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Characterization of the separativity of ultradifferentiable classes*

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§ 1. Introduction.

Recently, the problems in the mathematics are often considered in the Gevrey classes and sometimes in more general "ultradifferentiable classes". Here, we say that a linear subspace of $C^\infty(\Omega)$ $[\Omega \subseteq \boldsymbol{R}^l$, open$]$ is the ultradifferentiable class with weight $\{M_n\}_{n=0}^\infty$, when each element $f(x)$ satisfies the following condition;

$\forall K$: compact set in Ω, $\exists R>0$, $\exists C>0$ depending on K and $f(x)$, such that,

$$|f^{(\alpha)}(x)| \leqq CR^{|\alpha|} M_{|\alpha|} \quad \text{on } K \quad \text{for } \forall \alpha \in \boldsymbol{Z}_+^l,$$

$$\Big[\, f^{(\alpha)}(x) = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots\cdots \left(\frac{\partial}{\partial x_l}\right)^{\alpha_l} f(x), \qquad |\alpha| = \alpha_1 + \cdots\cdots + \alpha_l, \text{ and}$$
$$\boldsymbol{Z}_+ = \{0, 1, 2, \cdots\cdots\} \,\Big].$$

We write it $C\{M_n\}(\Omega)\,[=\mathscr{E}\{M_n\}(\Omega)]$. We call especially $C\{n!^\upsilon\}$ $(\upsilon>0)$ the Gevrey class of order υ.

Considering the problems in $C\{M_n\}(\Omega)$, sometimes the following condition is assumed:

$$\text{(S)} \qquad \exists R>1, \quad M_{n+m} \leqq R^{n+m} M_n M_m \quad \text{for } \forall m,\ n \in \boldsymbol{Z}_+.$$

We call it "the separativity condition".

It is sometimes called "stability under ultradifferential operators". However, it seems too long. The reason of our naming originates from the following conclusion of (S);

For the open set Ω_i in $\boldsymbol{R}^{l_i}$ (i=1, 2), we set

$C\{M_n, \bar{M}_n\}(\Omega_1 \times \Omega_2) = \{f(x, y) \in C^\infty(\Omega_{1x} \times \Omega_{2y})$; $\forall K_1$, $\forall K_2$; compacts in Ω_1 and Ω_2, respectively, $\exists C>0$, $\exists \boldsymbol{R}>0$ such that

$|f^{(\alpha_1, \alpha_2)}(x, y)| \leqq C R^{|\alpha_1|+|\alpha_2|} M_{|\alpha_1|} \bar{M}_{|\alpha_2|}$ on $K_1 \times K_2$ for $\forall \alpha_i \in \boldsymbol{Z}_+^{l_i}$ $(i=1, 2)\}$.

If $f(x, y)$ in $C\{M_n\}(\Omega_1 \times \Omega_2)$ is separated in the form $f(x, y) = f_1(x) f_2(y)$, it belongs also to

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$C\{M_n, M_n\}(\Omega_1\times\Omega_2)$. However, if $\{M_n\}$ satisfies the condition (S), all $f(x, y)$ in $C\{M_n\}(\Omega_1\times\Omega_2)$ belong to $C\{M_n, M_n\}(\Omega_1\times\Omega_2)$ regardless of the form.

In this paper, we offer some equivalent conditions to the separativity condition or to a weaker one which will be introduced later on. Further, we try to characterize them by the order of $\{M_n\}$.

In order to make clear the inference, we consider the spaces $\mathscr{B}\{M_n\}(\boldsymbol{R}^l)$ and $\mathscr{B}\{M_n, \bar{M}_n\}(\boldsymbol{R}^{l_1}\times\boldsymbol{R}^{l_2})$ instead of $C\{M_n\}(\Omega)$ and $C\{M_n, \bar{M}_n\}(\Omega_1\times\Omega_2)$, where

$\mathscr{B}\{M_n\}(\boldsymbol{R}^l)=\{f(x)\in\mathscr{B}(\boldsymbol{R}^l);\ ^{\exists}C>0,\ ^{\exists}R>0$ depending on $f(x)$ such that

$$|f^{(\alpha)}(x)|\leqq CR^{|\alpha|}M_{|\alpha|}\quad\text{in}\quad \boldsymbol{R}^l\quad\text{for}\ ^{\forall}\alpha\in\boldsymbol{Z}_+^l\},$$

and

$$\begin{aligned}&\mathscr{B}\{M_n, \bar{M}_n\}(\boldsymbol{R}^{l_1}\times\boldsymbol{R}^{l_2})=\{f(x,y)\in\mathscr{B}(\boldsymbol{R}^{l_1+l_2});\\ &^{\exists}C>0,\ ^{\exists}R>0\ \text{depending on}\ f(x,y)\ \text{such that}\\ &|f^{(\alpha_1,\alpha_2)}(x,y)|\leqq CR^{|\alpha_1|+|\alpha_2|}M_{|\alpha_1|}\bar{M}_{|\alpha_2|}\ \text{in}\ \boldsymbol{R}^{l_1+l_2}\\ &\qquad\text{for}\ ^{\forall}\alpha_i\in\boldsymbol{Z}_+{}^{l_i}\ (i=1,2)\}.\end{aligned}$$

Theorem. (Kolmogoroff)

On the function $f(x)$ which is n-times differentiable on $\boldsymbol{R}^l$, we have the estimate;

If $M_0=\sup_x|f(x)|$ and $M_n=\sup_{x,|\alpha|=n}|f^{(\alpha)}(x)|$ are finite, $M_k=\sup_{x,|\alpha|=k}|f^{(\alpha)}(x)|$ is also finite and majorized as follows:

$$M_k\leqq(\pi/2)^l(M_0)^{1-(k/n)}(M_n)^{k/n}\quad(k=1, 2, \cdots, n-1).$$

The proof was given in A. Kolmogoroff [1]. (See also S. Mandelbrojt [2].) By this theorem, in case of $\mathscr{B}\{M_n\}(\boldsymbol{R}^l)$ and $\mathscr{B}\{M_n, \bar{M}_n\}(\boldsymbol{R}^{l_1}\times\boldsymbol{R}^{l_2})$, the sequences $\{M_n\}$ and $\{\bar{M}_n\}$ can be replaced to logarithmicly convex ones keeping the classes.

Remark. If $\{M_n\}$ satisfies

(1.1) $\qquad cn\leqq M_{n+1}/M_n\quad(n\gg 1)\quad$ for a positive number c,

also in case of $C\{M_n\}(\Omega)$ and $C\{M_n, \bar{M}_n\}(\Omega_1\times\Omega_2)$, the sequences $\{M_n\}$ and $\{\bar{M}_n\}$ can be replaced to logarithmicly convex ones keeping the classes. (See S. Mandelbrojt [2].) Then, if $\{M_n\}$ is originally logarithmicly convex or it satisfies (1.1), all of the results in this paper are valid for $C\{M_n\}(\Omega_1\times\Omega_2)$.

From now on, we assume that $\{M_n\}$ is logarithmicly convex. Further, we assume that $\lim_{n\to\infty}(M_n)^{1/n}=\infty$, because when $\lim_{n\to\infty}(M_n)^{1/n}<\infty$, the class $\mathscr{B}\{M_n\}(\boldsymbol{R}^l)$ is well characterized. (See, S. Mandelbrojt [2].) In conclusion, we may assume also that $\{(\log M_n)/n\}$ is increasing and diverging.

For the purpose of the application to the theory of pseudodifferential operators on the ultradifferentiable classes, we introduce two weaker notions.

We say that $\{M_n\}$ satisfies the weak separativity condition or the differentiability condition, according as the following holds:

$$\text{(W. S)} \quad \exists R>0,\ \exists \{N_m\}\ (N_m>0),\quad M_{n+m} \leqq R^n M_n N_m \quad \text{for } \forall n,\ m\in \boldsymbol{Z}_+,$$

or

$$\text{(D)} \qquad \exists C>0, \quad M_{n+1} \leqq C M_n \quad \text{for } \forall n\in \boldsymbol{Z}_+.$$

Remark. $\{N_m\}$ in (W. S) is not necessarily logarithmicly convex, but it can be replaced to a logarithmicly convex sequence.

Corresponding to the notions on $\{M_n\}$, we say that $\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2})$ is separative, weakly separative or differentiable according as

$$\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2}) \subseteqq \mathscr{B}\{M_n, M_n\}(\boldsymbol{R}^{l_1}\times\boldsymbol{R}^{l_2}),$$

$$\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2}) \subseteqq \mathscr{B}\{M_n, N_n\}(\boldsymbol{R}^{l_1}\times\boldsymbol{R}^{l_2}) \text{ for a suitable sequence } \{N_n\},$$

or $$\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2}) \subseteqq \mathscr{B}^1_y(\boldsymbol{R}^{l_2};\ \mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1}_x)),$$

respectively.

In the section 2, we announce the theorems. Since the differentiability of $\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2})$ was well characterized (See S. Mandelbrojt [2]), we only prove the theorems on the separativity and on the weak separativity in the sections 3, 4 and 5. We essentially follow the proof in case of the differentiability.

In the forthcoming paper, we will apply the results in this paper to show the impossibility of the "nice" theory of pseudodifferential operators on the ultradifferentiable classes larger than the Gevrey classes.

§ 2. Notation and results.

In order to describe the theorems, we need some functions linked with $\{M_n\}$. We set $\boldsymbol{R}_+ = \{x\in\boldsymbol{R};\ x>0\}$ and $\boldsymbol{Z}_+ = \{0, 1, 2, \cdots\cdots\}$. Let us set

$$T(r) = \sup_{n\geqq 0} r^n/M_n \quad (r>0), \tag{2.1}$$

and call it the associated function of $\{M_n\}$. By virtue of the logarithmic convexity of $\{M_n\}$, we have

$$M_n = \sup_{r>0} r^n/T(r) \quad (n\geqq 0). \tag{2.2}$$

Both in (2.1) and (2.2), in reality, "sup" can be replaced by "max" owing to the assumption of $\lim_{n\to\infty}(M_n)^{1/n}=\infty$. Further, n and r which attain the maximums in (2.1) and (2.2) are non-decreasing and diverging. In view of the logarithmic convexity of $\{M_n\}$, it is combinient to use the followings:

$$a_n=\log M_n, \quad H(t)=\sup_{n\geqq 0}\{nt-a_n\}. \tag{2.3}$$

We call $H(t)$ the trace function of $\{a_n\}$. The following relations hold.

$$T(r)=\exp H(\log r), \tag{2.4}$$

$$a_n=\sup_t \{nt-H(t)\}, \quad (n\geqq 0). \tag{2.5}$$

$\{(n, a_n)\}$ forms a convex polygon, which is called the Newton polygon of $\{a_n\}$. The trace function $H(t)$ is convex (or, more exactly, concave) and piecewise linear. We set

$$h(t)=\left(\frac{d}{dt}\right)_r H(t). \quad \left(\left(\frac{d}{dt}\right)_r \text{ is the right derivation.}\right) \tag{2.6}$$

The function $h(t)$ is obviously non-decreasing and piecewise constant. Of course, "sup" in (2.3) and (2.5) can be also replaced by "max", and the maximums are attained by $n=h(t)$ and $t=a_{n+1}-a_n$, respectively. Therefore, we easily see the following relations:

$$H(t)=th(t)-a_{h(t)}, \tag{2.7}$$

$$a_n=n(a_{n+1}-a_n)-H(a_{n+1}-a_n), \tag{2.8}$$

because of $-H(t)=\inf_n\{t(x-n)+a_n\}|_{x=0}$, and $-a_n=\inf_t\{n(x-n)+H(t)\}|_{x=0}$.

We say that $\{a_n\}$ is of order $p(n)$ or has a smaller order than $p(n)$, and we write $a_n=O(p(n))$ or $a_n=o(p(n))$, according as it satisfies the following:

$$\sup_{n\geqq 0} a_n/p(n)<\infty, \tag{2.9}$$

or

$$\lim_{n\to\infty} a_n/p(n)=0. \tag{2.10}$$

Remark. Even if we replace finite elements of $\{M_n\}$ and modify $T(r)$ on a bounded set, $\sup_n r^n/M_n$ and $\sup_r r^n/T(r)$ are invariant for large n and for large r, respectively, and then, the class $\mathscr{B}\{M_n\}$ is unchanged. Thus, according as we give assumptions on $\{M_n\}$ for large n or on $T(r)$ for large r, we may consider them to be valid for all n or for all r, respectively. This also holds on $\{a_n\}$ and on $H(t)$.

On the other hand, if there exist two positive constants R_1 and R_2, such that

$$R_1^n M_n \leqq N_n \leqq R_2^n M_n,$$

the classes $\mathscr{B}\{M_n\}(\boldsymbol{R}^l)$ and $\mathscr{B}\{N_n\}(\boldsymbol{R}^l)$ coincide. In such case, we say that $\{N_n\}$ is equivalent to $\{M_n\}$.

First, we give some equivalent conditions to the separativity of $\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2})$.

Theorem 1. (Separativity.)
The following conditions are all equivalent.
1) $\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2})$ is separative.
2) $\{M_n\}$ satisfies the separativity condition (S).
3) $\exists R>1,\ M_{2n}\leqq R^{2n}(M_n)^2,\ \ (n\gg 1)$.
3′) $\limsup_{n\to\infty}(a_{2n}-2a_n)/n<\infty$.
4) $\exists R>1,\ \{T(r/R)\}^2\leqq T(r),\ (r\gg 1)$.
4′) $\exists \gamma>0,\ 2H(t-\gamma)\leqq H(t),\ (t\gg 1)$.

We prove this theorem in the section 3. Now, we can show some necessary conditions to the separativity.

Theorem 2. *If $\{M_n\}$ satisfies the separativity condition (S), the following equivalent conditions are satisfied. The converse is not always true.*
i) $\exists \nu>0,\ M_n\leqq n!^\nu,\ (n\gg 1)$.
i′) $a_n=O(n\log n)$.
ii) $\exists \kappa>0,\ T(r)\geqq \exp r^\kappa,\ (r\gg 1)$.
ii′) $\liminf_{t\to\infty}(\log H(t))/t>0$.
iii) $\liminf_{t\to\infty}(\log h(t))/t>0$.

The theorem 2 implies the following;

Corollary 3. If $\{M_n\}$ satisfies the separativity condition (S), $\mathscr{B}\{M_n\}(\boldsymbol{R}^l)$ is a Gevrey class or its subclass. The converse is not always true.

The first half of the theorem 2 is proved in the section 4. Here, we give two examples of separative $\{M_n\}$ and show the latter half of the theorem 2 by constructing an example.

Example 1. $M_n=\{\prod_{j=0}^{k}(\log_j n)^{\nu_j}\}^n$, where $k\in\boldsymbol{Z}_+$, $\nu_0>0$, $\nu_j\in\boldsymbol{R}$ $(j\geqq 1)$, $\log_0 n=\max\{1,\ n\}$ and $\log_j n=\max\{1,\ \log(\log_{j-1}n)\}$ $(j\geqq 1)$.

Example 2. $M_n=n!^{\nu_0}\prod_{j=2}^{k}n!^{\nu_j/\log_j n}$ for $k\geqq 2$, $\nu_0>0$ and $\nu_j\in\boldsymbol{R}$ $(2\leqq j\leqq k)$.
These $\{M_n\}$ satisfy the condition (S), since

$$\log_j(p+q)\leqq \log_j p+q(\prod_{k=0}^{j-1}\log_k p)^{-1},\quad (j\geqq 1 \text{ and } p,\ q\geqq 1).$$

Especially, the Gevrey weight $n!^\nu$, which is equivalent to the case of

$k=0$ and $\nu_0=\nu$ in the example 1, satisfies the condition (S).

Proof of the theorem 2. (The latter half)

We give an example of $\{a_n\}$ which does not satisfy the condition 3′) in the theorem 1 but does $\log a_n = O(n \log n)$.

Fixing $\nu>0$, we set, for large n,

$$b_n = \{\nu-(\log\log n)^{-1}\} n \log n^{**)},$$
$$c_n = \nu\, n \log n,$$
$$s_n = (c_{2n}-b_n)/n \quad \text{and} \quad t_n = b_n - b_{n-1}.$$

It is seen that

$$s_n = \nu \log n + (\log n)/(\log\log n) + 2\nu \log 2, \tag{2.11}$$

$$t_n < \nu \log n - (\log n)/(\log\log n) + \nu, \quad (n \geqq {}^{\exists}n_0' > 0), \tag{2.12}$$

because of $(\log n) - 1/n - 1/n^2 < \log(n-1) < (\log n) - 1/n - 1/(2n^2)$, and $\log\log n - 2/(n \log n) < \log\log(n-1) < \log\log n - 1/(n \log n) \quad (n \geqq 4)$. Moreover, it holds that

$$b_{n+1} < c_n \quad (n \geqq {}^{\exists}n_0'' > 0). \tag{2.13}$$

Now we are in a position to define $a_n(=\log M_n)$ inductively. First, we take

$$\begin{cases} n_0 = \max\{n_0', n_0''\}, \quad a_{n_0} = b_{n_0}, \quad a_{2n_0} = c_{2n_0}, \\ f_0(x) = a_{n_0}(2n_0 - x)/n_0 + a_{2n_0}(x - n_0)/n_0. \end{cases} \tag{2.14}$$

Let n_1 be the smallest integer in $\{n\,; n \geqq 2n_0 \text{ and } f_0(n) < b_n\}$. By the relation (2.13), n_1 must be greater than $2n_0+1$. We set

$$\begin{cases} a_n = f_0(n) \quad \text{for } n \leqq n_1 - 1, \quad a_{n_1} = b_{n_1}, \quad a_{2n_1} = c_{2n_1}, \\ f_1(x) = a_{n_1}(2n_1 - x)/n_1 + a_{2n_1}(x - n_1)/n_1. \end{cases} \tag{2.15}$$

From the definition of n_1, the following is satisfied:

$$s_{n_0} < a_{n_1} - a_{n_1 - 1} \leqq t_{n_1} < s_{n_1}. \tag{2.16}$$

Let n_2 be the smallest number in $\{n\,; n \geqq 2\,n_1 \text{ and } f_1(n) < b_n\}$. It is seen that $n_2 \geqq 2n_1+2$ by (2.13). Setting

$$a_n = f_1(n) \quad \text{for } n_1 \leqq n \leqq n_2 - 1, \tag{2.17}$$

$\{a_n\}_{n=0}^{n_2-1}$ is convex.

Repeating this procedure, the sequence $\{a_n\}$ is convex and

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**) Both $\{b_n\}$ and $\{c_n\}$ satisfy the condtion 3′). $\{\exp b_n\}$ is equivalent to $\{M_n\}$ in the example 2 for $k=2$, $\nu_0=\nu$ and $\nu_2=-1$.

$\lim_{n\to\infty} a_n/(n \log n) = \nu < \infty$ but $(a_{2n_k} - 2a_{n_k})/n_k$ $[=2 \log n_k/(\log\log n_k) + 2\nu \log 2]$ tends to infinity. Q. E. D.

Next, we consider the weak separativity.

Theorem 4. (Weak separativity)
The following conditions are all equivalent.

a) $\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2})$ *is weakly separative.*
b) $\{M_n\}$ *satisfies the weak separativity condition.*
b′) $^\exists \gamma > 0$, $^\exists \{b_m\}_{m=0}^{\infty}$ $(b_m \geqq 0)$, $a_{n+m} \leqq a_n + n\gamma + b_m$.
c) $^\exists R > 0$ *independent of* m, $\limsup_{n\to\infty} (M_{n+m}/M_n)^{1/n} \leqq R$.
c′) $\sup_m \{\limsup_{n\to\infty} (a_{n+m} - a_n)/n\} < \infty$.
d) $\lim_{n\to\infty} (M_{n+m}/M_n)^{1/n} = 1$.
d′) $\lim_{n\to\infty} (a_{n+m} - a_n)/n = 0$.
e) $^\forall \varepsilon > 0$, $^\exists n_0 > 0$, $M_n \leqq \exp(\varepsilon n^2)$, $(n \geqq n_0)$.
e′) $a_n = o(n^2)$.
f) $^\forall K > 0$, $^\exists r_0 > 0$, $T(r) \geqq r^{K \log r}$, $(r \geqq r_0)$.
f′) $\lim_{t\to\infty} H(t)/t^2 = \infty$.
g) $\lim_{t\to\infty} h(t)/t = \infty$.
h) $^\exists R > 0$, $^\forall K > 0$, $T(r) \geqq r^K T(r/R)$, $(r \gg 1)$.
h′) $^\exists \gamma > 0$, $\lim_{t\to\infty} \{H(t) - H(t-\gamma)\}/t = \infty$.

Remark. In b′), $\{b_m\}$ is not necessarily convex.
We give the proof of this theorem in the section 5.
In comparison, we present the well-known result on the differentiability.

Theorem 5. (Differentiability)
The following conditions are equivalent.

α) $\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2})$ *is differentiable.*
β) $\{M_n\}$ *satisfies the differentiability condition.*
β') $\limsup_{n\to\infty} (a_{n+1} - a_n)/n < \infty$.
γ) $^\exists \nu > 0$, $M_n \leqq \exp(\nu n^2)$, $(n \gg 1)$.
γ') $a_n = O(n^2)$.
δ) $^\exists K > 0$, $T(r) \geqq r^{K \log r}$, $(r \gg 1)$.
δ') $\liminf_{t\to\infty} H(t)/t^2 > 0$.
ε) $\liminf_{t\to\infty} h(t)/t > 0$.
ζ) $^\forall m \in \boldsymbol{Z}_+$, $^\exists R = R(m) > 0$, $T(r) \geqq r^m T(r/R)$, $(r \gg 1)$.
ζ') $\lim_{\gamma\to\infty} \{\liminf_{t\to\infty} (H(t) - H(t-\gamma))/t\} = \infty$.

§ 3. Proof of the theorem 1.

In order to see the relations of the conditions on $\mathscr{B}\{M_n\}$ and those on $\{M_n\}$, the following proposition is available.

Proposition 6. *For two sequences of positive numbers* $\{L_n\}$ *and* $\{K_n\}$, *we assume that* $\{L_n\}$ *satisfies the logarithmic convexity and the conditions* $\lim_{n\to\infty}(L_n)^{1/n}=\infty$ *and*

$$\liminf_{n\to\infty}(K_n/L_n)^{1/n}=0.$$

Then, there exists a periodic function $f(x)$ *in* $\mathscr{B}\{L_n\}(\boldsymbol{R})$ *which does not belong to* $\mathscr{B}\{K_n\}(\boldsymbol{R})$.

Remark. The sequence $\{K_n\}$ need not necessarily satisfy the logarithmic convexity and the condition $\lim_{n\to\infty}(K_n)^{1/n}=\infty$. The proof of the above proposition was given, for example, in S. Mandelbrojt [2].

Now, we can start the proof of the theorem 1. The equivalences of the pairs of 3) and 3′) and of 4) and 4′) are obvious. Our program to show the rest is the following:

$$1)\Rightarrow 3)\Rightarrow 2)\Rightarrow 1) \quad \text{and} \quad 3')\Leftrightarrow 4').$$

1) $\Rightarrow$ 3). Assuming the assertion 1) and the condition $\liminf_{n\to\infty}\{(M_n)^2/M_{2n}\}^{1/n}=0$, we find a contradiction. We regard $K_{2k}=(M_k)^2$, $K_{2k+1}=M_kM_{k+1}$ $(k\in\boldsymbol{Z}_+)$ and $L_n=M_n$ $(n\in\boldsymbol{Z}_+)$. By the proposition 6, there exists a function $f(t)$ in $\mathscr{B}\{L_n\}(\boldsymbol{R})$ which does not belong to $\mathscr{B}\{K_n\}(\boldsymbol{R})$.

Let us set $g(x,y)=f(x_1+y_1)$, where $x=(x_1,\dots,x_{l_1})$ and $y=(y_1,\dots,y_{l_2})$. Then, $g(x,y)$ belongs also to $\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2})$. Owing to the assumption, $g(x,y)$ belongs also to $\mathscr{B}\{M_n, M_n\}(\boldsymbol{R}^{l_1}\times\boldsymbol{R}^{l_2})$, that is, there is a positive number R such that

$$\begin{aligned}(3.1)\qquad &\sup|(\partial/\partial x_1)^n(\partial/\partial y_1)^{n+e}g(x,y)|\,(=\sup|f^{(2n+e)}(t)|)\\ &\leqq CR^{2n+e}M_nM_{n+e}\,(=cR^{2n+e}K_{2n+e}),\quad (\forall n\in\boldsymbol{Z}_+,\ e=0 \text{ or } 1).\end{aligned}$$

This means that $f(t)$ belongs to $\mathscr{B}\{K_n\}(\boldsymbol{R})$ and we are led to a contradiction.

3) $\Rightarrow$ 2). We show the following inequality by the induction on m from n to 1.

$$(3.2)\qquad M_{n+m}\leqq R^{2n}M_nM_m.$$

i) The case of $m=n$ is just the assertion 3).

ii) Assuming that (3.2) holds for $m=k$ $(1\leqq k\leqq n)$, we consider the case of $m=k-1$. By virtue of the logarithmic convexity, it holds that

$$(3.3)\qquad M_{n+k-1}/M_{n+k}\leqq M_{k-1}/M_k.$$

Producting M_{n+k-1}/M_{n+k} to the either side of (3.2) for $m=k$ and applying (3.3), we have

$$(3.4)\qquad M_{n+k-1}\leqq R^{2n}M_nM_{k-1}.$$

Thus, (3.2) holds for $0 \leqq m \leqq n$.

Replacing R^2 to R', we see the assertion 2).

2) $\Rightarrow$ 1). This is trivial.

3') $\Rightarrow$ 4'). For some positive constant γ and $t > \gamma$, we have

$$H(t) = \sup_n \{nt - a_n\} \geqq \sup_n \{2nt - a_{2n}\} \geqq \sup_n \{2nt - 2a_n - 2n\gamma\}$$
$$= 2 \sup_n \{n(t-\gamma) - a_n\} = 2H(t-\gamma).$$

4') $\Rightarrow$ 3'). For sufficiently large n, it holds that

$$a_{2n} = \sup_{t>\gamma} \{2nt - H(t)\} \leqq \sup_{t>\gamma} \{2nt - 2H(t-\gamma)\}$$
$$= 2n\gamma + 2 \sup_{t>\gamma} \{n(t-\gamma) - H(t-\gamma)\} = 2n\gamma + 2a_n. \qquad \text{Q. E. D.}$$

§ 4. Proof of the theorem 2. (The first half)

The equivalences of the pairs of i) and i'), and of ii) and ii') are obvious. Then, we show first the assertion i') from 3') in the theorem 1 and next the equivalence of the assertions i'), ii') and iii').

3') $\Rightarrow$ i'). From 3'), it holds that

$$a_{2n} - 2a_n \leqq cn, \tag{4.1}$$

for a suitable positive constant c. (4.1) implies

$$2^{k-j} a_{2^j} \leqq 2^{k-j+1} a_{2^{j-1}} + 2^{k-1} c. \tag{4.2}$$

Summing up (4.2) on j from 1 to k, we have

$$a_{2^k} \leqq 2^k a_1 + k\, 2^{k-1} c. \tag{4.3}$$

As we may assume that $\{a_n\}$ is non-decreasing, the following majoration holds good:

$$a_m/(m \log m) \leqq \{2^{k+1} a_1 + (k+1) 2^k c\}/(k 2^k \log 2), \quad (2^k \leqq m \leqq 2^{k+1}), \tag{4.4}$$

and it implies $\limsup_{m\to\infty} a_m/(m \log m) \leqq c/\log 2$.

ii') $\Leftrightarrow$ iii'). As $h(t)$ is non-decreasing, we see that

$$H(0) + th(t) \geqq H(t) \equiv H(0) + \int_0^t h(s)ds \geqq H(0) + (t/2) h(t/2). \tag{4.5}$$

From (4.5), it follows that

$$\liminf_{t\to\infty} \{\log h(t)\}/t \geqq \liminf_{t\to\infty} \{\log H(t)\}/t \geqq \liminf_{t\to\infty} \{\log h(t)\}/t. \tag{4.6}$$

i') $\Rightarrow$ ii'). Owing to the assertion i'), it holds for a positive number ν that

$$a_n \leqq \nu\, n \log n. \tag{4.7}$$

Therefore, the following inequality holds for sufficiently large t:

$$(4.8)\quad H(t)=\sup_{n\geqq 0}(nt-a_n)\geqq \max_{\geqq 0}(nt-n\log n)\geqq \nu\exp\{(t/\nu)-1\}-\nu.$$

This implies

$$\liminf_{t\to\infty}\{\log H(t)\}/t\geqq \nu^{-1}.$$

ii′) $\Rightarrow$i′). Owing to ii′), it holds that

$$(4.9)\qquad \log H(t)\geqq \kappa t,$$

for a positive constant κ. Then, for sufficiently large n, we see that

$$(4.10)\quad a_n=\sup_t\{nt-H(t)\}\leqq \max_t(nt-e^{\kappa t})=\kappa^{-1}n\log n-\kappa^{-1}n(1+\log\kappa).$$

This implies

$$\limsup_{n\to\infty}\ a_n/(n\log n)\leqq \kappa^{-1}.\qquad \text{Q. E. D.}$$

§5. Proof of the theorem 4.

The equivalence of the pairs of x) and x') are obvious (x=b, c, d, e, f, and h). Then, we show the rest under the program:

a) $\Rightarrow$ c) $\Rightarrow$ b) $\Rightarrow$ a), c′) $\Rightarrow$ e′) $\Rightarrow$ f′) $\Rightarrow$ g) $\Rightarrow$ d′) $\Rightarrow$ c′) and d′ $\Leftrightarrow$ h′).

The last equivalence is shown by the similar way to the proof of the theorem 1.

First group. a) $\Rightarrow$ c).

Assuming the assertion a) and the condition:

$$(5.1)\qquad \sup_{m\geqq 0}\{\limsup_{n\to\infty}(M_{n+m}/M_n)^{1/n}\}=\infty,$$

we find a contradiction. (5.1) implies

$$(5.2)\qquad \sup_{m\geqq 0}\{\limsup_{n\to\infty}(M_{n+m}/M_nN_m)^{1/n}\}=\infty.$$

Due to the condition (5.2), we can find a sequence of positive integers $\{m(n)\}_{n=0}^{\infty}$ such that

$$(5.3)\qquad \begin{cases} m(n+1)\geqq m(n),\quad n\geqq m(n)\quad (n\geqq 0)\quad\text{and}\\ \limsup_{n\to\infty}(M_{n+m(n)}/M_nN_{m(n)})^{1/n}=\infty.\end{cases}$$

Setting $n(k)=k+\mathrm{m}(k)$, $\{n(k)\}_{k=0}^{\infty}$ is monotonically increasing. We take

$$(5.4)\qquad \begin{cases} K_{n(k)}=M_kN_{m(k)},\\ K_n=\max\{K_{n(k)},K_{n(k+1)}\}\quad \text{for } n(k)<n<n(k+1),\\ \text{and } L_n=M_n.\end{cases}$$

By virtue of (5.3), it follows that

$$\liminf_{n\to\infty}(K_n/L_n)^{1/n} \leqq \liminf_{k\to\infty}(M_k N_{m(k)}/M_{k+m(k)})^{1/(2k)} = 0. \tag{5.5}$$

Applying the proposition 6, there exists a function $f(t)$ in $\mathscr{B}\{L_n\}(\boldsymbol{R})$ which does not belong to $\mathscr{B}\{K_n\}(\boldsymbol{R})$.
Set $g(x, y) = f(x_1+y_1)$. As $g(x, y)$ belongs to $\mathscr{B}\{M_n\}(\boldsymbol{R}^{l_1+l_2})$, it also belongs to $\mathscr{B}\{M_n, N_n\}(\boldsymbol{R}^{l_1}\times\boldsymbol{R}^{l_2})$ by the assumption. Thus, we have

$$\sup_{x,y}|(\partial/\partial x_1)^k(\partial/\partial y_1)^m g(x, y)| \quad (\equiv \sup_t |f^{(k+m)}(t)|) \leqq CR^{k+m}M_k N_m, \tag{5.6}$$

for a suitable positive constants C and R.
We take $m=m(k)$, then, it holds

$$\sup_t |f^{(n(k))}(t)| \leqq CR^{n(k)}K_{n(k)}. \tag{5.7}$$

Applying Kolmogoroff's theorem, the following holds good:

$$\sup_t |f^{(n)}(t)| \leqq CR^n K_n, \quad (n\geqq 0).$$

Thus, we are led to a contradiction.
c) $\Rightarrow$ b). As $\limsup_{n\to\infty}(M_{n+m}/M_n)^{1/n}$ is non-decreasing on m, there exists a positive constant R independent of m such that

$$\limsup_{n\to\infty}(M_{n+m}/M_n)^{1/n} \leqq R. \tag{5.8}$$

Setting $\sup_{n\geqq k}(M_{n+m}/M_n)^{1/n} = R\,F(k, m)$, (5.8) means that

$$\limsup_{\to\infty} F(k, m) \leqq 1. \tag{5.9}$$

As $\lim_{k\to\infty}\{F(k, m)/2\}^k = 0$ owing to (5.9), we see that $N_m = \sup_{k\geqq 0}\{F(k, m)/2\}^k$ is finite for all m. Therefore, it holds that

$$M_{k+m} \leqq R^k M_k\{F(k, m)\}^k \leqq (2R)^k M_k N_m.$$

b) $\Rightarrow$ a). This is trivial.
Second group. The equivalence of e′), f′) and g) is shown by the similar way to the proof of the theorem 2. Then, we show here "c′) $\Rightarrow$ e′)" and "g) $\Rightarrow$ d′) $\Rightarrow$ c′)".
c′) $\Rightarrow$ e′). As $\limsup_{n\to\infty}(a_{n+m}-a_n)/n$ is non-decreasingon m, there are a positive number γ independent of m and an integer $n_0 = n_0(m)$ such that

$$(a_{n+m}-a_n) \leqq \gamma n \quad (n\geqq n_0). \tag{5.10}$$

Summing up (5.10) from n_0+1 to $n-1$, we have

$$\sum_{j=n}^{n+m-1} a_j - \sum_{j=n_0+1}^{n_0+m} a_j \;\leqq\; (n+n_0)(n-n_0-1)\gamma/2. \tag{5.11}$$

Since $\{a_j\}$ is increasing, it follows from (5. 11) that

$$m(a_n - a_{n_0+m}) \leqq \{n^2 - n - n_0(n_0+1)\}\gamma/2. \tag{5.12}$$

Thus, we have

$$(a_n/n^2) - \{a_{n_0+m}/(n^2-2n)\} \leqq \gamma/(2m), \quad \text{for } n \geqq n_0(n_0+1). \tag{5.13}$$

This implies

$$\limsup_{n\to\infty} a_n/n^2 \leqq \gamma/(2m). \tag{5.14}$$

By the arbitrarity of m, we obtain $\lim_{n\to\infty} a_n/n^2 = 0$.

g) $\Rightarrow$ d′). By the property of $h(t)$, the followng relation holds for $t = (a_{n+m} - a_n)/m$:

$$n < h(t) \leqq n+m. \tag{5.15}$$

Since we have the inequality:

$$(a_{n+m} - a_n)/n = (mt)/n \leqq \{t/h(t)\} m\{1 + (m/n)\},$$

it follows that

$$\limsup_{n\to\infty} (a_{n+m} - a_n)/n = m \limsup_{t\to\infty} t/h(t) = 0.$$

d′) $\Rightarrow$ c′). This is trivial Q. E. D.

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