furthermore, we have the following

Proposition 6. *We define* $W^{\pm,K}(x, y, \eta)$, $K\in \boldsymbol{Z}_+$, *as above.* $W^{\pm,K}$ *satisfy* $(34)^{\pm,K}$ *and we have*

$$(47)\qquad \sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_y^\alpha W^{\pm,I}\partial_y^\alpha W^{\mp,J}=\delta_{K,0}I_2,$$

$$(48)\qquad |\partial_\eta^\beta\partial_y^\gamma W^{\pm,K,k}(x, y, \eta)|\leqslant CR_1^{-2K-|\beta+\gamma|}|\eta_1|^{-K-|\gamma|}(K+|\beta+\gamma|)! \times\exp\{R_1^{-1}\varepsilon|\eta_1|\}$$

with some C, $R_1>0$ *independent of* ε *for* $K\in \boldsymbol{Z}_+$, β, $\gamma\in \boldsymbol{Z}_+^n$ *on* $\Omega_{\theta,l}^1$.

Proof. We only have to prove that $W^{-,K}$, $K\in \boldsymbol{Z}_+$, satisfy $(34)^{-,K}$. Other statements are direct concequences of the previous argument. Now it is easy to see that $W^{-,0}=W^{-,0,0}=(W^{+,0,0})^{-1}=(W^{+,0})^{-1}$ satisfies $(34)^{-,0}$. Assume that $K\geqslant 1$ and that $W^{-,K'}$, $K'\leqslant K-1$, satisfy $(34)^{-,K'}$. From (47) it follows that

$$\partial_x\Big\{\sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha W^{+,I}\partial_y^\alpha W^{-,J}\Big\}$$

$$=\sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\{\partial_\eta^\alpha\partial_x W^{+,I}\partial_y^\alpha W^{-,J}+\partial_\eta^\alpha W^{+,I}\partial_y^\alpha\partial_x W^{-,J}\}=0.$$

From $(34)^{+,K'}$, $K'\leqslant K$ and $(34)^{-,k'}$, $K'\leqslant K-1$, it follows that

$$W^{+,0}\partial_x W^{-,K}=(\mathrm{i})'+(\mathrm{ii})'+(\mathrm{iii})'+(\mathrm{iv})'$$

where

$$(\mathrm{i})'=-\sum_{H+I+J+|\alpha+\beta|=K}\frac{1}{\alpha!\beta!}\partial_\eta^\alpha\{\partial_\eta^\beta(x^q A^H-B^H)\partial_y^\beta W^{+,I}\}\partial_y^\alpha W^{-,J},$$

$$(\mathrm{ii})'=\sum_{J+J+|\alpha+\beta|=K}\frac{1}{\alpha!\beta!}\partial_\eta^\alpha\{\partial_\eta^\beta W^{+,I}\partial_y^\beta(x^q A^0-\bar{B}^0)\}\partial_y^\alpha W^{-,J},$$

$$(\mathrm{iii})'=\sum_{\substack{J+J+|\alpha+\beta|=K\\ J+|\beta|\neq K}}\frac{1}{\alpha!\beta!}\partial_\eta^\alpha W^{+,I}\partial_y^\alpha\{\partial_\eta^\beta(x^q A^0\bar{B}^0)\partial_y^\beta W^{-,J}\},$$

and

$$(\mathrm{iv})'=\sum_{\substack{H+I+J+|\alpha+\beta|=K\\ H+J+|\beta|\neq K}}\frac{1}{\alpha!\beta!}\partial_\eta^\alpha W^{+,I}\partial_y^\alpha\{\partial_\eta^\beta W^{-,J}\partial_y^\beta(x^q A^H-B^H)\}.$$

From (47) it follows that

$$\begin{aligned}(\mathrm{i})'&=-\sum_{H+I+J+|\alpha+\beta|=K}\frac{1}{\alpha!\beta!}\partial_\eta^\alpha(x^q A^H-B^H)\partial_y^\alpha(\partial_\eta^\beta W^{+,I}\partial_y^\beta W^{-,J})\\&=-(x^q A^K-B^K),\end{aligned}$$

$$\begin{aligned}(\mathrm{iv})'&=\sum_{H+I+J+|\alpha+\beta|=K}\frac{1}{\alpha!\beta!}\partial_\eta^\alpha W^{+,I}\partial_y^\alpha\{\partial_\eta^\beta W^{-,J}\partial_y^\beta(x^q A^H-B^H)\}\\&\quad-W^{+,0}\sum_{H+J+|\beta|=K}\frac{1}{\beta!}\partial_\eta^\beta W^{-,J}\partial_y^\beta(x^q A^H-B^H)\\&=(x^q A^K-B^K)-W^{+,0}\sum_{H+J+|\beta|=K}\frac{1}{\beta!}\partial_\eta^\beta W^{-,J}\partial_y^\beta(x^q A^H-B^H).\end{aligned}$$

Furthermore, it is easy to see that

$$(\mathrm{ii})'+(\mathrm{iii})'=W^{+,0}\sum_{J+|\beta|=K}\frac{1}{\beta!}\partial_\eta^\beta(x^q A^0-\bar{B}^0)\partial_y^\beta W^{-,J}.$$

Thus we obtain $(34)^{-,K}$. Q.E.D.

Now let us consider the equations $(36)^{\pm,K}$, $K\in \boldsymbol{Z}_+$. We solve these equations also by successive approximation. Let $K, k\in \boldsymbol{Z}_+$, and consider the following initial value problems:

$$(36)^{+,K,k}\qquad \begin{cases}\partial_x X^{+,K,k}+\bar{B}^0X^{+,K,k}=G^{+,K,k}\\ X^{+,K,k}(\mathring{x}_1^+,y,\eta)=\delta_{K,0}\delta_{k,0}I_2\end{cases}$$

$$(36)^{-,K,k}\qquad \begin{cases}\partial_x X^{-,K,k}-X^{-,K,k}\bar{B}^0=G^{-,K,k}\\ X^{-,K,k}(\mathring{x}_1^+,y,\eta)=\delta_{K,0}\delta_{k,0}I_2,\end{cases}$$

where

$$G^{+,K,k}=\begin{cases}0 & k=0\\ \sum_{\substack{J+|\alpha|=K\\ J\neq K}}\frac{1}{\alpha!}\{\partial_\eta^\alpha(x^qA^0-\bar{B}^0)\partial_y^\alpha X^{+,J,k-1}-\partial_\eta^\alpha X^{+,J,k-1}\partial_y^\alpha(x^qA^0)\} & k\geqslant 1,\end{cases}$$

$$G^{-,K,k}=\begin{cases}0 & k=0\\ \sum_{\substack{J+|\alpha|=K\\ J\neq K}}\frac{1}{\alpha!}\{\partial_\eta^\alpha(x^qA^0)\partial_y^\alpha X^{-,J,k-1}-\partial_\eta^\alpha X^{-,J,k-1}\partial_y^\alpha(x^qA^0-\bar{B}^0)\} & k\geqslant 1.\end{cases}$$

Let us solve the kth approximation $(36)^{+,K,k}$, $K\in \boldsymbol{Z}_+$, of $(36)^{+,K}$ themselves again by successive approximation. Let $K, k, j\in \boldsymbol{Z}_+$, and consider the following innitial value problems:

$$(36)^{\pm,K,k,j}\qquad \begin{cases}\partial_x X^{\pm,K,k,j}=G^{\pm,K,k,j}\\ X^{+,K,k,j}(\mathring{x}_1^+,y,\eta)=\delta_{K,0}\delta_{k,0}\delta_{j,0}I_2,\end{cases}$$

where

$$G^{+,K,k,j}=\begin{cases}0 & j=0\\ -\bar{B}^0X^{+,K,k,j-1} & j\geqslant 1,\ k=0\\ -\bar{B}^0X^{+,K,k,j-1}+\sum_{\substack{J+|\alpha|=K\\ J\neq K}}\frac{1}{\alpha!}\{\partial_\eta^\alpha(x^pA^0-\bar{B}^0)\partial_y^\alpha X^{K,J,k-1,j-1} & \\ \qquad -\partial_\eta^\alpha X^{+,J,k-1,j-1}\partial_y^\alpha(x^qA^0)\} & j,\ k\geqslant 1,\end{cases}$$

$$G^{+,K,k,j}=\begin{cases}0 & j=0\\ X^{-,K,k,j-1}\bar{B}^0 & j\geqslant 1,\ k=0\\ X^{-,K,k,j-1}\bar{B}^0+\sum_{\substack{J+|\alpha|=K\\ J\neq K}}\frac{1}{\alpha!}\{\partial_\eta^\alpha(x^qA^0)\partial_y^\alpha X^{-,J,k-1,j-1} & \\ \qquad -\partial_\eta^\alpha X^{-,J,k-1,j-1}\partial_y^\alpha(x^qA^0-\bar{B}^0)\} & j,\ k\geqslant 1.\end{cases}$$

Now we have the following

Lemma 8. *Assume that* $(x, y, \eta)\in\tilde{\Omega}_0^1$ *and that* $\arg x=\arg\mathring{x}_1^+$. *Then we have* $X^{\pm,K,k,j}=0$ *if* $K<k$ *or* $j<k$, *and*

$$(49)^{\pm,K,k,j}\qquad |\partial_y^\beta\partial_\eta^\gamma X^{\pm,K,k,j}(x,y,\eta)|$$
$$\leqslant CR_1^{-2K-j-|\beta+\gamma|}|\eta_1|^{-K+k-|\gamma|}(K+|\beta+\gamma|)!$$
$$\times\frac{|x|^{-(q-1-q')(j-k)}}{(j-k)!}\frac{(|x|-\varepsilon)^k}{k!}$$

with some C, $R_1>0$ *independent of* $\varepsilon>0$, *for* $K, k, j\in\boldsymbol{Z}_+$, $\beta, \gamma\in\boldsymbol{Z}_+^n$.

Proof. If $j=0$, $X^{\pm,K,k,j}=\delta_{K,0}\delta_{k,0}I_2$ and the statement is trivial. Assume that

$j \geqslant 1$ and that the statement is valid if $j' \leqslant j-1$. Then it is easy to see that $G^{\pm,K,k,j}=0$ if $K<k$, and thus

$$X^{\pm,K,k,j}=\int_{\dot{x}_1^+}^{x} G^{\pm,K,k,j}dx=0$$

if $K<k$. Now let us prove $(49)^{+,K,k,j}$, $K, k \in \boldsymbol{Z}_+$. From (31), (32), (33) and $(49)^{+,K,k,j-1}$, $K, k \in \boldsymbol{Z}_+$, we can prove that

$$\begin{aligned}&|\partial_y^\beta \partial_\eta^\gamma(\bar{B}^0 X^{+,K,k,j-1})|\\ &\leqslant \frac{4^n a R_1}{(1-4R_1/R)^2} C R_1^{-2K-j-|\beta+\gamma|}|\eta_1|^{-K+k-|\gamma|}(K+|\beta+\gamma|)!\\ &\quad\times \frac{|x|^{-(q-1-q')(j-k)-1}}{(j-k-1)!}\frac{(|x|-\varepsilon)^k}{k!}\end{aligned}$$

if $j-1 \geqslant k$, and that

$$\begin{aligned}&\left|\partial_y^\beta \partial_\eta^\gamma\left\{\sum_{\substack{J+|\alpha|=K\\ J\neq K}} \frac{1}{\alpha!}\partial_\eta^\alpha(x^q A^q-\bar{B}^0)\partial_y^\alpha X^{+,J,k-1,j-1}\right\}\right|,\\ &\left|\partial_y^\beta \partial_\eta^\gamma\left\{\sum_{\substack{J+|\alpha|=K\\ J\neq K}} \frac{1}{\alpha!}\partial_\eta^\alpha X^{+,J,k-1,j-1}\partial_y^\alpha(x^q A^0)\right\}\right|\\ &\leqslant \frac{8^n a R_1}{(1-8R_1/R)^3} C R_1^{-2K-j-|\beta+\gamma|}|\eta_1|^{-K+k-|\gamma|}(K+|\beta+\gamma|)!\\ &\quad\times \frac{|x|^{-(q-1-q')(j-k)}}{(j-k)!}\frac{(|x|-\varepsilon)^{k-1}}{(k-1)!}\end{aligned}$$

if $k \geqslant 1$ (We have assumed that $R_1 \leqslant R/16$). Since

$$X^{+,K,k,j}(x, y, \eta)=\int_{\dot{x}_1^+}^{x} G^{+,K,k,j}(t, y, \eta)dt$$

we obtain $(49)^{+,K,k,j}$, $K, k \in \boldsymbol{Z}_+$, using

$$\left|\int_{\dot{x}_1^+}^{x} \frac{|t|^{-(q-1-q')(j-k)-1}}{(j-k-1)!}dt\right| \leqslant \frac{|x|^{-(q-1-q')(j-k)}}{(j-k)!}$$

if $j-1 \geqslant k$, and

$$\left|\int_{\dot{x}_1^+}^{x} \frac{(|t|-\varepsilon)^{k-1}}{(k-1)!}dt\right| \leqslant \frac{(|x|-\varepsilon)^k}{k!}$$

if $k-1 \geqslant 0$. We can prove $(49)^{-,K,k,j}$, $K, k \in \boldsymbol{Z}_+$, similarly. Q.E.D.

Now let us estimate $X^{\pm,K,k,j}(x, y, \eta)$ on $\tilde{\Omega}_\theta^1$:

Lemma 9. *If* $(x, y, \eta) \in \tilde{\Omega}_\theta^1$, *we have*

$$\begin{aligned}&|\partial_y^\beta \partial_\eta^\gamma X^{\pm,K,k,j}(x,y,\eta)|\\ &\leqslant CR_1^{-2K-j-|\beta+\gamma|}|\eta_1|^{-K+k-|\gamma|}(K+|\beta+\gamma|)!\\ &\quad\times\sum_{j'=0}^{j-k}\sum_{k'=0}^{k}\left\{\frac{|x|^{-(q-1-q')(j-k-j')}}{(j-k-j')!}\frac{(|x|-\varepsilon)^{k-k'}}{(k-k')!}\right.\\ &\qquad\left.\times|x|^{-(q-1-q')j'+(q+1)k'}\frac{|\arg x-\arg \mathring{x}_1^+|^{j'+k'}}{(j'+k')!}\right\}\end{aligned}$$

with some C, $R_1>0$ *independent of* $\varepsilon>0$ *for* $K, k, j\in \boldsymbol{Z}_+$, $\beta, \gamma\in \boldsymbol{Z}_+^n$.

Proof. Though the estimate looks rather complicated, we can prove *Lemma 9* just in the same way as *Lemma 8*. We note that $X^{\pm,K,k,j}$ can be given inductively by

$$X^{\pm,K,k,j}(x,y,\eta)=\delta_{K,0}\delta_{k,0}\delta_{j,0}I_2+\int_{\mathring{x}_1^+}^{x}G^{\pm,K,k,j}(t,y,\eta)dt.$$

We take the path of integration to be the union of $\{t\in \boldsymbol{C};\ \arg t=\arg \mathring{x}_1^+,\ \varepsilon\geqslant|t|\geqslant|x|\}$ and $\{t\in \boldsymbol{C};\ |t|=|x|,\ |\arg t-\arg \mathring{x}_1^+|\leqslant|\arg x-\arg \mathring{x}_1^+|\}$. *Lemma 9* is a consequence of direct calculation as *Lemma 8*. Q. E. D.

We define $X^{\pm,K,k}$, $K, k\in \boldsymbol{Z}_+$, by

$$X^{\pm,K,k}=\sum_{j=0}^{\infty}X^{\pm,K,k,j}.$$

Then we have the following

Corollary. *Assume that* $\varepsilon>0$ *is small enough. If* $(x,y,\eta)\in\Omega_{\theta,l}^1$, *we have* $X^{\pm,K,k}=0$ *if* $K<k$. $X^{\pm,K,k}$, $K, k\in \boldsymbol{Z}_+$, *satisfy* $(36)^{\pm,K,k}$ *and*

$$(50)^{K,k}\qquad \begin{aligned}&|\partial_y^\beta\partial_\eta^\gamma X^{\pm,K,k}(x,y,\eta)|\\ &\leqslant C_1R_1^{-2K-|\beta+\gamma|}(R_1^{-1}\varepsilon)^k|\eta_1|^{-K+k-|\gamma|}\frac{(K+|\beta+\gamma|)!}{k!}\\ &\quad\times\exp\{2^{-1}|\eta_1|^{-(q-1q')/)2q-q')}\}\end{aligned}$$

with some C_1, $R_1>0$ *independent of* $\varepsilon>0$ *for* $K, k\in \boldsymbol{Z}_+$, $\beta, \gamma\in \boldsymbol{Z}_+^n$.

Proof. Since $|\arg x-\arg \mathring{x}_1^+|<\dfrac{3\pi}{2(q+1)}$ on $\Omega_{\theta,l}^1$, this is a direct consequence of *Lemma 9*. Q. E. D.

Now let us define $X^{\pm,K}$, $K\in \boldsymbol{Z}_+$, by

$$(51)^K\qquad X^{\pm,K}=\sum_{k=0}^{K}X^{\pm,K,k}.$$

These power series converge on $\Omega_{\theta,l}^1$, and we have the following

Proposition 7. *Assume that* $\varepsilon>0$ *is small enough.* $X^{\pm,K}$, $K\in \boldsymbol{Z}_+$, *satisfy* $(36)^{\pm,K}$ *and*

$$(52)^K\qquad \sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha X^{\pm,I}\partial_y^\alpha X^{\mp,J}=\delta_{K,0}I_2.$$

Furthermore, they are written in the form $(51)^K$ *using* 2×2 *matrices* $X^{\pm,K,k}$ *which satisfy* $(50)^{K,k}$ *on* $\Omega^1_{\theta,l}$.

Proof. We only have to prove $(52)^K$. Since $X^{\pm,0}$ satisfy $(36)^{\pm,K}$ and $\dot{X}^{\pm,0}(\mathring{x}_1^+, y, \eta)=I_2$, we obtain $(52)^0$. Now let $K\geqslant 1$ and assume that $(52)^{K'}$, $K'\leqslant K-1$, are valid. Then we can prove that

$$\partial_x\Big\{\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha X^{\pm,I}\partial_y X^{\mp,J}\Big\}=0$$

using $(36)^{\pm,K'}$, $k'\leqslant K$, and $(52)^{K'}$, $K'\leqslant K-1$. Furthermore we have

$$\Big[\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha X^{\pm,I}\partial_y^\alpha X^{\mp,J}\Big]_{x=\mathring{x}_1^+}=0.$$

Thus we obtain $(52)^K$. Q. E. D.

Now we give the

Proof of Proposition 4. Let us define $V^{\pm,K,k}$, K, $k\in \boldsymbol{Z}_+$, by

$$V^{+,K,k}=\sum_{\substack{I+J+|\alpha|=K\\ i+j=k}}\frac{1}{\alpha!}\partial_\eta^\alpha W^{+,I,i}\partial_y^\alpha X^{+,J,j}$$

$$V^{-,K,k}=\sum_{\substack{I+J+|\alpha|=K\\ i+j=k}}\frac{1}{\alpha!}\partial_\eta^\alpha X^{-,J,j}\partial_y^\alpha W^{-,I,i}.$$

Then we have $V^{\pm,K,k}=0$ if $K<k$ and

$$|\partial_y^\beta\partial_\eta^\gamma V^{\pm,K,k}(x,y,\eta)|\leqslant CR_1^{-2K-|\beta+\gamma|}(R_1^{-1}\varepsilon)^k|\eta_1|^{-K+k-|\gamma|}$$
$$\times(K-k+|\beta+\gamma|)!\exp\{2^{-1}|\eta_1|^{(q-1-q')/(2q-q')}\}$$

with some C, $R_1>0$ independent of $\varepsilon>0$ for K, $k\in \boldsymbol{Z}_+$, β, $\gamma\in \boldsymbol{Z}_+^n$ on $\Omega^1_{\theta,l}$. Thus we may define $V^{\pm,K}$, $K\in \boldsymbol{Z}_+$, by $V^{\pm,K}=\sum_{k=0}^{K}V^{\pm,K,k}$. They satisfy $(29)^{\pm,K}$, and we have

$$\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha V^{\pm,I}\partial_y^\alpha V^{\mp,J}=\delta_{K,0}I_2.$$

Using *Lemma 1*, *Lemma 2*, and *Lemma 4*, we may write the transformations which transform $V^{\pm,K}$, $K\in \boldsymbol{Z}_+$, to $U^{\pm,K}$, $K\in \boldsymbol{Z}_+$, in the form

$$\begin{cases} U^{+,K}=\displaystyle\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha\tilde{S}^{+,I}\partial_y^\alpha V^{+,J}\\ U^{-,K}=\displaystyle\sum_{I+J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha V^{-,J}\partial_y^\alpha\tilde{S}^{-,I}\end{cases}$$

with some 2×2 matrices $\tilde{S}^{\pm,K}$, $K\in \boldsymbol{Z}_+$, which satisfy

$$|\partial_y^\beta\partial_\eta^\gamma\tilde{S}^{\pm,K}|\leqslant a|x|^{-2(q-q')}R^{-K-|\beta+\gamma|}|\eta_1|^{-K-|\gamma|}(K+|\beta+\gamma|)!$$

with some a, $R>0$ for $K\in \boldsymbol{Z}_+$, β $\gamma\in \boldsymbol{Z}_+^n$ on $\Omega^1_{\theta,l}$. Furthermore, we have

$$\sum_{I+J+|\alpha|=K} \frac{1}{\alpha!} \partial_\eta^\alpha \tilde{S}^{\pm,I} \partial_y^\alpha \tilde{S}^{\mp,J} = \delta_{K,0} I_2.$$

We define $U^{\pm,K,k}$, K, $k \in \boldsymbol{Z}_+$, by

$$U^{+,K,k} = \sum_{I+J+|\alpha|=K} \frac{1}{\alpha!} \partial_\eta^\alpha \tilde{S}^{+,I} \partial_y^\alpha V^{+,J,k}$$

$$U^{-,K,k} = \sum_{I+J+|\alpha|=K} \frac{1}{\alpha!} \partial_\eta^\alpha V^{-,J,k} \partial_y^\alpha \tilde{S}^{-,I}.$$

Then we can verify that $(16)^{K,k}$, $(17)^{\pm,K}$, and $(18)^K$, K, $k \in \boldsymbol{Z}_+$, are satisfied.

Q.E.D.

Now let us construct $U^\pm(x, y, \eta)$ on $\Omega^1_{\theta,l}$.

Proposition 8. *Assume that ε, $R>0$ are small enough and that $\varepsilon \ll R$. Then there exist 2×2 matrices $U^\pm(x, y, \eta)$ holomorphic on $\Omega^1_{\theta,l}$ such that*

$$(53) \qquad |U^\pm(x, y, \eta)| \leqslant C_1 \exp\{|\eta_1|^{(q-1-q')/(2q-q')}\}$$

with some $C_1>0$ on $\Omega^1_{\theta,l}$,

$$(54)^+ \qquad \Big|\partial_x U^+ - \sum_{|\alpha|<K} \frac{1}{\alpha!}\{\partial_y^\alpha \sigma(x^q A - A)\partial_y^\alpha U^+ - \partial_\eta^\alpha U^+ \partial_y^\alpha \sigma(x^q A)\}\Big|$$
$$\leqslant C_1 R^{-K}|\eta_1|^{-K} K! \exp\{R^{-1}\varepsilon|\eta_1| + |\eta_1|^{(q-1-q')/(2q-q')}\},$$

$$(54)^- \qquad \Big|\partial_x U^- - \sum_{|\alpha|<K} \frac{1}{\alpha!}\{\partial_\eta^\alpha \sigma(x^q A)\partial_y^\alpha U^- - \partial_\eta^\alpha U^- \partial_y^\alpha \sigma(x^q A - A)\}\Big|$$
$$\leqslant C_1 R^{-K}|\eta_1|^{-K} K! \exp\{R^{-1}\varepsilon|\eta_1| + |\eta_1|^{(q-1-q')/(2q-q')}\},$$

and

$$(55) \qquad \Big|\sum_{|\alpha|<K} \frac{1}{\alpha!} \partial_\eta^\alpha U^\pm \partial_y^\alpha U^\mp - I_2\Big|$$
$$\leqslant C_1 R^{-K}|\eta_1|^{-K} K! \exp\{R^{-1}\varepsilon|\eta_1| + |\eta_1|^{(q-1-q')/(2q-q')}\}$$

with some $C_1>0$ for $K=1, 2,\dots,$ on $\Omega^1_{\theta,l}$.

Proof. Since

$$|U^{\pm,K}| \leqslant C(R_1/2)^{-2K}|\eta_1|^{-K} K! \exp\{R_1^{-1}\varepsilon|\eta_1| + |\eta_1|^{(q-1-q')/(2q-q')}\}$$

for $K \in \boldsymbol{Z}_+$, there exist 2×2 matrices $U^\pm(x, y, \eta)$ holomorphic on $\Omega^1_{\theta,l}$ such that

$$(56) \qquad |U^\pm - \sum_{J=0}^{K-1} U^{\pm,J}|$$
$$\leqslant C R^{-K}|\eta_1|^{-K} K! \exp\{R^{-1}\varepsilon|\eta_1| + |\eta_1|^{(q-1-q')/(2q-q')}\}$$

for $K=1, 2,\dots,$ on $\Omega^1_{\theta,l}$, provided $R>0$ is small enough. Here we can choose $\varepsilon>0$ as small as we like, and $R>0$ is some constant which does not depend on ε. Now let us define $\bar{U}^{\pm,K}$, $K \in \boldsymbol{Z}_+$, by

$$\bar{U}^{\pm,K}=\sum_{J-j=K}U^{\pm,J,j}.$$

Since we may assume that $\varepsilon<R_1^3/2$, we obtain

$$|\bar{U}^{\pm,K}|\leqslant 2CR_1^{-2K}|\eta_1|^{-K}K!\exp\{|\eta_1|^{(q-1-q')/(2q-q')}\}$$

for $K\in\boldsymbol{Z}_+$ on $\Omega^1_{\theta,l}$. Let us remark that

$$(57)\qquad |\sum_{J=0}^{K-1}(U^{\pm,J}-\bar{U}^{\pm,J})|=|\sum_{\substack{0<J-j<K-1\\ K<J}}U^{\pm,J,j}|$$

$$\leqslant\frac{C}{1-R_1^{-3}\varepsilon}(R_1/2)^{-2K}|\eta_1|^{-K}K!\exp\{R_1^{-3}\varepsilon|\eta_1|+|\eta_1|^{(q-1-q')/(2q-q')}\}$$

kor $K=1, 2,\ldots,$ on $\Omega^1_{\theta,l}$. From (56) and (57), it follows that

$$|U^{\pm}-\sum_{J=0}^{K-1}\bar{U}^{\pm,J}|\leqslant 2CR^{-K}|\eta_1|^{-K}K!\exp\{R^{-1}\varepsilon|\eta_1|+|\eta_1|^{(q-1-q')/(2q-q')}\}$$

for $K=1, 2,\ldots,$ on $\Omega^1_{\theta,l}$, if we have assumed $R<R_1^3, (R_1/2)^2$. Since we may also assume that $\varepsilon<R^3$ and that $R\ll 1$, we obtain (53). To prove $(54)^{\pm}$ and (55), we must estimate the derivatives of U^+ with respect to y and η, and for this purpose we must rewrite $R/2$ and $\varepsilon/2$ as R and ε, respectively. We define the domain $\Omega^1_{\theta,l}$ using these new constants. Then $(54)^{\pm}$ (resp. (55)) follows easily from (56) and $(17)^{\pm,K}$ (resp. $(18)^K$), $K\in\boldsymbol{Z}_+$. Q.E.D.

§5. Construction of $U^{\pm}(x, y, \eta)$ on Ω^2_{θ}.

In this section, we investigate $U^{\pm}(x, y, \eta)$ on Ω^2_{θ}, which are already constructed on $\Omega^1_{\theta,l}$ in the last two sections.

Now let us define a 2×2 matrix $T(\eta_1)$ by

$$T=\begin{pmatrix}1 & \\ & \eta_1^{-(q-q')/(2q-q')}\end{pmatrix}.$$

Let $U^{\pm,K}(x, y, \eta)$, $K\in\boldsymbol{Z}_+$, be as in §4. We define 2×2 matrices $Y^{\pm,K}(x, y, \eta)$, $K\in\boldsymbol{Z}_+$, by

$$Y^{+,K}=\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha T\partial_y^\alpha U^{+,J},\quad Y^{-,K}=U^{-,K}T^{-1}.$$

We have the following

Lemma 10. 1) *We have*

$$U^{+,K}=\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha T^{-1}\partial_y^\alpha Y^{+,J},\quad U^{-,K}=Y^{-,K}T$$

for $K\in\boldsymbol{Z}_+$.

2) $(17)^{\pm,K}$, $K\in\boldsymbol{Z}_+$, *are equivalent to the following* $(58)^{\pm,K}$, $K\in\boldsymbol{Z}_+$:

(58)$^{+,K}$
$$\partial_x Y^{+,K} - \sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha C^I \partial_y^\alpha Y^{+,J} + \sum_{J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha Y^{+,J}\partial_y^\alpha \sigma(x^q \Lambda) = 0,$$

(58)$^{-,K}$
$$\partial_x Y^{-,K} - \sum_{J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha \sigma(x^q \Lambda)\partial_y^\alpha Y^{-,J} + \sum_{I+J+|\alpha|=K} \frac{1}{\alpha!}\partial_\eta^\alpha Y^{-,J}\partial_y^\alpha C^I = 0.$$

Here $C^I(x, y, \eta)$, $I \in \boldsymbol{Z}_+$, are given by

(59)
$$C^I = \sum_{|\alpha|=I} \frac{1}{\alpha!}\partial_\eta^\alpha T \partial_y^\alpha \sigma(x^q \Lambda - A) T^{-1}.$$

3) *We have*

$$|\partial_y^\beta \partial_\eta^\gamma C^I| \leqslant a R^{-K-|\beta+\gamma|} |\eta_1|^{\{(q-q')/(2q-q')-I-|\gamma|\}} (I+|\beta+\gamma|)!,$$
$$|\partial_y^\beta \partial_\eta^\gamma \sigma(x^q \Lambda)| \leqslant a R^{-|\beta+\gamma|} |\eta_1|^{\{(q-q')/(2q-q')-|\gamma|\}} |\beta+\gamma|!$$

with some a, $R>0$ for $I \in \boldsymbol{Z}_+$, $\beta, \gamma \in \boldsymbol{Z}_+^n$ if $|x| < 6\left(\varepsilon \sin \frac{\pi}{12}\right)^{-1} |\eta_1|^{-1/(2q-q')}$ and (y, η) satisfies the condition (2).

Proof. We only have to prove 3). From (59) it follows that

$$C^I_{(1,1)} = \delta_{K,0} x^q \lambda_1(x, y, \eta), \quad C^I_{(1,2)} = \delta_{K,0},$$

$$C^I_{(2,1)} = -\frac{1}{I!}\frac{\partial}{\partial \eta_1}(\eta_1^{-(q-q')/(2q-q')}) \frac{\partial}{\partial y_1}\{x^{q'}\sigma(\mu)(x, y, \eta) + a_{0,0}(x, y)\}$$

$$C^I_{(2,2)} = \frac{1}{I!}\frac{\partial}{\partial \eta_1}(\eta_1^{-(q-q')/(2q-q')}) \frac{\partial}{\partial y_1}\{x^q \lambda_2(x, y, \eta) + a_{1,0}(x, y)\} \times \eta_1^{(q-q')/(2q-q')}.$$

(59) is a consequence of direct calculation. Q. E. D.

Let us define $\mathring{x} \in \boldsymbol{C}$ by

$$\mathring{x} = 2\left(\varepsilon \sin \frac{\pi}{12}\right)^{-1} \eta_1^{-1/(2q-q')} \exp\left\{\frac{\sqrt{-1}}{q+1}\left(\theta_l + \theta + \frac{\pi}{2}\right) - \frac{\theta}{2q-q'}\right\}.$$

If (y, η) satisfies the condition (2), it follows that $(\mathring{x}, y, \eta) \in \Omega^1_{\theta,l} \cup \Omega^2_\theta$

Now for some holomorphic function $f(x, y, \eta)$ defined at $(\mathring{x}, \mathring{y}, \mathring{\eta})$, where $\mathring{y}$ and $\mathring{\eta}$ are some points of $\boldsymbol{C}^n$, we define $\bar{\partial}_\eta^\alpha f_j(\mathring{x}, y, \eta)$, $j \in \boldsymbol{Z}_+$, $\alpha \in \boldsymbol{Z}_+^n$, by

$$\bar{\partial}_\eta^\alpha f_j(\mathring{x}, y, \eta) = [\partial_y^j \partial_\eta^\alpha f(x, y, \eta)]_{x=\mathring{x}}.$$

Since $Y^{\pm,K}(x, y, \eta)$, $K \in \boldsymbol{Z}_+$, satisfy (58)$^{\pm,K}$ at $x = \mathring{x}$, we obtain

$$(60)^{+,K} \qquad (j+1)(Y^{+,K})_{j+1}=\sum_{\substack{I+J+|\alpha|=K\\ j'+j''=j}}\frac{1}{\alpha!}\bar{\partial}_\eta^\alpha(C^I)_{j'}\partial_\eta^\alpha(Y^{+,J})_{j''}$$

$$-\sum_{\substack{J+|\alpha|=K\\ j'+j''=j}}\frac{1}{\alpha!}\bar{\partial}_\eta^\alpha(Y^{+,J})_{j''}\partial_y^\alpha\sigma(x^q\Lambda)_{j'},$$

$$(60)^{-,K} \qquad (j+1)(Y^{-,K})_{j+1}=\sum_{\substack{J+|\alpha|=K\\ j'+j''=j}}\frac{1}{\alpha!}\bar{\partial}_\eta^\alpha\sigma(x^q\Lambda)_{j'}\partial_y^\alpha(Y^{-,J})_{j''}$$

$$-\sum_{\substack{I+J+|\alpha|=K\\ j'+j''=j}}\frac{1}{\alpha!}\bar{\partial}_\eta^\alpha(Y^{-,J})_{j''}\partial_y^\alpha(C^I)_{j'}$$

for $K, j\in \boldsymbol{Z}_+$. Now we can prove the following lemma directly from 3) of *Lemma 10*:

Lemma 11. *If* $|x|<6\left(\varepsilon\sin\frac{\pi}{12}\right)^{-1}|\eta_1|^{-1/(2q-q')}$ *and that* (y,η) *satisfies the condition* (2), *then we have*

$$|\partial_y^\beta\bar{\partial}_\eta^\gamma(C^K)_j|\leqslant a\left(\frac{\varepsilon\sin\pi/12}{6}\right)^j R^{-K-|\beta+\gamma|}|\eta_1|^{\{(q-q'+j)/(2q-q')-K-|\gamma|\}}$$

$$\times(K+|\beta+\gamma|)!,$$

$$|\partial_y^\beta\bar{\partial}_\eta^\gamma\sigma(x^q\Lambda)_j|\leqslant a\left(\frac{\sin\pi/12}{6}\right)^j R^{-|\beta+\gamma|}|\eta_1|^{\{(q-q'+j)/(2q-q')-|\gamma|\}}$$

$$\times|\beta+\gamma|!$$

with some a, $R>0$ *for* $K, j\in\boldsymbol{Z}_+$, $\beta,\gamma\in\boldsymbol{Z}_+^n$.

Now we have the following

Proposition 9. *Assume that* $|x|<6\left(\varepsilon\sin\frac{\pi}{12}\right)^{-1}|\eta_1|^{-1/(2q-q')}$ *and that* (y,η) *satisfies the condition* (2). *Then there exist* 2×2 *matrices* $(Y^{\pm,K,k})_j(\mathring{x},y,\eta)$, K, k, $j\in\boldsymbol{Z}_+$, *such that* $(Y^{\pm,K,k})_j=0$ *if* $K<k$, *and that*

$$(61)_j \qquad (Y^{\pm,K})_j=\sum_{k=0}^K(Y^{\pm,K,k})_j,$$

$$(62)_j \qquad |\partial_y^\beta\bar{\partial}_\eta^\gamma(Y^{\pm,K,k})_j|\leqslant C\left(\frac{\varepsilon\sin\pi/12}{5}\right)^j R_1^{-2K-|\beta+\gamma|}(\varepsilon/R_1)^k$$

$$\times|\eta_1|^{-K+k-|\gamma|+j/(2q-q')}(K-k+|\beta+\gamma|)!$$

$$\times\sum_{i=0}^j\frac{1}{i!}(C|\eta_1|^{(q-1-q')/(2q-q')})$$

$$\times\exp\{|\eta_1|^{(q-1-q')/(2q-q')}\}$$

with some C, $R_1>0$ *for* $K, k, j\in\boldsymbol{Z}_+$, $\beta,\gamma\in\boldsymbol{Z}_+^n$.

Proof. If $j=0$, this is nothing but *Proposition 4*. Assume that $j\geqslant 1$ and that there exist 2×2 matrices $(Y^{\pm,K,k})_{j'}$, K, $k\in \boldsymbol{Z}_+$, $k\leqslant K$, $j'\leqslant j-1$, such that $(61)_{j'}$ and $(62)_{j'}$ are valid. Let us define $(Y^{\pm,K,k})_j$ by

$$(Y^{+,K,k})_j=\frac{1}{j}\Big\{\sum_{\substack{I+J+|\alpha|=K\\ j'+j''=j-1}}\frac{1}{\alpha!}\bar{\partial}_\eta^\alpha(C^I)_{j'}\partial_y^\alpha(Y^{+,K,k})_{j''}$$

$$-\sum_{\substack{J+|\alpha|=K\\ j'+j''=j-1}}\frac{1}{\alpha!}\bar{\partial}_\eta^\alpha(Y^{+,K,k})_{j''}\partial_y^\alpha\sigma(x^q\Lambda)_{j'}\Big\},$$

$$(Y^{-,K,k})_j=\frac{1}{j}\Big\{\sum_{\substack{J+|\alpha|=K\\ j'+j''=j-1}}\frac{1}{\alpha!}\bar{\partial}_\eta^\alpha\sigma(x^q\Lambda)_{j'}\partial_y^\alpha(Y^{-,J,k})_{j''}$$

$$-\sum_{\substack{I+J+|\alpha|=K\\ j'+j''=j-1}}\frac{1}{\alpha!}\bar{\partial}_\eta^\alpha(Y^{-,J,k})_{j''}\partial_y^\alpha(C^I)_{j'}\Big\}.$$

Then $(61)_j$ and $(62)_j$ are direct consequences of $(61)_{j'}$, $(62)_{j'}$, $j'\leqslant j-1$, and *Lemma 10.* Q. E. D.

Corollary. *If* $|x-\mathring{x}|<4\left(\varepsilon\sin\frac{\pi}{12}\right)^{-1}|\eta_1|^{-1/(2q-q')}$, *the power series*

$$Y^{\pm,K,k}(x, y, \eta)=\sum_{j=0}^{\infty}(x-\mathring{x})^j(Y^{\pm,K,k})_j(\mathring{x}, y, \eta)$$

converege. We have $Y^{\pm,K,k}=0$ *if* $K<k$, *and*

$$|\partial_y^\beta\partial_\eta^\gamma Y^{\pm,K,k}|\leqslant 5CR_1^{-2K-|\beta+\gamma|}(\varepsilon/R_1)^k|\eta_1|^{-K+k-|\gamma|}(K-k+|\beta+\gamma|)!$$
$$\times\exp\{(C+1)|\eta_1|^{(q-1-q')/(2q-q')}\}$$

with some C, $R_1>0$ *for* K, $k\in\boldsymbol{Z}_+$, β, $\gamma\in\boldsymbol{Z}_+^n$.

Now we have the following

Proposition 10. *There exist* 2×2 *matrices* $Z^{\pm,K,k}(x, y, \eta)$ *such that*

$$|\partial_y^\beta\partial_\eta^\gamma Z^{\pm,K,k}|\leqslant CR_1^{-2K-|\beta+\gamma|}(\varepsilon/R_1)^k|\eta_1|^{-K+k-|\gamma|}(K+k+|\beta+\gamma|)! \tag{63}$$
$$\times\exp\{(C+1)|\eta_1|^{-(q-1-q')/(2q-q')}\}$$

with some C, $R_1>0$ *for* K, $k\in\boldsymbol{Z}_+$, β, $\gamma\in\boldsymbol{Z}_+^n$ *on* Ω_θ^2. *Here* C *and* R_1 *do not depend on* $\varepsilon>0$. *Furthermore, we have* $Z^{\pm,K,k}=0$ *if* $K<k$, *and*

$$U^{\pm,K}=\sum_{k=0}^{K}Z^{\pm,K,k}\Big(=\sum_{k=0}^{K}U^{\pm,K,k}\Big) \tag{64}$$

on $\Omega_{\theta,l}^1\cap\Omega_\theta^2$. *Thus* $U^{\pm,K}$ *satisfy* $(17)^{\pm,K}$ *and* $(18)^K$ *also on* Ω_θ^2.

Proof. We define $Z^{\pm,K,k}$, K, $k\in\boldsymbol{Z}_+$, ρy

$$Z^{+,K,k}=\sum_{J+|\alpha|=K}\frac{1}{\alpha!}\partial_\eta^\alpha T^{-1}\partial_y^\alpha Y^{+,J,k},\quad Z^{-,K,k}=Y^{-,K,k}T.$$

Since $|x-\mathring{x}|<4\left(\varepsilon\sin\frac{\pi}{12}\right)^{-1}|\eta_1|^{-1/(2q-q')}$ on Ω_θ^2, (63) follows directly from *Corollary* of *Proposition 9.* It is easy to see that $Z^{\pm,K,k}=0$ if $K<k$. Now let us remark that $\sum_{k=0}^{K} Y^{\pm,K,k}$ satisfy $(17)^{\pm,K}$ and that $\sum_{k=0}^{K} Y^{\pm,K,k}(\mathring{x}, y, \text{G})=U^{\pm,K}(\mathring{x}, y, \eta)$. Thus we obtain (64). Q. E. D.

This means that $U^{\pm,K}$, $K\in \boldsymbol{Z}_+$, satisfy analogous conditions as *Proposition 4* also on Ω_θ^2, this time $U^{\pm,K,k}$, $K, k\in \boldsymbol{Z}_+$ replaced by $Z^{\pm,K,k}$. We define $U^\pm$ on $\Omega_{\theta,2}^1\cup\Omega_\theta^2$ by (56). Then arguing just in the same way as *Proposition 8*, $U^\pm(x, y, \eta)$ satisfy $(54)^\pm$, (55), and the following (53)′ on $\Omega_{\theta,l}^1\cup\Omega_\theta^2$:

$$(53)' \qquad |U^\pm(x, y, \eta)|<C_1\exp\{C_1|\eta_1|^{(q-1-q')/(2q-q')}\}.$$

Thus we have proved 1° of *Main Lemma.*

§6. Construction of $E(x, y, \eta)$.

In this section we construct a 2×2 diagonal matrix $E(x, y, \eta)$ mentioned in 2° of *Main Lemma.* Let us define $\bar{\varphi}_\nu(x, y, \eta)$, $\nu=1, 2$, by

$$\begin{cases} \partial_x\bar{\varphi}_\nu-x^q\lambda_\nu(x, y, \eta+\nabla_y\bar{\varphi}(x, y, \eta))=0 \\ \bar{\varphi}_\nu(0, y, \eta)=0. \end{cases}$$

These problems are easy to solve and we obtain holomorphic functions $\bar{\varphi}_\nu(x, y, \eta)$ defined on (10), homogeneous of degree 1 in η. It is easy to see that

$$\varphi_\nu(x, y)\eta_1=[\bar{\varphi}_\nu(x, y, \eta)]_{\eta'=0}+y_1\eta_1, \qquad \nu=1, 2.$$

We have written $E(x, y, \eta)$ in the form $E(x, y, \eta)=E_0(x, y, \eta)E_1(x, y, \eta)$ in *Main Lemma.* Now consider 2×2 matrices $\bar{E}_0(x, y, \eta)$ and $E_1(x, y, \eta)$ which satisfy

$$\bar{E}_0(x, y, \eta)=\exp\begin{pmatrix}\bar{\varphi}_1(x, y, \eta) & \\ & \bar{\varphi}_2(x, y, \eta)\end{pmatrix},$$

$$\bar{E}_1(x, y, \eta)=E(x, y, \eta)(\bar{E}_0(x, y, \eta))^{-1}.$$

Now let us construct $\bar{E}_1(x, y, \eta)$. At first we prepare the following lemma which we can prove just in the same way as *Lemma 5* of Uchikoshi [8]:

Lemma 12. *Let $\alpha\in \boldsymbol{Z}_+^n$ and $k\in \boldsymbol{Z}_+$ satisfy $0\leqslant k\leqslant|\alpha|$. Then there exist 2×2 matrices $\bar{E}_0^{\alpha,k}(x, y, \eta)$ holomorphic on* (10) *which satisfy*

$$|\partial_y^\beta\partial_\eta^\gamma\bar{E}_0^{\alpha,k}|\leqslant aR^{-2|\alpha|-|\beta+\gamma|}(\varepsilon|\eta_1|)^{|\alpha|-k}|\eta_1|^{-|\gamma|}(k+|\beta+\gamma|)!$$

with some a, $R>0$ on (10),

$$\partial_y^\alpha\bar{E}^0=\sum_{k=0}^{|\alpha|}\bar{E}_0^{\alpha,k}\bar{E}_0, \quad \bar{E}_0^{\alpha,0}=\prod_{j=1}^{n}\begin{pmatrix}\frac{\partial}{\partial y_j}\bar{\varphi}_1 & \\ & \frac{\partial}{\partial y_j}\bar{\varphi}_2\end{pmatrix}^{\alpha_j}.$$

Now let us consider the following equations $(65)^K$, $K\in \boldsymbol{Z}_+$:

$$(65)^K \qquad \partial_x \bar{E}_1^K - \sum_{\substack{J+|\alpha+\beta|=K\\ k+|\beta|\neq 0}} \frac{1}{\alpha!\beta!}\partial_\eta^{\alpha+\beta}\sigma(x^q\Lambda)\bar{E}_0^{\alpha,k}\partial_y^\beta \bar{E}_1^J=0.$$

We have the following

Proposition 11. *There exist* 2×2 *matrices* $\bar{E}_1^{K,k}(x, y, \eta)$, K, $k\in \boldsymbol{Z}_+$, *such that* $\bar{E}_1^{K,k}=0$ *if* $K<k$, $\sum_{k=0}^{K} \bar{E}_1^{K,k}$, $K\in\boldsymbol{Z}_+$, *satisfy* $(65)^K$, *and that*

$$(66)^K \qquad |\partial_y^\gamma\partial_\eta^\delta \bar{E}_1^{K,k}|\leqslant CR_1^{-2K-|\gamma+\delta|}(\varepsilon|\eta_1|)^{K-k}|\eta_1|^{-K-|\delta|}(k+|\gamma+\delta|)!$$

with some C, $R_1>0$ *for* K, $k\in\boldsymbol{Z}_+$, γ, $\delta\in\boldsymbol{Z}_+^n$ *on* (10).

Proof. At first we remark that in the right-hand side of $(65)^K$, we have $J\leqslant K-1$. In fact, if $J=K$, we must have $\alpha=\beta=0$, thus $k\leqslant|\alpha|=0$, and it follows that $k+|\beta|=0$. Thus we can solve $(65)^K$ by induction on $K=0, 1, 2,\dots$.

If $K=0$, it is enough to take $\bar{E}_1^{0,0}=I_2$. Assume that $K\geqslant 1$, and that we have already constructed $\bar{E}_1^{J,j}$ for $0\leqslant j\leqslant J\leqslant K-1$. We define $\bar{E}_1^{K,k}$, $0\leqslant k\leqslant K$, by

$$\bar{E}_1^{K,k}=\int_0^x\Bigg\{\sum_{\substack{J+|\alpha+\beta|=K\\ i+j+|\beta|=k+1\\ i+|\beta|\neq 0}}\frac{1}{\alpha!\beta!}\partial_\eta^{\alpha+\beta}\sigma(x^q\Lambda)\bar{E}_0^{\alpha,i}\partial_y^\beta \bar{E}_1^{J,j}dx\Bigg\}.$$

It is easy to see that $\bar{E}_1^{K,k}=0$ if $K<k$, and that $\bar{E}_1^K=\sum_{k=0}^{K}\bar{E}_1^{K,k}$ satisfies $(65)^K$. We can prove $(66)^K$ using $(66)^J$, $J\leqslant K-1$, and *Lemma 12*. Q.E.D.

Corollary. *We have*

$$|\partial_y^\beta\partial_\eta^\gamma \bar{E}_1^K|\leqslant CR_1^{-2K-|\beta+\gamma|}|\eta_1|^{-K-|\gamma|}(K+|\beta+\gamma|)!\exp\{\varepsilon|\eta_1|\}$$

on (10).

Now let us define $\bar{E}_1(x, y, \eta)$ by $\bar{E}_1\sim\sum_{K=0}^{\infty}\bar{E}_1^K$ on (10). Then it is easy to see that $E(x, y, \eta)=\bar{E}_0(x, y, \eta)\bar{E}_1(x, y, \eta)$ satisfies

$$\partial_x E\sim\sum_{j=0}^{\infty}\Big(\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\eta^\alpha\sigma(x^q\Lambda)\partial_y^\alpha E\Big)$$

on (10). Since we have

$$|(\varphi_2(x, y)-y_1)\eta_1|,\ |\bar{\varphi}_2(x, y, \eta)|\leqslant\varepsilon\sum_{j=2}^{n}|\eta_j|$$

on (10), arguing just in the same way as *Proposition 8* we can verify that $E(x, y, \eta)$ satisfies the requirements of *Main Lemma*.

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References

[1] T. Aoki, Invertibility for microdifferential operators of infinite order, Publ. Res. Inst. Math. Sci., Kyoto Univ., **18** (1982), 421–449.

[2] L. Boutet de Monvel, Hypoelliptic operators with double characteristics and related pseudo-differential operators, Comm. Pure Appl. Math., **27** (1974), 585–639.

[3] M. Iwano and Y. Shibuya, Reduction of the order of a linear ordinary differential equation containing a small parameter, Kōdai Math. Sem. Rep., **15** (1963), 1–28.

[4] K. Kataoka, On the theory of Radon transformations of hyperfunctions, J. Fac. Sci. Univ. of Tokyo Sect. IA **28** (1981), 331–413.

[5] S. Nakane, Propagation of Singularities and Uniqueness in the Cauchy problem at a doubly characteristic point, Comm. in Part. Diff. Eq., **6** (1981), 917–921.

[6] M. Sato, T. Kawai and M. Kashiwara, Microfunctions and pseudo-differential equations, Lecture Notes in Math. 287, Springer, pp. 265–529.

[7] K. Takasaki, Singular Cauchy problems for a class of weakly hyperbolic differential operators, to appear in Comm. in Part. Diff. Eq.

[8] K. Uchikoshi, Microlocal analysis of partial differential operators with irregular singularities, to appear.

[9] J. Urabe, On the theorem of Hamada for a linear second order equation with variable multiplicities, J. of Math. Kyoto Univ., **19** (1978), 153–169.