J. Math. Kyoto Univ. (JMKYAZ)
24-4 (1984) 593-609

On the Cauchy-Kowalewski theorem for general system of differential equations

By

Takashi SADAMATSU

(Communicated by Prof. Mizohata March 7, 1983)

§ 1. Introduction.

Let $A(t, z; \partial_t, \partial_z)=(a_{ij}(t, z; \partial_t, \partial_z))_{1\leqq i,j\leqq N}$ be a matrix whose entries are partial differential operators with holomorphic coefficients in a neighbourhood of the origin in $\boldsymbol{C}^n$ $(n\leqq 3)$. We are concerned with the Cauchy-Kowalewski theorem for the Cauchy problem:

$$\text{(C. P.)}\quad \begin{cases} A(t, z; \partial_t, \partial_z)u=f(t, z) \\ \partial_t^k u_j|_{t=t_0}=\phi_{j,k}(z) \quad k=0, 1, \dots, m_j-1, j=1, 2, \dots, N \end{cases}$$

where $\{m_1, m_2, \dots, m_N\}$ is a given collection of non-negative integers and we denote $z=(x, y)$, $\partial_t=\frac{\partial}{\partial t}$ and $\partial_z=\frac{\partial}{\partial z}\left(=\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right)$.

The purpose of this article is to show that if the Cauchy-Kowalewski theorem holds for the Cauchy problem (C. P.) in a neighbourhood of the origin, then (C. P.) is equivalent to the Cauchy problem for a $(m_1, m_2, \dots, m_N)$-normal system in ∂_t under some assumptions. Therefore if $m_i>0$ $(i=1, 2, \dots, N)$, the Cauchy problem for general system is equivalent to the Cauchy problem for the first order system in ∂_t.

Concerning this subject, M. Miyake [3] treated the ordinary differential equations. K. Kitagawa and T. Sadamatsu [2] treated the partial differential equations under some assumptions. In case of the constant coefficients T. Sadamatsu [4] gave the necessary and sufficient condition for the Cauchy-Kowalewski theorem to hold.

Our arguments are based on the treatments by K. Kitagawa and T. Sadamatsu [2].

The author would like to thank Professor K. Kitagawa for his helpful suggestions and stimulating discussion.

§ 2. Assumptions and Results.

First of all we define the formal order of $A(t, z; \partial_t, \partial_z)$ by

$$\max_{\pi} \sum_{i=1}^{N} \text{order } a_{i\pi(i)}(t, z; \partial_t, \partial_z)$$

where π is a permutation of $\{1, 2, \dots, N\}$ and order $a_{ij}(t, z; \partial_t, \partial_z)$ is defined by $\sup_{t,z}$ {degree of $a_{ij}(t, z; \tau, \zeta)$ as a polynomial of τ and ζ} and order $a_{ij}(t, z; \partial_t, \partial_z) = -\infty$ if $a_{ij}(t, z; \partial_t, \partial_z) \equiv 0$. We assume

Assumption I The formal order of $A(t, z; \partial_t, \partial_z) = m \geqq 0$.

If the Assumption I holds, then by Volevic's lemma [5, 6], there exists a system of integers $\{t_j, s_i\}$ such that

(i) order $a_{ij}(t, z; \partial_t, \partial_z) \leqq t_j - s_i$
(ii) $\sum t_i - \sum s_i = m$.

Let $\mathring{a}_{ij}(t, z; \partial_t, \partial_z)$ be the homogeneous part of order $t_j - s_i$ of $a_{ij}(t, z; \partial_t, \partial_z)$. We assume

Assumption II The hyperplane $t=0$ is non-characteristic for $A(t, z; \partial_t, \partial_z)$, that is, $\det \mathring{A}(0, 0; 1, 0) \neq 0$ *where* $\mathring{A}(t, z; \tau, \zeta) = (\mathring{a}_{ij}(t, z; \tau, \zeta))_{1 \leq i, j \leq N}$

Under the Assumptions I and II, we consider the Cauchy problem:

$$\text{(C. P.)} \quad \begin{cases} A(t, z; \partial_t, \partial_z)u = f(t, z) \\ \partial_t^k u_j|_{t=t_0} = \phi_{j,k}(z) \quad k=0, 1, \dots, m_j - 1, j=1, 2, \dots, N \end{cases}$$

in $\Omega = I \times \mathcal{O}$ where $m_1, m_2, \dots, m_N$ are given non-negative integers satifying $m_1 + m_2 + \dots + m_N = m$ (=the formal order of $A(t, z; \partial_t, \partial_z)$), $\mathcal{O}$ is a neighbourhood of the origin in $\boldsymbol{C}^2$ and $t_0 \in I = \{t \in \boldsymbol{C}; |t| < \delta_0 \ \ \delta_0: \text{small}\}$.

We introduce the following definitions.

Definition 1. We say that the Cauchy-Kowalewski theorem holds for the Cauchy problem (C. P.) at a point (t_0, z_0) if there exists a unique solution $u(t, z) \in H^N(U'(t_0, z_0))$ of (C. P.) for any $f(t, z) \in H^N(U(t_0, z_0))$ and any $\phi_{j,k}(z) \in H(U(t_0, z_0) \cap \{t = t_0\})$, where $U(t_0, z_0)$ and $U'(t_0, z_0)$ are neighbourhoods of (t_0, z_0) in Ω and $H(U(t_0, z_0))$ denotes the space of holomorphic functions in $U(t_0, z_0)$. If the Cauchy-Kowalewski theorem holds for (C. P.) at any point in Ω, then we say that the Cauchy-Kowalewski theorem holds for (C. P.) in Ω.

Definition 2. $A(t, z; \partial_t, \partial_z)$ is said to be $(m_1, m_2, \dots, m_N)$-normal in ∂_t when

$$a_{ij}(t, z; \partial_t, \partial_z) = \delta_{ij}\partial_t^{m_i} + b_{ij}(t, z; \partial_t, \partial_z) \quad i, j=1, 2, \dots, N$$

where $\text{order}_{\partial_t} b_{ij}(t, z; \partial_t, \partial_z) < m_j$ for any i, j and δ_{ij} is Kronecker's δ.

Definition 3. We say that $A(z; \partial_z)u = f(z)$ is uniquely solvable at z_0 if there exists a unique solution $u(z) \in H^N(U'(z_0))$ of $A(z; \partial_z)u = f(z)$ for any $f(z)\in H^N(U(z_0))$. If $A(z;\partial_z)u=f(z)$ is uniquely solvable at any point in $\mathcal{O}$, then we say that $A(z;\partial_z)u=f(z)$ is uniquely solvable in $\mathcal{O}$.

Definition 4. We say that $A(z;\partial_z)$ is invertible in $\mathcal{O}$ if there exists a matrix $B(z;\partial_z)$ whose entries are partial differential operators with holomorphic coefficients in $\mathcal{O}$ such that $A(z;\partial_z)B(z;\partial_z)=B(z;\partial_z)A(z;\partial_z)$ $=$identity holds in $\mathcal{O}$. We denote $B(z;\partial_z)$ by $A^{-1}(z;\partial_z)$.

Our aim is to show that

Theorem 1. *Under the assumption I and II, if the Cauchy-Kowalewski theorem holds for the Cauchy problem (C. P.) in Ω, then there exists an invertible matrix $R(t, z;\partial_t, \partial_z)$ such that $R(t, z;\partial_t, \partial_z)A(t, z;\partial_t, \partial_z)$ is $(m_1, m_2, \dots, m_N)$-normal in ∂_t.*

To prove theorem 1, we need the following theorems.

Theorem 2. *In order that $A(z;\partial_z)u=f(z)$ is uniquely solvable in $\mathcal{O}$, it is necessary and sufficient that $A(z;\partial_z)$ is invertible in $\mathcal{O}$.*

Theorem 3. *If the formal order of $A(z;\partial_z)$ is $-\infty$, then $A(z;\partial_z)u=f(z)$ has not a solution for some $f(z)$.*

§ 3. Preliminaries.

Let $\{t_j, s_i\}$ be a fixed system of integers and

$$E_i\equiv\sum_{j=1}^{N}a_{ij}(t, z;\partial_t, \partial_z)u_j=f_i(t, z)\qquad i=1, 2, \dots, N.$$

We set

$$a_{ij}(t, z;\partial_t, \partial_z)=\mathring{a}_{ij}(t, z;\partial_t, \partial_z)+b_{ij}(t, z;\partial_t, \partial_z)$$

$$\mathring{a}_{ij}(t, z;\partial_t, \partial_z)=\sum_{k=0}^{t_j-s_i}\mathring{a}_{ij}^{(k)}(t, z;\partial_z)\partial_t^{t_j-s_i-k}$$

and

$$b_{ij}(t, z;\partial_t, \partial_z)=\sum_{k=1}^{t_j-s_i}b_{ij}^{(k-1)}(t, z;\partial_z)\partial_t^{t_j-s_i-k}$$

where order $\mathring{a}_{ij}^{(k)}(t, z;\partial_z)=k$ and order $b_{ij}^{(k)}(t, z;\partial_z)\leqq k$. Let us remark that $\mathring{A}(t, z;1, 0)=(\mathring{a}_{ij}^{(0)}(t, z;\partial_z))=(\mathring{a}_{ij}(t, z))$.

We differentiate E_i up to (s_i-1)-times

$$\partial_t^l E_i=\sum_{j=1}^{N}\{\mathring{a}_{ij}(t, z)\partial_t^{t_j-s_i+l}+\sum_{k=1}^{t_j-s_i+l}b_{ij}^{(k,l)}(t, z;\partial_z)\partial_t^{t_j-s_i+l-k}\}u_j$$
$$0\leqq l\leqq s_i-1,\ i=1, 2, \dots, N$$

where $b_{ij}^{(k,l)}(t, z;\partial_z)=\sum_{h+p=k}\binom{l}{h}\partial_t^h(\mathring{a}_{ij}^{(p)}(t, z;\partial_z)+b_{ij}^{(p-1)}(t, z;\partial_z))$ and $\binom{l}{h}$ denotes the binomial coefficient. Without loss of generality we may assume $t_j>m_j$, $s_i>0$ for any i, j.

$$\partial_t^l E_i = \sum_{t_j - s_i + l \geqq m_j} \mathring{a}_{ij}(t, z)\partial_t^{t_j - s_i + l} u_j + \sum_{j=1}^{N} \sum_{k=m_j}^{t_j - s_i + l - 1} b_{ij}^{(t_j - s_i + l - k, l)}(t, z; \partial_z)\partial_t^k u_j$$
$$+ \sum_{t_j - s_i + l < m_j} \mathring{a}_{ij}(t, z)\partial_t^{t_j - s_i + l} u_j + \sum_{j=1}^{N} \sum_{k=0}^{m_j - 1} b_{ij}^{(t_j - s_i + l - k, l)}(t, z; \partial_z)\partial_t^k u_j$$
$$0 \leqq l \leqq s_i - 1,\ i = 1, 2, \dots, N.$$

We denote the system $\partial_t^l E_i = \partial_t^l f_i$ $(l = 0, 1, \dots, s_i - 1, i = 1, 2, \dots, N)$ by

$$\boldsymbol{A}(t, z; \partial_z)U(t, z) + \boldsymbol{B}(t, z; \partial_t, \partial_z)u(t, z) = \hat{f}(t, z)$$

where $U(t, z) = {}^t(\partial_t^{m_1} u_1, \dots, \partial_t^{m_N} u_N, \partial_t^{m_1+1} u_1, \dots, \partial_t^{m_N+1} u_N, \dots, \partial_t^{t_{j'}-1} u_{j'})$, $\hat{f} = {}^t(f_1, \dots, f_N, \partial_t f_1, \dots, \partial_t^{s_{i'}-1} f_{i'})$ $j' = \max_j \{i; t_j = \max_k t_k\}$ $i' = \max_i \{i; s_i = \max_k s_k\}$, $\boldsymbol{A}(t, z; \partial_z)$ is a $\sum s_i \times (\sum t_j - m)$-square matrix whose entries are partial differential operators with holomorphic coefficients and $\boldsymbol{B}(t, z; \partial_t, \partial_z)$ is composed of linear combinations of $u_j, \partial_t u_j, \dots, \partial_t^{m_j - 1} u_j (j = 1, 2, \dots, N)$ with coefficients of differential operators in ∂_z.

When we put $t = t_0$, we have

$$\boldsymbol{A}(t_0, z; \partial_z)U(t_0, z) = F(t_0, z)$$

where $F(t_0, z) = \hat{f}(t_0, z) - \boldsymbol{B}(t_0, z; \partial_t, \partial_z)u(t_0, z)$ in which we replace $\partial_t^k u_j(t_0, z)$ by the initial data $\phi_{j,k}(z)$ $(k = 0, 1, \dots, m_j - 1, j = 1, 2, \dots, N)$.

For the sake of simplicity, let us say that the Cauchy problem (C. P.) is well-posed in Ω when the Cauchy-Kowalewski theorem holds for (C. P.) in Ω. Then we have

Proposition 1. *In order that the Cauchy problem (C. P.) is well-posed in Ω it is necessary and sufficient that $\boldsymbol{A}(t_0, z; \partial_z)U(t_0, z) = F(t_0, z)$ is uniquely solvable in $\mathcal{O}$ for any $t_0(|t_0| < \delta_0)$.*

Proof. Let t_0 be fixed. For any $F(t_0, z)$ we take $\phi_{j,k}(z) = 0$ and the corresponding $f(t, z)$. Let $u(t, z)$ be a unique solution of the Cauchy problem;

$$\begin{cases} A(t, z; \partial_t, \partial_z)u = f(t, z) \\ \partial_t^k u_j|_{t=t_0} = 0 \quad k = 0, 1, \dots, m_j - 1, j = 1, 2, \dots, N, \end{cases}$$

then $(\partial_t^{m_1} u_1, \dots, \partial_t^{m_N} u_N, \partial_t^{m_1+1} u_1, \dots, \partial_t^{m_N+1} u_N, \dots, \partial_t^{t_{j'}-1} u_{j'})|_{t=t_0}$ is a solution of $\boldsymbol{A}(t_0, z; \partial_z)U(t_0, z) = F(t_0, z)$. Conversely, for any $f(t, z)$, $\phi_{j,k}(z)$ we take the corresponding $F(t_0, z)$. Let $U(t_0, z)$ be a unique solution of $\boldsymbol{A}(t_0, z; \partial_z)U(t_0, z) = F(t_0, z)$.

The Cauchy problem:

$$\begin{cases} \partial_t^{s_i} E_i = \partial_t^{s_i} f_i(t, z) & i = 1, 2, \dots, N \\ \partial_t^k u_j|_{t=t_0} = \phi_{j,k}(z) & 0 \leqq k \leqq m_j - 1 \ j = 1, 2, \dots, N \\ \partial_t^k u_j|_{t=t_0} = \text{the corresponding elements of } U(t_0, z) & m_j \leqq k \leqq t_j - 1 \quad 1 \leqq j \leqq N \end{cases}$$

has a unique solution $u(t, z)$, since the system $\partial_t^{s_i}E_i=\partial_t^{s_i}f_i$ $(i=1,\dots,N)$ is a $(t_1, t_2,\dots, t_N)$-normal system and $t=t_0$ is non-characteristic. This solution $u(t, z)$ is also a solution of the Cauchy problem:

$$\begin{cases} A(t, z; \partial_t, \partial_z)u=f(t, z) \\ \partial_t^k u_j|_{t=t_0}=\phi_{j,k}(z) \quad k=0, 1,\dots, m_j-1, j=1, 2,\dots, N. \end{cases}$$

Q. E. D.

§ 4. Proof of theorem 2.

In this section we prove theorem 2. Since it is clear that $A(z; \partial_z)u=f(z)$ is uniquely solvable in $\mathcal{O}$ if $A(z; \partial_z)$ is invertible in $\mathcal{O}$, we show that the inversibility of $A(z; \partial_z)$ in $\mathcal{O}$ follows from the unique solvabilty of $A(z; \partial_z)u=f(z)$. Henceforth we assume that the formal order of $A(z; \partial_z)$ is non-negative. As snown in §6, we need not this assumption.

Lemma 1. *If $A(z; \partial_z)u=f(z)$ is uniquely solvable in $\mathcal{O}$ and the formal order of $A(z; \partial_z)$ is positive, then* det $\mathring{A}(z; \zeta)=0$ *for any $z\in\mathcal{O}$ and $\zeta\in\boldsymbol{C}^2$.*

Proof. Let the formal order of $A(z; \partial_z)$ be $m(>0)$. If there exists $z^0\in\mathcal{O}$ and $\zeta^0\in\boldsymbol{C}^2$ such that det $\mathring{A}(z^0; \zeta^0)\neq 0$, then by the suitable change of variables, we may suppose that the formal order of $A(z; \partial_z)$ is equal to m and that $x=0$ is non-characteristic for $A(z; \partial_z)$. According to C, Wagschal ([6] théorème 4. 1), there exists non-negative integers n_1, $n_2,\dots, n_N$ satisfying $n_1+n_2+\dots+n_N=m$ such that the Cauchy problem:

$$\begin{cases} A(z; \partial_z)u=f(z) \\ \partial_x^k u_j|_{x=0}=\phi_{j,k}(y) \quad k=0, 1,\dots, n_j-1, \ j=1, 2,\dots, N \end{cases}$$

has a unique solution for any $f(z)$ and $\phi_{j,k}(y)$. Therefore the solution of $A(z; \partial_z)u=f(z)$ is not unique. Q. E. D.

Let $l(z; \zeta)=(l_1(z; \zeta), l_2(z; \zeta),\dots, l_N(z; \zeta))$ be a left null vector of $\mathring{A}(z; \zeta)$ and $l_1(z; \zeta), l_2(z; \zeta),\dots, l_N(z; \zeta)$ be homogeneous polynomials in ζ and irreducible. We put $l_i(z; \zeta)=\sum_{|\alpha|=r_i} l_{i\alpha}(z)\zeta^\alpha$ $(i=1, 2,\dots, N)$, then the following degree relations

$$r_1+t_j-s_1=r_2+t_j-s_2=\dots=r_N+t_j-s_N \quad (j=1, 2,\dots, N)$$

hold, where we drop the terms $r_k+t_j-s_k$ if $l_k(z; \zeta)\equiv 0$.

$l(z; \zeta)$ can be divided the following two cases:

1^0) there exists i_0 such that $l_{i_0}(z; \zeta)=l_{i_0}(z)\not\equiv 0$ holds

2^0) degree $l_i(z; \zeta)>0$ holds for any i if $l_i(z; \zeta)\not\equiv 0$.

At first we treate the case 1^0).

Let

$$P(z\,;\,\zeta)=\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ \dfrac{l_1(z\,;\,\zeta)}{l_{i_0}(z\,;\,\zeta)} & \cdots & 1 & \cdots & \dfrac{l_N(z\,;\,\zeta)}{l_{i_0}(z\,;\,\zeta)} \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} < i_0$$

and $P(z\,;\,\partial_z)=P(z\,;\,\zeta)\,|_{\zeta=\partial_z}$, then $P(z\,;\,\partial_z)$ is invertible as far as $l_{i_0}(z)\neq 0$ and the order of the (i_0, j)-entry in $P(z\,;\,\partial_z)A(z\,;\,\partial_z)$ is less than $t_j-s_{i_0}$ $(j=1, 2,\dots, N)$. Therefore the formal order of $P(z\,;\,\partial_z)A(z\,;\,\partial_z)$ is less than that of $A(z\,;\,\partial_z)$. If we put $A'(z\,;\,\partial_z)=P(z\,;\,\partial_z)A(z\,;\,\partial_z)$, then $A(z\,;\,\partial_z)u=f(z)$ and $A'(z\,;\,\partial_z)v=g(z)$ are equivalent as far as $l_{i_0}(z)\neq 0$. If a solution of $A'(z\,;\,\partial_z)v=g(z)$ is represented by $R(z\,;\,\partial_z)g(z)$, then $R(z\,;\,\partial_z)P(z\,;\,\partial_z)f(z)$ is a solution of $A(z\,;\,\partial_z)u=f(z)$ as far as $l_{i_0}(z)\neq 0$ is satisfied where $R(z\,;\,\partial_z)$ is a matrix of size N whose entries are partial differential operators.

Secondly, we treat the case 2^0). There exists i_0 such that $r_{i_0}=\min\{r_i\,;\,l_{i(r_i,0)}(z)\not\equiv 0\}$. In fact if $l_{i(r_i,0)}(z)\equiv 0$ for any i, then $l_1(z\,;\,\zeta)$, $l_2(z\,;\,\zeta),\dots$ and $l_N(z\,;\,\zeta)$ have a common divisor η. This contradicts that l_1, l_2, $\dots$, l_N are irreducible.

Let

$$Q(z\,;\,\zeta)=\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ \dfrac{l_{1(r_1,0)}(z)}{l_{i_0\delta}(z)}\xi^{r_1-r_{i_0}} & \cdots & 1 & \cdots & \dfrac{l_{N(r_N,0)}(z)}{l_{i_0\delta}(z)}\xi^{r_N-r_{i_0}} \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} < i_0$$

then $Q(z\,;\,\zeta)$ and $Q(z\,;\,\partial_z)=Q(z\,;\,\zeta)\,|_{\zeta=\partial_z}$ are invertible as far as $l_{i_0\delta}(z)\neq 0$ is satisfied, where $\delta=(r_{i_0}, 0)$ and $l_{i_0\delta}(z)$ is the coefficient of $\xi^{r_{i_0}}=\zeta^\delta$ in $l_{i_0}(z\,;\,\zeta)$. We define $l^{(1)}(z\,;\,\zeta)=l(z\,;\,\zeta)Q^{-1}(z\,;\,\zeta)$, then i-th component of $l^{(1)}(z\,;\,\zeta)$ has a form $\eta\sum_{|\beta|=r_i-1} l'_{i\beta}(z)\zeta^\beta$ $(i\neq i_0)$ and the degree of $l_i^{(1)}(z\,;\,\zeta)\leqq r_i$. Hence $l_1^{(1)}(z\,;\,\zeta),\dots, l_{i_0-1}^{(1)}(z\,;\,\zeta)$, $l_{i_0+1}^{(1)}(z\,;\,\zeta),\dots, l_N^{(1)}(z\,;\,\zeta)$ have a common divisor η.

On the otherhand, the degree of the (i_0, j)-entries of $Q(z\,;\,\zeta)A(z\,;\,\zeta)$ are $t_j-s_{i_0}$ by the degree relations. If we define $A^{(1)}(z\,;\,\partial_z)=Q(z\,;\,\partial_z)$ $A(z\,;\,\partial_z)$, then the formal order of $A^{(1)}(z\,;\,\partial_z)$ is equal to that of $A(z;\,\partial_z)$ and we may define $\mathring{A}^{(1)}(z\,;\,\zeta)=Q(z\,;\,\zeta)\mathring{A}(z\,;\,\zeta)$.

Let

$$\mathring{A}^{(1)}(z\,;\,\zeta)=\begin{pmatrix} \mathring{a}_1^{(1)}(z\,;\,\zeta) \\ \cdots \\ \mathring{a}_N^{(1)}(z\,;\,\zeta)\end{pmatrix}$$ where $\mathring{a}_i^{(1)}(z\,;\,\zeta)$ is a row vector $(i=1, 2,\dots, N)$.

Since $l^{(1)}(z;\zeta)$ is a left null vector of $\mathring{A}^{(1)}(z;\zeta)$,

$$l_{i_0}^{(1)}(z;\zeta)\mathring{a}_{i_0}^{(1)}(z;\zeta) = -\sum_{j\neq i_0} l_j^{(1)}(z;\zeta)\mathring{a}_j(z;\zeta) = -\sum_{j\neq i_0} l'_j(z;\zeta)\mathring{a}_j^{(1)}(z;\zeta)$$

holds where $l'_j(z;\zeta)=l_j^{(1)}(z;\zeta)/\eta$ $(j\neq i_0)$ are polynomials in ζ. Therefore we can represent $\mathring{a}_{i_0}^{(1)}(z;\zeta)=\eta\mathring{\alpha}(z;\zeta)$ and $(l'_1(z;\zeta), l'_2(z;\zeta),\dots, l'_N(z;\zeta))$ $(l'_{i_0}(z;\zeta)=l_{i_0}(z;\zeta))$ is a left null vector of a matrix $\begin{pmatrix} \mathring{a}_1^{(1)}(z;\zeta) \\ \cdots \\ \mathring{\alpha}(z\ \zeta) \\ \cdots \\ \mathring{a}_N^{(1)}(z;\zeta) \end{pmatrix}$.

For $A^{(1)}(z;\partial_z) = \begin{pmatrix} \mathring{a}_1^{(1)}(z;\partial_z)+b_1(z;\partial_z) \\ \cdots \\ \partial_y\mathring{\alpha}(z;\partial_z)+b_{i_0}(z;\partial_z) \\ \cdots \\ \mathring{a}_N^{(1)}(z;\partial_z)+b_N(z;\partial_z) \end{pmatrix}$, we define an extended matrix of size $(N+1)$: $\mathscr{A}_1(z;\partial_z) = \begin{pmatrix} \mathring{a}_1^{(1)}(z;\partial_z)+b_1(z;\partial_z) & 0 \\ \cdots & \\ \mathring{\alpha}(z;\partial_z) & -1 \\ \cdots & \\ \mathring{a}_N^{(1)}(z;\partial_z)+b_N(z;\partial_z) & 0 \\ b_{i_0}(z;\partial_z) & \partial_y \end{pmatrix}$, then we may define $\mathring{\mathscr{A}}_1(z;\zeta) = \begin{pmatrix} \mathring{a}_1^{(1)}(z;\zeta) & 0 \\ \cdots & \\ \mathring{\alpha}(z;\zeta) & 0 \\ \cdots & \\ \mathring{a}_N^{(1)}(z;\zeta) & 0 \\ 0 & \eta \end{pmatrix}$.

The formal order of $\mathscr{A}_1(z;\partial_z)$ is equal to that of $A^{(1)}(z;\partial_z)$ and $\det\mathring{\mathscr{A}}_1(z;\zeta)=\det\mathring{A}^{(1)}(z;\zeta)=\det\mathring{A}(z;\zeta)\equiv 0$. Furthermore $(l'_1(z;\zeta), l'_2(z;\zeta),\dots, l'_N(z;\zeta), 0)$ is a left null vector of $\mathring{\mathscr{A}}_1(z;\zeta)$ and the degree of $l'_i(z;\zeta)$ is less than that of $l_i(z;\zeta)$ $(i\neq i_0)$.

Continuing the above procedures, we have finally an extended matrix

$$\mathscr{A}(z;\partial_z) = \begin{pmatrix} \alpha_1(z;\partial_z) & & * \\ \cdots & & \\ \alpha_N(z;\partial_z) & & \\ \beta_1(z;\partial_z) & k_1(z;\partial_z) & 0 \\ \cdots & & \ddots \\ \beta_l(z;\partial_z) & 0 & k_l(z;\partial_z) \end{pmatrix}$$

of $A(z;\partial_z)$ and a left null vector $(\hat{l}_1(z;\partial_z), l_2(z;\partial_z),\dots, l_N(z;\partial_z), 0,\dots, 0)$ of

$$\mathscr{A}(z;\zeta)=\begin{pmatrix} \mathring{\alpha}_1(z;\zeta) & & & & \\ \cdots & & 0 & & \\ \mathring{\alpha}_N(z;\zeta) & & & & \\ & k_1(z;\zeta) & & & \\ 0 & & \ddots & & \\ & & & & k_l(z;\zeta) \end{pmatrix}$$ which satisfies $\mathring{l}_{i_0}(z;\zeta)=l_{i_0}(z)$ $\neq 0$ for some i_0, where $\alpha_i(z;\zeta)$ and $\beta_j(z;\zeta)$ are row vectors of length N.

We define

$$\mathscr{P}(z;\zeta)=\begin{pmatrix} 1 & & & & 0 & & & \\ & \ddots & & & & & & \\ \frac{l_1(z;\zeta)}{l_{i_0}(z)} & \cdots & 1 & \cdots & \frac{l_N(z;\zeta)}{\mathring{l}_{i_0}(z)} & 0 & \cdots & 0 \\ & & & \ddots & & & & \\ & & & & & 1 & & \\ 0 & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} < i_0$$

and $\mathscr{P}(z;\partial_z)=\mathscr{P}(z;\zeta)|_{\zeta=\partial_z}$, then according to the case 1^0), the formal order of $\mathscr{P}(z;\partial_z)\mathscr{A}(z;\partial_z)$ is less than that of $A(z;\partial_z)$ and $\mathscr{P}(z;\partial_z)$ is invertible in $\mathcal{O}$ except certain finite analytic hypersurfaces.

Consequently we have the following proposition which plays an essential role in our considerations.

Proposition 2. *If* det $\mathring{A}(z;\zeta)\equiv 0$, *then we can degrade the formal order of* $A(z;\partial_z)$.

Exactly speaking, there exists a matrix $P(z;\partial_z)$ *or* $\mathscr{P}(z;\partial_z)$ *such that the formal order of* $P(z;\partial_z)A(z;\partial_z)$ *or* $\mathscr{P}(z;\partial_z)\mathscr{A}(z;\partial_z)$ *is less than that of* $A(z;\partial_z)$, *where* $P(z;\partial_z)$ *and* $\mathscr{P}(z;\partial_z)$ *are invertible in* $\mathcal{O}$ *except certain finite analytic hypersurfaces and* $\mathscr{A}(z;\partial_z)$ *is an extended matrix of* $A(z;\partial_z)$.

Remark 1. In particular, in case of a matrix of ordinary differential operators, we can degrade the formal order without the change of the size of the matrix under considerations.

Concerning the preceding proposition, we add the following property as lemma, which we use in §6.

Lemma 2. *In order that* $A(z;\partial_z)u=f(z)$ *is uniquely solvable in* $\mathcal{O}$, *it is necessary and sufficient that* $\mathscr{A}(z;\partial_z)U=F(z)$ *is uniquely solvable in* $\mathcal{O}$ *as far as* $l_{i_0}(z)\neq 0$ *are satisfied.*

Proof. It suffices to prove this lemma in case that $A(z;\partial_z)=A^{(1)}(z;\partial_z)$ and $\mathscr{A}(z;\partial_z)=\mathscr{A}_1(z;\partial_z)$. For any $F(z)={}^t(F_1(z), F_2(z),\dots, F_{N+1}(z))$ we take $f_i(z)=F_i(z)$ $(i\neq i_0)$ and $f_{i_0}(z)=\partial_y F_{i_0}(z)+F_{N+1}(z)$. Let $u(z)$ be a unique solution of $A^{(1)}(z;\partial_z)u=f(z)$, then $U(z)=(u(z), \mathring{\alpha}(z;\partial_z)u(z)-$

$F_{i_0}(z))$ is a solution of $\mathscr{A}_1(z;\ \partial_z)U=F(z)$.

Conversely for any $f(z)$ we take $F_i(z)=f_i(z)$ $(i\neq i_0)$ and $F_{N+1}(z)+\partial_y F_{i_0}(z)=f_{i_0}(z)$ and let $U(z)={}^t(U_1(z),\dots, U_{N+1}(z))$ be a unique solution of $\mathscr{A}_1(z;\ \partial_z)U=F(z)$, then $u(z)={}^t(U_1(z),\dots, U_N(z))$ is a solution of $\mathscr{A}^{(1)}(z;\ \partial_z)u=f(z)$. Q. E. D.

Remark 2. Let $\mathscr{A}_1(z;\ \partial_z)U(z)=F(z)$ have a solution of the form $R'(z;\ \partial_z)F(z)$ for any $F(z)$ where $R'(z;\ \partial_z)$ is a matrix of order $(N+1)$, whose entries are partial differential operators. If we take $F(z)={}^t(f_1(z),\dots, \overset{\overset{i_0}{\vee}}{0},\dots, f_N(z),\ f_{i_0}(z))$, then $\mathscr{A}_1(z;\ \partial_z)U=F(z)$ has a solution of the form $R'(z;\ \partial_z)F(z)=R''(z;\ \partial_z)f(z)$ and $u(z)=\begin{pmatrix} I_N & 0\\ 0 & 0\end{pmatrix}U(z)=\begin{pmatrix} I_N & 0\\ 0 & 0\end{pmatrix}R''(z;\ \partial_z)f(z)=R(z;\ \partial_z)f(z)$ is a solution of $A^{(1)}(z;\ \partial_z)u=f(z)$ where I_N is an identity matrix of order N and $R(z;\ \partial_z)$ is a matrix of order N whose entries are partial differential operators.

Proof of Theorem 2. As shown in §6, the formal order of $P(z;\ \partial_z)A(z;\ \partial_z)$ or $\mathscr{P}(z;\ \partial_z)\mathscr{A}(z;\ \partial_z)$ is never $-\infty$. We apply Lemma 1 and Proposition 2 repeatedly until that the formal order becomes 0.

Let $\mathscr{A}_0(z;\ \partial_z)$ be of the formal order 0, then according to K. Kitagawa and T. Sadamatsu ([2], proposition 1) we have

$$\mathscr{A}_0(z;\ \partial_z)\sim\begin{pmatrix} A_1(z) & & & * \\ & A_2(z) & & \\ & & \ddots & \\ 0 & & & A_p(z)\end{pmatrix},$$

where $A_1(z),\dots, A_p(z)$ are square matrces whose entries are functions and we denote $A\sim B$ when a matrix B is obtained from a matrix by the exchange of rows and columns of A. Hence det $\mathscr{A}_0(z;\zeta)$ is independent of ζ.

If $A(z;\ \partial_z)u=f(z)$ is uniquely solvable in $\mathcal{O}$, it must be det $\mathscr{A}_0(z;\zeta)\not\equiv 0$. In fact, by lemma 2, $\mathscr{A}_0(z;\ \partial_z)U=F(z)$ must have a unique solution for any $F(z)$. It is clear that $\mathscr{A}_0(z;\ \partial_z)U=F(z)$ has not a solution for some $F(z)$ provided that det $\mathscr{A}_0(z;\zeta)\equiv 0$. Incidentally, if det $\mathscr{A}_0(z;\zeta)\not\equiv 0$, then using the fact mentioned above, $\mathscr{A}_0(z;\ \partial_z)$ is invertible in $\mathcal{O}$ except certain finite analytic hypersurfaces and $U(z)=\mathscr{A}_0^{-1}(z;\ \partial_z)F(z)$ is a unique solution of $\mathscr{A}_0(z;\ \partial_z)U=F(z)$.

Taking account of Remark 2, there exists a matrix $R(z;\ \partial_z)$ of order N such that $u(z)=R(z;\ \partial_z)f(z)$ is a solution of $A(z;\ \partial_z)u=f(z)$. The coefficients of partial differential operators in the entries of $R(z;\ \partial_z)$ are holomorphic in $\mathcal{O}$. In fact, the construction of $R(z;\ \partial_z)$ shows that the cofficients are meromorphic in $\mathcal{O}$. On the otherhand by the assumption, $A(z;\ \partial_z)u=f(z)$ has a unique solution $u(z)\in H^N(U'(z_0))$ for any $f(z)\in H^N(U(z_0))$ at every z_0 in $\mathcal{O}$. Therefore the coefficients are holomorphic in $\mathcal{O}$.

Since $u(z)=R(z;\partial_z)f(z)$ is a solution of $A(z;\partial_z)u=f(z)$, it is easy to show that $R(z;\partial_z)A(z;\partial_z)=A(z;\partial_z)R(z;\partial_z)=I_N$ holds in $\mathcal{O}$, namely, $A(z;\partial_z)$ is invertible in $\mathcal{O}$.

§5. Proof of Tgeorem 1.

In this section we prove theorem 1. If the Cauchy problem (C. P.) is well-posed in Ω, then it follows from Proposition 1 that $\boldsymbol{A}(t_0, z;\partial_z)U=F(t_0, z)$ is uniquely solvable in $\mathcal{O}$ for any $t_0\in I$ and from Theorem 2, there exists an invertible matrix $\boldsymbol{R}(t_0, z;\partial_z^)$ such that $U(t_0, z)=\boldsymbol{R}(t_0, z;\partial_z)F(t_0, z)$ is a unique solution of $\boldsymbol{A}(t_0, z;\partial_z)U=F(t_0, z)$.

The coefficients of the entries in $\boldsymbol{R}(t_0, z;\partial_z)$ are holomorphic in t_0 in the same manner as those of $R(z;\partial_z)$ in the preceding section. Here let us remark that we take the left null vector $l(z;\zeta)$ in §4 that of $\boldsymbol{A}(\mathrm{t}_0, z;\zeta)$ restricted at $t=t_0$.

Now then we had in §3

$$\boldsymbol{A}(t, z;\partial_z)U(t, z)+\boldsymbol{B}(t, z;\partial_t, \partial_z)u(t, z)=\tilde{f}(t, z)$$

where the entries of $\boldsymbol{B}(t, z;\partial_t, \partial_z)$ were the linear combinations of $u_j, \partial_t u_j, \ldots, \partial_t^{m_j-1}$ $(j=1, \ldots, N)$ with the coefficients of differential operators in ∂_z. For any $t\in I$ we apply $\boldsymbol{R}(t, z;\partial_z)$ on the above system of equations, we obtain

$$U(t, z)+\boldsymbol{R}(t, z;\partial_z)\boldsymbol{B}(t, z;\partial_t, \partial_z)u(t, z)=\boldsymbol{R}(t, z;\partial_z)\tilde{f}(t, z).$$

The first N components of this system of equations can be represented by

$$\partial_t^{m_i}u_i+\sum_{j=1}^{N}b_{ij}(t, z;\partial_t, \partial_z)u_j=g_i(t, z) \qquad (i=1, 2, \ldots, N)$$

where $\underset{\partial_t}{\mathrm{order}}\, b_{ij}(t, z;\partial_t, \partial_z)<m_j$ for any i, j and $g_1, g_2, \ldots, g_N$ are the first N components of $\boldsymbol{R}(t, z;\partial_z)\tilde{f}(t, z)$.

Let $\boldsymbol{R}(t, z;\partial_z)=(\tilde{r}_{ij}(t, z;\partial_z))_{1\leq i, j\leq s}$, we define $r_{ij}(t, z;\partial_t, \partial_z)$ by

$$\sum_{k=1}^{s}\tilde{r}_{ik}(t, z;\partial_z)\tilde{f}_k(t, z)=\sum_{j=1}^{N}r_{ij}(t, z;\partial_t, \partial_z)f_j(t, z) \quad (i=1, 2, \ldots, N)$$

and $R(t, z;\partial_t, \partial_z)=(r_{ij}(t, z;\partial_t, \partial_z))_{1\leq i, j\leq N}$, then $R(t, z;\partial_t, \partial_z)$ is invertible in Ω (K. Kitagawa and T. Sadamatsu [2], proposition 4) and $R(t, z;\partial_t, \partial_z)A(t, z;\partial_t, \partial_z)$ is $(m_1, m_2, \ldots, m_N)$-normal in ∂_t.

§6. The format order $-\infty$.

In this section we treat a matrix $A(z;\partial_z)$ whose formal order is $-\infty$. Before the proof of Theorem 3, we show the remainder of the proof of Theorem 2, namely, the formal order of $P(z;\partial_z)A(z;\partial_z)$ or

$\mathscr{P}(z;\partial_z)\mathscr{A}(z;\partial_z)$ never becomes $-\infty$ provided that the unique solvability of $A(z;\partial_z)u=f(z)$.

At first we prepare two lemmas.

Lemma 3 (*G. Hufford* [1], *Thoerem* 4).
If the formal order of $A(z;\partial_z)$ is $-\infty$, then we have

$$A(z;\partial_z)\sim\begin{pmatrix}\tilde{A}_1(z;\partial_z) & B(z;\partial_z)\\ 0 & \tilde{A}_2(z;\partial_z)\end{pmatrix}$$

where $\tilde{A}_1(z;\partial_z)$ and $\tilde{A}_2(z;\partial_z)$ are $N_1\times(N_1+r)-$ and $(N_2+r)\times N_2$-matrices respectively and $N_1+N_2+r=$ the size of a matrix $A(z;\partial_z)$ $(r\geqq 1,\ N_1,\ N_2\geqq 0)$.

Lemma 4.
Let $\tilde{A}(z;\partial_z)$ be a $(N+1)\times N-$matrix of the form $\begin{pmatrix}c_1\ldots c_N\\ A(z;\partial_z)\end{pmatrix}$ whose entries are partial differential operators. If the formal order of $A(z;\partial_z)=m\geqq 0$ and $\det \mathring{A}(z;\zeta)\not\equiv 0$, then there exists z^0 at which $\tilde{A}(z;\partial_z)v=g(z)$ has not a solution for some $g(z)$.

Proof. The solution $v(z)={}^t(v_1(z),\ v_2(z),\ldots,v_N(z))$ of $\tilde{A}(z;\partial_z)v=g(z)$ satisfies $\sum c_j(z;\partial_z)v_j=g_0(z)$ and $A(z;\partial_z)v={}^t(g_1(z),\ldots,g_N(z))$ where $g(z)={}^t(g_0(z),\ g_1(z),\ldots,g_N(z))$. Let $g(z)={}^t(g_0(z),\ 0,\ldots,0)$ and $v(z)$ be a solution of $A(z;\partial_z)v=0$. If the formal order of $A(z;\partial_z)=m\geqq 0$ and $\det \mathring{A}(z^0;\zeta^0)\neq 0$, then by the suitable change of variables we may suppose that the formal order of $A(z;\partial_z)=m$ and $\det \mathring{A}(0,0;1,0)\neq 0$. According to K. Kitagawa and T. Sadamatsu ([2], Théorème 3), there exists non-negative integers $n_1,n_2,\ldots,n_N$ satisfying $n_1+n_2+\ldots+n_N=m$, so that $v(z)$ is a solution of the Cauchy problem:

$$\begin{cases}\partial_x^{n_i}v_i+\sum_{j=1}^{N}b_{ij}(x,y;\ \partial_x,\ \partial_y)v_j=0 & i=1,\ 2,\ldots,N\\ \partial_x^k v_j|_{x=0}=\phi_{j,k}(y) & k=0,\ 1,\ldots,n_j-1,\ j=1,\ 2,\ldots,N\end{cases}$$

where $\operatorname{order}_{\partial_x} b_{ij}<n_j$ for any $i,\ j$ and vice versa. Therefore it suffices to prove this lemma when $A(z;\partial_z)$ is a first order system in ∂_x. Let $v(z)$ be a solution

$$\begin{cases}\partial_x v_i+\sum_{j=1}^{m}c_{ij}(x,y;\partial_y)v_j=0\\ v_i(0,y)=\varphi_i(y)\end{cases}\qquad i=1,\ 2,\ldots,m$$

then $\partial_x^k v_i(0,y)$ $(k=1,\ 2,\ldots,\ i=1,\ 2,\ldots,m)$ are uniquely determined by $\{\varphi_j(y)\}_{j=1}^m$.

On the otherhand, let E be the left hand side of the first equation of $\tilde{A}(z;\partial_z)v=g(z)$ and we differentiate E up to m-times with respect to x:

$$\partial_x^k E=\sum_{j=1}^{m}c_j^{(k)}(x,y;\partial_y)v_j=\partial_x^k g_0(x,y)\qquad k=0,\ 1,\ldots,m$$

If $\tilde{A}(z;\ \partial_z)v={}^t(g_0(z),\ 0,\dots,\ 0)$ has a solution,

$$\sum_{j=1}^{m} c_j^{(k)}(x,\ y;\ \partial_y)v_j\,|_{x=0}=\partial_x^k g_0(0,\ y) \qquad k=0,\ 1,\dots,\ m$$

must hold. We substitude $\partial_x^k v_i(0, y)$ by the functions which are determined above, then we have the system of ordinary differential equations:

$$\sum_{j=1}^{m} h_{kj}(y,\ D_y)\phi_j(y)=\partial_x^k g_0(0,\ y) \qquad k=0,\ 1,\dots,\ m. \quad \left(D_y=\frac{d}{dy}\right)$$

Since we can take $\partial_x^k\partial_y^l g_0(0,\ y^0)$ $(k=0,\ 1,\dots,\ m,\ l=0,\ 1,\dots)$ arbitrary, taking $A(z;\ \partial_z)=(h_{kj}(y,\ D_y))_{1\leq k,j\leq m}$ and repeating the above reasoning, the above relations are not compatible. Q. E. D.

From now on we treat the case that the formal order of $A(z;\ \partial_z)=m\geqq 0$ and that there exists $P(z;\ \partial_z)$ or $\mathscr{P}(z;\ \partial_z)$ so that the formal order of $P(z;\ \partial_z)A(z;\ \partial_z)$ or $\mathscr{P}(z;\ \partial_z)\mathscr{A}(z;\ \partial_z)$ becomes $-\infty$, where $\mathscr{A}(z;\ \partial_z)$ is an extended matrix of $A(z;\ \partial_z)$ appeared in §4.

Case 1^0)

Let

$$P(z;\ \partial_z)=\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ \dfrac{l_1(z;\ \partial_z)}{l_{i_0}(z)} & \cdots & 1 & \cdots & \dfrac{l_N(z;\ \partial_z)}{l_{i_0}(z)} \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} < i_0$$

and $P(z;\ \partial_z)A(z;\ \partial_z)$ be of formal order $-\infty$, then the $(i_0,\ \pi(i_0))$-entry of $P(z;\ \partial_z)A(z;\ \partial_z)$ must be 0, where $\pi\in\Pi_0$ and Π_0 is the set of permutations of $\{1,\ 2,\dots,\ N\}$ satisfying $\sum_{i=1}^{N}$ order $a_{i\pi(i)}(z;\ \partial_z)=m$. In fact, if $(i_0,\ \pi(i_0))$-entry is not 0, the formal order of $P(z;\ \partial_z)A(z;\ \partial_z)$ is non-negative because of the i-th row of $P(z;\ \partial_z)A(z;\ \partial_z)$ to be that of $A(z;\ \partial_z)$ $(i\neq i_0)$. This contradicts that the formal order of $P(z;\ \partial_z)A(z;\ \partial_z)$ is $-\infty$. Hence we have by Lemma 3,

$$P(z;\ \partial_z)A(z;\ \partial_z)\sim\begin{pmatrix} \tilde{A}_1(z;\ \partial_z) & B(z;\ \partial_z) \\ 0 & \tilde{A}_2(z;\ \partial_z) \end{pmatrix}$$

where $\tilde{A}_1(z;\ \partial_z)=\begin{pmatrix} A_1(z;\ \partial_z) & \begin{matrix} a_1 \\ \vdots \\ a_{N_1} \end{matrix} \end{pmatrix}$, $\tilde{A}_2(z;\ \partial_z)=\begin{pmatrix} b_1 \cdots b_{N_2} \\ A_2(z;\ \partial_z) \end{pmatrix}$, $A_1(z;\ \partial_z)$ and $A_2(z;\ \partial_z)$ are the square matrices of the size N_1 and N_2 respectively and $N_1+N_2+1=N$ $(r=1)$. Here we may suppose that the $(i_0,\ \pi_0(i_0))$-entry of $A(z;\ \partial_z)$ is transformed to the $(N_1+1,\ N_1+1)$-entry by the exchange of rows and columns, where π_0 is a fixed permutation in Π_0.

Further we remark that the $(i,\ \pi_0(i))$-entries $(i\neq i_0)$ of $A(z;\ \partial_z)$ are transformed to those of $A_1(z;\partial_z)$ or $A_2(z;\partial_z)$ and consequently the formal order of $A_i(z;\partial_z)$ is non-negative if $\tilde{A}_i(z;\partial_z)$ is not empty $(i=1, 2)$. Since it is clear that $A(z;\partial_z)u=f(z)$ has not a solution if $\tilde{A}_2(z;\partial_z)$ is empty, henceforth we assume that $\tilde{A}_2(z;\partial_z)$ is not empty. Let us remark that the formal order of $A_2(z;\partial_z)\leqq m$ and the size of $A_2(z;\partial_z)$ is less than that of $A(z;\partial_z)$.

Furthermore we have $\det \mathring{A}_2(z;\zeta)\equiv 0$. In fact, if $\det \mathring{A}_2(z;\zeta)\not\equiv 0$, then by lemma 4, there exists z^0 at which $\tilde{A}_2(z;\partial_z)v=g(z)$ has not a solution for some $g(z)$. This contradicts the solvability of $A(z;\partial_z)u=f(z)$. If $\det \mathring{A}_2(z;\zeta)\equiv 0$, we can degrade the formal order of $A_2(z;\partial_z)$ by means of Proposition 2.

case 2^0)

For $A(z;\partial_z)$ whose formal order is m, we construct an extended matrix of $A(z;\partial_z)$:

$$\mathscr{A}(z;\partial_z)=\begin{pmatrix} A'(z;\partial_z) & * \\ B(z;\partial_z) & K(z;\partial_z) \end{pmatrix} \quad \left(K(z;\partial_z)=\begin{pmatrix} k_1(z;\partial_z) & & 0 \\ & \ddots & \\ 0 & & k_l(z;\partial_z) \end{pmatrix}\right)$$

and

$$\mathscr{P}(z;\partial_z)=\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & 0 & & \\ \dfrac{l_1(z;\partial_z)}{l_{i_0}(z)} & \cdots & 1 & \cdots & \dfrac{l_N(z;\partial_z)}{l_{i_0}(z)} & 0 & \cdots & 0 \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ & 0 & & & & \ddots & & \\ & & & & & & & 1 \end{pmatrix} < i_0,$$

and let $\mathscr{P}(z;\partial_z)\mathscr{A}(z;\partial_z)$ be of formal order $-\infty$.

According to lemma 3, we have

$$\mathscr{P}(z;\partial_z)\mathscr{A}(z;\partial_z)\sim\begin{pmatrix} \tilde{\mathscr{A}}_1(z;\partial_z) & * \\ 0 & \tilde{\mathscr{A}}_2(z;\partial_z) \end{pmatrix},$$

where $\tilde{\mathscr{A}}_1(z;\partial_z)=\begin{pmatrix} \mathscr{A}_1 & * & a_1 \\ & & \vdots \\ * & K_1 & a_{N_1+p} \end{pmatrix}$, $\tilde{\mathscr{A}}_2(z;\partial_z)=\begin{pmatrix} b_1 & \cdots & b_{N_2+q} \\ \mathscr{A}_2 & & * \\ * & & K_2 \end{pmatrix}$,

$K_1=\begin{pmatrix} k_{i_1} & & 0 \\ & \ddots & \\ 0 & & k_{i_p} \end{pmatrix}$, and $K_2=\begin{pmatrix} k_{j_1} & & 0 \\ & \ddots & \\ 0 & & k_{j_q} \end{pmatrix}$, here $\{i_1,\dots,i_p\}\cap\{i_1,\dots,j_q\}=\phi$, $\{i_1,\dots,i_p\}\cup\{j_1,\dots,j_q\}=\{1, 2,\dots,l\}$ and $\mathscr{A}_i(z;\partial_z)$ are the square matrices of the size $N_i(N_1+N_2+1=N=$ the size of $A(z;\partial_z))$.

Similar to case 1^0), we may assume that the $(i_0, \pi_0(i_0))$-entry of $\mathscr{A}(z;\partial_z)$ is transformed to the (N_1+p+1, N_1+p+1)-entry and that the formal order of $\mathscr{A}_i(z;\partial_z)$ is non-negative $(i=1, 2)$. Let us remark that the formal order of $\begin{pmatrix} \mathscr{A}_2 & * \\ 0 & K_2 \end{pmatrix} \leqq m$ and that the size of $\mathscr{A}_2(z;\partial_z)$ is less than that of $A(z;\partial_z)$.

Further $\det \begin{pmatrix} \mathscr{A}_2(z;\zeta) & 0 \\ 0 & K_2(z;\zeta) \end{pmatrix} \equiv 0$ and in this case we can degrade the formal order of $\begin{pmatrix} \mathscr{A}_2 & * \\ * & K_2 \end{pmatrix}$ by means of Proposition 2.

Continuing the above arguments, we finally reach to the three cases whether $\tilde{A}(z;\partial_z)v=g(z)$ has a solution or not:

(i) the formal order of $A(z;\partial_z)$ is 0

(ii) $A(z;\partial_z)=a(z;\partial_z)$ is a scalar differential operator

(iii) $A(z;\partial_z)$ is of the form $\begin{pmatrix} a & * & \\ & k_1 & & 0 \\ * & & \ddots & \\ & 0 & & k_{l'} \end{pmatrix}$,

where $\tilde{A}(z;\partial_z)=\begin{pmatrix} c(z;\partial_z) \\ A(z;\partial_z) \end{pmatrix}$ and $a(z;\partial_z)\not\equiv 0$.

It is obvious that $\tilde{A}(z;\partial_z)v=g(z)$ has not a solution in each case. Consequeently we can conclude that $P(z;\partial_z)A(z;\partial_z)$ or $\mathscr{P}(z;\partial_z)\mathscr{A}(z;\partial_z)$ is not of formal order $-\infty$.

At the same time we have

Proposition 3. *Let $A(z;\partial_z)$ be a $N'\times N$-matrix whose entries are partial differential operators with holomorphic coefficients in $\mathcal{O}$, then $A(z;\partial_z)u=f(z)$ has not a solution for some $f(z)$, where $N'>N$.*

Proof.

Let $A(z;\partial_z)=\begin{pmatrix} a_1(z;\partial_z) \\ \cdots \\ a_{N'}(z;\partial_z) \end{pmatrix}$, $\tilde{A}(z;\partial_z)=\begin{pmatrix} a_{i_0}(z;\partial_z) \\ \cdots \\ a_{i_N}(z;\partial_z) \end{pmatrix}$ and $A'(z;\partial_z)=\begin{pmatrix} a_{i_1} \\ \cdots \\ a_{i_N} \end{pmatrix}$ where $a_i(z;\partial_z)$ are row vectors of length N and $1\leqq i_0<i_1<\dots<i_N\leqq N'$.

If $A(z;\partial_z)u=f(z)$ has a solution, $\tilde{A}(z;\partial_z)v=g(z)$ has also a solution where $g(z)=(f_{i_1}(z),\dots,f_{i_N}(z))$. Let the formal order of $A'(z;\partial_z)$ is non-negative, then by Lemma 4 there exists $z^0\in\mathcal{O}$ at which $\tilde{A}(z;\partial_z)v=g(z)$ has not a solution for some $g(z)$. If the formal order of $A'(z;\partial_z)$ is $-\infty$, then applying Lemma 3 repeatedly until that the formal order of $A_0(z;\partial_z)\geqq 0$, $\tilde{A}_2(z;\partial_z)v=g(z)$ has not a solution, where $\tilde{A}_2(z;\partial_z)=\begin{pmatrix} C(z;\partial_z) \\ A_0(z;\partial_z) \end{pmatrix}$ The exceptional case is that $\tilde{A}_2(z;\partial_z)$ is empty, and in this case it is clear that $\tilde{A}(z;\partial_z)v=g(z)$ has not a solution. Q.E.D.

Using Lemma 3 and Proposition 3, it is easy to prove Theorem 3.

§ 7. Examples.

Lastly we give two examples. We call the type of Cauchy data of (C. P.) in the preceding sections the $(m_1, m_2, \ldots, m_N)$-type.

Example 1.

$$A=\begin{pmatrix} 2\partial_t^2+(\partial_y+1)\partial_t+y^2\partial_x^2+1 & \partial_t^2+(y\partial_x-1)\partial_t-x(x-y)\partial_x^2+1 \\ \partial_t+\partial_x & \partial_t+x\partial_x-1 \end{pmatrix}$$

The formal order of A is 3 and the hyperplane $t=0$ is non-characteristic for A. The 4 types of the Cauchy data are possible.

The (3, 0)-type.

Let $\{t_1, t_2; s_1, s_2\}$ be $\{4, 4; 2; 3\}$, then

$$\boldsymbol{A}=\begin{pmatrix} 0 & -x(x-y)\partial_x^2+1 & y\partial_x-1 & 1 & 0 \\ 0 & x\partial_x-1 & 1 & 0 & 0 \\ 2 & 0 & -x(x-y)\partial_x^2+1 & y\partial_x-1 & 1 \\ 0 & 0 & x\partial_x-1 & 1 & 0 \\ 1 & 0 & 0 & x\partial_x-1 & 1 \end{pmatrix}$$

is invertible and the Cauchy problem of (3, 0)-type is well-posed. We have an invertible

$$R=\begin{pmatrix} a(x, y; \partial) & -a(x, y; \partial)b(x, y; \partial)-1 \\ 1 & -b(x, y; \partial) \end{pmatrix}$$

$$(R^{-1}=\begin{pmatrix} -b(x, y; \partial) & b(x, y; \partial)a(x, y; \partial)+1 \\ -1 & a(x, y; \partial) \end{pmatrix})$$

where $a(x, y; \partial)=\partial_t+x\partial_x-1$ and $b(x, y; \partial)=\partial_t-(x-y)\partial_x$ and

$$RA=\begin{pmatrix} \partial_t^3+\{(2x-y-1)\partial_x+\partial_y\}\partial_t^2+\ldots & 0 \\ \partial_t^2+\{(x-v-1)\partial_x+\partial_y+1\}\partial_t+\ldots & 1 \end{pmatrix} \text{ is (3, 0)-normal in } \partial_t.$$

The (2, 1)-type.

Let $\{t_1, t_2; s_1, s_2\}$ be $\{3, 3; 1, 2\}$, then

$$\boldsymbol{A}=\begin{pmatrix} 2 & y\partial_x-1 & 1 \\ 0 & 1 & 0 \\ 1 & x\partial_x-1 & 1 \end{pmatrix}$$ is invertible and the Cauchy problem of (2, 1)-type is well-posed. We have an invertible

$$R=\begin{pmatrix} 1 & -\partial_t+(x-y)\partial_x \\ 0 & 1 \end{pmatrix} \quad (R^{-1}=\begin{pmatrix} 1 & \partial_t-(x-y)\partial_x \\ 0 & 1 \end{pmatrix})$$

and

$$RA=\begin{pmatrix} \partial_t^2+\{(x-y-1)\partial_x+\partial_y+1\}\partial_t+\ldots & 1 \\ \partial_t+\partial_x & \partial_t+x\partial_x-1 \end{pmatrix} \text{ is (2, 1)-normal in } \partial_t.$$

The (1, 2)-type.

$\boldsymbol{A}=\begin{pmatrix} \partial_y+1 & 1 & 2 \\ 1 & 0 & 0 \\ \partial_x & 1 & 1 \end{pmatrix}$ is invertible and the Cauchy problem of (1, 2)-type is well-posed. We have

$R=\begin{pmatrix} 0 & 1 \\ -1 & 2\partial_t-2\partial_x+\partial_y+1 \end{pmatrix}$ $\left(R^{-1}=\begin{pmatrix} 2\partial_t-2\partial_x+\partial_y+1 & -1 \\ 1 & 0 \end{pmatrix}\right)$ and

$RA=\begin{pmatrix} \partial_t+\partial_x & \partial_t+x\partial_x-1 \\ -(y^2+2)\partial_x^2+\partial_x\partial_y+\partial_x-1 & \partial_t^2+\{(2x-y-2)\partial_x+\partial_y\}\partial_t+\dots \end{pmatrix}$ is (1, 2)-normal in ∂_t.

The (0, 3)-type.

Let $\{t_1, t_2; s_1, s_2\}$ be $\{4, 4; 3, 2\}$, then $\boldsymbol{A}$ is not innvertible. In fact, the formal order or $\boldsymbol{A}(z; \partial_z)$ is 2 and det $\mathring{\boldsymbol{A}}(z; \zeta)=(y^2+2)\xi^2-\xi\eta\not\equiv 0$. Hence the Cauchy problem of (0, 3)-type is not well-posed.

Example 2.

$$A=\begin{pmatrix} \partial_x^2 & \partial_x\partial_y-1 \\ \partial_x\partial_y+1 & \partial_y^2 \end{pmatrix}$$

A is invertible $\left(A^{-1}=\begin{pmatrix} \partial_y^2 & -(\partial_x\partial_y-1) \\ -(\partial_x\partial_y+1) & \partial_x^2 \end{pmatrix}\right)$, namely, the Cauchy problem of (0, 0)-type for A is well-posed.

Let $w=\partial_x u+\partial_y v$, then $A^t(u, v)={}^t(f, g)$ is equivalent to

$$\mathscr{A}U=\begin{pmatrix} 0 & -1 & \partial_x \\ 1 & 0 & \partial_y \\ \partial_x & \partial_y & -1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}=\begin{pmatrix} f \\ g \\ 0 \end{pmatrix}.$$

The formal order of $\mathscr{A}$ is 2 and $(\eta, -\xi, 1)$ is a left null vector of $\mathring{\mathscr{A}}$.

Taking $\mathscr{P}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_y & -\partial_x & 1 \end{pmatrix}$, we have $\mathscr{P}\mathscr{A}=\mathscr{A}'=\begin{pmatrix} 0 & -1 & \partial_x \\ 1 & 0 & \partial_y \\ 0 & 0 & -1 \end{pmatrix}$ and $\mathscr{A}'$ is invertible. Further we have

$$u=\partial_x^2 f-(\partial_x\partial_y-1)g \quad \text{and} \quad v=-(\partial_x\partial_y+1)f+\partial_y^2 g.$$

Department of Applied Mathematics,
Ehime University

Bibliography

[1] G. Hufford, On the characteristic matrix of a matrix of differential operators, J. Differential Equations, **1** (1965), 27-38.
[2] K. Kitagawa and T. Sadamatsu, Sur le problème de Cauchy pour un système d'équations aux dérivées partielles, J. Math. Kyoto Univ., **21** (1981), 85-103.
[3] M. Miyake, Remarks on the formulations of the Cauchy problem for general system of

ordinary differential equations, Tohoku Math. J., **32** (1980), 79-89.

[4] T. Sadamatsu, On the Cauchy-Kowalewski theorem for general system of differential equations with constant coefficients, Bull. Soc. Roy. Sci. Liège. 50^e année (1981), 107-114.

[5] L. R. Volevic, On the general systems of differential equations, Soviet Math. Dokl., **1** (1960), 458-461.

[6] C. Wagschal, Diverses formulations du problème de Cauchy pour un système d'équations aux dérivées partielles, J. Math. Pures Appl., **53** (1974), 51-70.