J. Math. Kyoto Univ. (JMKYAZ)
25-1 (1985) 209-219

Mixed problem for evolution systems

By

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(Communicated by Prof. S. Mizohata, December 14, 1983)

In the preceding paper ([1]), we studied a general initial-boundary value problem for single evolution equations. For example, such a type of equations are derived from the system of equations in complessible fluid dynamics. But, it seems more natural to construct a theory of mixed problems for systems of evolution equations, which we shall consider in this paper, making use of the analysis in the preceding paper. The idea of the framework for systems owes to the one for elliptic systems ([2]).

§1. Problems & Assumptions

1.1. Problems. Let $L(t, x, y; D_t, D_x, D_y)$ be a $N\times N$-matrix $B(t, y; D_t, D_x, D_y)$ be a $m_+\times N$-matrix, whose entries are linear partial differential operators, that is,

$$L(t, x, y; \tau, \xi, \eta)=(l_{ij}(t, x, y; \tau, \xi, \eta))_{i,j=1,\cdots,N},$$

$$B(t, y; \tau, \xi, \eta)=(b_{ij}(t, y; \tau, \xi, \eta))_{i=1,\cdots,m,\ j=1,\cdots,N},$$

where l_{ij}, b_{ij} are polynomials with respect to (τ, ξ, η) with $\mathcal{B}^\infty$-coefficients in $(t, x, y)\in R^1\times R^1\times R^{n-1}$, where we assume these coefficients are constant outside a ball in R^{n+1}.

Our problem is to seek a vector valued solution with length N, satisfying

$$\text{(P)}\qquad\begin{cases} L(t, x, y; D_t, D_x, D_y)u=f & \text{in}\quad (-\infty, T)\times R^n_+, \\ B(t, y; D_t, D_x, D_y)u|_{x=0}=g & \text{on}\quad (-\infty, T)\times R^{n-1}, \\ u=0 \quad \text{for}\quad t<0. \end{cases}$$

where f, g are vecter valued given functions with length $\{N, m_+\}$ satisfying $f=0$, $g=0$ for $t<0$. Our main result is

Theorem. *Under the assumptions* (A.1)-(A.4), (B.1)-(B.3), (A*.1) *and* (B*.1)-(B*.2), *the problem* (P) *is H^∞-well posed.*

1.2. Pincipal part of L. Let $\{p^{(i)}\}_{i=1,\cdots,l}$ be given integers satisfying

$$p^{(1)}>p^{(2)}>\cdots>p^{(l)}\geqq 1.$$

Let P be a set of polygons in $\{(\sigma, s)\in R^2; \sigma\geqq 0, s\geqq 0\}$ whose sides are constructed of straight lines $p^{(i)}\sigma+s=\text{const.}$ $(i=1, \cdots, l)$, $\sigma=\text{const.}$ and $s=\text{const.}$, involving $\sigma=0$ and $s=0$. Let $N\in$ P, then its vertices are represented by (σ, s) coordinates:

$$(0, 0), (\mu^{(1)}+\cdots+\mu^{(l)}, 0), (\mu^{(2)}+\cdots+\mu^{(l)}, m^{(1)}), \cdots$$
$$\cdots, (\mu^{(l)}, m^{(1)}+\cdots+m^{(l-1)}), (0, m^{(1)}+\cdots+m^{(l)}),$$

some of which may coinside, where $m^{(i)}/\mu^{(i)}=p^{(i)}$ or $\mu^{(i)}=m^{(i)}=0$. Here we denote $\sigma(N)=\{m^{(1)}, \cdots, m^{(l)}\}$. Moreover we denote

$$N^0=\bigcup_{i=1}^{l} N^{(1)}, \qquad N^{(i)}=\partial N\cap\{p^{(i)}\sigma+s=\text{const.}\}.$$

Let

$$A(\tau, \xi, \eta)=\sum a_{\sigma\mu\nu}\tau^\sigma\xi^\mu\eta^\nu$$

be a non-zero polynomial, then we define the Nowton polygon N_A of A by the comvex hull of $\Delta_A\cup\Delta'_A$, where

$$\Delta_A=\{(\sigma, \mu+|\nu|); a_{\sigma\mu\nu}\neq 0\}$$

$$\Delta'_A=\{(\sigma, 0); (\sigma, s)\in\Delta_A\}\cup\{(0, s); (\sigma, s)\in\Delta_A\}\cup\{(0, 0)\}$$

Now, let us consider $N\times N$-matrix $L(\tau, \xi, \eta)=(l_{ij}(\tau, \xi, \eta))$ with polynomial entries. Denoting $N_{l_{ij}}$ for the Newton polygon of $l_{ij}(\tau, \xi, \eta)$, we assume

Assumption (A.1). For $i, j=1, \cdots, N$, there exists a polygon $S_i\in P$, containing $N_{l_{ij}}$, where $\sigma(S_i)=\{s_i^{(1)}, \cdots, s_i^{(l)}\}$.

Denoting

$$l_{ij}(\tau, \xi, \eta)=\sum_{(\sigma, \mu+|\nu|)\in S_i} c_{ij\sigma\mu\nu}\tau^\sigma\xi^\mu\eta^\nu$$

and

$$l_{ij}^0(\tau, \xi, \eta)=\sum_{(\sigma, \mu+|\nu|)\in S_i^0} c_{ij\sigma\mu\nu}\tau^\sigma\xi^\mu\eta^\nu,$$

we define the principal part of L with respect to $\{S_i\}$ by

$$L_0(\tau, \xi, \eta)=(l_{ij}^0(\tau, \xi, \eta)).$$

Now let us define for $N\in P$

$$\sigma(N)=\left\{\sum_{i=1}^{N} s_i^{(k)}\right\}_k=\{m^{(k)}\}_k,$$

where $m^{(k)}/\mu^{(k)}=p^{(k)}$, $\sum_{k=1}^{l} m^{(k)}=m$, $\sum_{k=1}^{l}\mu^{(k)}=\mu$. Denoting

$$A(\tau, \xi, \eta)=\det L_0(\tau, \xi, \eta)=\sum_{(\sigma, \mu+|\nu|)\in N} a_{\sigma\mu\nu}\tau^\sigma\xi^\mu\eta^\nu$$

and

$$A_0(\tau, \xi, \eta)=\sum_{(\sigma, \mu+|\nu|)\in N^0} a_{\sigma\mu\nu}\tau^\sigma\xi^\mu\eta^\nu,$$

we assume

Assumption (A.2).

i) $A_0(\tau, \xi, \eta)\neq 0$ if $\mathrm{Im}\,\tau<-K$ and $(\xi, \eta)\in R^n$,

ii) $A_0(0, 1, 0)\neq 0$.

Under this assumption, zeros of $A_0(\tau, \xi, \eta)$ with respect to ξ are non-real if $\mathrm{Im}\,\tau<-K$ and $\eta\in R^{n-1}$. We write m_+ (resp. m_-) for the number of zeros of $A_0(\tau, \xi, \eta)$ with respect to ξ with positive (resp. negative) imaginary parts if $\mathrm{Im}\,\tau<-K$ and $\eta\in R^{n-1}$.

Now, let $M(\tau, \xi, \eta)$ be a cofactor matrix of $L_0(\tau, \xi, \eta)$, i.e.

$$M(\tau, \xi, \eta)L_0(\tau, \xi, \eta)=\begin{pmatrix} A(\tau, \xi, \eta) & & \\ & \ddots & \\ & & A(\tau, \xi, \eta) \end{pmatrix}.$$

Let $d(\tau, \xi, \eta)$ be a polynomial and $\tilde{M}(\tau, \xi, \eta)$ be a $N\times N$-matrix with polynomial entries such that

$$M_0(\tau, \xi, \eta)=(d(\tau, \xi, \eta)\tilde{M}(\tau, \xi, \eta))_0,$$

where M_0 (resp. $(d\tilde{M})_0$) is the principal part of M (resp. $d\tilde{M}$) with respect to $\{T_j\}$, where

$$\sigma(T_j)=\{m^{(k)}-s_j^{(k)}\},$$

then we have

$$\tilde{M}(\tau, \xi, \eta)L_0(\tau, \xi, \eta)=\tilde{L}(\tau, \xi, \eta),$$

$$\tilde{L}_0(\tau, \xi, \eta)=\begin{pmatrix} \tilde{A}(\tau, \xi, \eta) & & \\ & \ddots & \\ & & \tilde{A}(\tau, \xi, \eta) \end{pmatrix},$$

$$\tilde{A}(\tau, \xi, \eta)=\sum_{(\sigma, \mu+|\nu|)\in\tilde{N}^0}\tilde{a}_{\sigma\mu\nu}\tau^\sigma\xi^\mu\eta^\nu,$$

where $\tilde{L}_0$ is the principal part of $\tilde{L}$ with respect to $\tilde{N}$, where

$$\sigma(\tilde{N})=\{\tilde{m}^{(1)}, \cdots, \tilde{m}^{(l)}\} \qquad (\tilde{m}^{(1)}+\cdots+\tilde{m}^{(l)}=\tilde{m}),$$

$$\tilde{m}^{(i)}/\tilde{\mu}^{(i)}=p^{(i)} \qquad (\tilde{\mu}^{(1)}+\cdots+\tilde{\mu}^{(l)}=\tilde{\mu}).$$

Let $\tilde{m}_+$ (resp. $\tilde{m}_-$) zeros of $\tilde{A}(\tau, \xi, \eta)$ with respect to ξ have positive (resp. negative) imaginary parts for $\mathrm{Im}\,\tau<-K$ and $\eta\in R^{n-1}$, which we denote $\{\xi_j^+\}$ (resp. $\{\xi_j^-\}$) and

$$\tilde{A}_\pm(\tau, \xi, \eta)=\prod_{j=1}^{\tilde{m}_\pm}(\xi-\xi_j^\pm(\tau, \eta)).$$

Moreover we denote the i-th part of $\tilde{A}$ with respect to $\tilde{N}$ by

$$\tilde{A}^{(i)}(\tau, \xi, \eta)=\sum_{(\sigma, \mu+|\nu|)\in\tilde{N}^{(i)}}\tilde{a}_{\sigma\mu\nu}\tau^\sigma\xi^\mu\eta^\nu\tau^{-(\tilde{\mu}^{(i+1)}+\cdots+\tilde{\mu}^{(l)})}$$

and we assume

Assumption (A.3).

i) $\tilde{A}(0, \xi, \eta)\neq 0$ for $(\xi, \eta)\in S^{n-1}$ $(i=1, \cdots, l-1)$ and $\tilde{A}^{(l)}(0, 1, 0)\neq 0$,

ii) $p^{(i)}$ is even and $\tilde{A}^{(i)}(\tau, \xi, \eta)\neq 0$ for $\mathrm{Im}\,\tau\leqq 0$ and $(\xi, \eta)\in S^{n-1}$ $(i=1, \cdots, l-1)$,

iii) $p^{(l)}=1$ and zeros of $\tilde{A}^{(l)}(\tau, \xi, \eta)$ with respect to τ are real and distinct for $(\xi, \eta)\in S^{n-1}$.

In general, let

$$p(\tau, \xi, \eta)=(p_1(\tau, \xi, \eta), \cdots, p_N(\tau, \xi, \eta))$$

satisfying

$$N_{p_i}\subset R\subset R_0 \quad (R, R_0\in P),$$

then we define the standardization of p with respect to $(R; R_0)$ by

$$p^{\#}(\tau, \xi, \eta)=(p_1^{\#}(\tau, \xi, \eta), \cdots, p_N^{\#}(\tau, \xi, \eta)),$$

where

$$p_i^{\#}(\tau, \xi, \eta)=\prod_{k=1}^{l}(\tau^{q_k}-i|\eta|)^{\beta^{(k)}-\alpha^{(k)}}p_i(\tau, \xi, \eta),$$

where

$$\sigma(I_{R_0}(R))=\{\alpha^{(k)}\}, \qquad \sigma(R_0)=\{\beta^{(k)}\}$$

and

$$I_{R_0}(R)=\bigcap_{(\alpha,\beta)\in I}\{(R_0-(\alpha, \beta))\cap R_0\}, \qquad I=\{(\alpha, \beta); R_0-(\alpha, \beta)\supset R\}.$$

Let

$$P(\tau, \xi, \eta)=\begin{bmatrix} p_{11}(\tau, \xi, \eta) & \cdots & p_{1N}(\tau, \xi, \eta) \\ \vdots & & \vdots \\ p_{H1}(\tau, \xi, \eta) & \cdots & p_{HN}(\tau, \xi, \eta)\end{bmatrix}=\begin{bmatrix} p_1(\tau, \xi, \eta) \\ \vdots \\ p_H(\tau, \xi, \eta)\end{bmatrix},$$

satisfying

$$N_{p_{ij}}\subset R_i\subset R_0 \quad (R_i, R_0\in P),$$

then we define the standardization of P with respect to $(R_1, \cdots, R_H; R_0)$ by

$$P^{\#}=\begin{pmatrix} p_1^{\#} \\ \vdots \\ p_H^{\#}\end{pmatrix},$$

where $p_i^{\#}$ is the standardization of p_i with respect to $(R_i; R_0)$.

Now, denoting $l_i=(l_{i1}\cdots l_{iN})$, we define

$$\boldsymbol{L}=\begin{pmatrix} L_1 \\ \vdots \\ L_N\end{pmatrix}, \qquad \boldsymbol{L}_i=\begin{pmatrix} l_i \\ \xi l_i \\ \vdots \\ \xi^{\tilde{m}-1-s_i}l_i\end{pmatrix} \quad \left(s_i=\sum_{k=1}^{l}s_i^{(k)}\right)$$

and $\boldsymbol{L}_i^{\#}$ be the standardization of $\boldsymbol{L}$ with respect to

$$(S_i, S_i+(0, 1), \cdots, S_i+(0, \tilde{m}-s_i); \tilde{N}').$$

Moreover, denoting

$$\mathscr{B}_L^{(k)}(\tau, \eta)=\left[\frac{1}{2\tau i}\int\frac{\boldsymbol{L}^{\#(k)}(\tau, \xi, \eta)\xi^{j-1}}{\tilde{A}_+^{(k)}(\tau, \xi, \eta)}d\xi\right]_{j=1,\cdots,\tilde{m}_l^++\cdots+\tilde{m}_k^+},$$

$$b_L^{(k)}=\left[\frac{1}{2\pi i}\int\frac{\boldsymbol{L}^{\#(k)}(1, \xi, 0)\xi^{j-1}}{\tilde{a}_+^{(k)}(1, \xi)}d\xi\right]_{j=1,\cdots,\tilde{m}_k^+},$$

we define

$$\mathcal{R}_L^{(k)}(\tau,\ \eta)=[\mathcal{B}_L^{(k)}(\tau,\ \eta)b_L^{(k+1)} \cdots b_L^{(l)}]\,,$$

where

$$\tilde{A}_+^{(k)}(\tau,\ \xi,\ 0)=\xi^{\tilde{m}_1^+ + \cdots + \tilde{m}_{k-1}^+}\tilde{a}_+^{(k)}(\tau,\ \xi)\,.$$

Then we have

Lemma 1.1. *rank* $\mathcal{R}_L^{(k)}(\tau,\ \eta)=\tilde{m}_+N-m_+$ for $\mathrm{Im}\,\tau<0$, $\eta\in R^{n-1}$.

Proof. For fixed $(\tau,\ \eta)$, we have

$$P^{(k)}(\tau,\ \eta\,;\ \xi)L^{(k)}(\tau,\ \xi,\ \eta)Q^{(k)}(\tau,\ \eta\,;\ \xi)=D^{(k)}(\tau,\ \eta\,;\ \xi)\,,$$

$$D^{(k)}(\tau,\ \eta\,;\ \xi)=\begin{bmatrix} e_1^{(k)}(\tau,\ \eta\,;\ \xi) & & \\ & \ddots & \\ & & e_N^{(k)}(\tau,\ \eta\,;\ \xi) \end{bmatrix},$$

where $\{e_i^{(k)}(\tau,\ \eta\,;\ \xi)\}$ are elementary divisors of $L^{(k)}(\tau,\ \xi,\ \eta)$ as polynomials with respect to ξ and

$$\pm\prod e_i^{(k)}(\tau,\ \eta\,;\ \xi)=A^{(k)}(\tau,\ \xi,\ \eta)\,,$$

where $A^{(k)}(\tau,\ \xi,\ \eta)$ is divisible by $\tilde{A}^{(k)}(\tau,\ \xi,\ \eta)$. Replacing $\boldsymbol{L}$ by $\boldsymbol{D}$ in $\mathcal{R}_L^{(k)}$, we can prove the lemma easily (see [2]). ■

Assumption (A.4).

$$\mathrm{rank}\,\mathcal{R}_L^{(k)}(\tau,\ \eta)=\tilde{m}_+N-m_+$$

for $\mathrm{Im}\,\tau\leqq 0$ and $\eta\in S^{n-2}$ $(k=0,\ 1,\ \cdots,\ l)$, where $\mathcal{R}_L^{(0)}=\mathcal{R}_L^{(1)}(1,\ 0)$.

1.3. Principal part of B. Let us consider boundary operators

$$B(\tau,\ \xi,\ \eta)=[b_{ij}(\tau,\ \xi,\ \eta)]_{i=1,\cdots,m_+,\ j=1,\cdots,m}$$

where we assume

Assumption (B.1). $N_{b_{ij}}\subset\tilde{N}'$, therefore there exists $R_i\in P$ such that $N_{b_{ij}}\subset R_i$, where

$$\sigma(R_i)=\{r_i^{(1)},\ \cdots,\ r_i^{(l)}\}\,,$$

where

$$r_i^{(1)}\leqq\tilde{m}^{(1)}-1\,,\qquad r_i^{(k)}\leqq\tilde{m}^{(k)}\quad(k=2,\ \cdots,\ l)\,.$$

Denoting

$$b_{ij}(\tau,\ \xi,\ \eta)=\sum_{(\sigma,\ \mu+|\nu|)\in R_i} b_{ij\sigma\mu\nu}\tau^\sigma\xi^\mu\eta^\nu\,,$$

$$b_{ij}^0(\tau,\ \xi,\ \eta)=\sum_{(\sigma,\ \mu+|\nu|)\in R_i^0} b_{ij\sigma\mu\nu}\tau^\sigma\xi^\mu\eta^\nu\,,$$

we define

$$B_0(\tau,\ \xi,\ \eta)=(b_{ij}^0(\tau,\ \xi,\ \eta))_{i=1,\cdots,m_+,\ j=1,\cdots,N}\,.$$

From Assumption (A.2)-(ii), we have $\det L_0(0,\ 1,\ 0)\neq 0$, that is, the row vectors of $\boldsymbol{L}(0,\ \xi,\ 0)$ spans a $(N\tilde{m}-m)$-dimensional subspace in the space spaned by $\{(\overbrace{0,\ \cdots\ 0}^{i-1},\ \xi^{j-1},\ \overbrace{0,\ \cdots,\ 0}^{N-i-1})\}_{i=1,\cdots,N\ j=1,\cdots,\tilde{m}}$. Here we say that B is normal, if the space spaned by the row vectors of

$$\begin{bmatrix} B_0(0, \xi, 0) \\ \boldsymbol{L}(0, \xi, 0) \end{bmatrix}$$

spans a $(N\tilde{m}-m_-)$-dimensional subspace. Here we assume

Assumption (B.2). B is normal.

Denoting

$$s_i=\sum_{k=1}^{l} s_i^{(k)},$$

we define $\{s_1', \cdots, s_m'\}$ the complement of the set

$$\{s_1, s_1+1, \cdots, \tilde{m}-1, s_2, s_2+1, \cdots, \tilde{m}-1, \cdots, s_N, s_N+1, \cdots, \tilde{m}-1\}$$

in the set

$$\{\overbrace{0, \cdots, 0}^{N}, \overbrace{1, \cdots, 1}^{N}, \cdots, \overbrace{\tilde{m}-1, \cdots, \tilde{m}-1}^{N}\}$$

Denoting

$$r_i=\sum_{k=1}^{l} r_i^{(k)} \quad (i=1, \cdots, m_+),$$

and

$$\{r_1, \cdots, r_{m_+}\} \cup \{r_{m_++1}, \cdots, r_m\} = \{s_1', \cdots, s_m'\},$$

we can find

$$C(\xi)=\begin{pmatrix} \xi^{r_{m_++1}} & & \\ & \ddots & \\ & & \xi^{r_m} \end{pmatrix}\begin{pmatrix} c_{11} & \cdots & c_{1N} \\ \vdots & & \vdots \\ c_{m_-1} & \cdots & c_{m_-N} \end{pmatrix}$$

such that the row vectors of

$$\begin{pmatrix} B_0(0, \xi, 0) \\ \boldsymbol{L}(0, \xi, 0) \\ C(\xi) \end{pmatrix}$$

are linearly independent.

Let us denote

$$\boldsymbol{B}=\begin{bmatrix} B \\ \boldsymbol{L} \end{bmatrix}, \qquad \boldsymbol{B}^{\#}=\begin{bmatrix} B^{\#} \\ \boldsymbol{L}^{\#} \end{bmatrix},$$

where $B^{\#}$ be the standardization of B with respect to $(R_1, \cdots, R_{m_+}; \tilde{N}')$.

We define the k-th part of Lopatinski matrix of (A, B) by

$$\mathcal{R}^{(k)}(\tau, \eta)=(\mathcal{R}^{(k)}(\tau, \eta)b^{(k+1)} \cdots b^{(l)}),$$

where

$$\mathcal{B}^{(k)}(\tau, \eta)=\left(\frac{1}{2\pi i}\int\frac{\boldsymbol{B}^{\#(k)}(\tau, \xi, \eta)\xi^{j-1}}{\tilde{A}(\tau, \xi, \eta)}d\xi\right)_{j=1,\cdots \tilde{m}_1^+ + \cdots + \tilde{m}_k^+}$$

and

$$b^{(k)}=\left(\frac{1}{2\pi i}\int\frac{\boldsymbol{B}^{\#(k)}(1,\ \xi,\ 0)\xi^{j-1}}{\tilde{a}^{(k)}{}_{+}(1,\ \xi)}d\xi\right)_{j=1,\cdots,\tilde{m}_{k}^{+}},$$

where

$$A_{+}^{(k)}(\tau,\ \xi,\ 0)=\xi^{\tilde{m}_{1}^{+}+\cdots+\tilde{m}_{k+1}^{+}}\tilde{a}_{+}^{(k)}(\tau,\ \xi)\,.$$

Denotiong $\mathcal{R}^{(0)}=\mathcal{R}^{(1)}(1,\ 0)$, we assume

Assumption (B.3). $\operatorname{rank}\mathcal{R}^{(k)}(\tau,\ \eta)=\tilde{m}_{+}N$ for $\operatorname{Im}\tau\leqq 0$, $\eta\in S^{n-2}$ $(k=0,\ 1,\ \cdots,\ l)$.

1.4. Example. Let us consider a linearized operator for compressible fluid without viscosity, that is,

$$L(\tau,\ \xi)=\begin{pmatrix}\tau+a\xi & \alpha\xi_1 & \alpha\xi_2 & \alpha\xi_3 & 0\\ \alpha\xi_1 & \tau+a\xi & 0 & 0 & \beta\xi_1\\ \alpha\xi_2 & 0 & \tau+a\xi & 0 & \beta\xi_2\\ \alpha\xi_3 & 0 & 0 & \tau+a\xi & \beta\xi_3\\ 0 & \beta\xi_1 & \beta\xi_2 & \beta\xi_3 & \tau+a\xi-i\kappa|\xi|^2\end{pmatrix}$$

where $\xi=(\xi_1,\ \xi_2,\ \xi_3)$, $a=(a_1,\ a_2,\ a_3)\in R^3$ $(a_1<0)$, $\alpha>0$, $\kappa>0$, $|a_1|<\alpha$. Let $p^{(1)}=2$, $p^{(2)}=1$,

$$\{s_1^{(1)}=\cdots=s_4^{(1)}=0,\ s_5^{(1)}=2\},\qquad \{s_1^{(2)}=\cdots=s_4^{(2)}=1,\ s_5^{(2)}=0\},$$

then we have

$$L_0=\begin{pmatrix}H_2 & \alpha\xi_1 & \beta\xi_2 & \alpha\xi_3 & 0\\ \alpha\xi_1 & H_2 & 0 & 0 & \beta\xi_1\\ \alpha\xi_2 & 0 & H_2 & 0 & \beta\xi_2\\ \alpha\xi_3 & 0 & 0 & H_2 & \beta\xi_3\\ 0 & 0 & 0 & 0 & H_1\end{pmatrix}$$

$$\tilde{M}=\begin{pmatrix}H_1H_2^2 & -\alpha\xi_1H_1H_2 & =\alpha\xi_2H_1H_2 & -\alpha\xi_3H_1H_2 & *\\ -\alpha\xi_1H_1H_2 & (H_3+\alpha^2\xi_1^2)H_1 & \alpha^2\xi_1\xi_2H_1 & \alpha^2\xi_1\xi_3H_1 & *\\ -\alpha\xi_2H_1H_2 & \alpha^2\xi_2\xi_1H_1 & (H_3-\alpha^2\xi_2^2)H_1 & \alpha^2\xi_2\xi_3H_1 & *\\ -\alpha\xi_3H_1H_2 & \alpha^2\xi_3\xi_1H_1 & \alpha^2\xi_3\xi_2H_1 & (H_3+\alpha^2\xi_3^2)H_1 & *\\ 0 & 0 & 0 & 0 & H_3\end{pmatrix}$$

where

$$H_1=\tau-i\kappa|\xi|^2\,,\quad H_2=\tau=a\xi\,,\quad H_3=(\tau-a\xi)^2-\alpha_3|\xi|^2\,,$$

therefore we have

$$A=\det L_0=H_1H_2^2H_3\quad (m=6,\ m_+=2,\ m^-=4)\,,$$

$$\tilde{A}=H_1H_2H_3\quad (\tilde{m}=5,\ \tilde{m}_+=2,\ \tilde{m}^-=3)\,,$$

$$\tilde{A}^{(1)}=H_1\,,\quad \tilde{A}^{(2)}=-i\kappa|\xi|^2H_2H_3\,.$$

Let

$$B=\begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix},$$

then the Assumptions (A), (B) are satisfied.

§ 2. Energy inequalities.

2.1. Norms. Let $\tilde{N}$ be the Newton polygon of $\tilde{A}$ and

$$\tilde{N}'=\{\tilde{N}-(0,1)\}\cap\tilde{N},$$

then we define basic norms for a scalor function $u(t, x, y)\in H^{\infty}(R^1\times R_+^1\times R^{n-1})$:

$$|||u|||^2=\sum_{(\sigma,s+\mu)\in\tilde{N}'}\|\langle D_t\rangle^{\sigma}\langle D_y\rangle^{s}D_x^{\mu}u\|^2,$$

$$\langle\!\langle u\rangle\!\rangle^2=\sum_{(\sigma,s+\mu)\in\tilde{N}'}\langle\langle D_t\rangle^{\sigma}\langle D_y\rangle^{s}D_x^{\mu}u\rangle^2,$$

where

$$\langle D_t\rangle^2=|D_t|^2+\gamma^2,\qquad \langle D_y\rangle^2=|D_y|^2+1,$$

$$\|u\|^2=\int_{R^1\times R_+^n}|u|^2dtdxdy,\qquad \langle u\rangle^2=\int_{R^1\times\{x=0\}\times R^{n-1}}|u|^2dtdy.$$

Now let us define

$$s_i^{(k)}+\tilde{s}_i^{(k)}=\tilde{m}^{(k)}\quad(k=1,\cdots,l),$$

$$r_i^{(1)}+\tilde{r}_i^{(1)}=\tilde{m}^{(1)}-1,\qquad r_i^{(k)}+\tilde{r}_i^{(k)}=\tilde{m}^{(k)}\quad(k=2,\cdots,l),$$

and we define polygons, belonging to P, by

$$\sigma(\tilde{S}_i)=\{\tilde{s}_i^{(1)},\cdots,\tilde{s}_i^{(l)}\}\quad(i=1,\cdots,N),$$

$$\sigma(\tilde{R}_i)=\{\tilde{r}_i^{(1)},\cdots,\tilde{r}_i^{(l)}\}\quad(i=1,\cdots,m_+),$$

Now we define a norm weighted by $\tilde{S}$ for a vector valued function

$$f=(f_1,\cdots,f_N)\in H^{\infty}(R^1\times R_+^1\times R^{n-1})$$

$$\|\Lambda_{\tilde{S}}f\|^2=\sum_{j=1}^{N}\sum_{(\sigma,s+\mu)\in\tilde{S}_j}\|\langle D_t\rangle^{\sigma}\langle D_y\rangle^{s}D_x^{\mu}f_j\|^2,$$

and a norm weighted by $\tilde{R}$ for $g=(g_1,\cdots,g_{m_+})\in H^{\infty}(R^1\times R^{n-1})$

$$\langle\dot{\Lambda}_{\tilde{R}}g\rangle^2=\sum_{j=1}^{m_+}\sum_{(\sigma,s)\in\tilde{R}_j}\langle\langle D_t\rangle^{\sigma}\langle D_y\rangle^{s}g_j\rangle^2.$$

2.2. Problem ($\tilde{R}$). Let u satisfy

$$(P)\quad\begin{cases}Lu=f & \text{in}\quad R^1\times R_+^n,\\ Bu|_{x=0}=g & \text{on}\quad R^1\times R^{n-1},\end{cases}$$

then u satisfies also

$$(\tilde{P})\quad\begin{cases}\tilde{L}u=\tilde{M}f & \text{in}\quad R^1\times R_+^n,\\ \boldsymbol{B}u|_{x=0}=\boldsymbol{g} & \text{on}\quad R^1\times R^{n-1},\end{cases}$$

where

$$\tilde{L}=\tilde{M}L,\qquad \boldsymbol{B}=\begin{bmatrix}B\\ \boldsymbol{L}\end{bmatrix},\qquad \boldsymbol{g}=\begin{bmatrix}g\\ \boldsymbol{L}_1f_1|_{x=0}\\ \vdots\\ \boldsymbol{L}_Nf_N|_{x=0}\end{bmatrix}.$$

Now let us denote

$$\tilde{L}=\tilde{L}_0+\tilde{L}_1, \qquad \boldsymbol{B}^{\#}=\boldsymbol{B}_0^{\#}+\boldsymbol{B}_1^{\#},$$

where $\tilde{L}_0$ (resp. $\boldsymbol{B}_0^{\#}$) is a principal parts of $\tilde{L}$ (resp. $\boldsymbol{B}^{\#}$) and

$$\tilde{L}_0=\begin{bmatrix}\tilde{A} & & \\ & \ddots & \\ & & \tilde{A}\end{bmatrix},$$

then we have the basic energy inequality:

Lemma 2.1.
$$\gamma^{q_1/2}|||u|||+\langle\!\langle u\rangle\!\rangle$$
$$\leqq C(\gamma^{-q_1/2}\|\tilde{L}_0(D_t-i\gamma,\ D_x,\ D_y)u\|+\langle\boldsymbol{B}_0^{\#}(D_t-i\gamma,\ D_x,\ D_y)u\rangle)$$

The proof of this lemma is omitted, because it is essentially proved in [1]. On the other hand, we have by the definitions of norms

Lemma 2.2.
$$\|\tilde{L}_1(D_t-i\gamma,\ D_x,\ D_y)u\|\leqq C|||u|||,$$
$$\langle\boldsymbol{B}_1^{\#}(D_t-i\gamma,\ D_x,\ D_y)u\rangle\leqq C\gamma^{-q_1}\langle\!\langle u\rangle\!\rangle.$$

Hence we have from Lemma 2.1 and Lemma 2.2.

Proposition 2.3. *There exist $\gamma_0>0$ and $C>0$ such that*

$$\gamma^{(1/2)q_1}|||u|||+\langle\!\langle u\rangle\!\rangle$$
$$\leqq C\{\gamma^{-(1/2)q_1}\|\tilde{L}(D_t-i\gamma,\ D_x,\ D_y)u\|+\langle\boldsymbol{B}^{\#}(D_t-i\gamma,\ D_x,\ D_y)u\rangle\}$$

for $\gamma>\gamma_0$ and $u\in H^{\infty}(R^1\times R_+^n)$.

Therefore we have

Corollary. *There exist $\gamma_0>0$ and $C>0$ such that*

$$\gamma^{(1/2)q_1}|||u|||+\langle\!\langle u\rangle\!\rangle$$
$$\leqq C\{\gamma^{-(1/2)q_1}\|\Lambda_{\tilde{S}}L(D_t-i\gamma,\ D_x,\ D_y)u\|+\langle\dot{\Lambda}_{\tilde{R}}B(D_t-i\gamma,\ D_x,\ D_y)u\rangle\}$$

for $\gamma>\gamma_0$ and $u\in H^{\infty}(R^1\times R_+^n)$.

2.3. Adjoint problem ($\boldsymbol{P}^*$). Denoting

$$L(D_t,\ D_x,\ D_y)=\sum_{j=0}^{s}\mathcal{L}_j(D_t,\ D_y)D_x^{s-j},$$
$$(\ ,\ )=(\ ,\ )_{L^2(R^1\times R_+^n)}, \qquad \langle\ ,\ \rangle=\langle\ ,\ \rangle_{L^2(R^1\times R^{n-1})},$$

we have

$$(L(D_t-i\gamma,\ D_x,\ D_y)u,\ v)-(u,\ L^*(D_t+i\gamma,\ D_x,\ D_y)v)$$
$$=\sum_{j=0}^{s}\{(\mathcal{L}_j(D_t-i\gamma,\ D_y)D_x^{s-j}u,\ v)-(u,\ D_x^{s-j}\mathcal{L}_j^*(D_t+i\gamma,\ D_y)v)\}$$

$$=i\sum_{j=0}^{s-1}\sum_{k=0}^{s-j-1}\langle D_x^k u,\ D_x^{s-j-1-k}\mathcal{L}_j^*(D_t+i\gamma,\ D_y)v\rangle$$

$$=i\sum_{k=0}^{s-1}\langle D_x^k u,\ \sum_{j=0}^{s-k-1}D_x^{s-j-1-k}\mathcal{L}_j^*(D_t+i\gamma,\ D_y)v\rangle$$

$$=i\left\langle\begin{bmatrix}\boldsymbol{B}(D_t-i\gamma,\ D_xD_y)\\ C(D_y)\end{bmatrix}u,\ \ \begin{bmatrix}C'(D_t+i\gamma,\ D_x,\ D_y)\\ B'(D_t+i\gamma,\ D_x,\ D_y)\end{bmatrix}v\right\rangle.$$

Hence we have the adjoint problem:

$$(P^*)_\gamma\begin{cases}L^*(D_t+i\gamma,\ D_x,\ D_y)v=\phi & \text{in}\quad R^1\times R_+^n\,,\\ B'(D_t+i\gamma,\ D_x,\ D_y)v|_{x=0}=\psi & \text{on}\quad R^1\times R^{n-1}\,,\end{cases}$$

where we assume the Assumption (A.1) is satisfied also by ${}^tL$ (: Ass. (A*.1)). Denoting

$$B'(\bar\tau,\ \bar\xi,\ \bar\eta)=(b'_{ij}(\bar\tau,\ \bar\xi,\ \bar\eta))_{i=1,\cdots,m_-\ j=1,\cdots,N}\,,$$

we assume

Assumption (B*.1). $\qquad N_{b'_{ij}}\subset\tilde N'\,.$

Denoting $\boldsymbol{B}'=\begin{bmatrix}B'\\ \boldsymbol{L}^*\end{bmatrix}$, we define for $\operatorname{Im}\bar\tau\geqq0$ and $\bar\eta\in R^{n-1}$

$$\mathcal{B}'^{(k)}(\bar\tau,\ \bar\eta)=\left[\frac{1}{2\pi i}\int\frac{\boldsymbol{B}'^{\#(k)}(\bar\tau,\ \bar\xi,\ \bar\eta)\bar\xi^{j-1}}{\tilde A_-^{*(k)}(\bar\tau,\ \bar\xi,\ \bar\eta)}d\bar\eta\right]_{j=1,\cdots,\tilde m_1^-+\cdots+\tilde m_k^-}$$

$$b'^{(k)}=\left[\frac{1}{2\pi i}\int\frac{\boldsymbol{B}'^{\#(k)}(1,\ \bar\xi,\ 0)\bar\xi^{j-1}}{\tilde a_-^{*(k)}(1,\ \bar\xi)}d\bar\xi\right]_{j=1,\cdots,\tilde m_k^-},$$

and

$$\mathcal{R}'^{(k)}(\bar\tau,\ \bar\eta)=(\mathcal{B}'^{(k)}(\bar\tau,\ \bar\eta)b'^{(k+1)}\cdots b'^{(l)})\,,$$

where

$$\tilde A_-^{*(k)}(\bar\tau,\ \bar\xi,\ \bar\eta)=\overline{\tilde A_-^{(k)}(\tau,\ \xi,\ \eta)}\,,$$

$$\tilde a_-^{*(k)}(\bar\tau,\ \bar\xi)=\overline{\tilde a_-^{(k)}(\tau,\ \xi)}\,.$$

Now we assume

Assumption (B*.2). $\qquad \operatorname{rank}\mathcal{R}'^{(k)}(\bar\tau,\ \bar\eta)=\tilde m_-N$

for $\operatorname{Im}\bar\tau\geqq0$, $\bar\eta\in S^{n-2}(k=0,\ 1,\ \cdots,\ l)$.

Here we have

Proposition 2.5. *Under the assumptions* (A*.1), (A.2), (A.3) *and* (B*.1)–(B*.2), *we have the energy inequality for the problem* $(P^*)_\gamma$:

$$\gamma^{(1/2)q_1}|||v|||+\langle\!\langle v\rangle\!\rangle\leqq C\{\gamma^{-(1/2)q_1}\|\Lambda_{\tilde S}L^*(D_t+i\gamma,\ D_x,\ D_y)v\|$$
$$+\langle\Lambda_{\tilde R'}B'(D_t+i\gamma,\ D_x,\ D_y)v\rangle\}\quad(\gamma\geqq\gamma_0)\,.$$

By the usual technique, it follows the theorem stated in 1.1 from Proposition 2.3 and Proposition 2.5.

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