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Some remarks on my paper "On the Cauchy problem for some non-kowalewskian equations with distinct characteristic roots"

(Schrödinger equations and generalizations, I)

By

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§1. Introduction.

In our paper [10], we made a mistake in reasoning and proofs of Theorems 1.1 and 1.2. Until now, under the condition (A), it should seem difficult to the author to obtain the conclusion of the Theorems 1.1 and 1.2 from the only Condition (B) or (B′). (In one space variable, Theorems 1.1 and 1.2 are true. cf. Examples 6.1 and 6.2 of [10].)

Prof. S. Mizohata proposed necessary conditions and sufficient conditions for an operator given by Example 6.3 of [10] to be L^2-wellposed. (See [5], [6], [7]. In [5], a necessary condition was given for more general operators.) In this paper following the inference of Mizohata, we show that for an operator with constant leading coefficients the conclusion of the Theorem 1.2 of [10] holds under the additional conditions.

§2. Statement of results.

Consider a linear partial differential operator defined on $(x, t)\in \boldsymbol{R}^n\times \boldsymbol{R}^1$:

$$P(x, D_x, D_t) = D_t^m + a_1(x, D_x) D_t^{m-1} + \cdots + a_m(x, D_x), \tag{2.1}$$

where

$$a_j(x, D_x) = \sum_{|\alpha|\leqq 2j} a_{\alpha j}(x) D_x^\alpha \qquad (1\leqq j\leqq m) \tag{2.2}$$

with coefficients $a_{\alpha j}(x)\in \mathscr{B}^\infty(\boldsymbol{R}^n)$. $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $D_x = \frac{1}{i}\frac{\partial}{\partial x}$. We are concerned with the "two-sided" Cauchy problem for $P(x, D_x, D_t)$:

$$(2.3)\qquad \begin{cases} P(x, D_x, D_t)u(x,t) = f(x,t) & \text{in } \boldsymbol{R}^n \times [-T, T], \\ D_t^{j-1}u(x,0) = g_j(x), & (1 \leqq j \leqq m). \end{cases}$$

We seek for the conditions for the Cauchy problem (2.3) to be H^s-wellposed.

Our conditions are as follows.

Condition (A. 1). $a_{\alpha j}(x) = a_{\alpha j}$*(conctant) for* $|\alpha| = 2j,\ 1 \leqq j \leqq m$.

Denote the principal symbol of $a_j(x, D_x)$ by $a_j^0(\xi)$ and the subprincipal symbol of $a_j(x, D_x)$ by $a_j^1(x, \xi)$, i. e.,

$$(2.4)\qquad a_j^0(\xi) = \sum_{|\alpha|=2j} a_{\alpha j}\xi^\alpha, \quad a_j^1(x, \xi) = \sum_{|\alpha|=2j-1} a_{\alpha j}(x)\xi^\alpha \qquad (1 \leqq j \leqq m).$$

Denote the principal symbol of $P(x, D_x, D_t)$ as 2-evolution in the sense of Petrowski [8] by $P^0(\xi, \tau)$, i. e.,

$$(2.5)\qquad P^0(\xi, \tau) = \tau^m + a_1^0(\xi)\tau^{m-1} + \cdots + a_m^0(\xi).$$

The second condition is the following.

Condition (A. 2). The roots of $P^0(\xi, \tau) = 0$ *are non-zero, real, distinct for* $\xi \in \boldsymbol{R}^n \setminus \{0\}$, i. e.,

$$(2.6)\qquad P^0(\xi, \tau) = \prod_{j=1}^{m} (\tau - \lambda_j(\xi)), \lambda_j(\xi) \neq \lambda_k(\xi), \quad (j \neq k, \xi \neq 0).$$

Remark 1. $\lambda_j(\xi)$ is homogeneous of degree 2 in ξ.

Remark 2. It is necessary for the Cauchy problem (2.3) to be $H^\infty(\boldsymbol{R}^n)$-wellposed that the roots $\lambda_j(\xi)(1 \leqq j \leqq m)$ are real for $\xi \in \boldsymbol{R}^n$ (Petrowski [8], Mizohata [4]).

Remark 3. Conditions (A. 1) and (A. 2) are the same as Condition (A) in the previous paper [10].

We put

$$(2.7)\qquad Q^0(x, \xi, \tau) = a_1^1(x, \xi)\tau^{m-1} + \cdots + a_m^1(x, \xi).$$

We replace the Condition (B) of [10] by the following.

Condition (B_0):

$$\begin{cases} \textit{For any } (x, \omega, t) \in \boldsymbol{R}^n \times S^{n-1} \times \boldsymbol{R}^1, \\ \displaystyle\int_0^t \operatorname{Im} Q^0(x + s(\nabla_\xi \lambda_j)(\omega), \omega, \lambda_j(\omega))ds \\ \textit{remains bounded}, (1 \leqq j \leqq m). \end{cases}$$

For any multi-index $\alpha(|\alpha| \geqq 1)$,

Condition (B_α):

$$\begin{cases} \textit{For any } (x,\omega)\in \boldsymbol{R}^n\times S^{n-1}, \\ \int_0^\infty |D_x^\alpha\, Q^0(x+s(\nabla_\xi\lambda_j)(\omega),\omega,\lambda_j(\omega))|\,ds \\ \textit{remains bounded}, \ (1\leqq j\leqq m). \end{cases}$$

Remark 4. Under the condition (A. 2), we have, by Euler's identity, $\xi\cdot\nabla_\xi\lambda_j(\xi)=2\lambda_j(\xi)\neq 0$ for $\xi\neq 0$, i. e., $\nabla_\xi\lambda_j(\xi)\neq 0$ for $\xi\neq 0$.

Remark 5. The above conditions are essentially the same as the conditions given by Mizohata in the case $m=1$. (cf. [6], [7]).

Our results are the following theorems.

Theorem 1. *Assume that the conditions (A. 1), (A. 2), (B_0) and (B_α) for all $\alpha(|\alpha|\geqq 1)$ hold. Then the Cauchy problem (2.3) is H^s-wellposed for any $s\in \boldsymbol{R}^1$, $s\geqq 0$, that is, for any $(g_1(x),\dots,g_m(x))\in H^{s+2m}\times H^{s+2(m-1)}\times\dots\times H^{s+2}$ and any $f(x,t)\in C_t^1([-T,T];H^s)$, there exists a unique solution $u(x,t)$ of (2.3): $u(x,t)\in C_t^0([T-,T]:H^{s+2m})\cap C_t^1([-T,T]:H^{s+2(m-1)})\cap\dots\dots\cap C_t^{m-1}([-T,T];H^{s+2})$ and the following energy inequality holds*:

$$(2.8)\quad |||u(t)|||_{(s)}\leqq C(s,T)\left\{|||u(0)|||_{(s)}+\left|\int_0^t\|f(t')\|_{(s)}dt'\right|\right\},\ t\in[-T,T].$$

Here

$$(2.9)\qquad |||u(t)|||_{(s)}^2=\sum_{j=1}^m\|(1-\Delta_x)^{m-j}D_t^{j-1}u(t)\|_{(s)}^2,$$

$$(2.10)\qquad \|u(t)\|_{(s)}^2=(2\pi)^{-n}\int(1+|\xi|^2)^s|\hat{u}(\xi,t)|^2d\xi,$$

$$(2.11)\qquad \hat{u}(\xi,t)=\mathscr{F}u(x,t)=\int e^{-ix\xi}u(x,t)dx.$$

As a special case of the conditions (B_0) and $(B_\alpha)(|\alpha|\geqq 1)$, we consider the following conditions.

Condition $(B_0)'$:

$$\begin{cases} \textit{For any } (x,\omega,t)\in \boldsymbol{R}^n\times S^{n-1}\times\boldsymbol{R}^1, \\ \int_0^t \mathrm{Im}\ a_k^1(x+s(\nabla_\xi\lambda_j)(\omega),\omega)\,ds \\ \textit{remains bounded}, \ (1\leqq j,k\leqq m). \end{cases}$$

For any multi-index α, $|\alpha|\geqq 1$,

Condition $(B_\alpha)'$:

$$\begin{cases} \textit{For any } (x,\omega)\in \boldsymbol{R}^n\times S^{n-1}, \\ \int_0^\infty |D_x^\alpha a_k^1(x+s(\nabla_\xi\lambda_j)(\omega),\omega)|\,ds \\ \textit{remains bounded}, \ (1\leqq j,k\leqq m). \end{cases}$$

Corollary of Theorem 1. *Assume that the conditions* (*A.* 1), (*A.* 2), $(B_0)'$ *and* $(B_\alpha)'$ *for all* α $(|\alpha|\geqq 1)$ *hold. Then the Cauchy problem* (2. 3) *is* $H^s(\boldsymbol{R}^n)$*-wellposed for any* $s\in\boldsymbol{R}^1$, $s\geqq 0$.

Remark 6. Assume that the conditions (A. 1) and (A. 2) hold, and that $\operatorname{Im} a_j^1(x, \xi)\equiv 0$, $(1\leqq j\leqq m)$. Then it is easy to see that the conclusion of Theorem 1 holds without the condition $(B_\alpha)(|\alpha|\geqq 1)$. In this case, the condition (B_0) is automatically satisfied.

Concerning the necessary condition, we assume the following weaker condition (A. 2)′ than (A. 2).

Condition (A. 2)′. The roots of $P^0(\xi, \tau)=0$ *are real, distinct for* $\xi\in\boldsymbol{R}^n\setminus\{0\}$:

$$P^0(\xi, \tau)=\prod_{j=1}^{m}(\tau-\lambda_j(\xi)),\ \lambda_j(\xi)\neq\lambda_k(\xi),\ (j\neq k,\ \xi\neq 0). \tag{2.6}$$

Remark that $\lambda_j(\xi)$ may be zero for some ξ or for all ξ.

Theorem 2. *Assume that the conditions (A.* 1) *and (A.* 2)′ *hold. Then the condition* (B_0) *is necessary for the Cauchy problem* (2. 3) *to be* $H^l(\boldsymbol{R}^n)$*-wellposed for any non-negative integer* l.

§ 3. Proof of Theorem 1.

To make this paper self-contained, we renew the proof from the beginning.

3. 1. Reduction to a system and its diagonalization.

Let $P(x, D_x, D_t)$ be a differential operator of the form (2. 1). Assume that the conditions (A. 1), (A. 2), (B_0) and (B_α) for all α $(|\alpha|\geqq 1)$ hold. We consider the Cauchy problem (2. 3). We put

$$u_j(x, t)=(1-\Delta)^{m-j}D_t^{j-1}u(x, t),\ (1\leqq j\leqq m), \tag{3.1}$$

$$U(x, t)={}^t(u_1(x, t), \dots, u_m(x, t)). \tag{3.2}$$

Then we have a system of the following form:

$$\begin{cases} D_tU(x, t)=M(x, D_x)U(x, t)+F(x, t), \\ U(x, 0)=G(x). \end{cases} \tag{3.3}$$

Here $M(x, D_x)=M_2(D_x)+M_1(x, D_x)+M_0(x, D_x)$ is a pseudo-differential operator of order 2. The symbol $M_j(x, \xi)$ of $M_j(x, D_x)(j=1, 2)$ has the following form:

$$M_2(\xi)=\begin{pmatrix} 0 & 1 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ -a_m^0(\xi/|\xi|) & \cdots & \cdots & \cdots & -a_1^0(\xi/|\xi|) \end{pmatrix}|\xi|^2, \tag{3.4}$$

$$(3.5)\quad M_1(x,\xi)=\begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots & 0 \\ -a_1^1(x,\xi/|\xi|) & \cdots\cdots & -a_m^1(x,\xi/|\xi|) \end{pmatrix}|\xi|,$$

$M_0(x, D_x)$ is a pseudo-differential operator of order 0.

$$(3.6)\qquad F(x,t)={}^t(0,\dots,0,f(x,t)),$$

$$(3.7)\qquad G(x)={}^t((1-\Delta)^{m-1}g_1(x),(1-\Delta)^{m-2}g_2(x),\dots,g_m(x)).$$

From the condition (A. 2), $M_2(\xi)$ has distinct eigenvalues $\lambda_1(\xi),\dots,\lambda_m(\xi)$ for $\xi\neq 0$. Thus the system (3.3) is diagonaizable as follows.

Lemma 3. 1. *There exist a diagonal pseudo-differential operator $\mathscr{D}(x, D_x)\in OPS^2_{1,0}$ and an invertible pseudo-differential operator $N(x, D_x)\in OPS^0_{1,0}$ such that*

$$(3.8)\quad N(x,D_x)(D_t-M(x,D_x))\equiv(D_t-\mathscr{D}(x,D_x))N(x,D_x)\quad(\text{mod. } OPS^0_{1,0}).$$

Proof. At first, consider the equation

$$(3.9)\qquad N(x,D_x)M(x,D_x)\equiv\mathscr{D}(x,D_x)N(x,D_x)\quad(\text{mod. } OPS^1_{1,0}).$$

We put

$$N(x,\xi)=N_0(\xi)+N_{-1}(x,\xi),$$
$$\mathscr{D}(x,\xi)=\mathscr{D}_2(\xi)+\mathscr{D}_1(x,\xi),$$

where N_j, $\mathscr{D}_j$ are homogeneous of degree j in ξ.
Then (3.9) implies that

$$(3.10)\qquad N_0(\xi)M_2(\xi)=\mathscr{D}_2(\xi)N_0(\xi).$$

Since

$$(3.11)\qquad \det(\tau I-M_2(\xi))=P^0(\xi,\tau)=\prod_{j=1}^m(\tau-\lambda_j(\xi)),$$

we have

$$(3.12)\qquad \mathscr{D}_2(\xi)=\begin{pmatrix}\lambda_1(\xi) & & 0\\ & \ddots & \\ 0 & & \lambda_m(\xi)\end{pmatrix},$$

and

$$(3.13)\qquad N_0(\xi)=\begin{pmatrix} l_1(\xi)\\ \vdots\\ l_m(\xi)\end{pmatrix},$$

where $l_i(\xi)$ is a left null vector of $\lambda_i(\xi)I-M_2(\xi)$ and of homogeneous degree 0 in ξ, $(1\leqq i\leqq m)$.
Obviously, we have $N_0(\xi)\in S^0_{1,0}$ and $|\det N_0(\xi)|\geqq\delta>0$ for $\xi\neq0$.
Next, consider the equation (3.8) (mod. $OPS^0_{1,0}$), that is,

$$(3.14)\qquad N_0(x, D_x)M_1(x, D_x)+N_{-1}(x, D_x)M_2(D_x) \equiv \mathscr{D}_2(D_x)N_{-1}(x, D_x)+\mathscr{D}_1(x, D_x)N_0(x, D_x)\quad (\text{mod. } OPS^0_{1,0}).$$

It follows from (3.14) that

$$(3.15)\qquad N_0(\xi)M_1(x, \xi)+N_{-1}(x, \xi)M_2(\xi) = \mathscr{D}_2(\xi)N_{-1}(x, \xi)+\mathscr{D}_1(x, \xi)N_0(\xi).$$

We put $N_{-1}(x, \xi)N_0(\xi)^{-1}=\tilde{N}_{-1}(x, \xi)=(\tilde{n}_{ij}(x, \xi))$, then we have

$$(3.16)\qquad \tilde{N}_{-1}(x, \xi)\mathscr{D}_2(\xi)-\mathscr{D}_2(\xi)\tilde{N}_{-1}(x, \xi) = \mathscr{D}_1(x, \xi)-N_0(\xi)M_1(x, \xi)N_0(\xi)^{-1}.$$

We put $R_1(x, \xi)=N_0(\xi)M_1(x, \xi)N_0(\xi)^{-1}=(r_{ij}(x, \xi))$.
Then we choose $\mathscr{D}_1(x, \xi)$ such that

$$(3.17)\qquad \mathscr{D}_1(x, \xi)=\text{diagonal of } R_1(x, \xi).$$

Define

$$(3.18)\qquad \tilde{n}_{ij}(x, \xi)=\begin{cases}(\lambda_i(\xi)-\lambda_j(\xi))^{-1}r_{ij}(x, \xi) & (i\neq j)\\ 0 & (i=j).\end{cases}$$

Then $\mathscr{D}_1(x, \xi)$ and $N_{-1}(x, \xi)=\tilde{N}_{-1}(x, \xi)N_0(\xi)$ satisfy (3.15). Thus $\mathscr{D}(x, D_x)=\mathscr{D}_2(D_x)+\mathscr{D}_1(x, D_x)$ and $N(x, D_x)=(I+\tilde{N}_{-1}(x, D_x))N_0(D_x)$ satisfy the equation (3.8). Furthermore, we can choose $\tilde{N}_{-1}(x, \xi)$ such that $\tilde{N}_{-1}(x, \xi)$ belongs to $S^{-1}_{1,0}$ and for each $s\in \boldsymbol{R}^1(s\geqq0)$, operator norm $\|\tilde{N}_{-1}(x, D_x)\|_{\mathscr{L}(H^s,H^s)}$ can be as small as one wishes by modifying $\tilde{N}_{-1}(x, \xi)$ in $|\xi|\leqq R_0$. Then

$$(3.19)\qquad (I+\tilde{N}_{-1}(x, D_x))^{-1}=I-\tilde{N}_{-1}+\tilde{N}^2_{-1}+\cdots+(-1)^k\tilde{N}^k_{-1}+\cdots$$

exists and belongs to $OPS^0_{1,0}$ (cf. Kumano-go [3], Appendix I). Thus $N(x, D_x)$ is invertible in $OPS^0_{1,0}$ and $N(x, D_x)^{-1}=N_0(D_x)^{-1}(I+\tilde{N}_{-1}(x, D_x))^{-1}$. This completes the proof of Lemma 3. 1.

From the proof of Lemma 3. 1 we have more explicit formula for $\mathscr{D}_1(x, \xi)=(\delta_{ij}\lambda^{(1)}_j(x, \xi))$.

Lemma 3. 2. *In Lemma* 3. 1, *we have*

$$(3.20)\qquad \lambda^{(1)}_i(x, \xi)=-Q^0(x, \xi, \lambda_i(\xi))/\prod_{j\neq i}(\lambda_i(\xi)-\lambda_j(\xi)),$$

where $Q^0(x, \xi, \tau)$ *is defined by* (2.7), $(1\leqq i\leqq m)$.

Proof. A left null vector $l_i(\xi)$ of

$$\lambda_i(\xi)I-M_2(\xi)=[\lambda_i(\xi/|\xi|)I-M_2(\xi/|\xi|)]|\xi|^2 \quad \text{in } (3.13)$$

is a constant multiple of $(l_{i1}(\omega),\dots,\lambda_{im}(\omega))$ $(\omega=\xi/|\xi|)$ where

$$(3.21)\qquad \begin{cases} l_{i1}(\omega)=\lambda_i(\omega)^{m-1}+a_1^0(\omega)\lambda_i(\omega)^{m-2}+\cdots+a_{m-1}^0(\omega) \\ l_{i2}(\omega)=\lambda_i(\omega)^{m-2}+a_1^0(\omega)\lambda_i(\omega)^{m-3}+\cdots+a_{m-2}^0(\omega) \\ \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \\ l_{im}(\omega)=1. \end{cases}$$

On the other hand, a right null vector $r_j(\xi)$ of $\lambda_j(\xi)I-M_2(\xi)$ is a constant multipel of ${}^t(1, \lambda_j(\omega), \lambda_j(\omega)^2,\dots, \lambda_j(\omega)^{m-1})$ $(\omega=\xi/|\xi|)$. We define $l_i(\xi)=l_i(\xi/|\xi|)=(l_{i1}(\xi/|\xi|),\dots, l_{im}(\xi/|\xi|))$ by (3.21) and choose $r_i(\xi)$ such that $l_i(\xi)r_i(\xi)=1$. Thus we have

$$r_i(\xi)={}^t(1, \lambda_i(\xi/|\xi|),\dots, \lambda_i(\xi/|\xi|)^{m-1})/\frac{\partial P^0}{\partial\tau}(\xi/|\xi|, \lambda_i(\xi/|\xi|)).$$

From the fact that $\lambda_1(\xi),\dots,\lambda_m(\xi)$ are distinct, it follows that $l_i(\xi)r_j(\xi)=\delta_{ij}$ $(1\leqq i, j\leqq m)$. Thus we have

$$N_0(\xi)=\begin{pmatrix} l_1(\xi) \\ \vdots \\ l_m(\xi)\end{pmatrix}, \quad N_0(\xi)^{-1}=[r_1(\xi)\dots r_m(\xi)]$$

and

$$\begin{aligned}\lambda_i^{(1)}(x,\xi)&=l_i(\xi)M_1(x,\xi)r_i(\xi)\\ &=-(a_1^1(x,\omega)\lambda_i(\omega)^{m-1}+\cdots+a_m^1(x,\omega))|\xi|/\frac{\partial P^0}{\partial\tau}(\omega,\lambda_i(\omega))\\ &=-Q^0(x,\omega,\lambda_i(\omega))|\xi|/\prod_{j\neq i}(\lambda_i(\omega)-\lambda_j(\omega))\\ &=-Q^0(x,\xi,\lambda_i(\xi))/\prod_{j\neq j}(\lambda_i(\xi)-\lambda_j(\xi)), \quad (\omega=\xi/|\xi|).\end{aligned}$$

3. 2. Proof of Theorem 1. Following the inference of Mizohata [6], [7], we transform $D_t-\mathscr{D}(x, D_x)$ to an operator without the first order term $\mathscr{D}_1(x, D_x)$. For this end, we put $V(x, t)=N(x, D_x)U(x, t)$ and

$$(3.22)\qquad \begin{aligned} V(x,t)&=K(x,t,D_x)W(x,t)\\ &=\begin{pmatrix} k_1(x,t,D_x) & & 0 \\ & \ddots & \\ 0 & & k_m(x,t,D_x)\end{pmatrix} W(x,t),\end{aligned}$$

where $K(x, t, D_x)$ is a diagonal pseudo-differential operator. Applying $D_t-\mathscr{D}(x, D_x)$ to $V(x, t)$, we have

$$(3.23)\qquad \begin{aligned}&(D_t-\mathscr{D}(x,D_x))[K(x,t,D_x)W(x,t)]\\ &=[D_tK(x,t,D_x)-\sum_{j=1}^n \mathscr{D}_2^{(j)}(D_x)K_{(j)}(x,t,D_x)-\\ &\quad -\mathscr{D}_1(x,D_x)K(x,t,D_x)]W(x,t)\end{aligned}$$

$$+K(x, t, D_x)[D_t-\mathscr{D}_2(D_x)]W(x, t)$$
$$-\tilde{\mathscr{D}}_0(x, t, D_x)K(x, t, D_x)W(x, t).$$

We define the symbol $K(x, t, \xi)$ of $K(x, t, D_x)$ by the solution of the equation

$$(3.24)\quad \begin{cases} [D_t-\sum_{j=1}^{n}\frac{\partial}{\partial\xi_j}\mathscr{D}_2(\xi)D_j-\mathscr{D}_1(x, \xi)]K(x, t, \xi)=0, \\ K(x, 0, \xi)=I, \end{cases}$$

that is,

$$(3.25)\quad \begin{cases} [D_t-\nabla_\xi\lambda_j(\xi)\cdot D_x+Q^0(x, \xi, \lambda_j(\xi))/\prod_{i\neq j}(\lambda_j(\xi)-\lambda_i(\xi))]\times \\ \quad\times k_j(x, t, \xi)=0, \\ k_j(x, 0, \xi)=1, \quad (1\leqq j\leqq m). \end{cases}$$

The solution of (3.25) is the following form:

$$(3.26)\quad \begin{cases} k_j(x, t, \xi)=\exp[i\phi_j(x, t, \xi)], \\ \phi_j(x, t, \xi)=\dfrac{-\int_0^t Q^0(x+s\nabla_\xi\lambda_j(\xi), \xi, \lambda_j(\xi))ds}{\prod_{i\neq j}(\lambda_j(\xi)-\lambda_i(\xi))}, \\ \quad(1\leqq j\leqq m). \end{cases}$$

It follows from (3.3) and (3.23) that

$$(3.27)\quad \begin{aligned} &K(x, t, D_x)[D_t-\mathscr{D}_2(D_x)]W(x, t) \\ &\quad -K_1(x, t, D_x)W(x, t)-\tilde{M}_0(x, D_x)N(x, D_x)^{-1}V(x, t) \\ &\quad =N(x, D_x)F(x, t) \end{aligned}$$

where $K_1(x, t, D_x)=\tilde{\mathscr{D}}_0(x, t, D_x)K(x, t, D_x)$.
By the conditions (B_0) and $(B_\alpha)(|\alpha|\geqq 1)$, $k_j(x, t, \xi)$ has the following estimate (Mizohata [6], [7]).

$$(3.28)\quad |D_x^\beta D_\xi^\alpha k_j(x, t, \xi)|\leqq M_{\alpha,\beta}|t|^{|\alpha|}, \quad (1\leqq j\leqq m).$$

$k_j(x, t, \xi)$ belongs to $S^0_{0,0}$ (t: parameter). By the theorem of Calderon-Vaillancourt [1], $K(x, t, D_x)$ and $K_1(x, t, D_x)$ are bounded operators on $[H^s(\boldsymbol{R}^n)]^m$. On the other hand, we have

$$(3.29)\quad \exp[-i\phi_j(x, t, D_x)]\exp[i\phi_j(x, t, D_x)]=I+R_j(x, t, D_x),$$

and from (3.28) we have

$$(3.30)\quad \|R_j(x, t, D_x)\|\leqq \text{const. } |t|.$$

By the theorem of Calderon-Vaillancourt, $(I+R_j(x, t, D_x))$ is invertible in $OPS^0_{0,0}$ when $|t|$ is small. Thus $K(x, t, D_x)$ is invertible in $OPS^0_{0,0}$ when $|t|$ is small. Finally from (3.27) we have

$$(3.31)\quad \begin{cases} [D_t-\mathscr{D}_2(D_x)-\tilde{K}_0(x,t,D_x)]W(x,t)=\tilde{\tilde{K}}_0(x,t,D_x)F(x,t) \\ W(x,0)=N(x,D_x)G(x) \end{cases}$$

where $\tilde{K}_0$, $\tilde{\tilde{K}}_0\in OPS^0_{0,0}$ for small $|t|$.
The Cauchy problem (3.31) is H^s-wellposed and $N(x, D_x)$ and $K(x, t, D_x)$ are bounded and invertible on H^s for small $|t|$. Thus the proof of Theorem 1 is complete when T is small. Repeating the above argument step by step we complete the proof of Theorem 1. (Q. E. D.)

§4. Proof of Theorem 2.

4. 1. Asymptotic solutions. We construct the asymptotic solution of the following form:

$$(4.1)\qquad u(x,t,\xi)=e^{i\varphi(x,t,\xi)}v(x,t,\xi),\quad \xi\in\boldsymbol{R}^n\setminus\{0\}.$$

Applying $P(x, D_x, D_t)$ to $u(x, t, \xi)$, we have

$$(4.2)\qquad \begin{aligned} &e^{-i\varphi(x,t,\xi)}P(x,D_x,D_t)(e^{i\varphi(x,t,\xi)}v(x,t,\xi)) \\ &=P\Big(x, D_x+\frac{\partial\varphi}{\partial x}, D_t+\frac{\partial\varphi}{\partial t}\Big)v(x,t,\xi) \\ &=P^0\Big(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial t}\Big)v(x,t,\xi) \\ &+\Big[\frac{\partial P^0}{\partial\tau}\Big(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial t}\Big)D_t+\sum_{k=1}^n\frac{\partial P^0}{\partial\xi_k}\Big(\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial t}\Big)D_k \\ &\qquad+Q^0\Big(x,\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial t}\Big)\Big]v(x,t,\xi) \\ &+R\Big(x,D_x,D_t,\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial t}\Big)v(x,t,\xi). \end{aligned}$$

We define the phase function $\varphi(x, t, \xi)$ by

$$(4.3)\qquad \varphi_j(x,t,\xi)=\xi x+\lambda_j(\xi)t,\quad \xi\in\boldsymbol{R}^n\setminus\{0\},\ (1\leqq j\leqq m),$$

so that $\varphi_j(x, t, \xi)$ satisfies $P^0\Big(\dfrac{\partial\varphi}{\partial x},\dfrac{\partial\varphi}{\partial t}\Big)=0$.
We solve the transport equation:

$$(4.4)\quad \begin{cases} \Big[\dfrac{\partial P^0}{\partial\tau}(\xi,\lambda_j(\xi))D_t+\displaystyle\sum_{k=1}^n\dfrac{\partial P^0}{\partial\xi_k}(\xi,\lambda_j(\xi))D_k+Q^0(x,\xi,\lambda_j(\xi))\Big]\times \\ \qquad\qquad\times v_j(x,t,\xi)=0, \\ v_j(x,0,\xi)=g(x). \end{cases}$$

Differentiate $P^0(\xi, \lambda_j(\xi))=0$ with respect to ξ_k,
we have $\dfrac{\partial P^0}{\partial\xi_k}(\xi,\lambda_j(\xi))+\dfrac{\partial P^0}{\partial\tau}(\xi,\lambda_j(\xi))\dfrac{\partial\lambda_j}{\partial\xi_k}=0.$
Since $\dfrac{\partial P^0}{\partial\tau}(\xi,\lambda_j(\xi))=\displaystyle\prod_{\substack{i=1\\ i\neq j}}^m(\lambda_j(\xi)-\lambda_i(\xi))\neq 0$ from (2.6),
we have

$$(4.5)\quad \begin{cases} \left[D_t-\sum_{k=1}^{n}\frac{\partial\lambda_j(\xi)}{\partial\xi_k}D_k+\frac{Q^0(x,\xi,\lambda_j(\xi))}{\prod_{\substack{i=1\\ i\neq j}}^{m}(\lambda_j(\xi)-\lambda_i(\xi))}\right]v_j(x,t,\xi)=0,\\ v_j(x,0,\xi)=g(x).\end{cases}$$

The solution $v_j(x, t, \xi)$ has the following form:

$$(4.6)\qquad v_j(x,t,\xi)=\exp[i\phi_j(x,t,\xi)]w_j(x,t,\xi),$$

where

$$(4.7)\qquad \phi_j(x,\mathrm{t},\xi)=-\int_0^t\frac{Q^0(x+s(\nabla_\xi\lambda_j)(\xi),\xi,\lambda_j(\xi))}{\prod_{\substack{i=1\\ i\neq j}}^{m}(\lambda_j(\xi)-\lambda_i(\xi))}ds$$

and

$$(4.8)\qquad w_j(x,t,\xi)=g(x+t(\nabla_\xi\lambda_j)(\xi)).$$

Remark that $\phi_j(x, t, \xi)$ is the same function as the function in (3.26).

Note that $\phi_j(x, t, \xi)=\phi_j(x, |\xi|t, \xi/|\xi|)$ and $w_j(x, t, \xi)=w_j(x, |\xi|t, \xi/|\xi|)$ so that $v_j(x, t, \xi)=v_j(x, |\xi|t, \xi/|\xi|)$. Thus we have the following solutions $u_j(x, t, \xi)$ of the equation

$$(4.9)\qquad \begin{cases} P(x,D_x,D_t)u_j(x,t,\xi)=f_j(x,t,\xi),\\ u_j(x,0,\xi)=e^{ix\xi}g(x),\end{cases}$$

where

$$(4.10)\qquad \begin{aligned} u_j(x,t,\xi)&=e^{i\varphi_j(x,t,\xi)}e^{i\phi_j(x,t,\xi)}w_j(x,t,\xi)\\ &=\exp i\left[\xi x+\lambda_j(\xi)t-\int_0^t\frac{Q^0(x+s(\nabla_\xi\lambda_j)(\xi),\xi,\lambda_j(\xi))}{\prod_{\substack{1\leq i\leq m\\ i\neq j}}(\lambda_j(\xi)-\lambda_i(\xi))}ds\right]\times\\ &\qquad\times g(x+t(\nabla_\xi\lambda_j)(\xi)),\end{aligned}$$

$$(4.11)\qquad \begin{aligned} f_j(x,t,\xi)&=e^{i\varphi_j(x,t,\xi)}R_j(x,D_x,D_t,\xi)(e^{i\phi_j(x,t,\xi)}w_j(x,t,\xi))\\ &=e^{i\varphi_j(x,t,\xi)}e^{i\phi_j(x,t,\xi)}R_j\left(x,D_x+\frac{\partial\phi_j}{\partial x},D_t+\frac{\partial\phi_j}{\partial t},\xi\right)w_j(x,t,\xi),\end{aligned}$$

$$(4.12)\qquad R_j(x,D_x,D_t,\xi)=R\left(x,D_x,D_t,\frac{\partial\varphi_j}{\partial x},\frac{\partial\varphi_j}{\partial t}\right).$$

Here R_j consists of the following three terms.

$$(4.13)\qquad \begin{aligned} &R_j(x,D_x,D_t,\xi)\\ &=\sum_{|\alpha|+i\geq 2}\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial\tau}\right)^{i}P^0(\xi,\tau)\Big]_{\tau=\lambda_j(\xi)}D_x^{\alpha}D_t^{i}\\ &\quad+\sum_{|\alpha|+i\geq 1}\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial\tau}\right)^{i}Q^0(x,\xi,\tau)\Big]_{\tau=\lambda_j(\xi)}D_x^{\alpha}D_t^{i}\end{aligned}$$

$$+\sum_{|\alpha|+i\geqq 0}\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial\tau}\right)^{i}\sum_{|\beta|+2l\leqq 2(m-1)}a_{\beta l}(x)\xi^{\beta}\tau^{l})\Big]_{\tau=\lambda_j(\xi)}D_x^{\alpha}D_t^{i}.$$

For $g(x)\in C_0^\infty(\boldsymbol{R}^n)$, we have the following estimates.

$$(4.14)\quad \begin{aligned}&\|R_j(x, D_x, D_t, \xi)w_j(x, t, \xi)\|\\ &\equiv\|R_j(x, D_x, D_t, \xi)g(x+t(\nabla_\xi\lambda_j)(\xi))\|\\ &\leqq const. \sum_{|\alpha|+i\geqq 2}\sum_{|\beta|+2l=2m}|a_{\beta l}||\xi|^{|\beta|-|\alpha|}|\lambda_j(\xi)|^{l-i}|(\nabla_\xi\lambda_j)(\xi)|^{i}\|g(x)\|_{(|\alpha|+i)}\\ &\quad+const. \sum_{|\alpha|+i\geqq 1}\sum_{|\beta|+2l=2m-1}|\xi|^{|\beta|-|\alpha|}|\lambda_j(\xi)|^{l-i}|(\nabla_\xi\lambda_j)(\xi)|^{i}\|g(x)\|_{(|\alpha|+i)}\\ &\quad+const. \sum_{|\alpha|+i\geqq 0}\sum_{|\beta|+2l\leqq 2(m-1)}|\xi|^{|\beta|-|\alpha|}|\lambda_j(\xi)|^{l-i}|(\nabla_\xi\lambda_j)(\xi)|^{i}\|g(x)\|_{(|\alpha|+i)}\\ &\leqq const. \sum_{|\alpha|+i\geqq 2}|\xi|^{2m-(|\alpha|+i)}\|g(x)\|_{(|\alpha|+i)}\\ &\quad+const. \sum_{|\alpha|+i\geqq 1}|\xi|^{2m-1-(|\alpha|+i)}\|g(x)\|_{(|\alpha|+i)}\\ &\quad+const. \sum_{|\alpha|+i\geqq 0}\sum_{|\beta|+2l\leqq 2(m-1)}|\xi|^{(|\beta|+2l)-(|\alpha|+i)}\|g(x)\|_{(|\alpha|+i)}\\ &\leqq const. |\xi|^{2(m-1)}\|g(x)\|_{(2m)}.\end{aligned}$$

4. 2. Proof of Theorem 2. Following Mizohata [6], [7], we show the necessity of (B_0). Let us suppose that the Cauchy problem (2.3) is H^l-wellposed and the inequality (2.8) holds for $s=l$. Suppose that (B_0) is violated. Then there exist $j_0\in\{1, 2,\dots, m\}$, $x^1\in\boldsymbol{R}^n$, $\omega^0\in S^{n-1}$ and $t_0>0$ such that

$$(4.15)\quad \begin{aligned}&\mathrm{Im}\int_0^{t_0}\frac{Q^0(x^1+s(\nabla_\xi\lambda_{j_0})(\omega^0), \omega^0, \lambda_{j_0}(\omega^0))}{\prod_{i\neq j_0}(\lambda_{j_0}(\omega^0)-\lambda_i(\omega^0))}ds\\ &\geqq 2\log(2C(l, T)),\end{aligned}$$

where $C(l, T)$ is the constant in (2.8) with $s=l$.

Put $x^0=x^1+t_0(\nabla_\xi\lambda_{j_0})(\omega^0)$, $\xi=\rho\omega^0$, $t_\rho=t_0/\rho$, ρ being positive parameter tending to infinity. Let the function $g(x)$ in (4.9) be a smooth function with small support around x^0.

We put $u_\rho(x, t)=u_{j_0}(x, t, \rho\omega^0)$, then we have

$$(4.16)\quad \begin{aligned}u_\rho(x, t_\rho)&=u_{j_0}(x, t_\rho, \rho\omega^0)\\ &=\exp i[\rho\varphi_{j_0}(x, t_0, \omega^0)+\psi_{j_0}(x, t_0, \omega^0)]w_{j_0}(x, t_0, \omega^0).\end{aligned}$$

Note that the support of $w_{j_0}(x, t_0, \omega^0)=g(x+t_0(\nabla_\xi\lambda_{j_0})(\omega^0))$ is concentrated around $x^1(=x^0-t_0(\nabla_\xi\lambda_{j_0})(\omega^0))$ and its diameter can be made small by schrinking the support of $g(x)$ to x^0.

It follows from (4.10), (4.15) and (4.16) that

$$(4.17)\quad |||u_\rho(\cdot, t_\rho)|||_{(l)}\gtrsim 2C(l, T)\rho^{2(m-1)+l}\|g(x)\|_{L^2(\boldsymbol{R}^n)}.$$

We put $f_\rho(x, t)=f_{j_0}(x, t, \rho\omega^0)$, then we have from (4.11)

$$(4.18)\quad f_\rho(x, t)=\exp i[\varphi_{j_0}(x, t, \rho\omega^0)+\psi_{j_0}(x, \rho t, \omega^0)]\times$$

$$\times R_{j_0}\Big(x,\ D_x+\frac{\partial\psi_{j_0}}{\partial x}(x,\rho t,\omega^0),\ D_t+\frac{\partial\psi_{j_0}}{\partial t}(x,\rho t,\omega^0),\ \rho\omega^0\Big)\times$$
$$\times w_{j_0}(x,\rho t,\omega^0).$$

(4.14) and (4.18) imply that

$$(4.19)\quad \int_0^{t_\rho}\|f_\rho(\cdot,t')\|_{(l)}dt'$$
$$\lesssim \rho^{2(m-1)+l-1}\int_0^{t_0}\|\exp[i\psi_{j_0}(x,t',\omega^0)]R_{j_0}\Big(x,D_x+\frac{\partial\psi_{j_0}}{\partial x}(x,t',\omega^0),$$
$$D_t+\frac{\partial\psi_{j_0}}{\partial t}(x,t',\omega^0),\omega^0\Big)\omega_{j_0}(x,t',\omega^0)\|_{L^2(R^n)}dt'.$$

Thus (2.8) implies the following inequality:

$$\rho^{2(m-1)+l}2C(l,T)\|g(x)\|_{L^2(R^n)}$$
$$\lesssim 2C(l,T)\rho^{2(m-1)+l}\Big\{\|g(x)\|_{L^2(R^n)}+\frac{1}{\rho}\int_0^{t_0}\|\tilde{f}(\cdot,t')\|_{L^2(R^n)}dt'\Big\},$$

where

$$\tilde{f}(x,t)=\exp[i\psi_{j_0}(x,t,\omega^0)]R_{0_j}\Big(x,D_x+\frac{\partial\psi_{j_0}}{\partial x}(x,t,\omega^0),$$
$$D_t+\frac{\partial\psi_{j_0}}{\partial t}(x,t,\omega^0),\omega^0\Big)w_{j_0}(x,t,w^0).$$

This is impossible when $\rho\longrightarrow+\infty$, which completes the proof of Theorem 2.

§5. Examples and concluding remarks.

5. 1. If the Cauchy problem (2.3) for an operator $P(x, D_x, D_t)$ of the form (2.1) is wellposed both for the future and for the past in some functional space, we would like to say that the operator $P(x, D_x, D_t)$ is *of Schrödinger type.*

Example 5. 1. (Mizohata [6], [7])

$$(5.1)\qquad P(x,D_x,D_t)=D_t+\frac{1}{2}\sum_{j=1}^{n}(D_j-b_j(x))^2.$$

The conditions (B_0) and (B_α) $(|\alpha|\geqq 1)$ have the following forms:

$$(B_0)\qquad \sup_{(x,\omega,t)\in R^n\times S^{n-1}\times R^1}\left|\int_0^t \mathrm{Im}\sum_{j=1}^{n}b_j(x+s\omega)\,\omega_j ds\right|<+\infty.$$

For any α $(|\alpha|\geqq 1)$,

$$(B_\alpha)\qquad \sup_{(x,\omega)\in R^n\times S^{n-1}}\int_0^\infty\left|\sum_{j=1}^{n}D_x^\alpha b_j(x+s\omega)\,\omega_j\right|ds<+\infty.$$

Then the conclusion of Theorem 1 holds.

We take the function $\phi(x, t, \xi)$ in (3.26) as follows:

$$\phi(x, t, \xi)=-\frac{1}{2}\int_0^t \sum_{j=1}^n b_j(x-s\xi)\xi_j\, ds. \tag{5.2}$$

Then the following equality holds:

$$\begin{aligned}&\Big[D_t+\frac{1}{2}\sum_{j=1}^n (D_j-b_j(x))^2\Big]e^{i\phi(x,t,D_x)}\\ &\equiv e^{i\phi(x,t,D_x)}\Big(D_t+\frac{1}{2}\sum_{j=1}^n D_j^2\Big),\quad (\text{mod. } OPS^0_{0,0}).\end{aligned} \tag{5.3}$$

Example 5. 2.

$$P(x, D_x, D_t)=D_t^2-|D_x|^4+\sum_{j=1}^n b_j(x)D_jD_t+\sum_{|\nu|=3} c_\nu(x)D_x^\nu. \tag{5.4}$$

$$P^0(\xi, \tau)=\tau^2-|\xi|^4=(\tau-|\xi|^2)(\tau+|\xi|^2). \tag{5.5}$$

It follows from (5.5) that the conditions (A. 1) and (A. 2) are satisfied. The conditions $(B_0)'$ and $(B_\alpha)'(|\alpha|\geqq 1)$ have the following forms:

$$(B_0)'\quad \begin{cases}\sup_{(x,\omega,t)\in R^n\times S^{n-1}\times R^1}\Big|\int_0^t \operatorname{Im}\sum_{j=1}^n b_j(x+s\omega)\,\omega_j ds\Big|<+\infty,\\ \sup_{(x,\omega,t)\in R^n\times S^{n-1}\times R^1}\Big|\int_0^t \operatorname{Im}\sum_{|\nu|=3} c_\nu(x+s\omega)\,\omega^\nu ds\Big|<+\infty.\end{cases}$$

$$(B_\alpha)'\quad \begin{cases}\sup_{(x,\omega)\in R^n\times S^{n-1}}\int_0^\infty \Big|\sum_{j=1}^n D_x^\alpha b_j(x+s\omega)\,\omega_j\Big| ds<+\infty,\\ \sup_{(x,\omega)\in R^n\times S^{n-1}}\int_0^\infty \Big|\sum_{|\nu|=3} D_x^\alpha c_\nu(x+s\omega)\,\omega^\nu\Big| ds<+\infty.\end{cases}$$

In this case, the condition (B_0) [resp. (B_α)] is equivalent to the condition $(B_0)'$ [resp. $(B_\alpha)'$]. Principal part of $P(x, D_x, D_t)$ is equal to $-\Big[\Big(\frac{\partial}{\partial t}\Big)^2+\Delta^2\Big]$ which is appeared in the equation of vibrating plate (cf. Courant-Hilbert [2, p. 252], Schrödinger [9, footnote]).

5. 2. Obviously, the results of this paper can be extended to some systems of linear partial differential equations.

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