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Gevrey well-posedness for a class of weakly hyperbolic equations

By

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§ 1. Introduction.

In this work we shall deal with the Cauchy problem

$$(1) \qquad \begin{cases} u_{tt}=\sum_{i,j}^{1,n}(a_{ij}(t,x)u_{x_j})_{x_i} & \text{on} \quad \boldsymbol{R}^n_x\times[0,\ T] \\ u(0,x)=\varphi(x) & \\ u_t(0,x)=\psi(x) & (a_{ij}=a_{ji}) \end{cases}$$

under the *weak hyperbolicity* condition

$$(2) \qquad \sum_{i,j}^{1,n} a_{ij}(t,x)\xi_i\xi_j\geqslant 0 \qquad \forall\xi\in\boldsymbol{R}^n.$$

We shall say that problem (1) is well-posed in some space $\mathscr{F}$ of functions or functionals on $\boldsymbol{R}^n$ if for any φ, ψ in $\mathscr{F}$ it admits one and only one solution u in $C^1([0,\ T],\ \mathscr{F})$.

It is known (see [2]) that the weakly hyperbolic equation $u_{tt}=a(t)u_{xx}$ may be not well-posed in C^∞, even if $a(t)\in C^\infty([0,\ T])$; therefore, we shall study problem (1) in the Gevrey classes $\gamma^{(s)}_{loc}$.

We shall prove the following

Theorem 1. *Let us consider problem* (1) *under the hypothesis* (2). *Let us suppose that the coefficients* $a_{ij}(t,x)$ *fulfill the following conditions*:

i) *There exists a* $\sigma\geqslant 1$ *such that*, $\forall K\Subset\boldsymbol{R}^n_x$, *the mapping*

$$\xi\longrightarrow[\sum_{i,j}^{1,n} a_{ij}(t,x)\xi_i\xi_j]^{1/\sigma}$$

is a continuous mapping from the sphere $S^n=\{\xi\in\boldsymbol{R}^n\colon |\xi|=1\}$ *into the space* $BV([0,\ T];\ L^\infty(K))$.

ii) $\forall K\Subset\boldsymbol{R}^n_x$ *there exist some positive constants* Λ_K, A_K *such that*

$$(3) \qquad ||D^\alpha_x a_{ij}(t,x)||_{L^\infty(K)}\leqslant\Lambda_K A_K^{|\alpha|}(|\alpha|!)^s$$

i.e. coefficients $a_{ij}(t, x)$ *belong to* $\gamma_{loc}^{(s)}(\boldsymbol{R}_x^n)$, *uniformly with respect to* t. *Then, problem* (1) *is well-posed in* $\gamma_{loc}^{(s)}$, *provided that*

$$(4) \qquad 1 \leqslant s < 1 + \frac{\sigma}{2}.$$

Let us briefly comment upon this theorem.

First of all, we observe that the hypothesis i) is equivalent to the following one:

i′) There exists a sequence of matrices $a_{ij}^{(h)}(t, x)$ in $C^1(\boldsymbol{R}_x^n \times [0, T])$, strictly positive defined and equibounded from above, such that $a_{ij}^{(h)} \longrightarrow a_{ij}$ in $L^1([0, T], L_{loc}^\infty)$ as $h \longrightarrow \infty$ and, $\forall K \Subset \boldsymbol{R}_x^n$,

$$(5) \qquad \sup_{|\xi|=1} \|\partial_t (\sum_{i,j}^{1,n} a_{ij}^{(h)}(t, x)\, \xi_i \xi_j)^{1/\sigma}\|_{L^\infty(K)} = \eta_k^{(h)}(t)$$

with

$$(6) \qquad \int_0^T \eta_k^{(h)}(t)\, dt \leqslant M_k < +\infty.$$

(This equivalence is a consequence of the fact that a function $f(t, x) \in BV([0, T]; L^\infty(\boldsymbol{R}_x^n))$ may be approximated by a sequence of functions $f^{(h)}(t, x)$ belonging to $C^1([0, T]; L^\infty(\boldsymbol{R}_x^n))$ such that

$$\int_0^T \|\partial_t f^{(h)}(t, x)\|_{L^\infty(\boldsymbol{R}_x^n)} dt \leqslant M < +\infty \quad).$$

As far as we know, the hypothesis i) has not been considered up to now in the theory of hyperbolic equations; therefore, we shall illustrate it by means of some examples.

Example 1. If $a_{ij}(t, x) = b(t) \cdot c_{ij}(x)$, with

$$(7) \qquad b(t) \geqslant 0, \quad b(t) \in C^{k,\alpha}([0, T])$$

and $0 \leqslant \sum_{i,j}^{1,n} c_{ij}(x) \xi_i \xi_j \leqslant \Lambda |\xi|^2$, then a_{ij} verify the hypothesis i) with $\sigma = k + \alpha$.

This is an immediate consequence of the fact that, if $b(t)$ verifies (7), then $[b(t)]^{\frac{1}{k+\alpha}} \in BV([0, T])$ (see Lemma 1 of [1]).

Example 2. If $a_{ij}(t, x) \in C^{1,\alpha}([0, T])$ uniformly with respect to x, then $a_{ij}(t, x)$ verify the hypothesis i) with $\sigma = 1 + \alpha$.

This is a consequence of the following

Lemma 1. *Let* $f(t)$ *be a non-negative function belonging to* $C^{1,\alpha}([0, T])$. *Then* $[f(t)]^{\frac{1}{1+\alpha}}$ *is a Lipschitz continuous function; moreover*

$$(8) \qquad \|f(t)^{\frac{1}{1+\alpha}}\|_{C^{0,1}([0,T])}^{(1+\alpha)} \leqslant C \|f(t)\|_{C^{1,\alpha}([0,T])}$$

where C *is a constant depending only on* α.

Let us consider for the moment Lemma 1 as though it was proved; we get that

$$\|\partial_t \{\sum_{i,j}^{1,n} a_{ij}(t, x)\, \xi_i \xi_j\}^{\frac{1}{1+\alpha}}\|_{L^\infty(R^n_x)} \leqslant M < +\infty;$$

so, a_{ij} verify hypothesis i′) (and, therefore, i)) with $\sigma = 1+\alpha$.

We point out that, in general, if $a_{ij}\,(t, x)$ belong to $C^k([0, T])$, $k \geqslant 2$, uniformly with respect to x, then a_{ij} verify the hypothesis i) with $\sigma = 2$.

For example, the function $a(t, x) = (t-x)^2$ is an holomorphic function, but it verifies hypothesis i) only for $\sigma \leqslant 2$.

On the other hand, there are coefficients discontinuous in t that verify hypothesis i) for larger values of σ; this means that, in general, there is no connection between hypothesis i) and high order regularity in t of the coefficients.

Now, for the sake of completeness, let us prove lemma 1; this proof is an adaptation to our case of a technique due to G. Glaeser (see [3]).

Proof of lemma 1. Let us extend $f(t)$ to a function $\tilde{f}(t): \boldsymbol{R} \longrightarrow \boldsymbol{R}^+$ such that $\tilde{f}(t) = f(t)$ for $t \in [0, T]$ and $\|\tilde{f}(t)\|_{C^{1,\alpha}(\boldsymbol{R})} = \|f(t)\|_{C^{1,\alpha}([0,T])}$.

The mean value theorem gives us

$$\begin{aligned} f(t) &= \tilde{f}(t_0) + \tilde{f}'(\xi)(t-t_0) = \\ &= \tilde{f}(t_0) + \tilde{f}'(t_0)(t-t_0) + [\tilde{f}'(\xi) - \tilde{f}'(t_0)](t-t_0) \end{aligned}$$

where ξ is a point between t and t_0.

Using the hölder continuity of $\tilde{f}'$ we get

$$0 \leqslant \tilde{f}(t) \leqslant \tilde{f}(t_0) + \tilde{f}'(t_0)(t-t_0) + k\,|t-t_0|^{1+\alpha} \tag{9}$$

The function $y(x) = k\,|x|^{1+\alpha} + \tilde{f}'(t_0)\,x + \tilde{f}(t_0)$ is a convex real function whose minimum value is

$$\min_{x \in \boldsymbol{R}} y(x) = \tilde{f}(t_0) - k\alpha \left[\frac{|\tilde{f}'(t_0)|}{k(1+\alpha)}\right]^{1+\frac{1}{\alpha}}$$

According to (9) we get

$$k\alpha \left[\frac{\tilde{f}'(t_0)}{k(1+\alpha)}\right]^{1+\frac{1}{\alpha}} \leqslant \tilde{f}(t_0) \tag{10}$$

Now (8) is an immediate consequence of (10) ■

Example 3. Let $n=1$; then problem (1) becomes

$$\begin{cases} u_{tt} = (a(t, x)\,u_x)_x & \text{on } \boldsymbol{R}^n_x \times [0, T] \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x) \end{cases}$$

Let $a(t, x) = [\alpha(t, x)]^\sigma$, $\sigma \geqslant 1$, where $\alpha(t, x)$ is a non-negative function in $C^1(\boldsymbol{R}_x \times [0, T])$. Then it is obvious that $a(t, x)$ fulfilles hypothesis i).

Let us return to problem (1). On the ground of theorem 1, we see that problem (1) may be well-posed also in the Gevrey classes $\gamma_{loc}^{(s)}$ with $s \geqslant 2$, provided that the coefficients $a_{ij}(t, x)$ fulfill hypothesis i) with $\sigma \geqslant 2$; in order to obtain results of this kind, the absence in problem (1) of lower order terms of the form $b_i(t, x) u_{x_i}$ is essential, since, as it's well known, the very simple equation $u_{tt} = u_x$ is not well-posed in $\gamma_{loc}^{(s)}$ for $s \geqslant 2$.

Theorem 1 may be also regarded as an extension of previous results, proved in [1], concerning weakly hyperbolic equations of the form $u_{tt} = \sum_{i,j}^{1,n} a_{ij}(t) u_{x_i x_j}$.

Moreover, a class of counterexamples in [1], §4, shows that, in general, the results of theorem 1 cannot be improved, in the sense that there exist $a(t)$ in $C^{k,\alpha}([0, T])$ (and, therefore, fulfilling hypothesis i) with $\sigma = k + \alpha$; see example 1 here above and lemma 1 of [1]) and $\varphi(x)$, $\psi(x)$ belonging to $\gamma_{loc}^{(s)}$ for any $s > 1 + \dfrac{k+\alpha}{2}$ for which the Cauchy problem

$$\begin{cases} u_{tt} = a(t) u_{xx} & \text{on } \boldsymbol{R}_x \times [0, T] \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x) \end{cases}$$

is not solvable in the space of distributions.

We remark that the case $s = 1$ (i.e. the well-posedness of problem (1) in the space of the real analytic functions) has been treated by the author in [4], where theorem 1 was proved under the sole hypothesis ii); therefore, in the present work we shall always suppose that $s > 1$.

Finally, we point out that remark 2 is devoted to a comparison among our results and the results of T. Nishitani, who recently has studied problem (1) with the addition of lower order terms, obtaining certain results of well-posedness in $\gamma_{loc}^{(s)}$, both in the case of strict hyperbolicity and weak hyperbolicity (see [6]).

In this remark we briefly show how, adopting our techniques, we are able to re-obtain the results of T. Nishitani under less restrictive hypotheses.

Notations. — $\gamma_{loc}^{(s)}$, for real $s \geqslant 1$, is the *t. v. s.* of Gevrey functions on $\boldsymbol{R}^n$ of order s, i.e. the C^∞ functions $f(x)$ verifying

$$|D^\alpha f(x)| \leqslant A_K A_K^{|\alpha|} (|\alpha|!)^s \quad \forall x \in K, \ \forall \alpha \in \boldsymbol{N}^n$$

for any compact subset $K \Subset \boldsymbol{R}^n$.

When $s = 1$, $\gamma_{loc}^{(1)}$ coincides with the space of the real analytic functions on $\boldsymbol{R}^n$.

— $\gamma_0^{(s)}$, for real $s > 1$, is the *t. v. s.* of Gevrey functions on $\boldsymbol{R}^n$ of order s having compact support.

– $(\gamma_0^{(s)})'$ is the dual space of $\gamma_0^{(s)}$, i. e. it is the *t. v. s.* of the Gevrey ultradistributions of order s on $\boldsymbol{R}^n$.

– $L^\infty([0, T], \gamma_{loc}^{(s)})$ is the *t. v. s.* of the measurable functions $f(t, x)$, defined on $[0, T]\times\boldsymbol{R}_x^n$, which, for any t, belong to $\gamma_{loc}^{(s)}(\boldsymbol{R}_x^n)$, uniformly with respect to t; this simply means that

$$|D_x^\alpha f(t, x)|\leqslant A_K A_K^{|\alpha|}(|\alpha|!)^s \quad \forall t\in[0, T],\ \forall x\in K,\ \forall\alpha\in\boldsymbol{N}^n$$

for any compact subset $K\Subset\boldsymbol{R}_x^n$.

– $C^{1,1}([0, T], \gamma_{loc}^{(s)})$ is the space of the functions $u:[0, T]\longrightarrow\gamma_{loc}^{(s)}$, belonging to $C^1([0, T], \gamma_{loc}^{(s)})$, whose second derivative belongs to $L^\infty([0, T], \gamma_{loc}^{(s)})$.

– For any multi-index $\alpha\in\boldsymbol{N}^n$, we shall denote by D^α the operator $\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. The symbol D^α will not involve derivatives with respect to t.

We shall often use u_t, u_{x_i} instead of $\partial_t u$, $\partial_{x_i} u$.

§ 2. The energy inequalities.

Let us consider problem (1) with $\varphi, \psi\in\gamma_{loc}^{(s)}$, $s>1$ (see the Introduction for the case $s=1$).

Owing to the finite speed of propagation of the solution, we can suppose that φ, $\psi\in\gamma_0^{(s)}$ and that, consequently, the solution u will have compact support too.

Therefore, from now on we shall suppose that

$$|D^\alpha\varphi(x)|+|D^\alpha\psi(x)|+|D^\alpha a_{ij}(t, x)|\leqslant AA^{|\alpha|}(|\alpha|!)^s \quad \forall t\in[0, T],\ \forall x\in\boldsymbol{R}^n,\ \forall\alpha\in N^n. \tag{11}$$

Moreover, in this section we shall suppose that

$$\sum_{i,j}^{1,n} a_{ij}(t, x)\xi_i\xi_j\geqslant\lambda|\xi|^2,\quad \lambda>0,\ \forall t, x;\ \forall\xi; \tag{12}$$

$$a_{ij}(t, x)\in C^1(\boldsymbol{R}_x^n\times[0, T]). \tag{13}$$

(These last two hypotheses will be removed in the next section).

The matrix $a_{ij}(t, x)$ fulfilles hypothesis i) of theorem 1; therefore, taking into account (12) and (13), we get

$$\sup_{|\xi|=1}\left\|\frac{\partial_t(\sum_{i,j}^{1,n} a_{ij}(t, x)\xi_i\xi_j)}{[\sum_{i,j}^{1,n} a_{ij}(t, x)\xi_i\xi_j]^{1-\frac{1}{\sigma}}}\right\|_{L^\infty(\boldsymbol{R}_x^n)}\leqslant\rho'(t) \tag{14}$$

with

$$\rho(t)=\int_0^t\rho'(s)ds\leqslant M<+\infty \quad \forall t\in[0, T]. \tag{15}$$

We explicitly remark that the constant M in (15) depends neither on the constant λ in (12) nor on the hypothesis (13), but it depends only on the hypothesis i) of theorem 1.

It's a well known fact that problem (1), under the additional hypotheses (12) and (13), maintaining the hypothesis ii) of theorem 1, is well-posed in $\gamma_0^{(s)}$ (one can use, with slight modifications, the technique exposed in [5]); now, the purpose of this section is to obtain a system of energy inequalities for the solution $u(t, x)$ of (1) and its derivatives, in such a way these inequalities does not depend on hypotheses (12) and (13).

For this end, we define, for any $h\in N$, $h\geqslant 1$,

$$(16)\quad E_h^2(t)=\sum_{|\alpha|=h-1}\left\{\int_{R_x^n}\left[\sum_{i,j}^{1,n}(a_{ij}(t,x)+\eta_\sigma(h,\lambda)h^{-\sigma}\delta_{ij})D^\alpha u_{x_i}(t,x)D^\alpha u_{x_j}(t,x)+\right.\right.$$
$$\left.\left.+h^2(D^\alpha u(t,x))^2+(D^\alpha u_t(t,x))^2\right]dx\right\}$$

where

$$\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}\quad;\quad \eta_\sigma(h,\lambda)=\begin{cases}1 & h^{-\sigma}>\lambda\\ 0 & h^{-\sigma}\leqslant\lambda\end{cases}$$

By derivation of problem (1), we get that $D^\alpha u$ solves the following equation:

$$(17)\quad \begin{aligned}D^\alpha u_{tt}=&\sum_{i,j}^{1,n}(a_{ij}D^\alpha u_{x_j})_{x_i}+\\&+\sum_{i,j}^{1,n}\sum_{\substack{k\leqslant\alpha\\|k|=1}}\binom{\alpha}{k}D^k a_{ij}(D^{\alpha+e_i-k}u)_{x_j}+\\&+\sum_{i,j}^{1,n}\sum_{\substack{k\leqslant\alpha+e_i\\|k|=2}}\binom{\alpha+e_i}{k}D^k a_{ij}(D^{\alpha+e_i-k}u)_{x_j}+\\&+\sum_{i,j}^{1,n}\sum_{\substack{k\leqslant\alpha+e_i\\|k|\geqslant3}}\binom{\alpha+e_i}{k}D^k a_{ij}(D^{\alpha+e_i-k}u)_{x_j}\end{aligned}$$

where $\alpha+e_i$ is the multi-index $(\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_n)$.

Now, let's derive (16) with respect to t. Taking into account (17) and the fact that u has compact support, we obtain

$$(18)\quad \begin{aligned}2E_h(t)E_h'(t)=\sum_{|\alpha|=h-1}&\left\{\int_{R_x^n}\left(\sum_{i,j}^{1,n}\partial_t a_{ij}D^\alpha u_{x_i}D^\alpha u_{x_j}\right)dx+\right.\\&+2\eta_\sigma(h,\lambda)h^{-\sigma}\int_{R_x^n}\left(\sum_i^{1,n}D^\alpha u_{x_i}D^\alpha u_{tx_i}\right)dx+\\&+2h^2\int_{R_x^n}(D^\alpha u\cdot D^\alpha u_t)\,dx+\\&+2\int_{R_x^n}\left[D^\alpha u_t\cdot\sum_{\substack{k\leqslant\alpha\\|k|=1}}\sum_{i,j}^{1,n}\binom{\alpha}{k}D^k a_{ij}(D^{\alpha+e_i-k}u)_{x_j}\right]dx+\end{aligned}$$

$$+2\int_{R^n_x}[D^\alpha u_t\cdot \sum_{\substack{k\leqslant\alpha+e_i\\|k|=2}}\sum_{i,j}^{1,n}\binom{\alpha+e_i}{k}D^k a_{ij}(D^{\alpha+e_i-k}u)_{x_j}]dx+$$

$$+2\int_{R^n_x}[D^\alpha u_t\cdot \sum_{\substack{k\leqslant\alpha+e_i\\|k|\geqslant 3}}\sum_{i,j}^{1,n}\binom{\alpha+e_i}{k}D^k a_{ij}(D^{\alpha+e_i-k}u)_{x_j}]dx\Big\}.$$

Let's examine in detail every single addendum in this sum.

From the inequality

$$|\int_{R^n_x}(\sum_{i,j}^{1,n}\partial_t a_{ij}D^\alpha u_{x_i}D^\alpha u_{x_j})dx|\leqslant$$

$$\leqslant\int_{R^n_x}\frac{|\sum_{i,j}^{1,n}\partial_t a_{ij}D^\alpha u_{x_i}D^\alpha u_{x_j}|\cdot\sum_{i,j}^{1,n}(a_{ij}+\eta_\sigma(h,\ \lambda)h^{-\sigma}\delta_{ij})D^\alpha u_{x_i}D^\alpha u_{x_j}}{(\sum_{i,j}^{1,n}a_{ij}D^\alpha u_{x_i}D^\alpha u_{x_j})^{1-1/\sigma}\cdot(h^{-\sigma}\sum_i^{1,n}(D^\alpha u_{x_i})^2)^{1/\sigma}}dx$$

it follows, taking into account (14), that

$$(19)\qquad \sum_{|\alpha|=h-1}|\int_{R^n_x}(\sum_{i,j}^{1,n}\partial_t a_{ij}D^\alpha u_{x_i}D^\alpha u_{x_j})dx|\leqslant h\rho'(t)E^2_h(t)$$

Moreover, we easily obtain

$$(20)\qquad \sum_{|\alpha|=h-1}2\eta_\sigma(h,\lambda)h^{-\sigma}\int_{R^n_x}(\sum_i^{1,n}D^\alpha u_{x_i}D^\alpha u_{tx_i})dx\leqslant$$

$$\leqslant 2\eta_\sigma(h,\ \lambda)h^{-\frac{\sigma}{2}}\sum_{|\alpha|=h-1}(\int_{R^n_x}h^{-\sigma}\sum_i^{1,n}(D^\alpha u_{x_i})^2dx)^{1/2}\cdot(\int_{R^n_x}\sum_i^{1,n}(D^\alpha u_{tx_i})^2dx)^{1/2}\leqslant$$

$$\leqslant 2\eta_\sigma(h,\ \lambda)h^{-\frac{\sigma}{2}}E_h(t)E_{h+1}(t);$$

$$(21)\qquad \sum_{|\alpha|=h-1}2h^2\int_{R^n_x}(D^\alpha u\cdot D^\alpha u_t)dx\leqslant$$

$$\leqslant 2h\sum_{|\alpha|=h-1}(\int_{R^n_x}(hD^\alpha u)^2dx)^{1/2}\cdot(\int_{R^n_x}(D^\alpha u_t)^2dx)^{1/2}\leqslant 2hE^2_h(t).$$

The term $2\int_{R^n_x}[D^\alpha u_t\cdot\sum_{\substack{k\leqslant\alpha\\|k|=1}}\sum_{i,j}^{1,n}\binom{\alpha}{k}D^k a_{ij}(D^{\alpha+e_i-k}u)_{x_j}]dx$ will be estimated using a lemma, concerning the non-negative defined matrices, due to O. A. Oleinik ([7], [8]), according to which for any function $v\in C^2(\boldsymbol{R}^n)$

$$(\sum_{i,j}^{1,n}\partial_{x_k}a_{ij}v_{x_ix_j})^2\leqslant C_1\sum_{i,j,h}^{1,n}a_{ij}v_{x_hx_i}v_{x_hx_j}$$

the constant C_1 depending only on the second derivatives (with respect to x) of the functions a_{ij}.

Having this lemma in mind, we get

$$(22)\qquad \sum_{|\alpha|=h-1}2\int_{R^n_x}[D^\alpha u_t\cdot\sum_{\substack{k\leq\alpha\\|k|=1}}\sum_{i,j}^{1,n}\binom{\alpha}{k}D^k a_{ij}(D^{\alpha-k}u)_{x_ix_j}]dx\leqslant$$

$$\leqslant 2\sum_{|\alpha|=h-1}(\int_{R^n_x}(D^\alpha u_t)^2dx)^{1/2}\cdot nC_1hE_h(t)\leqslant 2nC_1hE^2_h(t).$$

Moreover, we easily obtain

$$\sum_{|\alpha|=h-1} 2\int_{R_x^n} [D^\alpha u_t \cdot \sum_{\substack{k\leqslant\alpha+e_i \\ |k|=2}} \sum_{i,j}^{1,n} \binom{\alpha+e_i}{k} D^k a_{ij} (D^{\alpha+e_i-k} u)_{x_j}] dx \leqslant 2C_2 h E_h^2(t) \tag{23}$$

the constant C_2 depending only on the second derivatives (with respect to x) of the functions a_{ij};

$$\sum_{|\alpha|=h-1} 2\int_{R_x^n} [D^\alpha u_t \cdot \sum_{\substack{k\leqslant\alpha+e_i \\ |k|=\nu}} \sum_{i,j}^{1,n} \binom{\alpha+e_i}{k} D^k a_{ij} (D^{\alpha+e_i-k} u)_{x_j}] dx \leqslant \tag{24}$$

$$\leqslant 2\frac{\Lambda A^\nu \nu!^s}{h+2-\nu}\binom{h}{\nu} E_h(t) E_{h+2-\nu}(t) \qquad 3\leqslant\nu\leqslant h$$

where we used (11) and the inequality $\binom{\alpha}{\beta}\leqslant\binom{|\alpha|}{|\beta|}$, α, β being two multi-indexes of order n.

Taking into account the inequalities from (19) to (24), we get

$$\begin{aligned} E_h'(t) \leqslant & \left(\frac{\rho'(t)}{2} + 1 + nC_1 + C_2\right) h E_h(t) + \\ & + \eta_\sigma(h, \lambda) h^{-\frac{\sigma}{2}} E_{h+1}(t) + \\ & + \Lambda \sum_{j=0}^{h-3} \binom{h}{j} \frac{A^{h-j}(h-j)!^s}{j+2} E_{j+2}(t) \end{aligned} \tag{25}$$

where the last term appears only if $h\geqslant 3$.

If we define

$$\mu(t) = \frac{\rho(t)}{2} + (1 + nC_1 + C_2)t \tag{26}$$

we can write the system of differential inequalities (25) under the following form:

$$\begin{aligned} & E_1'(t) \leqslant \mu'(t) E_1(t) + E_2(t) \\ & E_2'(t) \leqslant 2\mu'(t) E_2(t) + 2^{-\frac{\sigma}{2}} E_3(t) \\ & E_3'(t) \leqslant 3\mu'(t) E_3(t) + 3^{-\frac{\sigma}{2}} E_4(t) + \Lambda \frac{A^3 \cdot 6^s}{2} E_2(t) \\ & \quad \cdots\cdots\cdots \\ & E_h'(t) \leqslant h\mu'(t) E_h(t) + \eta_\sigma(h, \lambda) h^{-\frac{\sigma}{2}} E_{h+1}(t) + \\ & \quad + \Lambda \sum_{j=0}^{h-3} \binom{h}{j} \frac{A^{h-j}(h-j)!^s}{j+2} E_{j+2}(t) \\ & \quad \cdots\cdots\cdots \end{aligned} \tag{27}$$

So, we have obtained a system (of infinite dimension) of energy inequalities, to manage which we need some very simple lemmas.

Lemma 2. *For any h, $j\in\boldsymbol{N}$, $j\leqslant h-2$, and for any $s\geqslant 1$, the following inequality is true:*

$$\binom{h}{j} \frac{(h-j)!^s (j+2)!^s}{h!^s (j+2)} \leqslant (j+2)(e^s)^{h-j} \tag{28}$$

Proof.

It's trivial that

$$(29)\qquad \frac{1}{j+2}\binom{h}{j}=\frac{(j+1)h!}{(j+2)!(h-j)!}\leqslant(j+1)\binom{h}{j+2};$$

$$(30)\qquad \frac{(h-j)!^s(j+2)!^s}{h!^s}\leqslant\frac{(h-j-2)!^s(j+2)!^s(h-j)^{2s}}{h!^s}\leqslant\binom{h}{j+2}^{-s}(e^s)^{h-j}$$

From (29) and (30) we obtain

$$\binom{h}{j}\frac{(h-j)!^s(j+2)!^s}{h!^s(j+2)}\leqslant(j+1)\binom{h}{j+2}^{1-s}(e^s)^{h-j}<(j+2)(e^s)^{h-j} \quad \blacksquare$$

Lemma 3. *Let* $\delta>0$, p, $q\in\boldsymbol{R}^+$, $q\geqslant 1$. *The function*

$$y(x)=px^{1-\delta}-qx \qquad x\in\boldsymbol{R}^+$$

satisfies the following estimation:

$$(31)\qquad y(x)\leqslant C_\delta\cdot p^{\frac{1}{\delta}}\cdot q$$

where C_δ *is a positive constant depending only on* δ.

Proof. The proof is trivial. We want only to point out that, obviously, (31) is not the best estimation for $y(x)$, but it is the most convenient for our purposes. $\blacksquare$

Let us return to system (27). We want to obtain an a priori estimate regarding the functions $E_h(t)$.

First of all, we remark that, taking into account (11) and (16), we have

$$(32)\qquad E_h(0)\leqslant\tilde{\Lambda}(eA)^h h!^s$$

where $\tilde{\Lambda}$ is a constant depending only on Λ and on the measure of the support of φ and ψ.

Now, let us define

$$(33)\qquad \alpha_h(t)=\frac{e^{-3h\mu(t)}E_h(t)}{B^h h!^s}$$

where B is a positive constant that we'll determine later.

We recall that, being $1<s<1+\frac{\sigma}{2}$, we can write

$$(34)\qquad s=1+\frac{\sigma}{2}-\delta$$

where $\delta>0$.

Taking into account (27), (33) and (34), we easily obtain the following system of integral inequalities:

$$(35)\qquad \alpha_h(t)\leqslant\alpha_h(0)-2h\int_0^t\mu'(v)\alpha_h(v)dv+$$

$$+2^{s}B\eta_{\sigma}(h,\ \lambda)\,h^{1-\delta}\int_{0}^{t}e^{3\mu(v)}\alpha_{h+1}(v)\,dv+$$
$$+\Lambda B^{2}\sum_{j=0}^{h-3}\binom{h}{j}\frac{(h-j)\,!^{s}(j+2)\,!^{s}}{h\,!^{s}(j+2)}\left(\frac{A}{B}\right)^{h-j}\int_{0}^{t}e^{-3(h-j-2)\mu(v)}\alpha_{j+2}(v)\,dv \qquad \forall h\in \boldsymbol{N}$$

Lemma 2 allows us to transform the system (35) into the simpler system

$$\alpha_{h}(t)\leqslant\alpha_{h}(0)-2h\int_{0}^{t}\mu'(v)\alpha_{h}(v)\,dv+ \tag{36}$$
$$+2^{s}B\eta_{\sigma}(h,\ \lambda)\,h^{1-\delta}\int_{0}^{t}e^{3\mu(v)}\alpha_{h+1}(v)\,dv+$$
$$+\Lambda B^{2}\sum_{j=0}^{h-3}\left(\frac{Ae^{s}}{B}\right)^{h-j}(j+2)\int_{0}^{t}\alpha_{j+2}(v)\,dv \qquad \forall h\in \boldsymbol{N}$$

where the last expression appears (as well as in (35)) only if $h\geqslant 3$.

Now, let us define

$$\beta(t)=\sum_{1}^{\infty}{}_{h}\,\alpha_{h}(t) \tag{37}$$

This series is well defined, the functions $\alpha_{h}(t)$ being positive. Taking into account (36), we obtain

$$\beta(t)\leqslant\beta(0)+ \tag{38}$$
$$+\sum_{1}^{[\lambda^{-1/\sigma}]}{}_{h}\int_{0}^{t}[-h\mu'(v)+2^{s}Be^{3\mu(v)}(h-1)^{(1-\delta)}]\alpha_{h}(v)\,dv+$$
$$+\sum_{h\geqslant 1}\int_{0}^{t}[-h\mu'(v)+\Lambda B^{2}h\sum_{p\geqslant 3}\left(\frac{Ae^{s}}{B}\right)^{p}]\alpha_{h}(v)\,dv$$

(there is no problem in grouping the terms of (36) in this way, because the first sum is extended only to a finite number of addenda).

Now we must estimate the expressions

$$-h\mu'(v)+2^{s}Be^{3\mu(v)}(h-1)^{(1-\delta)}; \tag{39}$$

$$-h\mu'(v)+\Lambda B^{2}h\sum_{p\geqslant 3}\left(\frac{Ae^{s}}{B}\right)^{p}. \tag{40}$$

By lemma 3 we get

$$-h\mu'(v)+2^{s}Be^{3\mu(v)}(h-1)^{(1-\delta)}\leqslant \tag{41}$$
$$\leqslant\frac{\delta}{3}C_{\delta}2^{s}Be^{\frac{3}{\delta}\mu(v)}\cdot\frac{3}{\delta}\mu'(v)=\theta'_{\delta}(v)$$

where $\theta'_{\delta}(v)$ is a positive function such that

$$\theta_{\delta}(t)=\int_{0}^{t}\theta'_{\delta}(v)\,dv=\frac{\delta}{3}C_{\delta}2^{s}B[e^{\frac{3}{\delta}\mu(t)}-1]. \tag{42}$$

As regards (40), we observe that $\mu'(v)\geqslant 1$, while

$$\Lambda B^{2}\sum_{p\geqslant 3}\left(\frac{Ae^{s}}{B}\right)^{p}=\frac{\Lambda A^{3}e^{3s}}{B}\sum_{p\geqslant 0}\left(\frac{Ae^{s}}{B}\right)^{p}; \tag{43}$$

therefore, as we can always suppose that $A \geqslant 1$ and $\Lambda \geqslant 1$, choosing

$$B = 2\Lambda A^3 e^{3s} \tag{44}$$

we immediately obtain that $\Lambda B^2 \sum_{p\geqslant 3}\left(\frac{Ae^s}{B}\right)^p \leqslant 1$. This means that

$$-h\mu'(v) + \Lambda B^2 h \sum_{p\geqslant 3}\left(\frac{Ae^s}{B}\right)^p \leqslant 0. \tag{45}$$

Substituting (41) and (45) into (38), we get

$$\beta(t) \leqslant \beta(0) + \int_0^t \theta'_\delta(v)\beta(v)dv \tag{46}$$

from which, using the Gronwall lemma, we obtain

$$\beta(t) \leqslant \beta(0)\cdot e^{\theta_\delta(t)}. \tag{47}$$

But

$$\beta(0) = \sum_1^\infty{}_h\, \alpha_h(0) = \sum_1^\infty{}_h\, E_h(0)B^{-h}h!^{-s} \leqslant \tilde{\Lambda}\sum_1^\infty{}_h\left(\frac{Ae}{B}\right)^h \leqslant \tilde{\Lambda}; \tag{48}$$

therefore substituting (48) into (47) we have

$$\alpha_h(t) \leqslant \tilde{\Lambda}\, e^{\theta_\delta(t)} \qquad \forall h \in \boldsymbol{N} \tag{49}$$

from which, taking into account (33), we finally derive the following *energy inequalities*:

$$E_h(t) \leqslant \tilde{\Lambda}\, e^{(\theta_\delta(t)+3h\mu(t))}\cdot B^h h!^s \tag{50}$$

We observe that these inequalities are independent of λ and of the derivatives with respect to t of the coefficients a_{ij}; more precisely, the estimations (50) depend only on the following elements:

—the constants Λ and A of (11);
—the measure of the support of the initial data φ and ψ;
—the constant M of (15);
—the constant δ of (34);
—some universal constants.

§ 3. Proof of theorem 1.

Existence of the solution Let $a_{ij}^{(\nu)}(t, x)$ be a sequence of strictly positive defined matrices, equibounded from above, fulfilling conditions (11) and (15) uniformly with respect to ν and such that $a_{ij}^{(\nu)}(t, x) \longrightarrow a_{ij}(t, x)$ in $L^1([0, T], L^\infty(\boldsymbol{R}_x^n))$

Such a sequence there always exists (see also the equivalence between hypothesis i) and hypothesis i′) in the Introduction); one can choose, for instance,

$$a_{ij}^{(\nu)}(t, x)=\nu\int_0^{+\infty}\tilde{a}_{ij}(t+s, x)\omega(\nu s)\,ds+\frac{1}{\nu}\delta_{ij} \tag{51}$$

where

$$\tilde{a}_{ij}(t, x)=\begin{cases} a_{ij}(t, x) & 0\leqslant t\leqslant T \\ a_{ij}(T, x) & t>T \end{cases}; \tag{52}$$

(53) $\omega(t)\in C^{\infty}(\boldsymbol{R}_t)$, $\omega\equiv 0$ on $(-\infty, 0]$ and on $[1, +\infty)$, and $\int_{-\infty}^{+\infty}\omega(s)\,ds=1$;

$$\delta_{ij}=\begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \tag{54}$$

Let $u^{(\nu)}(t, x)$ be the solutions of the problems

$$\begin{cases} u_{tt}^{(\nu)}=\sum_{ij}^{1,n}(a_{ij}^{(\nu)}(t, x)u_{x_j}^{(\nu)})_{x_i} & \text{on } \boldsymbol{R}_x^n\times[0, T] \\ u^{(\nu)}(0, x)=\varphi(x) \\ u_t^{(\nu)}(0, x)=\psi(x) \end{cases} \tag{55}$$

where φ, ψ belong to $\gamma_0^{(s)}$.

From the energy inequalities (50), we get that the sequence $\{u^{(\nu)}\}$ is bounded in $C^1([0, T], \gamma_0^{(s)})$; therefore, there exists a subsequence, that we shall denote again by $\{u^{(\nu)}\}$, such that

$$u^{(\nu)}\longrightarrow u \quad \text{in} \quad C([0, T], \gamma_0^{(s)}) \quad \text{when } \nu\longrightarrow+\infty. \tag{56}$$

Now, being $u^{(\nu)}$ solutions of (55), it's easy to see that the function $u(t, x)$, as a matter of fact, belongs to $C^{1,1}([0, T], \gamma_0^{(s)})$ and solves problem (1).

Uniqueness of the solution. We have found a solution $u\in C^{1,1}([0, T], \gamma_0^{(s)})$ of problem (1) by means of an approximation scheme; now, we must prove that this solution is indeed the only one.

In order to do this, let us define u as a solution of the problem

$$\begin{cases} u_{tt}=\sum_{i,j}^{1,n}(a_{ij}(t, x)u_{x_j})_{x_i} & \text{on } \boldsymbol{R}_x^n\times[0, T] \\ u(0, x)=0 \\ u_t(0, x)=0 \end{cases} \tag{57}$$

If we want that problem (57) makes sense, we must suppose at least that $u\in C([0, T], (\gamma_0^{(s)})')$. What we want to prove is that u is identically zero.

Let us consider the "dual" problem

$$\begin{cases} v_{tt}=\sum_{i,j}^{1,n}(a_{ij}(t, x)v_{x_j})_{x_i} & \text{on } \boldsymbol{R}_x^n\times[0, T^*] \\ v(T^*, x)=0 \\ v_t(T^*, x)=\eta(x) \end{cases} \tag{58}$$

where $T^* \in [0, T]$, $\eta(x) \in \gamma_0^{(s)}$ and v is a solution of (58) belonging to $C^{1,1}([0, T], \gamma_0^{(s)})$, the existence of a solution of this type being guaranteed by the first part of this proof.

We can multiply, in the duality $\langle\ ,\ \rangle$ between $\gamma_0^{(s)}$ and $(\gamma_0^{(s)})'$, problem (57) by v and problem (58) by u.

Integrating on $[0, T^*]$ we obtain

$$\int_0^{T^*} \{\langle u_{tt}, v\rangle - \langle v_{tt}, u\rangle\}\, dt = 0 \tag{59}$$

But $\langle u_{tt}, v\rangle - \langle v_{tt}, u\rangle =$
$= \partial_t(\langle u_t, v\rangle - \langle u, v_t\rangle)$; therefore we have

$$\langle u(T^*), \eta\rangle = 0. \tag{60}$$

The equality (60) holds for any $\eta \in \gamma_0^{(s)}$ and for any $T^* \in [0, T]$; this means that u, as an element of $C([0, T], (\gamma_0^{(s)})')$, is identically zero.

Initial data having no compact support. We have proved theorem 1 for $\varphi, \psi \in \gamma_0^{(s)}$; the case of $\varphi, \psi \in \gamma_{loc}^{(s)}$ may be treated, as usual, by means of a partition of unity, the solution of problem (1) having finite speed of propagation. This is a standard argument, and we'll not repeat it here.

Summing up, we have proved that, for any $\varphi, \psi \in \gamma_{loc}^{(s)}$ problem (1), under the hypotheses of theorem 1, has a solution $u \in C^{1,1}([0, T], \gamma_{loc}^{(s)})$; this solution is unique in $C([0, T], (\gamma_0^{(s)})')$ ■

Remark 1. Under the same assumptions of theorem 1, we can prove, by means of a duality process, that problem (1) is well-posed in $(\gamma_0^{(s)})'$, the space of the Gevrey ultradistributions of order $s < 1 + \frac{\sigma}{2}$.

This means, in particular, that if we choose the initial data φ, ψ in some Sobolev space, the problem (1) admits one and only one solution $u(t, x)$ as a Gevrey ultradistribution.

Remark 2. (Equations with lower order terms) Recently, T. Nishitani has studied in [6] the problem

$$\begin{cases} u_{tt} = \sum_{i,j}^{1,n} (a_{ij}(t, x) u_{x_j})_{x_i} + \sum_{i}^{1,n} b_i(t, x) u_{x_i} + c(t, x) u_t + d(t, x) u \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \boldsymbol{R}_x^n \times [0, T] \\ u(0, x) = \varphi(x) \\ u_t(0, x) = \psi(x) \end{cases} \tag{61}$$

He supposes that all the coefficients belong to $\gamma_{loc}^{(s)}$ in x (uniformly with respect to t), obtaining the following results:

Strictly hyperbolic case $\left(\sum_{i,j}^{1,n} a_{ij}(t, x)\xi_i\xi_j \geqslant \lambda|\xi|^2,\ \lambda > 0\right)$.

If the coefficients $a_{ij}(t, x)$ belong to $C^{0,\alpha}([0, T])$, uniformly with respect to x, then problem (61) is well-posed in $\gamma_{loc}^{(s)}$, provided that

(62) $$1 \leqslant s < \frac{1}{1-\alpha}$$

Weakly hyperbolic case $(\sum_{i,j}^{1,n} a_{ij}(t, x)\xi_i\xi_j \geqslant 0)$.

If the coefficients $a_{ij}(t, x)$ belong to $C^{k,\alpha}([0, T])$ with $k=0$ or $k=1$, uniformly with respect to x, then problem (61) is well-posed in $\gamma_{loc}^{(s)}$, provided that

(63) $$1 \leqslant s < 1 + \frac{k+\alpha}{2} < 2 \quad .$$

Now, we want briefly to show how, adopting our techniques, we can obtain these results under less restrictive hypotheses regarding the coefficients $a_{ij}(t, x)$.

More precisely, we state the following results:

Strictly hyperbolic case

If the coefficients $a_{ij}(t, x)$ verify the following condition

(64) $$\int_0^{T-\tau} \|a_{ij}(t+\tau, x) - a_{ij}(t, x)\|_{L^\infty(R_x^n)} dt \leqslant K\tau^\alpha, \quad 0 < \alpha < 1$$

then problem (61) is well-posed in $\gamma_{loc}^{(s)}$, provided that (62) holds, i. e.

$$1 \leqslant s < \frac{1}{1-\alpha}$$

Weakly hyperbolic case

If the coefficients $a_{ij}(t, x)$ verify (64), then problem (61) is well-posed in $\gamma_{loc}^{(s)}$, provided that

(65) $$1 \leqslant s < 1 + \frac{\alpha}{2};$$

if the coefficients $a_{ij}(t, x)$ verify hypothesis i) of theorem 1 with $1 \leqslant \sigma < 2$, then problem (61) is well-posed in $\gamma_{loc}^{(s)}$, provided that

(66) $$1 \leqslant s < 1 + \frac{\sigma}{2} < 2.$$

It's clear that (64) is weaker than the hypothesis of hölder-continuity of order α with respect to t, while hypothesis i) of theorem 1 with $1 \leqslant \sigma < 2$ is weaker than the hypothesis of continuity of order $C^{1,\alpha}$ with respect to t (see example 2 of the Introduction).

In order to re-obtain the results of [6] under these hypotheses, we can adopt our scheme of "approximated energies" that we have developed through this work.

To do this, we shall define

(67) $$\tilde{a}_{ij}(t, x) = \begin{cases} a_{ij}(t, x) & 0 \leqslant t \leqslant T \\ a_{ij}(T, x) & t > T \end{cases}$$

(68) $$a_{ij}^{(h)}(t, x) = h\int_0^{+\infty} \tilde{a}_{ij}(t+s)\omega(hs)\,ds$$

where $\omega(t)$ is defined in (53).

We point out that, when hypothesis (64) holds, the matrices $a_{ij}(t, x)$ satisfy the following estimations:

(69) $$\int_0^T ||\partial_t a_{ij}^{(h)}(t, x)||_{L^\infty(R_x^n)}\,dt \leqslant \tilde{K}h^{1-\alpha};$$

(70) $$\int_0^T ||a_{ij}^{(h)}(t, x) - a_{ij}(t, x)||_{L^\infty(R_x^n)}\,dt \leqslant \tilde{K}h^{-\alpha}.$$

Now, in the strict hyperbolic case we'll adopt the energies

(71) $$E_h^2(t) = \sum_{|p|=h-1}\Big\{\int_{R^n}\Big[\sum_{i,j}^{1,n}\big(a_{ij}^{(h^{\frac{1}{1-\alpha}})}(t, x)\big)\cdot D^p u_{x_i}(t, x)\, D^p u_{x_j}(t, x) + h^2(D^p u(t, x))^2 + (D^p u_t(t, x))^2\Big]dx\Big\}$$

In the weak hyperbolic case we'll adopt the energies

(72) $$E_h^2(t) = \sum_{|p|=h-1}\Big\{\int_{R^n}\Big[\sum_{i,j}^{1,n}\big(a_{ij}^{(h)}(t, x) + h^{-\alpha}\delta_{ij}\big)\cdot D^p u_{x_i}(t, x)\, D^p u_{x_j}(t, x) + h^2(D^p u(t, x))^2 + (D^p u_t(t, x))^2\Big]dx\Big\}$$

if the coefficients $a_{ij}(t, x)$ verify (64), while we'll adopt the energies defined by (16) if the coefficients $a_{ij}(t, x)$ verify hypothesis i) of theorem 1 with $1 \leqslant \sigma < 2$.

Taking into account (69) and (70), we can perform a proof similar to the one we've given for theorem 1, obtaining the results exposed here above.

We finally remark that, in the weak hyperbolic case, our method works out also in presence of lower order terms, because we confine ourselves to the Gevrey space of order $s<2$ (see also the Introduction).

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