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Isometric immersions of $SO(5)$

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Introduction.

Let $SO(n)$ be the rotation group endowed with a biinvariant Riemannian metric. In this paper we consider the problem of local or global isometric immersions of $SO(n)$ into the Euclidean spaces. For general n it is known that $SO(n)$ can be globally isometrically imbedded into $\boldsymbol{R}^{n^2}$, namely the canonical imbedding of $SO(n)$ into the set of real (n, n)-matrices $(\cong \boldsymbol{R}^{n^2})$ is isometric (Kobayashi [5]). On the other hand, in [1] Agaoka and Kaneda proved that $SO(n)$ cannot be isometrically immersed into $\boldsymbol{R}^{1/4\cdot(3n^2-3n-2[n/2])-1}$ even locally, by calculating the rank of the curvature transformation of $SO(n)$. This estimate is best possible in the cases $n=3$ and $n=4$ because $SO(3)$ and $SO(4)$ can be locally isometrically immersed into $\boldsymbol{R}^4$ and $\boldsymbol{R}^8$, respectively (see §1). But in the cases $n \geqq 5$, the estimate in [1] is not best possible. The first main purpose of this paper is to determine the least dimensional Euclidean space in which $SO(5)$ can be locally isometrically immersed.

If $SO(5)$ is locally isometrically immersed into some Euclidean space, then the curvature of $SO(5)$ satisfies the Gauss equation. We prove that in codimension 5 the Gauss equation of $SO(5)$ does not admit a solution and hence $SO(5)$ cannot be isometrically immersed into $\boldsymbol{R}^{15}$ even locally (Theorem 2. 1). On the other hand, it is known that the universal covering group $Spin(5)$ of $SO(5)$ is isomorphic to $Sp(2)$ and that $Sp(2)$ can be globally isometrically imbedded into $\boldsymbol{R}^{16}$ [5]. Therefore combining these two results, we know the best result on local isometric immersions of $SO(5)$ into the Euclidean spaces. As a corollary of Theorem 2. 1 we can prove that $SO(5)$ cannot be locally conformally immersed into $\boldsymbol{R}^{13}$.

The second main purpose of this paper is to prove the uniqueness of the solution of the Gauss equation of $SO(5)$ in codimension 6 (Theorem 2. 3). The Gauss equation is equivalent to a system of quadratic equations and hence it is in general difficult to prove the uniqueness of the solution of this equation. We prove Theorem 2. 3 by using elementary facts on the exterior algebra. But many calculations will be required (see §4).

As corollaries of this theorem we prove that any local isometric immersion of $SO(5)$ or $Sp(2)$ into $\boldsymbol{R}^{16}$ is uniquely determined up to the Euclidean transformation of $\boldsymbol{R}^{16}$ (Corollary 2.4) and that $SO(5)$ cannot be globally isometrically immersed into $\boldsymbol{R}^{16}$ (Corollary 2.5). Finally we prove that the non-compact dual space of $SO(5)$ cannot be locally isometrically immersed into $\boldsymbol{R}^{16}$ (Theorem 2.6).

In the case $n=6$ we can prove that $SO(6)$ cannot be locally isometrically immersed into $\boldsymbol{R}^{22}$, using a similar method developed in §3. But we do not know the least dimensional Euclidean space in which $SO(n)$ $(n\geqq 6)$ can be locally (or globally) isometrically immersed (see §5).

Throughout this paper we always assume the differentiability of class C^{∞}.

§ 1. Curvature of $SO(n)$ and the solutions of the Gauss equation.

We consider the Lie algebra $\mathfrak{o}(n)$ as a tangent space of $SO(n)$ at the identity element. We put $V=\mathfrak{o}(n)$ and $X_{ij}=E_{ij}-E_{ji}\in V$ $(i\neq j)$ where E_{ij} is the (n, n)-matrix such that the entry at the i-th row and the j-th column is 1 and other entries are all zero. Then $\{X_{ij}\}_{1\leqq i<j\leqq n}$ forms an orthonormal base of V with respect to a biinvariant Riemannian metric of $SO(n)$.

Since the Riemannian connection of $SO(n)$ is given by $\nabla_X Y=1/2\cdot[X, Y]$ for left invariant vector fields X and Y, the curvature $R\in\wedge^2 V^*\otimes V^*\otimes V$ of $SO(n)$ at the identity element is $\mathrm{R}(X, Y)Z=-1/4\cdot[[X, Y], Z]\in V$ for X, Y, $Z\in V$. Using the inner product, we may consider the curvature as a symmetric linear map R: $\wedge^2 V\longrightarrow\wedge^2 V$. Then by an easy calculation, the curvature transformation of $SO(n)$ is given by, up to a positive constant,

$$
(1.1)\qquad \begin{aligned} R(X_{ik}\wedge X_{il}) &= \sum_{p=1}^{n} X_{pk}\wedge X_{pl} \\ R(X_{ij}\wedge X_{kl}) &= 0 \end{aligned}
$$

for distinct i, j, k and l.

We assume that $SO(n)$ can be locally isometrically immersed into $\boldsymbol{R}^{1/2\cdot n(n-1)+N}$. We denote by $\boldsymbol{R}^N$ the normal space of $SO(n)$ at the identity element and by α: $V\times V\longrightarrow\boldsymbol{R}^N$ the second fundamental form of this immersion. Then α satisfies the Gauss equation $R(X, Y, Z, W)=\langle\alpha(X, Z), \alpha(Y, W)\rangle-\langle\alpha(X, W), \alpha(Y, Z)\rangle$ for X, Y, Z, $W\in V$. We fix an orthonormal base $\{\xi_i\}_{1\leqq i\leqq N}$ of $\boldsymbol{R}^N$ and define symmetric linear maps L_i: $V\longrightarrow V$ $(i=1, \cdots, N)$ by $\langle L_i(X), Y\rangle=\langle\alpha(X, Y), \xi_i\rangle$ for X, $Y\in V$. Then the above Gauss equation is equivalent to the equation $R=\sum_{i=1}^{N} L_i\wedge L_i$: $\wedge^2 V\longrightarrow\wedge^2 V$ (cf. [3]).

We now construct solutions of the Gauss equation of $SO(n)$ for $n\geqq 3$. We set $I=\{(i_1, i_2, i_3, i_4)\in\boldsymbol{Z}^4\,|\,0<i_1<i_2<i_3<i_4\leqq n\}$. For $(i)=(i_1, i_2, i_3, i_4)\in I$,

we define a linear map $L_{(i)}: V \longrightarrow V$ by

$$L_{(i)}(X_{i_p i_q}) = \begin{cases} 0 & \text{if } \{i_p, i_q\} \not\subset \{i_1, i_2, i_3, i_4\} \\ \varepsilon_{pqrs} X_{i_r i_s} & \text{if } \{i_p, i_q, i_r, i_s\} = \{i_1, i_2, i_3, i_4\} \end{cases}$$

where $\varepsilon_{pqrs} = \operatorname{sgn} \begin{pmatrix} 1 & 2 & 3 & 4 \\ p & q & r & s \end{pmatrix}$. Then we have

Proposition 1. 1. (1) *$L_{(i)}$ is a symmetric linear endomorphism of V for each $(i) \in I$.*

(2) *The equality*

$$R = id \wedge id + \sum_{(i) \in I} L_{(i)} \wedge L_{(i)} \tag{1.2}$$

holds.

Proof. We prove (2) only. The proof of (1) is immediate and left to the reader. We substitute the vector of the form $X_{pq} \wedge X_{rs} (p, q, r, s$ are all distinct) into (1.2). It is easy to see that $(i) = (i_1, i_2, i_3, i_4) \in I$ satisfies $L_{(i)}(X_{pq}) \wedge L_{(i)}(X_{rs}) \neq 0$ if and only if $\{i_1, \cdots, i_4\} = \{p, q, r, s\}$, and in this case $L_{(i)}(X_{pq}) = \varepsilon_{pqrs} X_{rs}$ and $L_{(i)}(X_{rs}) = \varepsilon_{rspq} X_{pq}$. Hence we have $L_{(i)}(X_{pq}) \wedge L_{(i)}(X_{rs}) = \varepsilon_{pqrs}{}^2 X_{rs} \wedge X_{pq} = -X_{pq} \wedge X_{rs}$. Therefore $(id \wedge id + \sum_{(i) \in I} L_{(i)} \wedge L_{(i)})(X_{pq} \wedge X_{rs}) = X_{pq} \wedge X_{rs} - X_{pq} \wedge X_{rs} = 0 = R(X_{pq} \wedge X_{rs})$. Next we substitute the vector of the form $X_{pq} \wedge X_{pr} (p, q, r$ are distinct) into (1.2). Then $(i) = (i_1, i_2, i_3, i_4) \in I$ satisfies $L_{(i)}(X_{pq}) \wedge L_{(i)}(X_{pr}) \neq 0$ if and only if $\{i_1, \cdots, i_4\} = \{p, q, r, s\}$ for some $s (\neq p, q, r)$. We fix such an (i). Then it holds $L_{(i)}(X_{pq}) = \varepsilon_{pqrs} X_{rs}$ and $L_{(i)}(X_{pr}) = \varepsilon_{prqs} X_{qs}$. Hence we have $L_{(i)}(X_{pq}) \wedge L_{(i)}(X_{pr}) = \varepsilon_{pqrs} \varepsilon_{prqs} X_{rs} \wedge X_{qs} = -X_{rs} \wedge X_{qs} = X_{sq} \wedge X_{sr}$. Therefore $(id \wedge id + \sum_{(i) \in I} L_{(i)} \wedge L_{(i)})(X_{pq} \wedge X_{pr}) = X_{pq} \wedge X_{pr} + \sum_{s \neq p, q, r} X_{sq} \wedge X_{sr} = \sum_{s=1}^{n} X_{sq} \wedge X_{sr} = R(X_{pq} \wedge X_{pr})$.

q. e. d.

Since $\# I = \binom{n}{4}$, this proposition implies that $SO(n)$ admits a solution of the Gauss equation (at one point of $SO(n)$) in codimension $\binom{n}{4} + 1$ $(n \geqq 3)$. We remark that $\binom{n}{4} + 1$ is smaller than the codimension of the canonical isometric imbedding of $SO(n)$ into $\boldsymbol{R}^{n^2}$ for $3 \leqq n \leqq 6$.

The spaces $SO(3)$ and $SO(4)$ are locally isometric to the sphere S^3 and the product of the spheres $S^3 \times S^3$ respectively and hence they can be locally isometrically immersed into $\boldsymbol{R}^4$ and $\boldsymbol{R}^8$. The universal covering space of $SO(5)$ is isometric to $Sp(2)$ and by Kobayashi [5], $Sp(2)$ can be globally isometrically imbedded into $\boldsymbol{R}^{16}$. Hence $SO(3)$, $SO(4)$ and $SO(5)$ admit solutions of the Gauss equation in codimension 1, 2 and 6, respectively. By an easy calculation we can verify that, if we choose a suitable orthonormal base of the normal space, these solutions coincide the ones constructed as above.

In the case $n = 5$, we rewrite the solution of the Gauss equation (1.2) in the form $\alpha: V\times V\longrightarrow \boldsymbol{R}^6$ ($V=\mathfrak{o}(5)$) for later use. Let $\{e_0, e_{1234}, e_{1235}, e_{1245}, e_{1345}, e_{2345}\}$ be an orthonormal base of $\boldsymbol{R}^6$ and we put $e_{ijkl}=\mathrm{sgn}\begin{pmatrix} i & j & k & l \\ p & q & r & s\end{pmatrix} e_{pqrs}$ for distinct i,j,k,l ($\leqq 5$). Then the solution (1.2) is expressed as:

$$\begin{aligned} \alpha(X_{ij}, X_{ij}) &= e_0 \\ \alpha(X_{ij}, X_{kl}) &= e_{ijkl} \\ \alpha(X_{ij}, X_{ik}) &= 0 \end{aligned} \tag{1.3}$$

for distinct i, j, k and l.

§ 2. Statement of results.

In this section we state the main results of this paper. The proofs of Theorems 2.1, 2.3 and 2.6 will be given in §3 and §4.

Theorem 2. 1. *Let* $R: \wedge^2 V\longrightarrow \wedge^2 V$ ($V=\mathfrak{o}(5)$) *be the curvature transformation of* $SO(5)$. *If there exist symmetric linear maps* L_i ($i=1, \cdots, k$) *such that* $R=\sum_{i=1}^{k}\varepsilon_i L_i\wedge L_i$ ($\varepsilon_i=1$ *or* -1), *then* $k\geqq 6$. *In particular* $SO(5)$ *cannot be isometrically immersed into* $\boldsymbol{R}^{15}$ *even locally.*

As a corollary of this theorem we have

Corollary 2. 2. $SO(5)$ *cannot be locally conformally immersed into* $\boldsymbol{R}^{13}$.

Proof. We assume that $SO(5)$ can be locally conformally immersed into $\boldsymbol{R}^{10+N}$. Then by a result of Moore [7], $SO(5)$ is locally isometrically immersed into $\boldsymbol{R}^{1,11+N}$, where $\boldsymbol{R}^{1,11+N}$ is the Minkowski space of signature $(-, \overbrace{+, +, \cdots, +}^{11+N})$. Let α be the second fundamental form of this isometric immersion and let $\{\xi_1, \cdots, \xi_{N+2}\}$ be a base of the normal space such that $\langle\xi_i, \xi_j\rangle=0$ ($i\neq j$) and $-\langle\xi_1, \xi_1\rangle=\langle\xi_2, \xi_2\rangle=\cdots=\langle\xi_{N+2}, \xi_{N+2}\rangle=1$. We define symmetric linear endomorphisms L_i ($i=1, \cdots, N+2$) of V by $\langle L_i(X), Y\rangle=\langle\alpha(X, Y), \xi_i\rangle$. Then the Gauss equation of this isometric immersion is expressed in the form: $R=-L_1\wedge L_1+\sum_{i=2}^{N+2}L_i\wedge L_i$. Hence by Theorem 2.1 we have $N+2\geqq 6$ and the corollary is proved. q. e. d.

Remark. Since the image of the canonical isometric imbedding of $Sp(2)$ into $\boldsymbol{R}^{16}$ (Kobayashi [5]) is contained in the sphere $S^{15}\subset\boldsymbol{R}^{16}$, $SO(5)$ can be locally conformally immersed into $\boldsymbol{R}^{15}$.

Theorem 2. 3. *A solution of the Gauss equation of* $SO(5)$ *in codimension 6 is unique up to the action of* $O(6)$ *on the normal space.*

Remark. The last statement in Theorem 2.1 follows immediately from this theorem. In fact if $SO(5)$ admits a solution α of the Gauss equation in codimension 5, then by the action of $O(6)$ it can be expressed in the form (1.3). Hence we have $\dim\{\alpha(X, Y)\mid X, Y\in V\}=6$, which contradicts the assumption.

As corollaries of Theorem 2.3 we have

Corollary 2.4. *Let U be a connected open Riemannian submanifold of $SO(5)$ or $Sp(2)$ and let f_1, f_2 be isometric immersions of U into $\boldsymbol{R}^{16}$. Then there exists a Euclidean transformation ϕ of $\boldsymbol{R}^{16}$ such that $\phi\circ f_1=f_2$.*

Proof. We denote by N_k the normal bundle of f_k and let $\alpha_k: TU\times TU \longrightarrow N_k$ be the second fundamental form of $f_k (k=1, 2)$. We define a bundle isomorphism $\Phi: N_1 \longrightarrow N_2$ by $\Phi\alpha_1(X, Y)=\alpha_2(X, Y)$ for X, $Y\in T_pU$ $(p\in U)$. We identify T_pU with V in a natural way. Then by the uniqueness of the solution of the Gauss equation, we may consider that α_1 and α_2 are expressed in the form (1.3). Hence Φ is well defined and Φ preserves the metrics and the second fundamental forms. Since vectors of the form $\alpha_1(X, Y)$ $(X, Y\in T_pU)$ span the normal space of f_1 for each $p\in U$, we can apply Theorem 2 in Nomizu [8] and obtain the desired result. q. e. d.

Remark. Since the canonical isometric imbedding of $Sp(2)$ into $\boldsymbol{R}^{16}$ is parallel (cf. [10]), the normal connection $\nabla^{\perp}$ of any isometric immersion of $U(\subset SO(5)$ or $Sp(2))$ into $\boldsymbol{R}^{16}$ is given by $\nabla_X^{\perp}\alpha(Y, Z)=\alpha(\nabla_XY, Z)+\alpha(Y, \nabla_XZ)=1/2\cdot\{\alpha([X, Y], Z)+\alpha(Y, [X, Z])\}$ for left invariant vector fields X, Y and Z.

Corollary 2.5. *$SO(5)$ cannot be globally isometrically immersed into $\boldsymbol{R}^{16}$.*

Proof. We assume that there exists a global isometric immersion $f: SO(5)\longrightarrow\boldsymbol{R}^{16}$. Then composing the double covering map $\pi: Sp(2)\longrightarrow SO(5)$, we have an isometric immersion $f\circ\pi$ of $Sp(2)$ into $\boldsymbol{R}^{16}$, which is not an imbedding. But by Corollary 2.4 this immersion must be congruent to the canonical isometric imbedding of $Sp(2)$, and hence a contradiction follows. q. e. d.

We remark that $SO(5)$ can be globally isometrically imbedded into $\boldsymbol{R}^{25}$, as stated in Introduction. But we do not know whether $SO(5)$ can be globally isometrically immersed into a lower dimensional Euclidean space.

For non-compact dual space of $SO(5)$, we have

Theorem 2.6. *Let $SO(5, \boldsymbol{C})/SO(5)$ be the non-compact dual space of $SO(5)$. Then $SO(5, \boldsymbol{C})/SO(5)$ cannot be locally isometrically immersed into $\boldsymbol{R}^{16}$.*

§3. Proof of Theorem 2.1.

In this section we prove Theorem 2.1. Let $X_1, \cdots, X_k, Y_1, \cdots, Y_k$ be elements of $V(=\mathfrak{o}(5))$. Then using the inner product of V, we consider

$\sum_{i=1}^{k} X_i \wedge Y_i \in \wedge^2 V$ as a skew symmetric linear endomorphism of V: $(\sum_{i=1}^{k} X_i \wedge Y_i) Z = \sum_{i=1}^{k} \{\langle X_i, Z\rangle Y_i - \langle Y_i, Z\rangle X_i\}$ for $Z \in V$. In the following we denote by $\mathrm{Im}(\sum_{i=1}^{k} X_i \wedge Y_i)$ the image of this linear map.

The following lemma is easy to verify (cf. p. 351 [6]).

Lemma 3. 1. *Let $X_1, \cdots, X_k, Y_1, \cdots, Y_k$ be elements of V.*

(1) *If $\{X_1, \cdots, X_k\}$ is linearly independent and $\sum_{i=1}^{k} X_i \wedge Y_i = 0$, then $Y_i \in \{X_1, \cdots, X_k\}$ for $i = 1, \cdots, k$.*

(2) *If* $\dim \{X_1, \cdots, X_k\} = k-1$ *and $\sum_{i=1}^{k} X_i \wedge Y_i = 0$, then there exists $Z \in V$ such that $Y_i \in \{X_1, \cdots, X_k, Z\}$ for $i = 1, \cdots, k$.*

Now we prove Theorem 2. 1. We assume that there exist symmetric linear maps $L_i : V \longrightarrow V$ $(i = 1, \cdots, 5)$ such that $R = \sum_{i=1}^{5} \varepsilon_i L_i \wedge L_i$ $(\varepsilon_i = 1$ or $-1)$. For $X \in V$ we define a subspace $V(X)$ of V by $V(X) = \{L_i(X)\}_{1 \leqq i \leqq 5}$. Then we have

$$(*) \qquad \dim V(X_{ij}) \leqq 4 \textit{ for any } X_{ij} \in V \ (i \neq j).$$

Proof. Suppose that $\dim V(X_{ij}) = 5$ for some X_{ij}. By the symmetry we may assume that $\dim V(X_{12}) = 5$. Then from the equation $0 = R(X_{12} \wedge X_{34}) = \sum_{i=1}^{5} \varepsilon_i L_i(X_{12}) \wedge L_i(X_{34})$ and from Lemma 3. 1 (1) we have $V(X_{34}) \subset V(X_{12})$. In the same way, from the equation $R(X_{12} \wedge X_{35}) = 0$ we have $V(X_{35}) \subset V(X_{12})$. On the other hand since $R(X_{34} \wedge X_{35}) = X_{14} \wedge X_{15} + X_{24} \wedge X_{25} + X_{34} \wedge X_{35} = \sum_{i=1}^{5} \varepsilon_i L_i(X_{34}) \wedge L_i(X_{35})$, we have $\mathrm{Im}\, R(X_{34} \wedge X_{35}) = \{X_{14}, X_{15}, X_{24}, X_{25}, X_{34}, X_{35}\} = \mathrm{Im}\, \{\sum_{i=1}^{5} \varepsilon_i L_i(X_{34}) \wedge L_i(X_{35})\} \subset V(X_{34}) + V(X_{35}) \subset V(X_{12})$. But this is impossible because $\dim \{X_{14}, \cdots, X_{35}\} = 6$ and $\dim V(X_{12}) = 5$.

q. e. d.

Next we prove

$$(**) \qquad \dim V(X_{ij}) \geqq 4 \textit{ for some } X_{ij} \in V.$$

Proof. We assume that $\dim V(X_{ij}) \leqq 3$ for all $X_{ij} \in V (i \neq j)$. From the equality $R(X_{12} \wedge X_{13}) = X_{12} \wedge X_{13} + X_{24} \wedge X_{34} + X_{25} \wedge X_{35} = \sum_{i=1}^{5} \varepsilon_i L_i(X_{12}) \wedge L_i(X_{13})$, we have $\mathrm{Im}\, R(X_{12} \wedge X_{13}) = \{X_{12}, X_{13}, X_{24}, X_{25}, X_{34}, X_{35}\} \subset V(X_{12}) + V(X_{13})$. But since $\dim V(X_{12}) \leqq 3$ and $\dim V(X_{13}) \leqq 3$, we have $\dim V(X_{12}) = \dim V(X_{13}) = 3$ and $\{X_{12}, \cdots, X_{35}\} = V(X_{12}) \oplus V(X_{13})$ (direct sum). In the same way, using the terms $R(X_{12} \wedge X_{14})$ and $R(X_{12} \wedge X_{15})$, we have $V(X_{12}) \oplus V(X_{14}) = \{X_{12}, X_{14}, X_{23}, X_{25}, X_{34}, X_{45}\}$ and $V(X_{12}) \oplus V(X_{15}) = \{X_{12}, X_{15}, X_{23}, X_{24}, X_{35}, X_{45}\}$. Hence we have $V(X_{12}) \subset \{X_{12}, X_{13}, X_{24}, X_{25}, X_{34}, X_{35}\} \cap \{X_{12}, X_{14}, X_{23}, X_{25}, X_{34}, X_{45}\} \cap \{X_{12}, X_{15}, X_{23}, X_{24}, X_{35}, X_{45}\} = \{X_{12}\}$, which contradicts $\dim V(X_{12}) = 3$. Therefore $\dim V(X_{ij}) \geqq 4$ for some $X_{ij} \in V$.

q. e. d.

From $(*)$ and $(**)$ we know that there exists some $X_{ij}\in V$ such that dim $V(X_{ij})=4$. By the symmetry we may assume that dim $V(X_{12})=4$. Since $R(X_{12}\wedge X_{34})=\sum_{i=1}^{5}\varepsilon_i L_i(X_{12})\wedge L_i(X_{34})=0$, there exists $Y_1\in V$ such that $V(X_{34})\subset V(X_{12})+\{Y_1\}$ (Lemma 3.1(2)). In the same way from the equation $R(X_{12}\wedge X_{35})=0$, there exists $Y_2\in V$ such that $V(X_{35})\subset V(X_{12})+\{Y_2\}$. Considering the image of the linear map $R(X_{34}\wedge X_{35})=X_{14}\wedge X_{15}+X_{24}\wedge X_{25}+X_{34}\wedge X_{35}$, we have $\{X_{14}, X_{15}, X_{24}, X_{25}, X_{34}, X_{35}\}\subset V(X_{34})+V(X_{35})\subset V(X_{12})+\{Y_1, Y_2\}$. Since dim $V(X_{12})=4$ and dim $\{X_{14},\cdots, X_{35}\}=6$, it follows that dim $\{V(X_{12})+\{Y_1, Y_2\}\}=6$ and $\{X_{14},\cdots, X_{35}\}=V(X_{12})\oplus\{Y_1\}\oplus\{Y_2\}$ (direct sum). In particular we have $V(X_{12})\subset\{X_{14},\cdots, X_{35}\}$. Similarly using the equalities $R(X_{12}\wedge X_{34})=R(X_{12}\wedge X_{45})=0$, we can prove that $V(X_{12})\subset\{X_{13}, X_{15}, X_{23}, X_{25}, X_{34}, X_{45}\}$. Hence we have $V(X_{12})\subset\{X_{14}, X_{15}, X_{24}, X_{25}, X_{34}, X_{35}\}\cap\{X_{13}, X_{15}, X_{23}, X_{25}, X_{34}, X_{45}\}=\{X_{15}, X_{25}, X_{34}\}$. But this is a contradiction because dim $V(X_{12})=4$. Therefore the curvature R of $SO(5)$ cannot be expressed in the form $R=\sum_{i=1}^{5}\varepsilon_i L_i\wedge L_i$ and we complete the proof of Theorem 2.1. q. e. d.

§4. Proof of Theorems 2. 3 and 2. 6.

In this section we prove Theorems 2.3 and 2.6 in parallel. The proof will be divided into several steps.

We denote by R the curvature transformation of $SO(5)$, as in the previous sections. Then the curvature of the non-compact dual space $SO(5, \boldsymbol{C})/SO(5)$ is given by $-R$. Now we assume that there exist symmetric linear maps L_i: $V\longrightarrow V$ $(i=1,\cdots, 6)$ satisfying the Gauss equation $\varepsilon R=\sum_{i=1}^{6}L_i\wedge L_i$ $(\varepsilon=1$ or $-1)$. For $X\in V$ we define a subspace $V(X)$ of V by $V(X)=\{L_i(X)\}_{1\leqq i\leqq 6}$, as in §3.

The following lemma is easy to prove.

Lemma 4. 1. *Let* $X_1,\cdots, X_k, Y_1,\cdots, Y_k$ *be elements of* V. *If* dim Im $(\sum_{i=1}^{k}X_i\wedge Y_i)\geqq 2m$, *then* dim $\{X_1,\cdots, X_k\}\geqq m$ *and* dim $\{Y_1,\cdots, Y_k\}\geqq m$.

We prove

Lemma 4. 2. dim $V(X_{ij})=4$ *for all* $X_{ij}\in V$ $(i\neq j)$.

Proof. Since $R(X_{12}\wedge(X_{13}+2X_{24}))=X_{12}\wedge X_{13}+X_{24}\wedge X_{34}+X_{25}\wedge X_{35}+2X_{12}\wedge X_{24}+2X_{13}\wedge X_{34}-2X_{15}\wedge X_{45}$, we have dim Im $R(X_{12}\wedge(X_{13}+2X_{24}))=8$. Then from the equation $\varepsilon R(X_{12}\wedge(X_{13}+2X_{24}))=\sum_{i=1}^{6}L_i(X_{12})\wedge L_i(X_{13}+2X_{24})$, we have dim $V(X_{12})\geqq 4$ (Lemma 4.1). By the symmetry we conclude that dim $V(X_{ij})\geqq 4$ for all $X_{ij}\in V$ $(i\neq j)$. Next we assume that dim V

$(X_{ij})\geqq 5$ for some $X_{ij}\in V$. Then by the symmetry we may assume dim $V(X_{12})\geqq 5$. Since $\varepsilon R(X_{12}\wedge X_{34})=\sum_{i=1}^{6}L_i(X_{12})\wedge L_i(X_{34})=0$, there exists a vector $Y_1\in V$ such that $V(X_{34})\subset V(X_{12})+\{Y_1\}$ (Lemma 3.1). Similarly using the equality $R(X_{12}\wedge X_{35})=R(X_{12}\wedge X_{45})=0$, we have $V(X_{35})\subset V(X_{12})+\{Y_2\}$ and $V(X_{45})\subset V(X_{12})+\{Y_3\}$ for some vectors Y_2, $Y_3\in V$. Then by the equation $\varepsilon R(X_{34}\wedge X_{35})=\varepsilon(X_{14}\wedge X_{15}+X_{24}\wedge X_{25}+X_{34}\wedge X_{35})=\sum_{i=1}^{6}L_i(X_{34})\wedge L_i(X_{35})$, we have $\{X_{14}, X_{15}, X_{24}, X_{25}, X_{34}, X_{35}\}\subset V(X_{34})+V(X_{35})\subset V(X_{12})+\{Y_1\}+\{Y_2\}$. Now we prove that there exists $Z_1\in V$ such that

$$V(X_{12})\subset\{X_{14}, X_{15}, X_{24}, X_{25}, X_{34}, X_{35}, Z_1\}. \tag{4.1}$$

If dim $V(X_{12})=6$, then we may put $Y_1=Y_2=0$ and hence $V(X_{12})=\{X_{14}, \cdots, X_{35}\}$ because dim $\{X_{14}, \cdots, X_{35}\}=6$. Hence if we set $Z_1=0$, (4.1) holds. If dim $V(X_{12})=5$, then the dimension of the space $V(X_{12})+\{Y_1\}+\{Y_2\}$ is at most 7. Hence there exists $Z_1\in V$ such that $\{X_{14}, \cdots, X_{35}, Z_1\}=V(X_{12})+\{Y_1\}+\{Y_2\}$ and in particular (4.1) holds. In the same way using the equation $\varepsilon R(X_{34}\wedge X_{45})=\sum_{i=1}^{6}L_i(X_{34})\wedge L_i(X_{45})=\varepsilon(-X_{13}\wedge X_{15}-X_{23}\wedge X_{25}+X_{34}\wedge X_{45})$, we can prove that there exists $Z_2\in V$ such that

$$V(X_{12})\subset\{X_{13}, X_{15}, X_{23}, X_{25}, X_{34}, X_{45}, Z_2\}. \tag{4.2}$$

We assume that dim $V(X_{12})=6$, then we may put $Z_1=Z_2=0$, and hence $V(X_{12})=\{X_{14}, \cdots, X_{35}\}=\{X_{13}, \cdots, X_{45}\}$, which is a contradiction. Hence we have dim $V(X_{12})=5$. Now we modify Z_1 and Z_2 such that Z_1 and Z_2 are contained in the orthogonal complements of $\{X_{14}, \cdots, X_{35}\}$ and $\{X_{13}, \cdots, X_{45}\}$, respectively. We assume that $Z_1\notin\{X_{13}, \cdots, X_{45}\}$. Then it can be easily proved that $\{X_{14}, \cdots, X_{35}, Z_1\}\cap\{X_{13}, \cdots, X_{45}\}=\{X_{15}, X_{25}, X_{34}\}$. Hence dim $\{X_{14}, \cdots, X_{35}, Z_1\}\cap\{X_{13}, \cdots, X_{45}, Z_2\}\leqq 4$, which contradicts (4.1), (4.2) and dim $V(X_{12})=5$. Therefore $Z_1\in\{X_{13}, \cdots, X_{45}\}$. Similarly we can prove that $Z_2\in\{X_{14}, \cdots, X_{35}\}$ and hence we have $V(X_{12})=\{X_{15}, X_{25}, X_{34}, Z_1, Z_2\}$. In particular $\{X_{15}, X_{25}, X_{34}\}\subset V(X_{12})$. Next using the equation $\varepsilon R(X_{35}\wedge X_{45})=\sum_{i=1}^{6}L_i(X_{35})\wedge L_i(X_{45})=\varepsilon(X_{13}\wedge X_{14}+X_{23}\wedge X_{24}+X_{35}\wedge X_{45})$, we can prove in the same way as above that there exists $Z_3\in V$ such that

$$V(X_{12})\subset\{X_{13}, X_{14}, X_{23}, X_{24}, X_{35}, X_{45}, Z_3\}. \tag{4.3}$$

Then from (4.1) and (4.3), we have $\{X_{14}, X_{24}, X_{35}\}\subset V(X_{12})$. Therefore combining the above results, we have $\{X_{15}, X_{25}, X_{34}\}\oplus\{X_{14}, X_{24}, X_{35}\}\subset V(X_{12})$. But this is impossible because dim $V(X_{12})=5$. Hence if we assume dim $V(X_{12})\geqq 5$, we obtain a contradiction. Therefore dim $V(X_{12})=4$. By the symmetry we conclude that dim $V(X_{ij})=4$ for all $X_{ij}\in V$. q. e. d.

We express the solution of the Gauss equation $\varepsilon R=\sum_{i=1}^{6}L_i\wedge L_i$ in the form $\alpha: V\times V\longrightarrow \boldsymbol{R}^6$. Then it can be easily verified that $V(X)^{\perp}=\{Y\in$

$V|\alpha(X, Y)=0\}$ for any $X\in V$. We prove

Lemma 4. 3. *$\alpha(X_{ij}, X_{ik})=0$ for distinct i, j and k.*

Proof. Let $\{v_1, \cdots, v_4\}$ be a base of $V(X_{12})$. Then we have $v_1\wedge\cdots\wedge v_4\wedge R(X_{12}\wedge Y)=0\in\wedge^6 V$ for any $Y\in V$ because $\varepsilon R(X_{12}\wedge Y)=\sum_{i=1}^{6} L_i(X_{12})\wedge L_i(Y)$. We prove that $\Phi\in\wedge^4 V$ satisfies $\Phi\wedge R(X_{12}\wedge Y)=0$ for all $Y\in V$ if and only if Φ is a constant multiple of $X_{12}\wedge X_{34}\wedge X_{35}\wedge X_{45}$. We express Φ in the form $\Phi=\sum a_{(i_1,i_2)\cdots(i_7,i_8)}X_{i_1i_2}\wedge\cdots\wedge X_{i_7i_8}\in\wedge^4 V$ $(i_1<i_2, \cdots, i_7<i_8)$. Since $R(X_{12}\wedge X_{13})=X_{12}\wedge X_{13}+X_{24}\wedge X_{34}+X_{25}\wedge X_{35}$ and $\Phi\wedge R(X_{12}\wedge X_{13})=0$, we have $\Phi\wedge R(X_{12}\wedge X_{13})\wedge X_{24}\wedge X_{25}=\Phi\wedge X_{12}\wedge X_{13}\wedge X_{24}\wedge X_{25}=0$. Hence if the indices $(i_1, i_2), \cdots, (i_7, i_8)$ are all contained in the set $\{(1, 4), (1, 5), (2, 3), (3, 4), (3, 5), (4, 5)\}$, then $a_{(i_1,i_2)\cdots(i_7,i_8)}=0$. We change $Y=X_{13}$ and $X_{24}\wedge X_{25}$ to other elements and repeat the same procedure as above. Then we know that most of the coefficients $a_{(i_1,i_2)\cdots(i_7,i_8)}$ of Φ are zero and finally it follows that Φ is contained in the 9-dimensional subspace $\{X_{12}\wedge X_{13}\wedge X_{14}\wedge X_{15}, X_{12}\wedge X_{23}\wedge X_{24}\wedge X_{25}, X_{13}\wedge X_{23}\wedge X_{34}\wedge X_{35}, X_{14}\wedge X_{24}\wedge X_{34}\wedge X_{45}, X_{15}\wedge X_{25}\wedge X_{35}\wedge X_{45}, X_{12}\wedge X_{34}\wedge X_{35}\wedge X_{45}, X_{12}\wedge X_{13}\wedge X_{23}\wedge X_{45}, X_{12}\wedge X_{14}\wedge X_{24}\wedge X_{35}, X_{12}\wedge X_{15}\wedge X_{25}\wedge X_{34}\}$ of $\wedge^4 V$. We express Φ as a linear combination of these 9 vectors and once we substitute Φ into the equations $\Phi\wedge R(X_{12}\wedge Y)=0$. Then it can be directly verified that Φ is contained in the 1-dimensional subspace $\{X_{12}\wedge X_{34}\wedge X_{35}\wedge X_{45}\}$ of $\wedge^4 V$. Therefore we know that $\{X_{12}, X_{34}, X_{35}, X_{45}\}$ is the base of $V(X_{12})$. Then by the symmetry we conclude that $V(X_{ij})=\{X_{ij}, X_{kl}, X_{kp}, X_{lp}\}$ for distinct integers i, j, k, l and p. Hence as remarked above, $\{Y\in V|\alpha(X_{ij}, Y)=0\}=V(X_{ij})^{\perp}=\{X_{ik}, X_{il}, X_{ip}, X_{jk}, X_{jl}, X_{jp}\}$. In particular we have $\alpha(X_{ij}, X_{ik})=0$. q. e. d.

Next we fix an orthonormal base of the normal space in the following way. We first remark that $\alpha(X_{ij}, X_{ij})$, $\alpha(X_{ij}, X_{kl})\neq 0$ for distinct i, j, k and l because $\varepsilon R(X_{ij}, X_{ik}, X_{ij}, X_{ik})=\langle\alpha(X_{ij}, X_{ij}), \alpha(X_{ik}, X_{ik})\rangle-\|\alpha(X_{ij}, X_{ik})\|^2=\langle\alpha(X_{ij}, X_{ij}), \alpha(X_{ik}, X_{ik})\rangle=1$ and $\varepsilon R(X_{ij}, X_{ik}, X_{jl}, X_{kl})=\langle\alpha(X_{ij}, X_{jl}), \alpha(X_{ik}, X_{kl})\rangle-\langle\alpha(X_{ij}, X_{kl}), \alpha(X_{ik}, X_{jl})\rangle=-\langle\alpha(X_{ij}, X_{kl}), \alpha(X_{ik}, X_{jl})\rangle=1$. Then using the Gauss equation directly, we can prove that the 6 vectors $\alpha(X_{12}, X_{12})$, $\alpha(X_{13}, X_{24})$, $\alpha(X_{15}, X_{23})$, $\alpha(X_{14}, X_{25})$, $\alpha(X_{13}, X_{45})$, $\alpha(X_{24}, X_{35})$ are orthogonal to each other. For example $\alpha(X_{12}, X_{12})\perp\alpha(X_{13}, X_{24})$ follows from the equation $\varepsilon R(X_{12}, X_{13}, X_{12}, X_{24})=\langle\alpha(X_{12}, X_{12}), \alpha(X_{13}, X_{24})\rangle-\langle\alpha(X_{12}, X_{24}), \alpha(X_{12}, X_{13})\rangle=\langle\alpha(X_{12}, X_{12}), \alpha(X_{13}, X_{24})\rangle=0$ and $\alpha(X_{13}, X_{24})\perp\alpha(X_{15}, X_{23})$ follows from the equation $\varepsilon R(X_{13}, X_{15}, X_{24}, X_{23})=\langle\alpha(X_{13}, X_{24}), \alpha(X_{15}, X_{23})\rangle=0$. Hence $\{\alpha(X_{12}, X_{12}), \cdots, \alpha(X_{24}, X_{35})\}$ forms an orthogonal base of the normal space $\boldsymbol{R}^6$. Let $\{e_0, e_{1234}, e_{1235}, e_{1245}, e_{1345}, e_{2345}\}$ be an orthonormal base of $\boldsymbol{R}^6$ such that

$$\begin{aligned}(4.4)\qquad &\alpha(X_{12}, X_{12})/\!/e_0,\ \alpha(X_{13}, X_{24})/\!/e_{1234},\ \alpha(X_{15}, X_{23})/\!/e_{1235}\\ &\alpha(X_{14}, X_{25})/\!/e_{1245},\ \alpha(X_{13}, X_{45})/\!/e_{1345},\ \alpha(X_{24}, X_{35})/\!/e_{2345}.\end{aligned}$$

We put $e_{ijkl}=\mathrm{sgn}\begin{pmatrix} p & q & r & s \\ i & j & k & l \end{pmatrix} e_{pqrs}$, as in §1. Using Lemma 4.3 and the Gauss equation in full detail, we prove

Lemma 4.4. *$\alpha(X_{ij}, X_{ij}) /\!/ e_0$ and $\alpha(X_{ij}, X_{kl}) /\!/ e_{ijkl}$ for distinct i, j, k and l.*

Proof. We first prove that $\alpha(X_{13}, X_{25}) /\!/ e_{1235}$. From the equalities $R(X_{12}, X_{13}, X_{12}, X_{25}) = R(X_{13}, X_{25}, X_{24}, X_{13}) = R(X_{14}, X_{25}, X_{25}, X_{13}) = R(X_{13}, X_{25}, X_{45}, X_{13}) = R(X_{24}, X_{13}, X_{35}, X_{25}) = 0$ and Lemma 4.3, we have easily $\alpha(X_{13}, X_{25}) \perp e_0, e_{1234}, e_{1245}, e_{1345}, e_{2345}$. Since the dimension of the normal space is 6, we have $\alpha(X_{13}, X_{25}) /\!/ e_{1235}$. In the same way, using the Gauss equation and Lemma 4.3, we can prove that

$$
\begin{aligned}
&\alpha(X_{14}, X_{23}) /\!/ e_{1234},\ \alpha(X_{14}, X_{35}) /\!/ e_{1345},\ \alpha(X_{15}, X_{24}) /\!/ e_{1245} \\
(4.5)\qquad &\alpha(X_{15}, X_{34}) /\!/ e_{1345},\ \alpha(X_{23}, X_{45}) /\!/ e_{2345},\ \alpha(X_{25}, X_{34}) /\!/ e_{2345} \\
&\alpha(X_{34}, X_{34}) /\!/ e_0.
\end{aligned}
$$

Next we use (4.4), (4.5), $\alpha(X_{13}, X_{25}) /\!/ e_{1235}$ and the Gauss equation once again. Then we obtain $\alpha(X_{12}, X_{35}) /\!/ e_{1235}$, $\alpha(X_{12}, X_{45}) /\!/ e_{1245}$ and $\alpha(X_{ij}, X_{ij}) /\!/ e_0$ $(i \neq j)$. For example from the equalities $R(X_{34}, X_{35}, X_{34}, X_{12}) = R(X_{13}, X_{12}, X_{24}, X_{35}) = R(X_{14}, X_{12}, X_{25}, X_{35}) = R(X_{13}, X_{35}, X_{45}, X_{12}) = R(X_{24}, X_{35}, X_{35}, X_{12}) = 0$ we have $\alpha(X_{12}, X_{35}) \perp e_0, e_{1234}, e_{1245}, e_{1345}, e_{2345}$ and hence $\alpha(X_{12}, X_{35}) /\!/ e_{1235}$. Also from the equalities $R(X_{14}, X_{13}, X_{23}, X_{13}) = R(X_{15}, X_{13}, X_{23}, X_{13}) = R(X_{14}, X_{13}, X_{25}, X_{13}) = R(X_{14}, X_{13}, X_{35}, X_{13}) = R(X_{24}, X_{13}, X_{35}, X_{13}) = 0$, we have $\alpha(X_{13}, X_{13}) /\!/ e_0$. Finally $\alpha(X_{12}, X_{34}) /\!/ e_{1234}$ can be proved in the same way, using the equalities $R(X_{13}, X_{12}, X_{13}, X_{34}) = R(X_{15}, X_{12}, X_{23}, X_{34}) = R(X_{14}, X_{12}, X_{25}, X_{34}) = R(X_{13}, X_{34}, X_{45}, X_{12}) = R(X_{24}, X_{34}, X_{35}, X_{12}) = 0$. Therefore we have $\alpha(X_{ij}, X_{ij}) /\!/ e_0$ and $\alpha(X_{ij}, X_{kl}) /\!/ e_{ijkl}$. q. e. d.

Now we prove Theorems 2.3 and 2.6. We put $\alpha(X_{ij}, X_{ij}) = a_{ij} e_0$ $(a_{ij} = a_{ji})$ and $\alpha(X_{ij}, X_{kl}) = b_{ijkl} e_{ijkl}$ $(b_{ijkl} = b_{jikl} = b_{ijlk} = b_{klij})$ for distinct i, j, k and l. We first assume that α satisfies the Gauss equation of $SO(5)$: $R(X, Y, Z, W) = \langle \alpha(X, Z), \alpha(Y, W) \rangle - \langle \alpha(X, W), \alpha(Y, Z) \rangle$. Then from the equation $R(X_{ij}, X_{ik}, X_{ij}, X_{ik}) = \langle \alpha(X_{ij}, X_{ij}), \alpha(X_{ik}, X_{ik}) \rangle = 1$, we have $a_{ij} a_{ik} = 1$. Then $1 = a_{ij} a_{ik} = a_{ij} a_{il}$ and we have $a_{ik} = a_{il}$ for distinct i, k, l and hence $a_{ik} a_{il} = a_{ik}^2 = 1$. Considering the action of $O(6)$ on the normal space, we may put $a_{ij} = 1$ for all i, j $(i \neq j)$. Next, from the equalities $R(X_{13}, X_{14}, X_{23}, X_{24}) = -R(X_{12}, X_{14}, X_{23}, X_{34}) = R(X_{12}, X_{13}, X_{24}, X_{34}) = 1$, we have $b_{1324} b_{1423} = b_{1234} b_{1423} = b_{1234} b_{1324} = 1$. Hence we have $b_{1234} = b_{1324} = b_{1423} = \pm 1$. By the action of $O(6)$ on the normal space we may set $b_{1234} = b_{1324} = b_{1423} = 1$. In the same way we have $b_{ijkl} = 1$ for distinct i, j, k and l. Then the solution α, which we obtain in this way, just coincides the one constructed at the end of §1 and therefore we complete the proof of Theorem 2.3. Next we assume that α satisfies the Gauss equation of $SO(5, \boldsymbol{C})/SO(5)$: $-R(X, Y, Z, W) = \langle \alpha(X, Z), \alpha(Y, W) \rangle - \langle \alpha(X, W), \alpha(Y, Z) \rangle$. Then from the equality $-R(X_{ij}, X_{ik}, X_{ij}, X_{ik})=\langle\alpha(X_{ij}, X_{ij}), \alpha(X_{ik}, X_{ik})\rangle=-1$, we have $a_{ij}a_{ik}=-1$. Hence in the same way as above we obtain $a_{ik}^2=-1$, which does not admit a real solution. Hence the Gauss equation of $SO(5, \boldsymbol{C})/SO(5)$ does not admit a real solution in codimension 6 and therefore $SO(5, \boldsymbol{C})/SO(5)$ cannot be isometrically immersed into $\boldsymbol{R}^{16}$ even locally.

q. e. d.

§ 5. Final remarks.

Let $\alpha : V\times V\longrightarrow\boldsymbol{R}^6$ $(V=\mathfrak{o}(5))$ be the solution of the Gauss equation of $SO(5)$ in codimension 6. Using an element $g\in SO(5)$, we define a new symmetric bi-linear map α_g: $V\times V\longrightarrow\boldsymbol{R}^6$ by $\alpha_g(X, Y)=\alpha(\mathrm{Ad}(g)\cdot X, \mathrm{Ad}(g)\cdot Y)$ for $X, Y\in V$. Then it can be easily verified that α_g is also a solution of the Gauss equation. Hence by Theorem 2. 3 there exists a Lie group homomorphism ρ: $SO(5)\longrightarrow O(6)$ such that $\alpha_g(X, Y)=\rho(g)\cdot\alpha(X, Y)$ for $X, Y\in V$ and $g\in SO(5)$. We differentiate this equality. Then we have

$$(5.1)\quad \alpha([X, Y], Z)+\alpha(Y, [X, Z])=\rho(X)\cdot\alpha(Y, Z) \quad \text{for } X, Y, Z\in V,$$

where ρ: $\mathfrak{o}(5)\longrightarrow\mathfrak{o}(6)$ is the differential of ρ: $SO(5)\longrightarrow O(6)$. By an easy calculation, we can prove that ρ is equivalent to a sum of the 1-dimensional trivial representation and the identity representation of $\mathfrak{o}(5)$.

The solution of the Gauss equation of $SO(3)$ (resp. $SO(4)$) in codimension 1 (resp. 2) is unique and hence it also satisfies (5. 1), where ρ is a trivial representation in this case. Therefore in the cases $n=3$, 4 and 5, the least codimensional solution of the Gauss equation of $SO(n)$ satisfies the condition (5. 1) for some representation ρ of $\mathfrak{o}(n)$.

In the case $n=6$, we can prove that in codimension $\leqq 15$ there does not exist a solution of the Gauss equation of $SO(6)$ satisfying the condition (5. 1). On the other hand the solution of the Gauss equation which we construct in §1 satisfies (5. 1) for any $n(\geqq 3)$, where ρ is a sum of the 1-dimensional trivial representation and the irreducible representation of degree $\binom{n}{4}$. In particular $SO(6)$ admits a solution of the Gauss equation in codimention 16 which satisfies the condition (5. 1). We can also prove that $SO(6)$ does not admit a solution in codimension $\leqq 7$ by a similar method as in §3. But at the present time we know neither the least codimension in which the Gauss equation of $SO(6)$ admits a solution nor the least dimensional Euclidean space in which $SO(6)$ can be locally isometrically immersed. (We remark that the double covering space of $SO(6)$ is isometric to $SU(4)$ and $SU(4)$ can be globally isometrically imbedded into $\boldsymbol{R}^{32}$ [5].)

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