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The induced and intrinsic Finsler connections of a hypersurface and Finslerien projective geometry

Dedicated to Professor Dr. Jōyō Kanitani on the occasion of his ninetieth birthday

By

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The theory of Finsler subspaces may be developed after the model of Riemannian geometry. In the early years of Finsler geometry, closely following É. Cartan, made M. Haimovici fundamental and essential contributions to the theory. Since then, various interesting results on Finsler subspaces have been found by O. Varga, H. Rund and others. It seems, however, to the present author that there had to be unevitable obstructions to develope the theory of Finsler subspaces analogously to the Riemannian theory, and, as a consequence, almost all the existing literatures are not easy to understand and confused notations sometimes bewilder the readers. The first among those obstructions is perhaps surviving of quantities which are derived from Cartan's C-tensor, given by (1.9), and cause, for instance, the non-symmetry property of the second fundamental tensor. The second, a consequence of the first, is that the induced connection, defined by the projection, does not generally coincide with the intrinsic connection, determined from the induced Finsler metric, and that the former is beyond the usual concept of connection appearing in Finsler geometry.

The quantites, derived from the C-tensor, are rather useful for enriching the Finslerian theory and, in fact, we have Brown's interesting work which was devoted to studying the behavior of those quantities. The problem of induced connections is just the initial motive for the author in beginning the theory of subspaces. Theorems 5.1 and 6.2 are satisfactory answers of the problem from an axiomatic standpoint, based on the author's theory of Finsler connections, and propose new important problems.

Now a Riemannian space of constant curvature, as is well known, is characterized among Riemannian spaces by the property that there exists a totally geodesic subspace at each subspace-element. M. Haimovici is the first who was concerned with some generalizations of this property to Finsler geometry. After thirteen years S. Kikuchi solved part of this Haimovici's problem and finally A. Rapcsák might show nearly perfect solutions. The second main purpose of the present paper is to give perfect proofs of Haimovici-Kikuchi-Rapcsák's results and, in particular, to conclude Theorem 9.1. As a consequence, together with the author's previous result, it is pointed out that a projectively flat Finsler space of dimension more than two is to realize the projective geometry with respect to a rectilinear coordinate system.

The terminology and notations are referred to the author's monoraphs ([26], [28]) and especially the quotation from the latter [28] is sometimes indicated only by putting asterisk.

§ 1. The induced Finsler metric.

We consider an n-dimensional Finsler space $F^n=(M^n, L(x, y))$, a differentiable n-manifold M^n equipped with fundamental function $L(x, y)$ which is assumed to be (1)p-homogeneous in $y=(y^i)$, $y^i=\dot{x}^i$ ((22.6) of [26]; *Definition 12.1), and to yield the regular fundamental tensor field $g_{ij}(x, y)=(\dot{\partial}_i\dot{\partial}_j L^2)/2$. (Throughout the present paper, Latin indices take values $1, \cdots, n$.) We put $l_i=\dot{\partial}_i L$ as usual, but *Cartan's C-tensor* $C_{ijk}=(\dot{\partial}_k g_{ij})/2$ *is denoted by* g_{ijk} to avoid confusion.

A hypersurface M^{n-1} of the M^n may be represented parametrically by the equations $x^i=x^i(u^\alpha)$, $\alpha=1, \cdots, n-1$, where u^α are Gaussian coordinates on M^{n-1}. (Greek indices run from 1 to $n-1$.) We usually assume the matrix consisting of the so-called projection factors $B^i_\alpha=\partial x^i/\partial u^\alpha$ is of rank $n-1$. Then $B_\alpha(u)=(B^i_\alpha(u))$ may be regarded as $n-1$ linearly independent vectors tangent to M^{n-1} at the point (u^α) and a vector X^i tangent to M^{n-1} at the point may be expressed uniquely in the form $X^i=B^i_\alpha X^\alpha$, where X^α are components of the vector with respect to the coordinate system (u^α).

To introduce a Finsler structure in M^{n-1}, the supporting element y^i at a point (u^α) of M^{n-1} is assumed to be tangential to M^{n-1}, so that we may write

$$y^i=B^i_\alpha(u)v^\alpha\,. \tag{1.1}$$

Thus v^α is thought of as the supporting element of M^{n-1} at the point (u^α). Denoting y^i of (1.1) by $y^i(u, v)$, the function

$$\underline{L}(u, v):=L(x(u), y(u, v)) \tag{1.2}$$

gives rise to a Finsler metric of M^{n-1}. Consequently we get an $(n-1)$-dimensional Finsler space $F^{n-1}=(M^{n-1}, \underline{L}(u, v))$. The fundamental function $\underline{L}(u, v)$ of this Finslerian hypersurface F^{n-1} of F^n is called the *induced metric* on F^{n-1}.

In the following we employ the notations

$$B^i_{\alpha\beta}:=\partial_\beta B^i_\alpha\,,\quad B^i_{0\beta}:=v^\alpha B^i_{\alpha\beta}\,,\quad B^{ij\cdots}_{\alpha\beta\cdots}:=B^i_\alpha B^j_\beta\cdots\,.$$

It then follows from (1.1) that

$$\partial_\alpha=B^i_\alpha\partial_i+B^i_{0\alpha}\dot{\partial}_i\,,\qquad \dot{\partial}_\alpha=B^i_\alpha\dot{\partial}_i\,. \tag{1.3}$$

The induced metric $\underline{L}(u, v)$ yields $l_\alpha=\dot{\partial}_\alpha\underline{L}$, the metric tensor $g_{\alpha\beta}=(\dot{\partial}_\alpha\dot{\partial}_\beta\underline{L}^2)/2$ and Cartan's C-tensor $g_{\alpha\beta\gamma}=(\dot{\partial}_\gamma g_{\alpha\beta})/2$ of F^{n-1}. Paying attention to $\dot{\partial}_\beta B^i_\alpha=0$, from (1.2) we get

(1.4) $$l_\alpha=l_iB^i_\alpha\,,\quad g_{\alpha\beta}=g_{ij}B^{ij}_{\alpha\beta}\,,\quad g_{\alpha\beta\gamma}=g_{ijk}B^{ijk}_{\alpha\beta\gamma}\,.$$

At each point (u^α) of the F^{n-1}, a unit normal vector $B^i(u, v)$ is defined by

(1.5) $$g_{ij}(x(u),\ y(u,\ v))B^i_\alpha(u)B^j=0\,,\qquad g_{ij}(x(u),\ y(u,\ v))B^iB^j=1\,.$$

This normal vector $B^i(u, v)$ depends clearly on the supporting element $y^i(u, v)$ and so it should be said that we have a normal cone $B^i(u, v)$ at the point (u^α).

As for the angular metric tensor $h_{ij}=g_{ij}-l_il_j$, (1.4) and $l_iB^i=0$ yield

(1.6) $$h_{ij}B^{ij}_{\alpha\beta}=h_{\alpha\beta}\,,\quad h_{ij}B^i_\alpha B^j=0\,,\quad h_{ij}B^iB^j=1\,.$$

Remark. Pay attention to n^i and n^{*i} of Rund [10], p. 372 and [13], p. 190. See also the remark below the equation (1.17) of Rund [17].

Thus we get the regular matrix (B^i_α, B^i). Let (B^α_i, B_i) be the inverse matrix of (B^i_α, B^i); we have

(1.7) $$B^i_\alpha B^\beta_i=\delta^\beta_\alpha\,,\quad B^i_\alpha B_i=0\,,\quad B^iB^\alpha_i=0\,,\quad B^iB_i=1\,,$$

and further

(1.8) $$B^i_\alpha B^\alpha_j+B^iB_j=\delta^i_j\,.$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get $B^\alpha_i=g^{\alpha\beta}g_{ij}B^j_\beta$, $B_i=g_{ij}B^j$, and quantities $B_{\alpha i}:=B^j_\alpha g_{ij}=B^\beta_i g_{\alpha\beta}$ and $B^{\alpha i}:=B^\alpha_j g^{ij}=B^i_\beta g^{\alpha\beta}$ will be used later on.

Lemma 1.1. *Let $\underline{X}_{\alpha\beta\gamma}(u, v)$ be the projection $X_{ijk}B^{ijk}_{\alpha\beta\gamma}$ of a tensor $X_{ijk}(x, y)$ into F^{n-1}, and put $\underline{X}_{\alpha\beta}=X_{ijk}B^{ij}_{\alpha\beta}B^k$, $\underline{X}_\alpha=X_{ijk}B^i_\alpha B^jB^k$ and $\underline{X}=X_{ijk}B^iB^jB^k$. Then we have*

$$X_{ijk}B^{ij}_{\alpha\beta}=\underline{X}_{\alpha\beta\gamma}B^\gamma_k+\underline{X}_{\alpha\beta}B_k\,,\qquad X_{ijk}B^i_\alpha B^k=\underline{X}_{\alpha\beta}B^\beta_j+\underline{X}_\alpha B_j\,,$$
$$X_{ijk}B^jB^k=\underline{X}_\alpha B^\alpha_i+\underline{X}B_i\,.$$

Proof. From (1.7) we have

$$\underline{X}_{\alpha\beta}B^\beta_j=(X_{ihk}B^{ih}_{\alpha\beta}B^k)B^\beta_j=X_{ihk}B^i_\alpha B^k(\delta^h_j-B^hB_j)=X_{ijk}B^i_\alpha B^k-\underline{X}_\alpha B^j\,.$$

The similar way shows the other two equations. Q.E.D.

We now introduce important tensors from Cartan's C-tensor g_{ijk}:

(1.9) $$M_{\alpha\beta}=g_{ijk}B^{ij}_{\alpha\beta}B^k\,,\quad M_\alpha=g_{ijk}B^i_\alpha B^jB^k\,,\quad M=g_{ijk}B^iB^jB^k\,.$$

From Lemma 1.1 we immediately get

(1.10) $$g_{ijk}B^{jk}_{\alpha\beta}=g_{\alpha\beta\gamma}B^\gamma_i+M_{\alpha\beta}B_i\,,\qquad g_{ijk}B^j_\alpha B^k=M_{\alpha\beta}B^\beta_i+M_\alpha B_i\,,$$
$$g_{ijk}B^jB^k=M_\alpha B^\alpha_i+MB_i\,.$$

Next, differentiating (1.5) by v^β, we get

$$2g_{ijk}B^{ik}_{\alpha\beta}B^j+g_{ij}B^i_\alpha\dot{\partial}_\beta B^j=0\,,\qquad g_{ijk}B^iB^jB^k_\beta+g_{ij}B^i\dot{\partial}_\beta B^j=0\,,$$

that is to say,

$$2M_{\alpha\beta}+B_{\alpha j}\dot{\partial}_\beta B^j=0\,,\qquad M_\beta+B_j\dot{\partial}_\beta B^j=0\,.$$

Thus, putting $M^\alpha_\beta=g^{\alpha\gamma}M_{\gamma\beta}$, we get

(1.11) $$\dot{\partial}_\beta B^j=-2M^\alpha_\beta B^j_\alpha-M_\beta B^j\,.$$

Proposition 1.1. (1) *The v-dependence of* B^α_i *and* B_i *are shown by*

$$\dot{\partial}_\beta B^\alpha_i=2M^\alpha_\beta B_i\,,\qquad \dot{\partial}_\beta B_i=M_\beta B_i\,.$$

(2) *Putting* $G=\det(g_{ij})$ *and* $\underline{G}=\det(g_{\alpha\beta})$, *we have* $M_\beta=\{\dot{\partial}_\beta\log(G/\underline{G})\}/2$.

Proof. (1) It follows from (1.10) that

$$\dot{\partial}_\beta B^\alpha_i=\dot{\partial}_\beta(g^{\alpha\gamma}g_{ij}B^j_\gamma)=(-2g^{\alpha\gamma}_\beta g_{ij}+2g^{\alpha\gamma}g_{ijk}B^k_\beta)B^j_\gamma$$
$$=-2g^{\alpha}_{\beta\gamma}B^\gamma_i+2g^{\alpha\gamma}(g_{\beta\gamma\delta}B^\delta_i+M_{\beta\gamma}B_i)=2M^\alpha_\beta B_i\,.$$

Similarly we have

$$\dot{\partial}_\beta B_i=\dot{\partial}_\beta(g_{ij}B^j)=2g_{ijk}B^k_\beta B^j+g_{ij}(-2M^\alpha_\beta B^j_\alpha-M_\beta B^j)$$
$$=2(M^\alpha_\beta B_{\alpha i}+M_\beta B_i)-2M^\alpha_\beta B_{\alpha i}-M_\beta B_i=M_\beta B_i\,.$$

(2) It is well-known that $(\dot{\partial}_i\log G)/2=g_i$ $(=g^j_{ij})$ (*(24.1)). From (1.8) we have

$$M_\beta=g_{ijk}B^i_\beta(g^{jk}-g^{\gamma\delta}B^j_\gamma B^k_\delta)=g_iB^i_\beta-g_{\beta\gamma\delta}g^{\gamma\delta}\,,$$

that is,

(1.12) $$M_\beta=g_iB^i_\beta-g_\beta\,,$$

which is the equation we should prove.

Finally we shall show

(1.13) $$\dot{\partial}_\gamma M^\alpha_\beta-\dot{\partial}_\beta M^\alpha_\gamma+M^\alpha_\beta M_\gamma-M^\alpha_\gamma M_\beta=0\,,$$

which is equivalent to

(1.13′) $$\dot{\partial}_\gamma M_{\alpha\beta}-\dot{\partial}_\beta M_{\alpha\gamma}+M_{\delta\gamma}(2g^{\,\delta}_{\alpha\beta}-\delta^\delta_\alpha M_\beta)-M_{\delta\beta}(2g^{\,\delta}_{\alpha\gamma}-\delta^\delta_\alpha M_\gamma)=0\,.$$

It follows from (1.11) that

$$\dot{\partial}_\gamma M_{\alpha\beta}=\dot{\partial}_\gamma(g_{hij}B^{hi}_{\alpha\beta}B^j)=\dot{\partial}_kg_{hij}B^{hik}_{\alpha\beta\gamma}B^j-2g^{\,\delta}_{\alpha\beta}M_{\delta\gamma}-M_{\alpha\beta}M_\gamma\,,$$

which implies (1.13′) immediately.

Remark. It seems that Rund [19] especially paid attention to great importance of M's defined by (1.9), and Brown [20] studied them in detail. It is noted that their M's are equal to our LM's, (0)p-homogenized tensors. (1.11) and facts mentioned in Proposition 1.1 were all shown by Brown [20]. As to (1.13), see Theorem 4.1 of [20].

It is hard to understand that the second term of the right-hand side of (1.11) does not appear in the equation (17) of Davies [8]. His reasoning for (17) seems to come from $\dot{\partial}_\beta B_i=0$ (in our notations), contrary to our result. Following Varga [14] and [15], if a hypersurface M^{n-1} is to be given by an equation $\Phi(x^1, \cdots, x^n)=0$, we get $\partial_\alpha \Phi(x(u))=\Phi_i B^i_\alpha=0$ $(\Phi_i=\partial_i \Phi)$, and so $B_i=\Phi_i(x(u))/C$ where C is the length $\{g^{ij}(x(u), y(u, v))\Phi_i \Phi_j\}^{1/2}$ of Φ_i relative to a supporting element v^α. Therefore B_i certainly depends on v^α.

§ 2. Induced Finsler connection.

We are concerned with a Finsler space $(F^n, F\Gamma)$ equipped with a Finsler connection $F\Gamma=(F^i_{jk}, N^i_j, C^i_{jk})$ (cf. [26], [28]). In this section we have not to do with any relation between the Finsler metric $L(x, y)$ and the connection $F\Gamma$. Simply speaking for the following use, the Finsler connection $F\Gamma$ is such that the absolute differential Dy^i of supporting element y^i is given by

$$Dy^i=dy^i+N^i_j(x, y)dx^j, \tag{2.1}$$

and the absolute differential DX^i of a Finsler vector field $X^i(x, y)$ is

$$DX^i=dX^i+X^j\{\Gamma^i_{jk}(x, y)dx^k+C^i_{jk}(x, y)dy^k\}, \tag{2.2}$$

where we put $\Gamma^i_{jk}=F^i_{jk}+C^i_{jh}N^h_k$. If $DX^i=0$ along a curve $(x^i(t), y^i(t))$ of the tangent bundle $T(M^n)$, X^i is said to be parallel along the curve $(x^i(t))$ of M^n with respect to the supporting element $y^i(t)$. In terms of dx^k and Dy^k we have

$$DX^i=X^i_{|k}dx^k+X^i|_k Dy^k. \tag{2.2'}$$

The h- and v-covariant derivatives $X^i_{|k}$, $X^i|_k$ of X^i are defined by

$$X^i_{|k}=\delta_k X^i+X^h F^i_{hk}, \qquad X^i|_k=\dot{\partial}_k X^i+X^h C^i_{hk}, \tag{2.3}$$

where $\delta_k=\partial_k-N^h_k\dot{\partial}_h$.

Let $F^{n-1}=(M^{n-1}, \underline{L}(u, v))$ be a Finslerian hypersurface of the F^n. It is noted that, along a curve $(u^\alpha(t))$ of F^{n-1}, dx^k and dy^k of (2.2) are written as $dx^k=B^k_\alpha(u)du^\alpha$ and $dy^k=B^k_{0\alpha}(u)du^\alpha+B^k_\alpha(u)dv^\alpha$.

Definition. The *induced* (*Finsler*) *connection* $IF\Gamma$ on a hypersurface M^{n-1} of a Finsler space $(F^n, F\Gamma)$ is such that the absolute differential Dv^α (resp. DX^α) of the supporting element v^α (resp. a Finsler vector field X^α) is given by $Dv^\alpha=B^\alpha_i Dy^i$ (resp. $DX^\alpha=B^\alpha_i DX^i$), where $y^i=B^i_\alpha v^\alpha$ (resp. $X^i=B^i_\alpha X^\alpha$).

Putting $IF\Gamma=(F^\alpha_{\beta\gamma}, N^\alpha_\beta, C^\alpha_{\beta\gamma})$ and $\Gamma^\alpha_{\beta\gamma}=F^\alpha_{\beta\gamma}+C^\alpha_{\beta\delta}N^\delta_\gamma$, we have $Dv^\alpha=dv^\alpha+N^\alpha_\beta du^\beta$, which is defined as

$$B^\alpha_i(dy^i+N^i_j dx^j)=B^\alpha_i\{(B^i_{0\beta}du^\beta+B^i_\beta dv^\beta)+N^i_j B^j_\beta du^\beta\},$$

which implies

(2.4) $$N^{\alpha}_{\beta}=B^{\alpha}_{i}(B^{i}_{0\beta}+N^{i}_{j}B^{j}_{\beta}).$$

Next $DX^{\alpha}=dX^{\alpha}+X^{\beta}(\Gamma^{\alpha}_{\beta\gamma}du^{\gamma}+C^{\alpha}_{\beta\gamma}dv^{\gamma})$ is written as

$$B^{\alpha}_{i}\{dX^{i}+X^{j}(\Gamma^{i}_{jk}dx^{k}+C^{i}_{jk}dy^{k})\}$$
$$=B^{\alpha}_{i}[(B^{i}_{\beta\gamma}X^{\beta}du^{\gamma}+B^{i}_{\beta}dX^{\beta})+B^{j}_{\beta}X^{\beta}\{\Gamma^{i}_{jk}B^{k}_{\gamma}du^{\gamma}+C^{i}_{jk}(B^{k}_{0\gamma}du^{\gamma}+B^{k}_{\gamma}dv^{\gamma})\}],$$

which implies

(2.5) $$\Gamma^{\alpha}_{\beta\gamma}=B^{\alpha}_{i}\{B^{i}_{\beta\gamma}+B^{j}_{\beta}(\Gamma^{i}_{jk}B^{k}_{\gamma}+C^{i}_{jk}B^{k}_{0\gamma})\},$$

(2.6) $$C^{\alpha}_{\beta\gamma}=B^{\alpha}_{i}C^{i}_{jk}B^{jk}_{\beta\gamma}.$$

If we put

(2.7) $$H_{\beta}:=B_{i}(B^{i}_{0\beta}+N^{i}_{j}B^{j}_{\beta}),$$

this, together with (2.4), yields

(2.8) $$B^{i}_{0\gamma}+N^{i}_{j}B^{j}_{\gamma}=N^{\alpha}_{\gamma}B^{i}_{\alpha}+H_{\gamma}B^{i}.$$

Therefore (2.5) is now rewritten as

(2.9) $$F^{\alpha}_{\beta\gamma}=B^{\alpha}_{i}\{B^{i}_{\beta\gamma}+B^{j}_{\beta}(F^{i}_{jk}B^{k}_{\gamma}+C^{i}_{jk}B^{k}H_{\gamma})\}.$$

Consequently we may conclude

Proposition 2.1. *The induced connection $IF\Gamma=(F^{\alpha}_{\beta\gamma}, N^{\alpha}_{\beta}, C^{\alpha}_{\beta\gamma})$ on a hypersurface M^{n-1} of the Finsler space $(F^{n}, F\Gamma)$ is given by* (2.9), (2.4) *and* (2.6).

Similarly to (2.7), if we put

(2.10) $$H_{\beta\gamma}:=B_{i}\{B^{i}_{\beta\gamma}+B^{j}_{\beta}(F^{i}_{jk}B^{k}_{\gamma}+C^{i}_{jk}B^{k}H_{\gamma})\},$$

this, together with (2.9), leads to

(2.11) $$B^{i}_{\beta\gamma}+B^{j}_{\beta}(F^{i}_{jk}B^{k}_{\gamma}+C^{i}_{jk}B^{k}H_{\gamma})=F^{\alpha}_{\beta\gamma}B^{i}_{\alpha}+H_{\beta\gamma}B^{i}.$$

The Finsler vector field H_{β}, defined by (2.7), will be called the *normal curvature vector* due to its geometrical property (cf. (7.5)). On the other hand, Finsler tensor field $H_{\beta\gamma}$, defined by (2.10), will be called the *second fundamental h-tensor* (cf. (3.5)) by the analogy of Riemannian geometry.

The tangential component $B^{\beta}_{i}Dy^{i}$ of Dy^{i} is by definition equal to Dv^{β}, while the normal component $B_{i}Dy^{i}$ is

$$B_{i}Dy^{i}=B_{i}(dy^{i}+N^{i}_{j}dx^{j})=B_{i}(B^{i}_{0\beta}+N^{i}_{j}B^{j}_{\beta})du^{\beta}.$$

Therefore from (2.7) we get

(2.12) $$Dy^{i}=Dv^{\beta}B^{i}_{\beta}+H_{\beta}du^{\beta}B^{i}.$$

Next, substituting from (2.8) and (2.11), we get

$$B^{i}_{\beta\gamma}+B^{j}_{\beta}(\Gamma^{i}_{jk}B^{k}_{\gamma}+C^{i}_{jk}B^{k}_{0\gamma})=C^{i}_{jk}B^{jk}_{\beta\delta}N^{\delta}_{\gamma}+F^{\alpha}_{\beta\gamma}B^{i}_{\alpha}+H_{\beta\gamma}B^{i},$$

which implies

$$B_i\{B^i_{\beta\gamma}+B^j_\beta(\Gamma^i_{jk}B^k_\gamma+C^i_{jk}B^k_{0\gamma})\}=B_iC^i_{jk}B^{jk}_{\beta\delta}N^\delta_\gamma+H_{\beta\gamma}\,.$$

Thus, if we put

$$K_{\beta\delta}:=B_iC^i_{jk}B^{jk}_{\beta\delta}\,, \tag{2.13}$$

the above, together with (2.5), yields

$$B^i_{\beta\gamma}+B^j_\beta(\Gamma^i_{jk}B^k_\gamma+C^i_{jk}B^k_{0\gamma})=\Gamma^\alpha_{\beta\gamma}B^i_\alpha+(H_{\beta\gamma}+K_{\beta\delta}N^\delta_\gamma)B^i\,.$$

Therefore, similarly to the case of Dy^i, we get

$$DX^i=DX^\alpha B^i_\alpha+X^\alpha(H_{\alpha\beta}du^\beta+K_{\alpha\beta}Dv^\beta)B^i\,. \tag{2.14}$$

The Finsler tensor field $K_{\beta\delta}$, defined by (2.13), will be called the *second fundamental v-tensor* (cf. (3.6)).

We shall be concerned with the torsion tensors and other important tensors of the induced connection $IF\Gamma$ and F^{n-1}. First, taking the skew-symmetric part of (2.11) in β and γ, we get

$$T^i_{jk}B^{jk}_{\beta\gamma}+C^i_{jk}(B^j_\beta H_\gamma-B^j_\gamma H_\beta)B^k=T^\alpha_{\beta\gamma}B^i_\alpha+(H_{\beta\gamma}-H_{\gamma\beta})B^i\,, \tag{2.15}$$

where $T^i_{jk}=F^i_{jk}-F^i_{kj}$ is the *(h)h-torsion tensor* of the original $F\Gamma$. Thus the $(h)h$-torsion tensor $T^\alpha_{\beta\gamma}=F^\alpha_{\beta\gamma}-F^\alpha_{\gamma\beta}$ of $IF\Gamma$ is given by

$$T^\alpha_{\beta\gamma}=B^\alpha_i\{T^i_{jk}B^{jk}_{\beta\gamma}+C^i_{jk}(B^j_\beta H_\gamma-B^j_\gamma H_\beta)B^k\}\,, \tag{2.16}$$

and we have

$$H_{\beta\gamma}-H_{\gamma\beta}=B_i\{T^i_{jk}B^{jk}_{\beta\gamma}+C^i_{jk}(B^j_\beta H_\gamma-B^j_\gamma H_\beta)B^k\}\,. \tag{2.17}$$

Secondly, contracting (2.11) by v^β and subtracting (2.8) from it, we get the *deflection tensor* $D^\alpha_\gamma=v^\beta F^\alpha_{\beta\gamma}-N^\alpha_\gamma$ and $H_{0\gamma}-H_\gamma(H_{0\gamma}:=v^\beta H_{\beta\gamma})$ as follows:

$$D^\alpha_\gamma=B^\alpha_i(D^i_kB^k_\gamma+C^i_{0k}B^kH_\gamma)\,, \tag{2.18}$$

$$H_{0\gamma}-H_\gamma=B_i(D^i_kB^k_\gamma+C^i_{0k}B^kH_\gamma)\,, \tag{2.19}$$

where D^i_k is the deflection tensor $y^jF^i_{jk}-N^i_k$ of $F\Gamma$.

Thirdly, differentiating (2.8) by v^β and substituting from (1.11), we get

$$B^i_{\beta\gamma}+\dot\partial_kN^i_jB^{jk}_{\gamma\beta}=(\dot\partial_\beta N^\alpha_\gamma-2M^\alpha_\beta H_\gamma)B^i_\alpha+(\dot\partial_\beta H_\gamma-M_\beta H_\gamma)B^i\,.$$

Therefore, comparing this with (2.11), we have the *(v)hv-torsion tensor* $P^\alpha_{\gamma\beta}=\dot\partial_\beta N^\alpha_\gamma-F^\alpha_{\beta\gamma}$ and $\dot\partial_\beta H_\gamma-H_{\beta\gamma}$ as follows:

$$P^\alpha_{\gamma\beta}=2H_\gamma M^\alpha_\beta+B^\alpha_i(P^i_{jk}B^{jk}_{\gamma\beta}-H_\gamma C^i_{jk}B^j_\beta B^k)\,, \tag{2.20}$$

$$\dot\partial_\beta H_\gamma-H_{\beta\gamma}=M_\beta H_\gamma+B_i(P^i_{jk}B^{jk}_{\gamma\beta}-H_\gamma C^i_{jk}B^j_\beta B^k)\,, \tag{2.21}$$

where P^i_{jk} is the $(v)hv$-torsion tensor $\dot\partial_kN^i_j-F^i_{kj}$ of $F\Gamma$.

In general a Finsler connection $F\Gamma$ has five torsion tensors ([26], [28]). The $(h)h$-torsion tensor $T=(T^i_{jk})$, *(h)hv-torsion tensor* $C=(C^i_{jk})$, and $(v)hv$-torsion tensor $P^1=(P^i_{jk})$ have appeared in the present section. The $(v)v$-torsion tensor $S^1=(S^i_{jk})$ is solely $S^i_{jk}=C^i_{jk}-C^i_{kj}$; it follows from (2.6) that the *(v)v-torsion tensor*

$S^{\alpha}_{\beta\gamma}$ of $IF\Gamma$ is given by

$$S^{\alpha}_{\beta\gamma}=B^{\alpha}_{i}S^{i}_{jk}B^{jk}_{\beta\gamma}. \tag{2.22}$$

Remark. We have interesting three pairs {(2.16), (2.17)}, {(2.18), (2.19)} and {(2.20), (2.21)} of equations. The first equations of these pairs concern $F^{\alpha}_{\beta\gamma}$ and N^{α}_{β}:

$$T^{\alpha}_{\beta\gamma}=F^{\alpha}_{\beta\gamma}-F^{\alpha}_{\gamma\beta}, \quad D^{\alpha}_{\beta}=F^{\alpha}_{0\beta}-N^{\alpha}_{\beta}, \quad P^{\alpha}_{\beta\gamma}=\dot{\partial}_{\gamma}N^{\alpha}_{\beta}-F^{\alpha}_{\gamma\beta},$$

and the second equations present the corresponding forms of $H_{\beta\gamma}$ and H_{β}:

$$H_{\beta\gamma}-H_{\gamma\beta}, \quad H_{0\beta}-H_{\beta}, \quad \dot{\partial}_{\gamma}H_{\beta}-H_{\gamma\beta}.$$

It should be remarked that these quantities survive in some Finsler connection (cf. §§ 5, 6), contrary to the Riemannian case.

§ 3. Relative covariant differentiations.

We have to consider the integrability conditions of equations (1.11), (2.8), (2.11) and so on, and then the curvature tensors of induced connection $IF\Gamma$ will appear in those conditions. To do so we shall introduce the so-called relative covariant differentiations.

The projection factors $B^{i}_{\alpha}(u)$ are quantities which behave as components of $n-1$ contravariant vectors $B_{\alpha}=(B^{i}_{\alpha})$ in F^{n} and also those of n covariant vectors $B^{i}=(B^{i}_{\alpha})$ in F^{n-1}. We are concerned with such a set Y of functions $Y^{i\alpha}_{j\beta}(u, v)$ defined along F^{n-1}. The relative h- and v-covariant derivatives of Y are defined as follows:

First the *relative h-covariant derivative* is

$$Y^{i\alpha}_{j\beta|\gamma}:=\delta_{\gamma}Y^{i\alpha}_{j\beta}+Y^{k\alpha}_{j\beta}F^{i}_{k\gamma}-Y^{i\alpha}_{k\beta}F^{k}_{j\gamma}+Y^{i\delta}_{j\beta}F^{\alpha}_{\delta\gamma}-Y^{i\alpha}_{j\delta}F^{\delta}_{\beta\gamma}, \tag{3.1}$$

where $\delta_{\gamma}=\partial_{\gamma}-N^{\delta}_{\gamma}\dot{\partial}_{\delta}$ are δ-differentiation with respect to the nonlinear connection N^{α}_{β} of the induced connection $IF\Gamma$, $F^{\alpha}_{\beta\gamma}$ are connection coefficients of $IF\Gamma$ and $F^{i}_{k\gamma}$ are so-called mixed connection coefficients given by

$$F^{i}_{k\gamma}:=F^{i}_{kj}B^{j}_{\gamma}+C^{i}_{kj}B^{j}H_{\gamma}, \tag{3.2}$$

which appear in (2.11). Secondly the *relative v-covariant derivative* is

$$Y^{i\alpha}_{j\beta}|_{\gamma}:=\dot{\partial}_{\gamma}Y^{i\alpha}_{j\beta}+Y^{k\alpha}_{j\beta}C^{i}_{k\gamma}-Y^{i\alpha}_{k\beta}C^{k}_{j\gamma}+Y^{i\delta}_{j\beta}C^{\alpha}_{\delta\gamma}-Y^{i\alpha}_{j\delta}C^{\delta}_{\beta\gamma}, \tag{3.3}$$

where $C^{\alpha}_{\beta\gamma}$ are connection coefficients of $IF\Gamma$ and $C^{i}_{k\gamma}$ are mixed connection coefficients given by

$$C^{i}_{k\gamma}:=C^{i}_{kj}B^{j}_{\gamma}. \tag{3.4}$$

Remark. The covariant differentiations with respect to both of connection of an enveloping space F^{n} and of induced connection were first studied in Finsler geometry by Hombu [4] and applied to various geometrical theories by Varga [7], Davies [8] and others. See (V.4.13) of Rund [13].

We first apply the h-covariant differentiation to the projection factors B^i_α and obtain

$$B^i_{\alpha|\beta}=H_{\alpha\beta}B^i\,, \tag{3.5}$$

which is nothing but (2.11). As to the v-covariant derivative $B^i_\alpha|_\beta$ of B^i_α, it is observed from $\dot{\partial}_\beta B^i_\alpha=0$ and (3.4) that $B^i_\beta|_\gamma=C^i_{jk}B^j_\beta B^k_\gamma-C^\alpha_{\beta\gamma}B^i_\alpha$. Therefore Lemma 1.1 and (2.13) lead to

$$B^i_\beta|_\gamma=K_{\beta\gamma}B^i\,. \tag{3.6}$$

Now we consider the δ-differentiation δ_β with respect to $IF\Gamma$. It follows from (1.3) and (2.8) that

$$\delta_\beta=B^i_\beta\delta_i+B^iH_\beta\dot{\partial}_i\,. \tag{3.7}$$

Let $Y^i_j(x, y)$ be a tensor field of F^n. Along F^{n-1} we easily obtain

$$Y^i_{j|\beta}=Y^i_{j|k}B^k_\beta+Y^i_j|_kB^kH_\beta\,,\qquad Y^i_j|_\beta=Y^i_j|_kB^k_\beta\,. \tag{3.8}$$

Applying this to the fundamental tensor g_{ij}, we get

$$g_{ij|\beta}=g_{ij|k}B^k_\beta+g_{ij}|_kB^kH_\beta\,,\qquad g_{ij}|_\beta=g_{ij}|_kB^k_\beta\,. \tag{3.9}$$

We now consider the relative covariant derivatives of the unit normal vector $B^i(u, v)$. (1.5) gives

$$g_{ij|\beta}B^i_\alpha B^j+g_{ij}H_{\alpha\beta}B^iB^j+B_{\alpha j}B^j_{|\beta}=0\,,\qquad g_{ij|\beta}B^iB^j+2B_jB^j_{|\beta}=0\,,$$

which imply

$$B^i_{|\beta}=-H_{\alpha\beta}B^{\alpha i}+g_{jk|\beta}(B^iB^j/2-g^{ij})B^k\,, \tag{3.10}$$

Similar way leads us to

$$B^i|_\beta=-K_{\alpha\beta}B^{\alpha i}+g_{jk}|_\beta(B^iB^j/2-g^{ij})B^k\,. \tag{3.11}$$

It is easily verified by means of Lemma 1.1 and (3.9) that (3.11) is equivalent to (1.11).

We shall deal with the induced metric $\underline{L}(u, v)$ of F^{n-1}. $\underline{L}_{|\alpha}=\delta_\alpha\underline{L}$ and (3.7) yield $\underline{L}_{|\beta}=B^i_\beta\delta_iL+B^iH_\beta\dot{\partial}_iL$. Since $l_i=\dot{\partial}_iL$ is orthogonal to the unit normal vector B^i, we get

$$\underline{L}_{|\beta}=B^i_\beta L_{|i}\,. \tag{3.12}$$

Next (1.4) and (3.5) give $g_{\alpha\beta|\gamma}=g_{ij|\gamma}B^{ij}_{\alpha\beta}$. Therefore (3.9) leads to the first equation of the following (3.13). Similarly we obtain

$$g_{\alpha\beta|\gamma}=g_{ij|k}B^{ijk}_{\alpha\beta\gamma}+g_{ij}|_kB^{ij}_{\alpha\beta}B^kH_\gamma\,,\qquad g_{\alpha\beta}|_\gamma=g_{ij}|_kB^{ijk}_{\alpha\beta\gamma}\,. \tag{3.13}$$

Proposition 3.1. (1) *If a Finsler connection $F\Gamma$ of the enveloping space F^n satisfies $L_{|i}=0$, the induced connection $IF\Gamma$ of any hypersurface F^{n-1} does $\underline{L}_{|\beta}=0$.*

(2) *If $F\Gamma$ is metrical ($g_{ij|k}=g_{ij}|_k=0$), $IF\Gamma$ is also metrical.*

§ 4. Generalization of the Gauss and Codazzi equations.

We are concerned with commutation formulas of relative covariant differentiations, generalizations of the Ricci identities. First we treat a scalar field $Y(x, y)$ of the enveloping space F^n, to which the Ricci identity with respect to the induced connection $IF\Gamma$ applied:

$$Y_{|\beta|\gamma}-Y_{|\gamma|\beta}=-Y_{|\alpha}T^{\alpha}_{\beta\gamma}-Y|_{\alpha}R^{\alpha}_{\beta\gamma}.$$

By means of (3.8) and (3.5) the left-hand side is written in the form

$$\begin{aligned}
&\mathfrak{A}_{(\beta\gamma)}\{(Y_{|j}B^j_\beta+Y|_k B^k H_\beta)_{|\gamma}\}\\
&\quad=\mathfrak{A}_{(\beta\gamma)}\{(Y_{|j|k}B^k_\gamma+Y_{|j}|_k B^k H_\gamma)B^j_\beta+Y_{|j}H_{\beta\gamma}B^j\\
&\qquad+(Y|_{k|j}B^j_\gamma+Y|_k|_j B^j H_\gamma)B^k H_\beta+Y|_k(B^k_{|\gamma}H_\beta+B^k H_{\beta|\gamma})\}\\
&\quad=(Y_{|j|k}-Y_{|k|j})B^{jk}_{\beta\gamma}+(Y_{|j}|_k-Y|_{k|j})(B^j_\beta H_\gamma-B^j_\gamma H_\beta)B^k+Y_{|j}(H_{\beta\gamma}-H_{\gamma\beta})B^j\\
&\qquad+Y|_k\mathfrak{A}_{(\beta\gamma)}\{H_\beta B^k_{|\gamma}+H_{\beta|\gamma}B^k\}. \qquad \text{(See *Remark 5.1.)}
\end{aligned}$$

Then, by applying the Ricci identities of $F\Gamma$, we have

$$\begin{aligned}
&=(-Y_{|i}T^i_{jk}-Y|_i R^i_{jk})B^{jk}_{\beta\gamma}+(-Y_{|i}C^i_{jk}-Y|_i P^i_{jk})(B^j_\beta H_\gamma-B^j_\gamma H_\beta)B^k\\
&\qquad+(H_{\beta\gamma}-H_{\gamma\beta})Y_{|i}B^i+Y|_i\mathfrak{A}_{(\beta\gamma)}\{H_\beta B^i_{|\gamma}+H_{\beta|\gamma}B^i\}.
\end{aligned}$$

On the other hand, the right-hand side is written as

$$-(Y_{|i}B^i_\alpha+Y|_i B^i H_\alpha)T^\alpha_{\beta\gamma}-Y|_i B^i_\alpha R^\alpha_{\beta\gamma}.$$

Therefore, equating the terms containing $Y_{|i}$, we obtain (2.15). Equating the terms containing $Y|_i$, we get

(4.1) $\quad B^i_\alpha R^\alpha_{\beta\gamma}+B^i H_\alpha T^\alpha_{\beta\gamma}=R^i_{jk}B^{jk}_{\beta\gamma}+P^i_{jk}(B^j_\beta H_\gamma-B^j_\gamma H_\beta)B^k-\mathfrak{A}_{(\beta\gamma)}\{H_\beta B^i_{|\gamma}+H_{\beta|\gamma}B^i\},$

where the tensor $B^i_{|\gamma}$ is already given by (3.10).

Secondly, from the Ricci identity

$$Y_{|\beta}|_\gamma-Y|_{\gamma|\beta}=-Y_{|\alpha}C^\alpha_{\beta\gamma}-Y|_\alpha P^\alpha_{\beta\gamma}$$

we obtain

(4.2) $\quad B^i_\alpha C^\alpha_{\beta\gamma}=C^i_{jk}B^{jk}_{\beta\gamma}-K_{\beta\gamma}B^i,$

(4.3) $\quad B^i_\alpha P^\alpha_{\beta\gamma}+B^i H_\alpha C^\alpha_{\beta\gamma}=P^i_{jk}B^{jk}_{\beta\gamma}+S^i_{jk}B^j H_\beta H^k_\gamma-B_\beta B^i|_\gamma-(H_\beta|_\gamma-H_{\gamma\beta})B^i.$

The former is clear from (2.6), (2.13) and Lemma 1.1. The term $B^i|_\gamma$ of the latter is already given by (3.11).

Thirdly the Ricci identity

$$Y|_\beta|_\gamma-Y|_\gamma|_\beta=-Y|_\alpha S^\alpha_{\beta\gamma}$$

immediately gives

(4.4) $\quad B^i_\alpha S^\alpha_{\beta\gamma}=S^i_{jk}B^{jk}_{\beta\gamma}-(K_{\beta\gamma}-K_{\gamma\beta})B^i,$

which is only a consequence of (4.2). Cf. (2.22).

Now we shall deal with a vector field Y^i_α, and first consider $Y^i_{\alpha|\beta|\gamma}-Y^i_{\alpha|\gamma|\beta}$. Direct calculation (cf. *(10.11)) leads to

$$Y^i_{\alpha|\beta|\gamma}-Y^i_{\alpha|\gamma|\beta}=Y^j_\alpha K_j{}^i{}_{\beta\gamma}-Y^i_\delta K_\alpha{}^\delta{}_{\beta\gamma}-Y^i_{\alpha|\delta}T^\delta_{\beta\gamma}-(\dot{\partial}_\delta Y^i_\alpha)R^\delta_{\beta\gamma},$$

where $K_\alpha{}^\delta{}_{\beta\gamma}$ is the K-tensor of $IF\Gamma$ (cf. *(10.17)), i.e.,

$$K_\alpha{}^\delta{}_{\beta\gamma}=\mathfrak{A}_{(\beta\gamma)}\{\delta_\gamma F_\alpha{}^\delta{}_\beta+F_\alpha{}^\varepsilon{}_\beta F_\varepsilon{}^\delta{}_\gamma\}=R_\alpha{}^\delta{}_{\beta\gamma}-C_\alpha{}^\delta{}_\varepsilon R^\varepsilon_{\beta\gamma},$$

and $K_j{}^i{}_{\beta\gamma}$ is an analogous tensor to the K-tensor which is constructed from $F_j{}^i{}_\beta$ defined by (3.2):

$$K_j{}^i{}_{\beta\gamma}:=\mathfrak{A}_{(\beta\gamma)}\{\delta_\gamma F_j{}^i{}_\beta+F_j{}^k{}_\beta F_k{}^i{}_\gamma\}. \tag{4.5}$$

To consider this tensor $K_j{}^i{}_{\beta\gamma}$, we refer to (3.2) and (3.7), and substitute from (2.15) and (4.1). Then we get

$$\begin{aligned}\mathfrak{A}_{(\beta\gamma)}\{\delta_\gamma F_j{}^i{}_\beta\}=&(\delta_h F_j{}^i{}_k-\delta_k F_j{}^i{}_h+C_j{}^i{}_l R^l_{kh})B^{kh}_{\beta\gamma}-C_j{}^i{}_\alpha R^\alpha_{\beta\gamma}\\&+(\dot{\partial}_h F_j{}^i{}_k-\delta_k C_j{}^i{}_h+C_j{}^i{}_l\dot{\partial}_h N^l_k)(B^k_\beta H_\gamma-B^k_\gamma H_\beta)B^h.\end{aligned}$$

Therefore we have

$$\begin{aligned}&K_j{}^i{}_{\beta\gamma}=R_j{}^i{}_{\beta\gamma}-C_j{}^i{}_\alpha R^\alpha_{\beta\gamma},\\&R_j{}^i{}_{\beta\gamma}:=R_j{}^i{}_{kh}B^{kh}_{\beta\gamma}+P_j{}^i{}_{kh}(B^k_\beta H_\gamma-B^k_\gamma H_\beta)B^h.\end{aligned} \tag{4.5'}$$

Consequently we get one of the relative Ricci identities:

$$Y^i_{\alpha|\beta|\gamma}-Y^i_{\alpha|\gamma|\beta}=Y^j_\alpha R_j{}^i{}_{\beta\gamma}-Y^i_\delta R_\alpha{}^\delta{}_{\beta\gamma}-Y^i_{\alpha|\delta}T^\delta_{\beta\gamma}-Y^i_\alpha|_\delta R^\delta_{\beta\gamma}. \tag{4.6}$$

This form is quite similar as in a general Finsler connection. The tensor $R_j{}^i{}_{\beta\gamma}$ may be called the *mixed h-curvature tensor* of $IF\Gamma$.

Secondly we compute $Y^i_{\alpha|\beta}|_\gamma-Y^i_\alpha|_{\gamma|\beta}$ directly and get

$$Y^i_{\alpha|\beta}|_\gamma-Y^i_\alpha|_{\gamma|\beta}=Y^j_\alpha P_j{}^i{}_{\beta\gamma}-Y^i_\delta P_\alpha{}^\delta{}_{\beta\gamma}-Y^i_{\alpha|\delta}C^\delta_{\beta\gamma}-Y^i_\alpha|_\delta P^\delta_{\beta\gamma}, \tag{4.7}$$

where $P_j{}^i{}_{\beta\gamma}$ is defined by

$$P_j{}^i{}_{\beta\gamma}:=\dot{\partial}_\gamma F_j{}^i{}_\beta-\delta_\beta C_j{}^i{}_\gamma+F_j{}^k{}_\beta C_k{}^i{}_\gamma-C_j{}^k{}_\gamma F_k{}^i{}_\beta+C_j{}^i{}_\delta\dot{\partial}_\gamma N^\delta_\beta, \tag{4.8}$$

similar to the hv-curvature tensor $P_j{}^i{}_{kh}$ (cf. *(10.16)). On account of (2.20) and (2.21) this may be written in the form

$$P_j{}^i{}_{\beta\gamma}=P_j{}^i{}_{kh}B^{kh}_{\beta\gamma}+S_j{}^i{}_{kh}B^k H_\beta B^h_\gamma, \tag{4.9}$$

called the *mixed hv-curvature tensor.*

Finally we easily get

$$Y^i_\alpha|_\beta|_\gamma-Y^i_\alpha|_\gamma|_\beta=Y^j_\alpha S_j{}^i{}_{\beta\gamma}-Y^i_\delta S_\alpha{}^\delta{}_{\beta\gamma}-Y^i_\alpha|_\delta S^\delta_{\beta\gamma}, \tag{4.10}$$

where we put

$$S_j{}^i{}_{\beta\gamma}:=\mathfrak{A}_{(\beta\gamma)}\{\dot{\partial}_\gamma C_j{}^i{}_\beta+C_j{}^k{}_\beta C_k{}^i{}_\gamma\}=S_j{}^i{}_{kh}B^{kh}_{\beta\gamma}, \tag{4.11}$$

called the *mixed v-curvature tensor.*

Now we apply these relative Ricci identities to B^i_α. (4.6), together with (3.5) and (3.6), yields

$$B^i_{\alpha|\beta|\gamma}-B^i_{\alpha|\gamma|\beta}=B^j_\alpha R^i_{j\beta\gamma}-B^i_\delta R^\delta_{\alpha\beta\gamma}-H_{\alpha\delta}B^iT^\delta_{\beta\gamma}-K_{\alpha\delta}B^iR^\delta_{\beta\gamma}.$$

On the other hand, the direct calculation and (3.10) change the left-hand side to

$$\begin{aligned}&(H_{\alpha\beta}B^i)_{|\gamma}-(H_{\alpha\gamma}B^i)_{|\beta}\\&=(H_{\alpha\beta|\gamma}-H_{\alpha\gamma|\beta})B^i-\mathfrak{A}_{(\beta\gamma)}[H_{\alpha\beta}\{H_{\delta\gamma}B^{\delta i}-(g_{jk|\gamma}B^jB^k/2)B^i+g^{ji}g_{jk|\gamma}B^k\}].\end{aligned}$$

Thus, equating the tangential component, we have

$$R^\delta_{\alpha\beta\gamma}-B^\delta_iR^i_{j\beta\gamma}B^j_\alpha=\mathfrak{A}_{(\beta\gamma)}\{H_{\alpha\beta}(B^{\delta i}g_{ij|\gamma}B^j+H^\delta_\gamma)\}. \tag{4.12}$$

Equating the normal component, we have

$$H_{\alpha\delta}T^\delta_{\beta\gamma}+K_{\alpha\delta}R^\delta_{\beta\gamma}-B_iR^i_{j\beta\gamma}B^j_\alpha=\mathfrak{A}_{(\beta\gamma)}\{H_{\alpha\beta}g_{ij|\gamma}B^iB^j/2-H_{\alpha\beta|\gamma}\}. \tag{4.13}$$

Next, applying (4.7) to B^i_α, we similarly get

$$P^\delta_{\alpha\beta\gamma}-B^\delta_iP^i_{j\beta\gamma}B^j_\alpha=H_{\alpha\beta}(B^{\delta i}g_{ij}|_\gamma B^j+K^\delta_\gamma)-K_{\alpha\gamma}(B^{\delta i}g_{ij|\beta}B^j+H^\delta_\beta), \tag{4.14}$$

$$\begin{aligned}&H_{\alpha\delta}C^\delta_{\beta\gamma}+K_{\alpha\delta}P^\delta_{\beta\gamma}-B_iP^i_{j\beta\gamma}B^j_\alpha\\&=(H_{\alpha\beta}g_{ij}|_\gamma B^iB^j/2-H_{\alpha\beta}|_\gamma)-(K_{\alpha\gamma}g_{ij|\beta}B^iB^j/2-K_{\alpha\gamma|\beta}).\end{aligned} \tag{4.15}$$

Finally, applying (4.10) to B^i_α, we get

$$S^\delta_{\alpha\beta\gamma}-B^\delta_iS^i_{j\beta\gamma}B^j_\alpha=\mathfrak{A}_{(\beta\gamma)}\{K_{\alpha\beta}(B^{\delta i}g_{ij}|_\gamma B^j+K^\delta_\gamma)\}, \tag{4.16}$$

$$K_{\alpha\delta}S^\delta_{\beta\gamma}-B_iS^i_{j\beta\gamma}B^j_\alpha=\mathfrak{A}_{(\beta\gamma)}\{K_{\alpha\beta}g_{ij}|_\gamma B^iB^j/2-K_{\alpha\beta}|_\gamma\}. \tag{4.17}$$

Remark. Applying the relative Ricci identities to the unit normal vector field B^i, we shall obtain the equations which are essentially the same with those obtained above; they will be rather complicated in case where the connection $F\Gamma$ is not metrical, because of (3.10) and (3.11).

The equations (4.12)~(4.17) are generalizations of the well-known Gauss and Codazzi equations in the Riemannian theory of subspaces. Each author who was concerned with the theory of subspaces got those equations or part of them in rather complicated form. For instance, see (2.23) and (2.24) of Rund [17]. In this paper Rund dealt with the Rund connection $R\Gamma$ in our sense (cf. §5). Compare his (1.15) with our (2.11). Thus his (2.23) is simpler than our (4.12); P^i_{jkh} in (4.7) reduced to our $F^i_{jkh}=\dot\partial_h\Gamma^{*i}_{jk}$ for $R\Gamma$ (*(18.2′)).

§5. Induced Cartan and Rund connections.

In almost all the existing literatures, the authors were concerned with an enveloping space $F^n=(M^n, L(x, y))$ which is to be endowed with the *Cartan connection* $C\Gamma=(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0j}, g^i_{jk})$ constructed from the fundamental function $L(x, y)$. According to the theory of Finsler connections due to the present author ([26], [28]), the $C\Gamma$ is determined from the axiomatic standpoint as follows:

Definition. There exists a unique Finsler connection $F\Gamma=(F^i_{jk}, N^i_j, C^i_{jk})$ which satisfies the following five conditions:

(C1) $g_{ij|k}=0$,
(C2) $(h)h$-torsion $T^i_{jk}(=F^i_{jk}-F^i_{kj})=0$,
(C3) deflection $D^i_j(=y^h F^i_{hj}-N^i_j)=0$,
(C4) $g_{ij}|_k=0$,
(C5) $(v)v$-torsion $S^i_{jk}(=C^i_{jk}-C^i_{kj})=0$.

This connection is called the *Cartan connection* and denoted by $C\Gamma=(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0j}, g^i_{jk})$.

The first three conditions give $F^i_{jk}=\Gamma^{*i}_{jk}$, $N^i_j=\Gamma^{*i}_{0j}$, and the remainder two lead to $C^i_{jk}=g^i_{jk}$. We shall denote by $IC\Gamma$ the connection of a hypersurface F^{n-1} induced from the Cartan connection $C\Gamma$ and indicate the quantities with respect to $IC\Gamma$ by putting "c" on them. Then (1.4) and (2.6) show $\overset{c}{C}{}^\alpha_{\beta\gamma}=g^\alpha_{\beta\gamma}$. But *we usually omit "c" on* N^α_β *and* H_β *only,* because these are common to four connections we treat in the present paper, as shown in the following. Thus $IC\Gamma=(\overset{c}{F}{}^\alpha_{\beta\gamma}, N^\alpha_\beta, g^\alpha_{\beta\gamma})$.

What sort of Finsler connection is the induced connection $IC\Gamma$? It is obviously metrical $(g_{\alpha\beta|\gamma}=g_{\alpha\beta}|_\gamma=0)$ from (3.13). Next (2.18) gives $\overset{c}{D}{}^\alpha_\gamma=0$. However (2.16) does not lead to $\overset{c}{T}{}^\alpha_{\beta\gamma}=0$, but from (1.9) we get

$$\overset{c}{T}{}^\alpha_{\beta\gamma}=M^\alpha_\beta H_\gamma-M^\alpha_\gamma H_\beta\,. \tag{5.1}$$

Thus, according to the theory of generalized Cartan connections due to Hashiguchi ([25], [28]), we have

Theorem 5.1. *The connection $IC\Gamma$ of a hypersurface of a Finsler space F^n, induced from the Cartan connection $C\Gamma$ of F^n, is a generalized Cartan connection which is uniquely determined from the induced metric $\underline{L}(u, v)$ by the following five conditions:*

(IC1) $g_{\alpha\beta|\gamma}=0$,
(IC2) *$(h)h$-torsion $\overset{c}{T}{}^\alpha_{\beta\gamma}$ is given by* (5.1),
(IC3) *deflection* $\overset{c}{D}{}^\alpha_\beta=0$,
(IC4) $g_{\alpha\beta}|_\gamma=0$,
(IC5) *$(v)v$-torsion* $\overset{c}{S}{}^\alpha_{\beta\gamma}=0$.

We shall apply the procedure to find a generalized Cartan connection to this $IC\Gamma$ (cf. *p. 165). Putting

$$2A_{\alpha\beta\gamma}=\overset{c}{T}_{\alpha\beta\gamma}-\overset{c}{T}_{\beta\gamma\alpha}+\overset{c}{T}_{\gamma\alpha\beta}=2(M_{\alpha\gamma}H_\beta-M_{\beta\gamma}H_\alpha)\,, \tag{*25.4}$$

*(25.3) leads to

$$\overset{c}{F}_{\alpha\beta\gamma}=\gamma_{\alpha\beta\gamma}-g_{\alpha\beta\delta}N^\delta_\gamma-g_{\beta\gamma\delta}N^\delta_\alpha+g_{\alpha\gamma\delta}N^\delta_\beta+M_{\alpha\gamma}H_\beta-M_{\beta\gamma}H_\alpha\,.$$

Thus we get

$$\overset{c}{F}_{0\beta\gamma}=\gamma_{0\beta\gamma}-g_{\beta\gamma\delta}N^{\delta}_{0}-M_{\beta\gamma}H_{0}, \qquad \overset{c}{F}_{0\beta 0}(=g_{\beta\alpha}N^{\alpha}_{0})=\gamma_{0\beta 0}.$$

Consequently, denoting by $C\underline{\Gamma}=(\Gamma^{*\alpha}_{\beta\gamma}, \Gamma^{*\alpha}_{0\beta}, g^{\alpha}_{\beta\gamma})$ the (intrinsic) Cartan connection of F^{n-1} determined from the induced metric $\underline{L}(u, v)$, we have

$$\begin{aligned} \overset{c}{F}{}^{\alpha}_{\beta\gamma}&=\Gamma^{*\alpha}_{\beta\gamma}+(g^{\alpha}_{\beta\delta}M^{\delta}_{\gamma}+g^{\alpha}_{\gamma\delta}M^{\delta}_{\beta}-g^{\delta}_{\beta\gamma}M^{\alpha}_{\delta})H_{0}+M_{\beta\gamma}H^{\alpha}-M^{\alpha}_{\gamma}H_{\beta}, \\ N^{\alpha}_{\beta}&=\Gamma^{*\alpha}_{0\beta}-M^{\alpha}_{\beta}H_{0}. \end{aligned} \tag{5.2}$$

Remark. Varga [7] already mentioned $\overset{c}{T}{}^{\alpha}_{\beta\gamma}\neq 0$ in the remark on his (3.21). Cf. (V.4.34) of Rund [13]. Our (5.2) is essentially the same with (41) of Varga [14], although the latter deals with Cartan's $\Gamma^{\alpha}_{\beta\gamma}$ (without *). The Cartan connection $C\underline{\Gamma}$ determined from the induced metric has been called the *intrinsic connection*. Cf. Haimovici [6], p.583 and Davies [8], pp.21, 22. We remember that to find a simple form of the above difference between $IC\Gamma$ and $C\underline{\Gamma}$ had been an interesting problem in the theory of Finslerian subspaces for a ling time. Cf. Varga [14] and the final remark on p.214 of Rund [13]. It seems to the author that (22) of Varga [16] and (26) of [18] are a little strange, even if we pay attention to the footnotes; the asterisk should be erased in these formulas. Our theory as mentioned above gives a good indication of a merit of the axiomatic standpoint.

From (2.4) and (2.9) we have

$$\begin{aligned} N^{\alpha}_{\beta}&=B^{\alpha}_{i}(B^{i}_{0\beta}+\Gamma^{*i}_{0j}B^{j}_{\beta}), \\ \overset{c}{F}{}^{\alpha}_{\beta\gamma}&=B^{\alpha}_{i}(B^{i}_{\beta\gamma}+B^{j}_{\beta}\Gamma^{*i}_{jk}B^{k}_{\gamma})+M^{\alpha}_{\beta}H_{\gamma}, \end{aligned} \tag{5.3}$$

and (2.7) gives the normal curvature vector

$$H_{\beta}=B_{i}(B^{i}_{0\beta}+\Gamma^{*i}_{0j}B^{j}_{\beta}). \tag{5.4}$$

The second fundamental h-tensor $\overset{c}{H}_{\beta\gamma}$ is given by (2.10):

$$\overset{c}{H}_{\beta\gamma}=B_{i}(B^{i}_{\beta\gamma}+B^{j}_{\beta}\Gamma^{*i}_{jk}B^{k}_{\gamma})+M_{\beta}H_{\gamma}. \tag{5.5}$$

The equation (2.17) shows that $\overset{c}{H}_{\beta\gamma}$ is generally not symmetric:

$$\overset{c}{H}_{\beta\gamma}-\overset{c}{H}_{\gamma\beta}=M_{\beta}H_{\gamma}-M_{\gamma}H_{\beta}, \tag{5.6}$$

which is analogous in form to (5.1). Further (2.19) yields

$$\overset{c}{H}_{0\gamma}=H_{\gamma}, \qquad \overset{c}{H}_{\gamma 0}=H_{\gamma}+M_{\gamma}H_{0}. \tag{5.7}$$

To consider (2.20) and (2.21), we remember that the $(v)hv$-torsion tensor $P^{1}=(P^{i}_{jk})$ of $C\Gamma$ is equal to $g^{i}_{jk|0}$ (cf. *(17.22)). Putting

$$Q_{\alpha\beta\gamma}=g_{ijk|0}B^{ijk}_{\alpha\beta\gamma}, \quad Q_{\alpha\beta}=g_{ijk|0}B^{ij}_{\alpha\beta}B^{k}, \quad Q_{\alpha}=g_{ijk|0}B^{i}_{\alpha}B^{j}B^{k}, \tag{5.8}$$

(2.20) and (2.21) are respectively written

$$\overset{c}{P}{}^{\alpha}_{\gamma\beta}=H_{\gamma}M^{\alpha}_{\beta}+Q^{\alpha}_{\gamma\beta}\,, \tag{5.9}$$

$$\dot{\partial}_{\beta}H_{\gamma}-\overset{c}{H}_{\beta\gamma}=Q_{\beta\gamma}\,. \tag{5.10}$$

From (5.10) and (5.7) we easily get

$$\dot{\partial}_{\beta}H_{0}=2H_{\beta}+M_{\beta}H_{0}\,. \tag{5.11}$$

Consequently we have

Proposition 5.1. *The normal curvature $H_0=H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β itself vanishes (cf.* § 7).

Theorem 5.2. *The induced connection $IC\Gamma$ of F^{n-1} coincides with the intrinsic Cartan connection $C\underline{\Gamma}$ of F^{n-1} if and only if* (1) $M_{\alpha\beta}=0$ *or* (2) $H_\beta=0$.

Proof. It is obvious from Theorem 5.1 that $IC\Gamma$ coincides with $C\underline{\Gamma}$ if and only if $\overset{c}{T}{}^{\alpha}_{\beta\gamma}=0$: $M^{\alpha}_{\beta}H_{\gamma}-M^{\alpha}_{\gamma}H_{\beta}=0$. If $H_\beta\neq 0$, we have quantities h^α satisfying $M^{\alpha}_{\gamma}=h^{\alpha}H_{\gamma}$. From $h_\alpha H_\gamma=h_\gamma H_\alpha$ we get a quantity h satisfying $h_\alpha=hH_\alpha$, and so $M_{\alpha\gamma}=hH_\alpha H_\gamma$. Then $M_{\alpha 0}=0$ leads to $hH_0=0$. Since $H_0=0$ implies $H_\alpha=0$ from Proposition 5.1, we get $h=0$, and so $M_{\alpha\gamma}=0$.

Remark. Theorem 5.2 as well as Proposition 5.1 were shown by Varga [14].

Next (3.5), (3.6), (3.10) and (3.11) for $IC\Gamma$ are written as

$$B^{i}_{\alpha\,|\,\beta}=\overset{c}{H}_{\alpha\beta}B^{i}\,,\qquad B^{i}_{\alpha}|_{\beta}=M_{\alpha\beta}B^{i}\,, \tag{5.12}$$

$$B^{i}{}_{|\,\beta}=-\overset{c}{H}_{\alpha\beta}B^{\alpha i}\,,\quad B^{i}|_{\beta}=-M_{\alpha\beta}B^{\alpha i}\,. \tag{5.13}$$

Thus the second fundamental v-tensor with respect to $C\Gamma$ is nothing but the M-tensor $M_{\alpha\beta}$ and these derivation equations are quite analogous to those of a Riemannian hypersurface.

We shall treat the Gauss and Codazzi equations in case of $C\Gamma$. First the equation (4.1), decomposed into tangential and normal components, is written

$$\overset{c}{R}_{\alpha\beta\gamma}=R_{ijk}B^{ijk}_{\alpha\beta\gamma}+\mathfrak{A}_{(\beta\gamma)}\{H_{\beta}(\overset{c}{H}_{\alpha\gamma}-Q_{\alpha\gamma})\}\,, \tag{5.14}$$

$$H_{\alpha}\overset{c}{T}{}^{\alpha}_{\beta\gamma}=R_{ijk}B^{i}B^{jk}_{\beta\gamma}+\mathfrak{A}_{(\beta\gamma)}\{Q_{\beta}H_{\gamma}-H_{\beta\,|\,\gamma}\}\,. \tag{5.15}$$

Thus it is observed that the Q-tensors play an important role. Next (4.3) solely gives (5.9) and (5.10).

Secondly (4.12) and (4.13) are written as

$$\overset{c}{R}_{\alpha\beta\gamma\delta}=R_{ij\gamma\delta}B^{ij}_{\alpha\beta}+(\overset{c}{H}_{\alpha\gamma}\overset{c}{H}_{\beta\delta}-\overset{c}{H}_{\alpha\delta}\overset{c}{H}_{\beta\gamma})\,, \tag{5.16}$$

$$\overset{c}{H}_{\alpha\delta}\overset{c}{T}{}^{\delta}_{\beta\gamma}+M_{\alpha\delta}\overset{c}{R}{}^{\delta}_{\beta\gamma}=R_{ij\beta\gamma}B^{i}_{\alpha}B^{j}-(\overset{c}{H}_{\alpha\beta\,|\,\gamma}-\overset{c}{H}_{\alpha\gamma\,|\,\beta})\,, \tag{5.17}$$

where it should be remarked that $R_{ij\beta\gamma}\neq R_{ijkh}B^{kh}_{\beta\gamma}$; from (4.5′) we have

$$R_{ij\beta\gamma}=R_{ijkh}B^{kh}_{\beta\gamma}+P_{ijkh}(B^{k}_{\beta}H_{\gamma}-B^{k}_{\gamma}H_{\beta})B^{h}. \tag{5.18}$$

It will be observed that (5.14) and (5.15) are consequences of (5.16) and (5.17) respectively by contracting by v^{α}(cf. *Theorem 13.3).

Thirdly (4.14) and (4.15) respectively yield

$$\overset{c}{P}_{\alpha\beta\gamma\delta}=P_{ij\gamma\delta}B^{ij}_{\alpha\beta}+(\overset{c}{H}_{\alpha\gamma}M_{\beta\delta}-M_{\alpha\delta}\overset{c}{H}_{\beta\gamma}). \tag{5.19}$$

$$\overset{c}{H}_{\alpha\delta}g^{\delta}_{\beta\gamma}=M_{\alpha\delta}\overset{c}{P}{}^{\delta}_{\beta\gamma}+P_{ij\beta\gamma}B^{i}_{\alpha}B^{j}-(\overset{c}{H}_{\alpha\beta}|_{\gamma}-M_{\alpha\gamma}|_{\beta}). \tag{5.20}$$

Here we also have to remark

$$P_{ij\beta\gamma}=P_{ijkh}B^{kh}_{\beta\gamma}+S_{ijkh}B^{k}H_{\beta}B^{h}_{\gamma}. \tag{5.21}$$

Finally (4.16) and (4.17) respectively give

$$\overset{c}{S}_{\alpha\beta\gamma\delta}=S_{ijkh}B^{ijkh}_{\alpha\beta\gamma\delta}+(M_{\alpha\gamma}M_{\beta\delta}-M_{\alpha\delta}M_{\beta\gamma}), \tag{5.22}$$

$$S_{ijkh}B^{i}_{\alpha}B^{j}B^{kh}_{\gamma\delta}-(M_{\alpha\gamma}|_{\delta}-M_{\alpha\delta}|_{\gamma})=0. \tag{5.23}$$

The former is a consequence of Lemma 1.1 and the fact that

$$\overset{c}{S}{}^{\beta}_{\alpha\gamma\delta}=g^{\varepsilon}_{\alpha\delta}g^{\beta}_{\varepsilon\gamma}-g^{\varepsilon}_{\alpha\gamma}g^{\beta}_{\varepsilon\delta}, \tag{5.24}$$

similar to the v-curvature tensor S^{i}_{hjk} of $C\Gamma$. The latter is also a consequence of

$$M_{\alpha\beta}|_{\gamma}=g_{hij}|_{k}B^{h}B^{ijk}_{\alpha\beta\gamma}+M_{\alpha\gamma}M_{\beta}+M_{\beta\gamma}M_{\alpha}-g^{\delta}_{\alpha\beta}M_{\delta\gamma}, \tag{5.25}$$

which is easily shown from (1.8) by (5.12) and (5.13).

Now we shall be concerned with the enveloping space $F^{n}=(M^{n}, L(x, y))$ which is to be endowed with the *Rund connection* $R\Gamma=(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0j}, 0)$. The first two connection coefficients of the $R\Gamma$ are same with those of the Cartan connection $C\Gamma$, while the third is equal to zero. Thus it may be said that $R\Gamma$ is derived from $C\Gamma$ by the *C-process* (*Definition 14.2: $F^{i}_{jk}\to{}'F^{i}_{jk}=F^{i}_{jk}$, $N^{i}_{j}\to{}'N^{i}_{j}=N^{i}_{j}$, $C^{i}_{jk}\to{}'C^{i}_{jk}=0$).

If we denote by $IR\Gamma$ the connection of a hypersurface F^{n-1} induced from the $R\Gamma$ and indicate the quantities with respect to $IR\Gamma$ by putting "r" on them, (2.4) and (5.3) show $\overset{r}{N}{}^{\alpha}_{\beta}=\overset{c}{N}{}^{\alpha}_{\beta}$, and (2.9) gives

$$\overset{r}{F}{}^{\alpha}_{\beta\gamma}=B^{\alpha}_{i}(B^{i}_{\beta\gamma}+\Gamma^{*i}_{jk}B^{jk}_{\beta\gamma}), \tag{5.26}$$

so that (5.3) leads to

$$\overset{r}{F}{}^{\alpha}_{\beta\gamma}=\overset{c}{F}{}^{\alpha}_{\beta\gamma}-M^{\alpha}_{\beta}H_{\gamma}, \tag{5.27}$$

As to $\overset{r}{H}_{\beta}$ and $\overset{r}{H}_{\beta\gamma}$, (2.7) and (5.4) show $\overset{r}{H}_{\beta}=\overset{c}{H}_{\beta}$, and (2.10) gives

$$\overset{r}{H}_{\beta\gamma}=B_{i}(B^{i}_{\beta\gamma}+\Gamma^{*i}_{jk}B^{jk}_{\beta\gamma}), \tag{5.28}$$

so that (5.5) shows

(5.29) $$\overset{r}{H}_{\beta\gamma}=\overset{c}{H}_{\beta\gamma}-M_\beta H_\gamma\,.$$

On the other hand, paying attention to $C^i_{jk}=0$ of $R\Gamma$, (2.16), (2.18) and (2.6) respectively give $\overset{r}{T}{}^\alpha_{\beta\gamma}=0$, $\overset{r}{D}{}^\alpha_\beta=0$ and $\overset{r}{C}{}^\alpha_{\beta\gamma}=0$. Further, from $g_{ij|k}=0$ and $g_{ij}|_k=2g_{ijk}$ for $R\Gamma$, (3.13) yields

(5.30) $$\overset{r}{g}_{\alpha\beta|\gamma}=2M_{\alpha\beta}H_\gamma\,,\qquad \overset{r}{g}_{\alpha\beta}|_\gamma=2g_{\alpha\beta\gamma}\,.$$

These observations enable us to conclude

Theorem 5.3. *The connection $IR\Gamma$ of a hypersurface of a Finsler space F^n, induced from the Rund connection $R\Gamma=(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0j}, 0)$ of F^n, is uniquely determined from the induced metric $\underline{L}(u, v)$ by the following four conditions*:

(IR1) $\overset{r}{g}_{\alpha\beta|\gamma}=2M_{\alpha\beta}H_\gamma$, (IR2) *(h)h-torsion* $\overset{r}{T}{}^\alpha_{\beta\gamma}=0$,

(IR3) *deflection* $\overset{r}{D}{}^\alpha_\beta=0$, (IR4) $\overset{r}{C}{}^\alpha_{\beta\gamma}=0$.

In fact, we apply the method by which *Theorem 17.2 is proved: (IR1) is written

$$\partial_\gamma g_{\alpha\beta}-2g_{\alpha\beta\delta}\overset{r}{N}{}^\delta_\gamma-\overset{r}{F}_{\alpha\beta\gamma}-\overset{r}{F}_{\beta\alpha\gamma}=2M_{\alpha\beta}H_\gamma\,,$$

from which we have

$$\gamma_{\alpha\beta\gamma}-g_{\alpha\beta\delta}\overset{r}{N}{}^\delta_\gamma-g_{\beta\gamma\delta}\overset{r}{N}{}^\delta_\alpha+g_{\gamma\alpha\delta}\overset{r}{N}{}^\delta_\beta-\overset{r}{F}_{\alpha\beta\gamma}=M_{\alpha\beta}H_\gamma+M_{\beta\gamma}H_\alpha-M_{\gamma\alpha}H_\beta\,,$$

$$\gamma_{0\beta\gamma}-g_{\beta\gamma\delta}\overset{r}{N}{}^\delta_0-\overset{r}{F}_{0\beta\gamma}=M_{\beta\gamma}H_0\,,\qquad \gamma_{0\beta0}-\overset{r}{F}_{0\beta0}=0\,.$$

Therefore we get $\overset{r}{N}{}^\alpha_\beta=\gamma^\alpha_{0\beta}-g^\alpha_{\beta\delta}\gamma^\delta_{00}-M^\alpha_\beta H_0=\overset{c}{N}{}^\alpha_\beta$ by (5.2) and

(5.31) $$\overset{r}{F}{}^\alpha_{\beta\gamma}=\Gamma^{*\alpha}_{\beta\gamma}+(g^\alpha_{\beta\delta}M^\delta_\gamma+g^\alpha_{\gamma\delta}M^\delta_\beta-g^\delta_{\beta\gamma}M^\alpha_\delta)H_0+M_{\beta\gamma}H^\alpha-M^\alpha_\beta H_\gamma-M^\alpha_\gamma H_\beta\,.$$

This and (5.2) lead to (5.27).

As to the second fundamental h-tensor $\overset{r}{H}_{\beta\gamma}$, (2.17) shows

(5.32) $$\overset{r}{H}_{\beta\gamma}=\overset{r}{H}_{\gamma\beta}\,,$$

which is also derived from (5.6) and (5.29). Next (2.19) shows

(5.33) $$\overset{r}{H}_{\gamma0}=\overset{r}{H}_{0\gamma}=H_\gamma\,.$$

Finally (2.20) and (2.21) lead to

(5.34) $$\overset{r}{P}{}^\alpha_{\gamma\beta}=2H_\gamma M^\alpha_\beta+Q^\alpha_{\beta\gamma}=\overset{c}{P}{}^\alpha_{\gamma\beta}+H_\gamma M^\alpha_\beta\,,$$

(5.35) $$\dot\partial_\beta H_\gamma-\overset{r}{H}_{\beta\gamma}=M_\beta H_\gamma+Q_{\beta\gamma}\,.$$

The former is also obvious from (5.9) and (5.27), and the latter is solely a consequence of (5.10) and (5.29).

Proposition 5.2. *The induced connection $IR\Gamma$ of F^{n-1} coincides with the intrinsic Rund connection $R\underline{\Gamma}$ of F^{n-1} if and only if* (1) $M_{\alpha\beta}=0$ *or* (2) $H_{\beta}=0$.

This is clear from Theorem 5.3, in particular, the first condition.

Remark. It seems that Rund treated the $R\Gamma$ only whenever he considered the induced connection, contrary to the case of Kikuchi, Rapcsák and Varga who were concerned with $C\Gamma$ alone. In fact, (V.3.10) of his book [13] as well as (1.15) of [17] coincide with our (5.26). Cf. (5.2) of Brown [20]. (She denotes our Γ^{*i}_{jk} by Γ^{i}_{jk}.) Her (5.8) is nothing but our (5.31). Cf. (V.9.25) of [13].

It may be a merit of the $R\Gamma$ that $\overset{r}{H}_{\alpha\beta}$ is symmetric, while $\overset{c}{H}_{\alpha\beta}$ is not so. In Rund [17] $\overset{r}{H}_{\alpha\beta}$ is denoted by $\tilde{\Omega}_{\alpha\beta}$ (cf. (1.17) of [17]). Compare our (5.28) with (3.4) of Brown [20].

§ 6. Induced Berwald and Hashiguchi connections.

In this section we first consider a Finsler space $F^n=(M^n, L(x, y))$ which is to be endowed with the *Berwald connection* $B\Gamma=(G^i_{jk}, G^i_j, 0)$. According to the general theory of Finsler connections due to the present author [28], the $B\Gamma$ is derived from the Rund connection $R\Gamma$ by the P^1-*process* (*Definition 15.2: $F^i_{jk}\to {}'F^i_{jk}=F^i_{jk}+P^i_{kj}$, $N^i_j\to{}'N^i_j=N^i_j$, $C^i_{jk}\to{}'C^i_{jk}=C^i_{jk}$). Thus $G^i_{jk}=\dot{\partial}_j\Gamma^{*i}_{0k}$ and $G^i_j=\Gamma^{*i}_{0j}$.

On the other hand, we have an axiomatic viewpoint of $B\Gamma$ shown by Okada ([33], [30]) and analogous to the case of $C\Gamma$:

Definition. There exists a unique Finsler connection which satisfies the following five conditions:

(B1) $L_{|i}=0$, (B2) $(h)h$-torsion $T^i_{jk}=0$,
(B3) deflection $D^i_j=0$, (B4) $(v)hv$-torsion $P^i_{jk}=0$,
(B5) $(h)hv$-torsion $C^i_{jk}=0$.

This connection is called the *Berwald connection* and denoted by $B\Gamma$.

In the following we denote by (;) and (.) the h- and v-covariant differentiations with respect to $B\Gamma$ respectively. Thus (.) is only $\dot{\partial}$ and (B5) shows $g_{ij\cdot k}=2g_{ijk}$. Further (B1) shows $L_{;i}=0$ and it is well-known [28] that $g_{ij;k}=-2g_{ijk|0}(g_{ijk|0}=P_{ijk}$ of $C\Gamma$).

Now we deal with the connection $IB\Gamma$ of a hypersurface F^{n-1}, induced from the Berwald connection $B\Gamma$ of F^n, and indicate the quantities with respect to $IB\Gamma$ by putting "b" on them. Then (3.12), (2.16), (2.18) and (2.6) show $\underline{L}_{;\alpha}=0$, $\overset{b}{T}{}^{\alpha}_{\beta\gamma}=0$, $\overset{b}{D}{}^{\alpha}_{\beta}=0$ and $\overset{b}{C}{}^{\alpha}_{\beta\gamma}=0$ respectively. However (2.20) gives

$$\overset{b}{P}{}^{\alpha}_{\beta\gamma}=2H_{\beta}M^{\alpha}_{\gamma}\,, \tag{6.1}$$

because $B\Gamma$ has the characteristic property (B4).

The above axiomatic definition of $B\Gamma$ has been generalized to Finsler connection of Berwald type with surviving $(h)h$-torsion tensor T^i_{jk} [30]. (Cf. Theorem 5.1.) But matters are different for $IB\Gamma$. Therefore we need another generalization of Berwald connection as follows, which will be studied by Aikou and Hashiguchi [24] in detail:

Theorem 6.1. *There exists a unique Finsler connection which satisfies the following five conditions*:

(P1) $L_{|i}=0$, (P2) *$(h)h$-torsion* $T^i_{jk}=0$,
(P3) *deflection* $D^i_j=0$, (P4) *$(v)hv$-torsion* P^i_{jk} *is given,*
(P5) *$(h)hv$-torsion* $C^i_{jk}=0$,

if and only if P^i_{jk} *satisfies*

(1) $P^i_{j0}=0$, (2) $y^h(\dot{\partial}_k Q^i_{hj}-\dot{\partial}_j Q^i_{hk})=0$,

where we put $Q^i_{hj}=P^i_{hj}-P^i_{jh}$.

We shall call this connection a *generalized Berwald P^1-connection.* Therefore the above observations of the $IB\Gamma$ enable us to assert

Theorem 6.2. *The connection $IB\Gamma$ of a hypersurface F^{n-1} of a Finsler space F^n, induced from the Berwald connection $B\Gamma$ of F^n, is a generalized Berwald P^1-connection such that the $(v)hv$-torsion tensor is given by* (6.1).

It should be noticed that this $\overset{b}{P}{}^\alpha_{\beta\gamma}=2H_\beta M^\alpha_\gamma$ satisfies the above Aikou-Hashiguchi conditions (1) and (2). In fact, $\overset{b}{P}{}^\alpha_{\beta 0}=0$ is obvious and

$$\begin{aligned}\mathfrak{A}_{(\beta\gamma)}[v^\delta\{\dot{\partial}_\gamma(H_\delta M^\alpha_\beta-H_\beta M^\alpha_\delta)\}]&=\mathfrak{A}_{(\beta\gamma)}\{\dot{\partial}_\gamma(H_0 M^\alpha_\beta)-(H_\gamma M^\alpha_\beta-H_\beta M^\alpha_\gamma)\}\\&=(\dot{\partial}_\gamma M^\alpha_\beta-\dot{\partial}_\beta M^\alpha_\gamma+M_\gamma M^\alpha_\beta-M_\beta M^\alpha_\gamma)H_0\,,\end{aligned}$$

which is really equal to zero from (1.13).

Theorem 6.3. *The induced connection $IB\Gamma$ of F^{n-1} coincides with the intrinsic Berwald connection $\underline{B\Gamma}$ of F^{n-1} if and only if* (1) $M_{\alpha\beta}=0$ *or* (2) $H_\alpha=0$.

This is obvious from Theorem 6.2, in particular (6.1).

We shall find the connection coefficients of $IB\Gamma=(\overset{b}{F}{}^\alpha_{\beta\delta}, \overset{b}{N}{}^\alpha_\beta, 0)$ owing to the procedure to prove Theorem 6.1. Denote by $\underline{B\Gamma}=(G^\alpha_{\beta\gamma}, G^\alpha_\beta, 0)$ the intrinsic Berwald connection determined from the induced metric $\underline{L}(u, v)$. Then it is well-known [28] that

$$2G_\alpha(=2g_{\alpha\beta}G^\beta)=v^\beta\dot{\partial}_\alpha\partial_\beta(\underline{L}^2/2)-\partial_\alpha(\underline{L}^2/2)\,.$$

Since the condition (P1): $\underline{L}_{;\alpha}=0$ is written as $\partial_\alpha\underline{L}=v_\beta\overset{b}{N}{}^\beta_\alpha/\underline{L}$, we have

$$2G_\alpha=v^\beta\dot{\partial}_\alpha(v_\gamma\overset{b}{N}{}^\gamma_\beta)-v_\gamma\overset{b}{N}{}^\gamma_\alpha\,.$$

Next, paying attention to the definition $\overset{b}{P}{}^{\alpha}_{\beta\gamma}=\dot{\partial}_{\gamma}\overset{b}{N}{}^{\alpha}_{\beta}-\overset{b}{F}{}^{\alpha}_{\gamma\beta}$, we get

$$2G_{\alpha}=g_{\alpha\gamma}\overset{b}{N}{}^{\gamma}_{0}+\overset{b}{P}{}^{0}_{0\alpha}+v_{\gamma}(v^{\beta}\overset{b}{F}{}^{\gamma}_{\alpha\beta}-\overset{b}{N}{}^{\gamma}_{\alpha}).$$

Therefore conditions (P2, 3) and $\overset{b}{P}{}^{0}_{0\alpha}=0$ yield $2G^{\alpha}=\overset{b}{N}{}^{\alpha}_{0}$. Then

$$2G^{\alpha}_{\beta}=\dot{\partial}_{\beta}\overset{b}{N}{}^{\alpha}_{0}=\overset{b}{N}{}^{\alpha}_{\beta}+v^{\gamma}(\overset{b}{P}{}^{\alpha}_{\gamma\beta}+\overset{b}{F}{}^{\alpha}_{\beta\gamma})=2(\overset{b}{N}{}^{\alpha}_{\beta}+M^{\alpha}_{\beta}H_{0}),$$

$$G^{\alpha}_{\beta\gamma}=\dot{\partial}_{\gamma}(\overset{b}{N}{}^{\alpha}_{\beta}+M^{\alpha}_{\beta}H_{0})=\overset{b}{F}{}^{\alpha}_{\gamma\beta}+\overset{b}{P}{}^{\alpha}_{\beta\gamma}+\dot{\partial}_{\gamma}(M^{\alpha}_{\beta}H_{0}).$$

Consequently we get

$$\begin{aligned}\overset{b}{F}{}^{\alpha}_{\beta\gamma}&=G^{\alpha}_{\beta\gamma}-2M^{\alpha}_{\gamma}H_{\beta}-\dot{\partial}_{\gamma}(M^{\alpha}_{\beta}H_{0}),\\ N^{\alpha}_{\beta}&=G^{\alpha}_{\beta}-M^{\alpha}_{\beta}H_{0}.\end{aligned}\tag{6.2}$$

This N^{α}_{β} coincides with $\overset{c}{N}{}^{\alpha}_{\beta}$ from (5.2) and $\Gamma^{*\alpha}_{0\beta}=G^{\alpha}_{\beta}$. $\overset{b}{F}{}^{\alpha}_{\beta\gamma}$ is rewritten as

$$\overset{b}{F}{}^{\alpha}_{\beta\gamma}=G^{\alpha}_{\beta\gamma}-2(M^{\alpha}_{\gamma}H_{\beta}+M^{\alpha}_{\beta}H_{\gamma})-(\dot{\partial}_{\gamma}M^{\alpha}_{\beta}+M^{\alpha}_{\beta}M_{\gamma})H_{0}.\tag{6.3}$$

Therefore (1.13) asserts that the second of Aikou-Hashiguchi conditions shows the symmetry property of $\overset{b}{F}{}^{\alpha}_{\beta\gamma}$.

Now (2.4) and (2.9) give

$$N^{\alpha}_{\beta}=B^{\alpha}_{i}(B^{i}_{0\beta}+G^{i}_{j}B^{j}_{\beta}),\qquad \overset{b}{F}{}^{\alpha}_{\beta\gamma}=B^{\alpha}_{i}(B^{i}_{\beta\gamma}+G^{i}_{jk}B^{jk}_{\beta\gamma}).\tag{6.4}$$

Therefore the well-known relation $G^{i}_{jk}=\Gamma^{*i}_{jk}+P^{i}_{kj}$ (*(18.4)), (5.26) and (5.27) lead to

$$\overset{b}{F}{}^{\alpha}_{\beta\gamma}=\overset{r}{F}{}^{\alpha}_{\beta\gamma}+Q^{\alpha}_{\beta\gamma}=\overset{c}{F}{}^{\alpha}_{\beta\gamma}-M^{\alpha}_{\beta}H_{\gamma}+Q^{\alpha}_{\beta\gamma}.\tag{6.5}$$

Next (2.7) gives $\overset{b}{H}_{\alpha}=\overset{r}{H}_{\alpha}$ and (2.10) does

$$\overset{b}{H}_{\beta\gamma}=B_{i}(B^{i}_{\beta\gamma}+G^{i}_{jk}B^{jk}_{\beta\gamma}).\tag{6.6}$$

Thus (5.28) and (5.29) give

$$\overset{b}{H}_{\beta\gamma}=\overset{r}{H}_{\beta\gamma}+Q_{\beta\gamma}=\overset{c}{H}_{\beta\gamma}-M_{\beta}H_{\gamma}+Q_{\beta\gamma}.\tag{6.7}$$

(2.17), (2.19) and (2.21) show

$$\overset{b}{H}_{\beta\gamma}=\overset{b}{H}_{\gamma\beta},\quad \overset{b}{H}_{0\gamma}=H_{\gamma},\quad \dot{\partial}_{\beta}H_{\gamma}-\overset{b}{H}_{\beta\gamma}=M_{\beta}H_{\gamma}.\tag{6.8}$$

Finally (3.13) shows

$$g_{\alpha\beta;\gamma}=2(M_{\alpha\beta}H_{\gamma}-Q_{\alpha\beta\gamma}),\qquad g_{\alpha\beta\cdot\gamma}=2g_{\alpha\beta\gamma}.\tag{6.9}$$

We shall be concerned with the Gauss and Codazzi equations with respect to $B\Gamma$. First (4.1) yields

$$\overset{b}{R}_{\alpha\beta\gamma}=R_{ijk}B^{ijk}_{\alpha\beta\gamma}+\mathfrak{A}_{(\beta\gamma)}\{H_{\beta}(\overset{b}{H}_{\alpha\gamma}-2Q_{\alpha\gamma})\},\tag{6.10}$$

$$R_{ijk}B^{i}B^{jk}_{\beta\gamma}=\mathfrak{A}_{(\beta\gamma)}\{H_{\beta;\gamma}+H_{\beta}Q_{\gamma}\}.\tag{6.11}$$

It is easily verified that these are essentially the same with (5.14) and (5.15) respectively. Next (4.12) and (4.13) give

(6.12) $$\overset{b}{R}_{\alpha\beta\gamma\delta}=H_{ij\gamma\delta}B^{ij}_{\alpha\beta}+\mathfrak{B}_{(\gamma\delta)}\{\overset{b}{H}_{\alpha\gamma}(\overset{b}{H}_{\beta\delta}-2Q_{\beta\delta}+2M_\beta H_\delta)\},$$

(6.13) $$H_{ij\beta\gamma}B^i_\alpha B^j=\mathfrak{A}_{(\beta\gamma)}\{\overset{b}{H}_{\alpha\beta;\gamma}+\overset{b}{H}_{\alpha\beta}(Q_\gamma-MH_\gamma)\},$$

where we put

$$H_{ij\gamma\delta}=H_{ijkh}B^{kh}_{\gamma\delta}+G_{ijkh}(B^k_\gamma H_\delta-B^k_\delta H_\gamma)B^h.$$

Finally (4.14) and (4.15) yield

(6.14) $$\overset{b}{P}_{\alpha\beta\gamma\delta}=G_{ijkh}B^{ijkh}_{\alpha\beta\gamma\delta}+2\overset{b}{H}_{\alpha\gamma}M_{\beta\delta},$$

(6.15) $$G_{ijkh}B^jB^{ikh}_{\alpha\beta\gamma}=\overset{b}{H}_{\alpha\beta}M_\gamma-\overset{b}{H}_{\alpha\beta\cdot\gamma}.$$

As it was mentioned above, $B\Gamma$ is derived from $R\Gamma$ by the P^1-process and $R\Gamma$ is done from $C\Gamma$ by the C-process. On the other hand, it was known ([26], [28]) that $C\Gamma$ yields a connection by the P^1-process, called the *Hashiguchi connection* $H\Gamma=(G^i_{jk}, G^i_j, g^i_{jk})$, and $B\Gamma$ is derived from $H\Gamma$ by the C-process. Then, inducing those connections on a hypersurface, the process to derive $IR\Gamma$ (resp. $IB\Gamma$) from $IC\Gamma$ (resp. $IR\Gamma$) may be called the *IC-process* (resp. *IP^1-process*). It has been already observed in the last and present sections that

(6.16) IC-process $IC\Gamma\rightarrow IR\Gamma$: $$\overset{r}{F}{}^\alpha_{\beta\gamma}=\overset{c}{F}{}^\alpha_{\beta\gamma}-M^\alpha_\beta H_\gamma,\quad \overset{r}{H}_{\beta\gamma}=\overset{c}{H}_{\beta\gamma}-M_\beta H_\gamma,$$

(6.17) IP^1-process $IR\Gamma\rightarrow IB\Gamma$: $$\overset{b}{F}{}^\alpha_{\beta\gamma}=\overset{r}{F}{}^\alpha_{\beta\gamma}+Q^\alpha_{\beta\gamma},\quad \overset{b}{H}_{\beta\gamma}=\overset{r}{H}_{\beta\gamma}+Q_{\beta\gamma}.$$

Then the connection $IH\Gamma$ of F^{n-1}, induced from $H\Gamma$, is derived from $IC\Gamma$ by IP^1-process. Thus, putting "h" on the quantities with respect to $IH\Gamma$, we have

(6.18) $$\overset{h}{F}{}^\alpha_{\beta\gamma}=\overset{c}{F}{}^\alpha_{\beta\gamma}+Q^\alpha_{\beta\gamma},\qquad \overset{h}{H}_{\beta\gamma}=\overset{c}{H}_{\beta\gamma}+Q_{\beta\gamma}.$$

Further we have $\overset{h}{N}{}^\alpha_\beta=\overset{c}{N}{}^\alpha_\beta$, $\overset{h}{H}_\beta=\overset{c}{H}_\beta$ and

(6.19) $$\overset{h}{H}_{\beta\gamma}-\overset{h}{H}_{\gamma\beta}=M_\beta H_\gamma-M_\gamma H_\beta.$$

(6.20) $$\overset{h}{H}_{0\gamma}=H_\gamma,\qquad \dot\partial_\beta H_\gamma-\overset{h}{H}_{\beta\gamma}=0.$$

Applying a recent result shown by Aikou-Hashiguchi [24], we have

Theorem 6.4. *The connection $IH\Gamma$ of a hypersurface F^{n-1} of a Finsler space F^n, induced from the Hashiguchi connection $H\Gamma$ of F^n, is uniquely determined from the induced metric $\underline{L}(u, v)$ by the following five conditions*:

(IH1) $\underline{L}_{;\alpha}=0$ (IH2) *(h)h-torsion* $\overset{h}{T}{}^\alpha_{\beta\gamma}=M^\alpha_\beta H_\gamma-M^\alpha_\gamma H_\beta$, (IH3) *deflection* $\overset{h}{D}{}^{\alpha}_{\beta}=0$, (IH4) *(v)hv-torsion* $\overset{h}{P}{}^{\alpha}_{\beta\gamma}=H_\beta M^{\alpha}_{\gamma}$,

(IH5) *(h)hv-torsion* $\overset{h}{C}{}^{\alpha}_{\beta\gamma}=g^{\alpha}_{\beta\gamma}$.

In fact, (3.12), (2.16), (2.18) and (2.6) give (IH1, 2, 3, 5) respectively, and (2.20) leads to (IH4) which differs a little from $\overset{b}{P}{}^{\alpha}_{\beta\gamma}$ given by (6.1), while (IH2) coincides with (5.1).

Proposition 6.1. *The induced connection* $IH\Gamma$ *of* F^{n-1} *coincides with the intrinsic Hashiguchi connection* $H\underline{\Gamma}$ *of* F^{n-1} *if and only if* (1) $M_{\alpha\beta}=0$ *or* (2) $H_\beta=0$.

Remark. Berwald [3] defined the second fundamental tensor as $\Omega_{\alpha\beta}=\dot\partial_\alpha\dot\partial_\beta H_0/2$. See (2.2) and (2.8) of [3], (5.16) of Varga [7], (90) of Davies [8], (V.6.15) of Rund [13] and (3.2) of Brown [20]. By means of (5.11) and (6.20) we easily get

$$\Omega_{\alpha\beta}=\overset{h}{H}_{\alpha\beta}+H_\alpha M_\beta+(\dot\partial_\alpha M_\beta+M_\alpha M_\beta)H_0/2\,. \tag{6.21}$$

This is nothing but Brown's result (3.8) of [20], where Brown's N and M_α are equal to our H_{00}/L^2 and LM_α respectively.

§ 7. Hyperplanes.

Following Kikuchi [9] and Rapcsák [12], we shall define three kinds of hyperplanes in a Finsler space $F^n=(M^n,\ L(x,\ y))$.

It is well-known that a geodesic curve of F^n, an extremal curve of the length integral $s=\int_{t_0}^{t_1}L(x(t),\ \dot x(t))dt$, is given by the differential equations

$$d^2x^i/ds^2+2G^i(x,\ dx/ds)=0\,, \tag{7.1}$$

where $G^i=g^{ij}G_j$ and

$$2G_j=y^k\dot\partial_j\partial_k F-\partial_j F\,,\qquad F=L^2/2\,. \tag{7.2}$$

Connection coefficients of the Cartan connection $C\Gamma=(\Gamma^{*i}_{jk},\ \Gamma^{*i}_{0j},\ g^i_{jk})$ and the Berwald connection $B\Gamma=(G^i_{jk},\ G^i_j,\ 0)$ of F^n are such that $2G^i=\Gamma^{*i}_{00}=G^i_{00}=G^i_0$.

We construct the corresponding quantities G^α of a hypersurface $F^{n-1}=(M^{n-1},\ \underline{L}(u,\ v))$. Putting $\underline{F}=\underline{L}^2/2$, (1.3) leads to

$$\begin{aligned}v^\beta\dot\partial_\alpha\partial_\beta\underline{F}&=v^\beta\dot\partial_\alpha(\partial_jFB^j_\beta+\dot\partial_iFB^i_{0\beta})\\&=y^j\dot\partial_i\partial_jFB^i_\alpha+\dot\partial_j\dot\partial_iFB^i_{00}B^j_\alpha+\dot\partial_iFB^i_{\alpha0}\,,\end{aligned}$$

which implies

$$2G_\alpha=v^\beta\dot\partial_\alpha\partial_\beta\underline{F}-\partial_\alpha\underline{F}=2G_iB^i_\alpha+g_{ij}B^i_{00}B^j_\alpha\,.$$

Thus (2.8) gives

$$2G^\alpha-N^\alpha_0=(2G^i-N^i_0)B^\alpha_i\,. \tag{7.3}$$

This equation holds for any nonlinear connection N^i_j and the induced N^α_β given by (2.4).

On the other hand, a path with respect to a Finsler connection $F\Gamma=(F_{j\,k}^{\,i}, N_j^i, C_{j\,k}^{\,i})$, defined as *Definition 39.2, is given by

(7.4) $$d^2x^i/ds^2+N_0^i(x, dx/ds)=0.$$

Therefore (7.3) shows

Proposition 7.1. *If each geodesic curve in a Finsler space* $F^n=(M^n, L(x, y))$ *endowed with a Finsler connection* $F\Gamma$ *is a path, then each geodesic curve of a hypersurface* F^{n-1} *with respect to the induced metric* $\underline{L}(u, v)$ *is a path with respect to the induced connection* $IF\Gamma$.

It is remarked that four typical Finsler connections, treated in §§5 and 6, have the common nonlinear connection and satisfy the assumption of Proposition 7.1.

Now the equation (7.4) of a path may be written as $D(dx^i/ds)/ds=0$ from (2.1), and (2.12) may be

(7.5) $$d^2x^i/ds^2+N_0^i(x, dx/ds)$$
$$=\{d^2u^\alpha/ds^2+N_0^\alpha(u, du/ds)\}B_\alpha^i+H_0(u, du/ds)B^i.$$

The quantity $H_0(u, du/ds)$ along the curve $u^\alpha=u^\alpha(s)$ is called the normal curvature at u^α. Paying attention to the fact that $H_\alpha(u, v)$ is (1)p-homogeneous, the quantity

(7.6) $$N(u, v)=H_0(u, v)/\underline{L}^2(u, v)$$

should be called the *normal curvature* of F^{n-1} at (u, v) and H_α is the *normal curvature vector.*

In general (2.19) gives

(7.7) $$H_{00}-H_0=B_i(D_0^i+C_{0\,k}^{\,i}B^kH_0),$$

and especially we easily get

(7.8) $$\overset{c}{H}_{00}=\overset{r}{H}_{00}=\overset{b}{H}_{00}=\overset{h}{H}_{00}=H_0$$

Remark. As to the equation (7.3), see (28) of Varga [14]. It seems that Berwald [3] first payed attention to the normal curvature. Cf. p. 22 of Davies [8], (V.7.18) and (V.7.26) of Rund [13] and Theorem 3.4 of Brown [20].

It follows from (7.5) that if a path of F^n is on the hypersurface F^{n-1}, it is a path of F^{n-1} with vanishing normal curvature.

Definition 1. If each path of a hypersurface F^{n-1} with respect to the induced connection $IF\Gamma$ is a path of the enveloping space F^n with respect to the $F\Gamma$, then F^{n-1} is called a *hyperplane of the first kind.*

Proposition 7.2. *A hypersurface F^{n-1} is a hyperplane of the 1st kind, if and only if the normal curvature vector $H_\alpha(u, v)$ vanishes identically.*

This a consequence of (7.5) and Proposition 5.1.

Corollary 7.1. *As to $C\Gamma$, $R\Gamma$, $B\Gamma$ and $H\Gamma$, the induced connections on a hyperplane of the 1st kind coincide with the respective intrinsic connections.*

Definition 1′. If each geodesic curve of a hypersurface F^{n-1} with respect to the induced metric $\underline{L}(u, v)$ is a geodesic curve of the enveloping space F^n, then F^{n-1} is called *totally geodesic.*

Theorem 7.1. *A hypersurface F^{n-1} is totally geodesic, if and only if with respect to the connections $C\Gamma$, $R\Gamma$, $B\Gamma$ and $H\Gamma$ the normal curvature vector H_α vanishes or the second fundamental h-tensors satisfy* (1) $\overset{c}{H}_{\alpha\beta}=-Q_{\alpha\beta}$, (2) $\overset{r}{H}_{\alpha\beta}=-Q_{\alpha\beta}$, (3) $\overset{b}{H}_{\alpha\beta}=0$, (4) $\overset{h}{H}_{\alpha\beta}=0$ *respectively.*

Proof. Since each path with respect to those connections is a geodesic curve, it is sufficient for the proof to show that these conditions are equivalent to $H_\alpha=0$. From (5.10) it follows that $H_\alpha=0$ implies the above (1). Conversely (1) and (5.7) lead to $H_\alpha=0$. Similar way will be applied to other cases.

Remark. "Hyperplane of the 1st kind" is the name given by Rapcsák [12]. Kikuchi [9] named it a totally extremal hypersuface, following Haimovici [6], p. 570. It is shown from (2.12) and indicated by Kikuchi [9] that, on such a hypersurface, $Dv^\alpha=0$ implies $Dy^i=0$. Cf. Theorem 6.2 of Brown [20].

It is noted that $\tilde{\Omega}_{\alpha\beta}$, appeared in Theorem 6.3 of [20], is equal to our $\overset{r}{H}_{\alpha\beta}$ and "affinely connected" means $g_{hij|k}=0$ (cf. *Theorem 25.2). Strictly speaking, "affinely connected" is too strong condition. That is, $\overset{r}{H}_{\alpha\beta}=0$ on a totally geodesic hypersurface of a *Landsberg space* ($g_{hij|0}=0$; *Theorem 25.3), because $g_{hij|0}=0$ implies $Q_{\alpha\beta}=0$ from (5.8).

Secondly we are concerned with an *h-path* with respect to a Finsler connection $F\Gamma=(F^i_{jk}, N^i_j, C^i_{jk})$, which is defined as *Definition 39.1 and given by

$$(7.9)\qquad \begin{aligned} &dy^i/ds+N^i_j(x, y)dx^j/ds=0\,, \\ &d^2x^i/ds^2+F^i_{jk}(x, y)(dx^i/ds)(dx^k/ds)=0\,. \end{aligned}$$

Therefore an *h*-path is a curve $(x^i(s), y^i(s))$ of the tangent bundle $T(M^n)$, or regarded as a curve $x^i(s)$ of M^n with a vector field $y^i(s)$. It is uniquely determined by giving initial values $x^i(0)$, $(dx^i/ds)_0$ and $y^i(0)$. In terms of the concept of parallelism, $y^i(s)$ is parallel along the curve $x^i(s)$ with respect to the nonlinear connection N^i_j and dx^i/ds is parallel along the curve with respect to the field of supporting element $y^i(s)$.

Remark. An h-path was first defined and called a *quasigeodesic curve* by Varga [35]. Cf. Rapcsák [12].

Two equations of (7.9) are written as $Dy^i/ds=0$ and $D(dx^i/ds)/ds=0$ from (2.1) and (2.2). Observing (2.12) and (2.14), we get

$$Dy^i/ds=(Dv^\alpha/ds)B^i_\alpha+H_\alpha(u,v)(du^\alpha/ds)B^i,$$

$$(7.10)\qquad D(dx^i/ds)/ds=\{D(du^\alpha/ds)/ds\}B^i_\alpha+\{H_{\alpha\beta}(u,v)(du^\alpha/ds)(du^\beta/ds)$$
$$+K_{\alpha\beta}(u,v)(du^\alpha/ds)(Dv^\beta/ds)\}B^i=0.$$

It then follows that if an h-path of F^n is on a hypersurface F^{n-1}, it is also an h-path of F^{n-1} and $H_\alpha(u,v)du^\alpha/ds=0$, $H_{\alpha\beta}(u,v)(du^\alpha/ds)(du^\beta/ds)=0$.

Definition 2. If each h-path of a hypersurface F^{n-1} with respect to the induced connection $IF\Gamma$ is an h-path of the enveloping space F^n with respect to a connection $F\Gamma$, then F^{n-1} is called a *hyperplane of the second kind.*

Proposition 7.3. *A hypersurface F^{n-1} is a hyperplane of the 2nd kind if and only if the normal curvature vector H_α and the symmetric part $(H_{\alpha\beta}+H_{\beta\alpha})/2$ of the second fundamental h-tensor vanish identically.*

This is obvious from (7.10). Then Theorem 7.1 leads to

Theorem 7.2. (1) *With respect to $B\Gamma$ and $H\Gamma$, the concepts of hyperplane of the 1st and 2nd kinds coincide with each other.* (2) *With respect to $C\Gamma$ and $R\Gamma$, a hypersurface is a hyperplane of the 2nd kind if and only if $H_\alpha=0$ and $Q_{\alpha\beta}=0$, and then we have $\overset{c}{H}_{\alpha\beta}=\overset{r}{H}_{\alpha\beta}=0$.*

Remark. "Hyperplane of the 2nd kind" is the name given by Rapcsák [12]. Kikuchi [9] named it a *weakly geodesic hypersurface.* It is shown from (2.12) and (2.14) and indicated by Kikuchi that, on such a hypersurface, $DX^\alpha=0$ implies $DX^i=0(X^i=B^i_\alpha X^\alpha)$ on the supposition $Dv^\alpha=0$.

Thirdly we shall deal with the displacement of the unit normal vector B^i of an F^{n-1}. It is easily shown that

$$(7.11)\qquad DB^i=B^i_{|\alpha}du^\alpha+B^i|_\alpha Dv^\alpha.$$

Definition 3. A hypersurface F^{n-1} of $F^n=(M^n, L(x,y))$, endowed with a Finsler connection $F\Gamma$, is called a *hyperplane of the third kind,* if the unit normal vector B^i of F^{n-1} with respect to the metric $L(x,y)$ is parallel along each curve $(u^\alpha(s), v^\alpha(s))$ on F^{n-1}.

Remark. It is noted that the field of supporting element $v^\alpha(s)$ of the curve in Definition 3 is to be tangential to F^{n-1}.

The naming "hyperplane of the 3rd kind" is due to Rapcsák [12]. Kikuchi

[9] said it to be totally geodesic, following Haimovici [6], p. 594, and indicated that, on such a hypersurface, $DX^{\alpha}=0$ implies $DX^{i}=0$ $(X^{i}=B^{i}_{\alpha}X^{\alpha})$.

From (7.11), together with (3.10) and (3.11), it follows that a necessary and sufficient condition for an F^{n-1} to be a hyperplane of the 3rd kind is to satisfy

$$-H_{\alpha\beta}B^{\alpha i}+g_{jk\mid\beta}(B^{i}B^{j}/2-g^{ij})B^{k}=0\,,$$

$$-K_{\alpha\beta}B^{\alpha i}+g_{jk}|_{\beta}(B^{i}B^{j}/2-g^{ij})B^{k}=0\,.$$

Equating the tangential and normal components of these left-hand side to zero, we respectively get

$$H_{\alpha\beta}+g_{jk\mid\beta}B^{j}_{\alpha}B^{k}=0\,,\qquad g_{jk\mid\beta}B^{j}B^{k}=0\,, \tag{7.12}$$

$$K_{\alpha\beta}+g_{jk}|_{\beta}B^{j}_{\alpha}B^{k}=0\,,\qquad g_{jk}|_{\beta}B^{j}B^{k}=0\,. \tag{7.13}$$

Proposition 7.4. *A hypersurface F^{n-1} is a hyperplane of the 3rd kind, if and only if* (7.12) *and* (7.13) *are satisfied.*

We consider these conditions for each of four typical connections. As to the Cartan connection $C\Gamma$, these equations easily reduce to $\overset{c}{H}_{\alpha\beta}=0$, $K_{\alpha\beta}\,(=M_{\alpha\beta})=0$ respectively. In this case we have $H_{\alpha}=0$ from (5.7) and $Q_{\alpha\beta}=0$ from (5.10). Conversely $H_{\alpha}=0$ and $Q_{\alpha\beta}=0$ imply $\overset{c}{H}_{\alpha\beta}=0$.

Secondly, as to the Rund connection $R\Gamma$, we have $g_{jk\mid\beta}=2g_{jkh}B^{h}H_{\beta}$, $g_{jk}|_{\beta}=2g_{jkh}B^{h}_{\beta}$ from (3.9) and $K_{\alpha\beta}=0$, so that (7.12) and (7.13) are of the form

$$\overset{r}{H}_{\alpha\beta}+2M_{\alpha}H_{\beta}=0\,,\quad MH_{\beta}=0\,,\quad M_{\alpha\beta}=0\,,\quad M_{\beta}=0\,.$$

Therefore we get $\overset{r}{H}_{\alpha\beta}=0$, and so $H_{\alpha}=0$ from (5.33) and $Q_{\alpha\beta}=0$ from (5.35).

Thirdly, as to the Berwald connection $B\Gamma$, we have $g_{jk\mid\beta}=-2P_{jkh}B^{h}_{\beta}+2g_{jkh}B^{h}H_{\beta}$, $g_{jk}|_{\beta}=2g_{jkh}B^{h}_{\beta}$ and $K_{\alpha\beta}=0$, so that (7.12) and (7.13) reduce to

$$\overset{b}{H}_{\alpha\beta}-2Q_{\alpha\beta}+2M_{\alpha}H_{\beta}=0\,,\quad Q_{\beta}-MH_{\beta}=0\,,\quad M_{\alpha\beta}=0\,,\quad M_{\beta}=0\,.$$

The first equation and (6.8) give $H_{\alpha}=0$ and $\overset{b}{H}_{\alpha\beta}=0$.

Finally, as to the Hashiguchi connection $H\Gamma$, we have $g_{jk\mid\beta}=-2P_{jkh}B^{h}_{\beta}$, $g_{jk}|_{\beta}=0$ and $K_{\alpha\beta}=M_{\alpha\beta}$, so that (7.12) and (7.13) become

$$\overset{h}{H}_{\alpha\beta}-2Q_{\alpha\beta}=0\,,\quad Q_{\beta}=0\,,\quad M_{\alpha\beta}=0\,.$$

The first equation and (6.18) give $H_{\alpha}=0$ and $\overset{h}{H}_{\alpha\beta}=0$.

Consequently we have

Theorem 7.3. *A hypersurface F^{n-1} is a hyperplane of the 3rd kind with respect to the connections $C\Gamma$, $R\Gamma$, $H\Gamma$ and $B\Gamma$ respectively, if and only if*

(1) $C\Gamma$: $M_{\alpha\beta}=Q_{\alpha\beta}=0$, $H_{\alpha}=0$,
(2) $R\Gamma$: $M_{\alpha\beta}=Q_{\alpha\beta}=0$, $H_{\alpha}=M_{\alpha}=0$,

(3) $H\Gamma$: $M_{\alpha\beta}=Q_{\alpha\beta}=0,\ H_\alpha=Q_\alpha=0,$

(4) $B\Gamma$: $M_{\alpha\beta}=Q_{\alpha\beta}=0,\ H_\alpha=M_\alpha=Q_\alpha=0.$

In every case the second fundamental h-tensors vanish.

As a conclusion of the present section we compare the concepts of hyperplane with respect to $C\Gamma$, $R\Gamma$, $H\Gamma$ and $B\Gamma$ with each other:

(1) The concepts of hyperplane of the 1st kind are identical with respect to these four connections, and characterized by $H_\alpha=0$.

(2) With respect to $C\Gamma$ and $R\Gamma$, the concepts of hyperplane of the 2nd kind are identical, and characterized by $H_\alpha=0$ and $Q_{\alpha\beta}=0$.

(3) With respect to $B\Gamma$ and $H\Gamma$, the concepts of hyperplane of the 2nd kind coincide with that of the 1st kind.

(4) With respect to $C\Gamma$, a hypersurface is a hyperplane of the 3rd kind, if and only if $M_{\alpha\beta}=Q_{\alpha\beta}=0$ and $H_\alpha=0$.

(5) The condition for a hypersurface to be a hyperplane of the 3rd kind with respect to other three connections is obtained by adding to that with respect to $C\Gamma$ the conditions $M_\alpha=0$ or $Q_\alpha=0$ or $M_\alpha=Q_\alpha=0$ as indicated by the diagram

$$\begin{array}{ccc} C\Gamma & \xrightarrow{M_\alpha=0} & R\Gamma \\ {\scriptstyle Q_\alpha=0}\downarrow & & \downarrow{\scriptstyle Q_\alpha=0} \\ H\Gamma & \xrightarrow[M_\alpha=0]{} & B\Gamma \end{array}$$

§ 8. Some examples.

We first deal with C-reducible Finsler spaces [28] which are characterized by the following special form of the C-tensor:

(8.1) $$g_{ijk}=(h_{ij}g_k+h_{jk}g_i+h_{ki}g_j)/(n+1)\,,$$

where $h_{ij}=g_{ij}-l_il_j$ is the angular metric tensor and $g_i=g^j{}_{ij}$. Thus, for a C-reducible space F^n, (1.4), (1.6) and (1.12) give

(8.2) $$g_{\alpha\beta\gamma}=(h_{\alpha\beta}g_\gamma+h_{\beta\gamma}g_\alpha+h_{\gamma\alpha}g_\beta)/n\,,$$
$$M_{\alpha\beta}=(g_iB^i)h_{\alpha\beta}/(n+1),\qquad M_\alpha=g_\alpha/n\,.$$

Therefore any hypersurface F^{n-1} of a C-reducible space F^n is also C-reducible.

Next (5.1) and (8.2) imply the existence of t_α such that the $(h)h$-torsion tensor $\overset{c}{T}{}^\alpha_{\beta\gamma}$ of the induced Cartan connection $IC\Gamma$ is written

(8.3) $$\overset{c}{T}{}^\alpha_{\beta\gamma}=h^\alpha_\beta t_\gamma-h^\alpha_\gamma t_\beta\,.$$

Similarly (6.1) is of the form

(8.4) $$\overset{b}{P}{}^\alpha_{\beta\gamma}=2t_\beta h^\alpha_\gamma\,.$$

Proposition 8.1. *Any hypersurface of a C-reducible Finsler space is also C-reducible and we have* (8.2). *The induced Cartan and Berwald connections satisfy* (8.3) *and* (8.4) *respectively.*

Remark. In viewpoint of Theorem 5.1, it will be an interesting problem to consider generalized Cartan connections with the $(h)h$-torsion tensor of the form (8.3). For a Wagner connection ([25], [28]), δ^α_β take place instead of h^α_β in (8.3). Next it will be also an interesting problem to study generalized Berwald P^1-connections with P^1 of the form (8.4).

On C-reducible spaces we know the conclusive theorem (cf. *p. 227): The metric of any C-reducible space is Randers or Kropina. That is, putting $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta(x, y) = b_i(x)y^i$, we have

Randers metric: $L(x, y) = \alpha(x, y) + \beta(x, y)$,
Kropina metric: $L(x, y) = \alpha^2(x, y)/\beta(x, y)$.

We are here concerned with a special Randers metric with a gradient $b_i(x) = \partial_i b$ for a scalar function $b(x)$, and consider a hypersurface $F^{n-1}(c)$ which is given by an equation $b(x) = c$ (constant). From parametrical equations $x^i = x^i(u)$ of $F^{n-1}(c)$ we get $\partial_\alpha b(x(u)) = 0 = b_i B^i_\alpha$, so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore, *along the* $F^{n-1}(c)$ we have

$$b_i B^i_\alpha = 0, \qquad b_i y^i = 0. \tag{8.5}$$

In general the induced metric $\underline{L}(u, v)$ from the Randers metric is given by

$$\underline{L}(u, v) = \sqrt{a_{ij}(x(u))B^{ij}_{\alpha\beta}v^\alpha v^\beta} + b_i(x(u))B^i_\alpha v^\alpha.$$

Therefore the induced metric of the $F^{n-1}(c)$ becomes

$$\underline{L}(u, v) = \sqrt{a_{\alpha\beta}(u)v^\alpha v^\beta}, \qquad a_{\alpha\beta}(u) = a_{ij}(x(u))B^{\alpha\beta}_{ij}. \tag{8.6}$$

This is Riemannian (cf. [23]).

Next it is known [28] that for a Randers metric we have

*(30.17)
$$g^{ij} = \alpha a^{ij}/L - \alpha(a^{ih}y^j + a^{jh}y^i)b_h/L^2 + (\alpha b^2 + \beta)y^i y^j/L^3,$$

where $b^2 = a^{ij}b_i b_j$. Then, along the $F^{n-1}(c)$, (8.5) leads to $b^2 = g^{ij}b_i b_j$. Thus we get

$$b_i(x(u)) = \sqrt{b^2}B_i, \qquad b^2 = a^{ij}b_i b_j. \tag{8.7}$$

Now the C-reducibility of the Randers metric comes from the equation

*(30.18)
$$2Lg_{ijk} = \mathfrak{S}_{(ijk)}\{h_{ij}(b_k - \beta a_{kh}y^h/\alpha^2)\},$$

which implies $g_i = (n+1)(b_i - \beta a_{ij}y^j/\alpha^2)/2L$. In particular, along the $F^{n-1}(c)$ we have $g_i = (n+1)b_i/2\alpha$, and so $g_i B^i = (n+1)\sqrt{b^2}/2$. Since $\underline{L}$ is Riemannian, we have $g_\alpha=0$. Therefore (8.2) yields

(8.8) $$M_{\alpha\beta}=\sqrt{b^2}h_{\alpha\beta}/2\alpha, \qquad M_\alpha=0.$$

It is noted that $M_\alpha=0$ implies the symmetry property of $\overset{c}{H}_{\alpha\beta}$ from (5.6).

Next, from $b_iB^i_\alpha=0$ we get $b_{i|\beta}B^i_\alpha+b_iB^i_{\alpha|\beta}=0$. Referring to the $C\Gamma$, this is written

$$(b_{i|j}B^j_\beta+b_i|_jB^jH_\beta)B^i_\alpha+b_i\overset{c}{H}_{\alpha\beta}B^i$$
$$=b_{i|j}B^{ij}_{\alpha\beta}+(-b_hg^h_{ij})B^i_\alpha B^jH_\beta+\sqrt{b^2}\overset{c}{H}_{\alpha\beta}=0.$$

Since $b_hg^h_{ij}B^i_\alpha B^j=\sqrt{b^2}M_\alpha=0$ from (8.7) and (8.8), we have

(8.9) $$\sqrt{b^2}\overset{c}{H}_{\alpha\beta}+b_{i|j}B^{ij}_{\alpha\beta}=0.$$

It is noted that $b_{i|j}$ is symmetric. Further we have

(8.10) $$\sqrt{b^2}H_\alpha+b_{i|j}B^i_\alpha y^j=0, \qquad \sqrt{b^2}H_0+b_{i|j}y^iy^j=0.$$

To consider the condition $H_0=0$ for the $F^{n-1}(c)$ to be a hyperplane of the 1st kind, we deal with $b_{i|j}y^iy^j$. This $b_{i|j}$ is the covariant derivative with respect to $C\Gamma$ of F^n, and so $b_{i|j}$ may depend on y^i. On the other hand, if we denote by $b_{i;j}$ the covariant derivative with respect to the Riemannian connection (γ^i_{jk}) constructed from $a_{ij}(x)$, then $b_{i;j}$ does not depend on y^i. We shall consider the difference $b_{i|j}-b_{i;j}$ in the following. The so-called difference tensor $D^i_{jk}=\Gamma^{*i}_{jk}-\gamma^i_{jk}$ has been found in a previous paper [27]. By means of (III) and (2.6′) of [27], along the $F^{n-1}(c)$ we have

$$D^k_{ij}=\alpha H_{hij}a^{hk}+(H_{ij}/\alpha-H_{\beta ij})y^k,$$

which implies $b_kD^k_{ij}=\alpha H_{hij}a^{hk}b_k$. It is easily seen that

$$2\alpha^2D^i_{0j}=\alpha E_{00}\overset{a}{h}{}^i_j+(2\alpha E_{j0}-E_{00}b_j)y^i,$$

where $\overset{a}{h}_{ij}$ is the angular metric tensor of the Riemannian $a_{ij}(x)$. Then we get

$$4\alpha^3L_{jkh}D^h_{0i}=-2E_{00}(\overset{a}{h}_{ik}\overset{a}{l}_j+\overset{a}{h}_{ij}\overset{a}{l}_k)+(2\alpha E_{i0}-E_{00}b_i)\overset{a}{h}_{jk},$$

where $\overset{a}{l}_i$ is the normalized supporting element of the Riemannian $a_{ij}(x)$. Thus, paying attention to $\overset{a}{h}_{ij}=h_{ij}$ along the $F^{n-1}(c)$, we finally get

(8.11) $$4\alpha^2b_kD^k_{ij}=-(2\alpha E_{\beta0}-E_{00}b^2)h_{ij}+2\alpha(E_{j0}b_i+E_{i0}b_j)-2E_{00}b_ib_j,$$

which implies

(8.12) $$b_kD^k_{i0}=E_{00}b_i/2\alpha, \qquad b_kD^k_{00}=0.$$

Consequently (8.10) may be written as

(8.10′) $$\sqrt{b^2}H_\alpha+b_{i;j}B^i_\alpha y^j=0, \qquad \sqrt{b^2}H_0+b_{i;j}y^iy^j=0.$$

Thus the condition $H_0=0$ is equivalent to $b_{i;j}y^iy^j=0$, where $b_{i;j}$ does not depend on y^i. Since y^i is to satisfy (8.5), the condition is written as $b_{i;j}y^iy^j=(b_iy^i)(c_jy^j)$ for some $c_j(x)$, so that we have

$$2b_{i;j}=b_ic_j+b_jc_i\,. \tag{8.13}$$

Then we have $b_{i;j}B^i_\alpha y^j=0$ from (8.5), and so $H_\alpha=0$ (cf. Proposition 5.1).

Now we have $E_{00}=0$ and $E_{\beta 0}=b^2c_0/2$, so that (8.9) reduces to

$$\sqrt{b^2}\overset{c}{H}_{\alpha\beta}+(b^2c_0/4\alpha)h_{\alpha\beta}=0\,. \tag{8.9$'$}$$

Further the condition $\overset{c}{H}_{\alpha\beta}=0$ for $F^{n-1}(c)$ is $c_0=c_i(x)y^i=0$, so that we have a function $e(x)$ satisfying $c_i(x)=e(x)b_i(x)$:

$$b_{i;j}=eb_ib_j\,. \tag{8.14}$$

Finally (8.8) and Theorem 7.3 show that this $F^{n-1}(c)$ does not become a hyperplane of the 3rd kind.

Summarizing up all the above, we have

Theorem 8.1. *Let F^n be a Randers Finsler space with a gradient $b_i(x)=\partial_i b(x)$ and let $F^{n-1}(c)$ be the hypersurface of F^n which is given by $b(x)=c$ (constant). Suppose the Riemannian metric $a_{ij}(x)dx^idx^j$ be positive-definite and b_i be nonzero field. Then the induced metric on $F^{n-1}(c)$ is a Riemannian metric, given by* (8.6), *and we have* (8.7) *and* (8.8). *In particular $\overset{c}{H}_{\alpha\beta}$ is symmetric.*

The condition for $F^{n-1}(c)$ to be a hyperplane of the 1st *kind is* (8.13) *and* (8.9′) *is satisfied. Next the condition for $F^{n-1}(c)$ to be a hyperplane of the* 2nd *kind is* (8.14). *$F^{n-1}(c)$ does not become a hyperplane of the* 3rd *kind.*

Remark. The condition (8.14) may be also shown by $Q_{\alpha\beta}=0$ and Proposition 4 of [27]. In fact, we then have $Q_{\alpha\beta}=(p_iB^i)g_{\alpha\beta}$ and (8.13) implies $p_iB^i=\sqrt{b^2}c_0/4\alpha$.

§ 9. Haimovici-Kikuchi-Rapcsák's theorems.

We consider hypersurfaces with respect to the Cartan connection $C\Gamma$. For a hyperplane of the 1st kind ($H_\alpha=0$), (5.15), (6.15) and Theorem 7.1 give

$$R_{ijk}B^iB^{jk}_{\beta\gamma}=0\,, \tag{9.1}$$

$$G^i_{jkh}B_iB^{jkh}_{\alpha\beta\gamma}=0\,. \tag{9.2}$$

For a hyperplane of the 2nd kind, we additionally have $Q_{\beta\gamma}=0$ from Theorem 7.2, that is

$$P_{ijk}B^iB^{jk}_{\beta\gamma}=0\,. \tag{9.3}$$

For a hyperplane of the 3rd kind, we further have $M_{\beta\gamma}=0$ from Theorem 7.3, that is,

$$g_{ijk}B^iB^{jk}_{\beta\gamma}=0. \tag{9.4}$$

Now in 1957 Rapcsák [12], following Haimovici [6] and Kikuchi [9], showed the following remarkable theorems:

(I) There exists a hyperplane of the 1st kind at each hypersurface element of a Finsler space $F^n(n>2)$, if and only if F^n is *projectively flat* and *of scalar curvature.*

(II) The similar circumstances hold for a hyperplane of the 2nd kind, if and only if F^n is *projectively flat, of scalar curvature* and *Landsberg.*

(III) The similar circumstances hold for a hyperplane of the 3rd kind, if and only if F^n is a *Riemannian space of constant curvature.*

Remark. "At each hypersurface element" means "at each point and in each direction".

Haimovici [6] first announced that such an F^n as in (I) is necessarily to be of scalar curvature and the condition for an F^n as in (III) is to be a Riemannian space of constant curvature. Kikuchi [9] showed the proofs of Haimovici's results.

Rapcsák's proofs of these theorems were based on an interesting lemma. It seems, however, to the author that the long proof of this lemma, simplified by Varga's help, is quite hard to understand and its application may be false. Moreover his proofs of the sufficiency hardly have geometrical meaning. We shall show some lemmas[1] in the following and give a new proof.

Lemma 9.1. *Suppose a tensor* $T^i_{j\cdots k}$ *of* (1, *r*)*-type be* (1) *indicatory, i.e.,* $T^0_{j\cdots k}=T^i_{0\cdots k}=\cdots=T^i_{j\cdots 0}=0$, (2) *symmetric in subscripts* $j, \cdots, k$, *and* (3) *for each hypersurface element* (B^i_α, B^i) *we have* $B_iT^i_{j\cdots k}B^{j\cdots k}_{\alpha\cdots\beta}=0$. *Then we have an indicatory and symmetric tensor* $\mathring{T}_{j\cdots k}$ *of* $(0, r-1)$*-type satisfying*

$$T^i_{j_1\cdots j_r}=h^i_{j_1}\mathring{T}_{j_2\cdots j_r}+\cdots+h^i_{j_r}\mathring{T}_{j_1\cdots j_{r-1}}, \tag{4}$$

where $h^i_j=\delta^i_j-l^il_j$.

Proof. It will be sufficient for the proof to deal with T^h_{ijk} of (1, 3)-type. Let $E(x, y)$ be the set of all the orthonormal frames $\{e^i_{a)}\}$ at a point (x, y) of F^n such that $e^i_{1)}=l^i(=y^i/L(x, y))$, and consider the scalar components of T^h_{ijk}:

$$T_{abcd}=T^h_{ijk}e_{a)h}e^i_{b)}e^j_{c)}e^k_{d)}.$$

Conditions (2) and (1) respectively show the symmetry property of T_{abcd} in b, c, d, and $T_{1bcd}=T_{a1cd}=0$. (3) further shows

$$T_{abcd}=0, \qquad a, b, c, d=2, \cdots, n;\ a\neq b, c, d. \tag{3'}$$

If we fix an $\mathring{e}(x, y)=\{e^i_{a)}\}\in E(x, y)$, any $\bar{e}(x, y)=\{\bar{e}^i_{a)}\}\in E(x, y)$ is written $\bar{e}^i_{a)}=t^b_ae^i_{b)}$ where $t^1_1=1$, $t^a_1=t^1_b=0$, $(t^a_b)\in O(n-1)$ for $a, b=2, \cdots, n$. We consider 1-parameter subgroup of $E(x, y)$ with a parameter t where $t=0$ corresponds to

[1] According to Professor L. Tamássy's recent communication to the author, Professor A. Rapcsák says that the correct form of his lemma is just our Lemma 9.4.

$\mathring{e}(x, y)$. If we put $O^a_b=(dt^a_b/dt)_0$, we have $O^1_1=O^a_1=O^1_b=0$, $O^a_b=-O^b_a$ for a, b as above. We then have relations

$$\overline{T}_{abcd}=T_{pqrs}t^p_a t^q_b t^r_c t^s_d=0, \qquad a, b, c, d=2, \cdots, n\,;\ a\neq b, c, d\,,$$

between the components $\overline{T}_{abcd}$(resp. T_{pqrs}) of T^h_{ijk} with respect to $\bar{e}(x, y)$ (resp. $\mathring{e}(x, y)$). Differentiating the relations by t and evaluating at $t=0$, we get

$$(5)\quad T_{pbcd}O^p_a+T_{aqcd}O^q_b+T_{abrd}O^r_c+T_{abcs}O^s_d=0\,,$$

where summation convention is made in p, q, r, s.

(i) $b, c, d\neq$: Owing to (3′), the surviving terms of (5) are

$$(T_{bbcd}O^b_a+T_{cbcd}O^c_a+T_{dbcd}O^d_a)+T_{aacd}O^a_b+T_{abad}O^a_c+T_{abca}O^a_d$$
$$=(T_{bbcd}-T_{aacd})O^b_a+(T_{ccbd}-T_{aabd})O^c_a+(T_{ddbc}-T_{aabc})O^d_a=0\,.$$

Arbitrariness of (O^b_a) leads to

$$(5\text{-}1)\quad T_{aacd}=T_{bbcd}, \qquad a, b, c, d=2, \cdots, n\,; \qquad a, b, c, d\neq\,.$$

(ii) $b\neq c=d$: Similarly (5) reduces to

$$(T_{bbcc}O^b_a+T_{cbcc}O^c_a)+T_{aacc}O^a_b+2T_{abac}O^a_c$$
$$=(T_{bbcc}-T_{aacc})O^b_a+(T_{cccb}-2T_{aacb})O^c_a=0,$$

which immediately implies

$$(5\text{-}2)\quad T_{aacc}=T_{bbcc}, \quad T_{cccb}=2T_{aacb}, \quad a, b, c=2, \cdots, n\,; \quad a, b, c\neq\,.$$

(iii) $b=c=d$: Similarly (5) reduces to

$$T_{bbbb}O^b_a+3T_{aabb}O^a_b=(T_{bbbb}-3T_{aabb})O^a_b=0\,,$$

which shows

$$(5\text{-}3)\quad T_{bbbb}=3T_{aabb}, \qquad a, b=2, \cdots, n\,; \qquad a\neq b\,.$$

Consequently (5) enables us to introduce quantities $T_{bc}(=T_{cb})$, $b, c=2, \cdots, n$ such that

$$(6)\quad T_{aabc}=T_{bc}, \qquad T_{aaab}=2T_{ab}, \qquad T_{aaaa}=3T_{aa}\,,$$
$$a, b, c=2, \cdots, n\,; \qquad a\neq b, c\,.$$

Now, denoting by Σ the summation from 2 to n and paying attention to surviving components, we have

$$T_{hijk}=\sum_{a\neq b, c}\{T_{aabc}e_{a)h}e_{a)i}e_{b)j}e_{c)k}+T_{abac}e_{a)h}e_{b)i}e_{a)j}e_{c)k}+T_{abca}e_{a)h}e_{b)i}e_{c)j}e_{a)k}\}$$
$$+\sum_{b\neq c}\{T_{bbbc}e_{b)h}e_{b)i}e_{b)j}e_{c)k}+T_{bbcb}e_{b)h}e_{b)i}e_{c)j}e_{b)k}+T_{bcbb}e_{b)h}e_{c)i}e_{b)j}e_{b)k}\}$$
$$+\sum_{b}\{T_{bbbb}e_{b)h}e_{b)i}e_{b)j}e_{b)k}\}\,.$$

Owing to (6), dividing the first $\{\cdots\}$ into two parts $(b=c)$ and $(b\neq c)$, and paying attention to $\sum_b \{e_{b)i}e_{b)j}\} = g_{ij} - e_{1)i}e_{1)j} = h_{ij}$, we have

$$
\begin{aligned}
T_{hijk} = & \sum_b T_{bb}[\{h_{hi} - e_{b)h}e_{b)i}\} e_{b)j}e_{b)k} + \{h_{hj} - e_{b)h}e_{b)j}\} e_{b)i}e_{b)k} \\
& + \{h_{hk} - e_{b)h}e_{b)k}\} e_{b)i}e_{b)j}] \\
& + \sum_{b\neq c} T_{bc}[\{h_{hi} - e_{b)h}e_{b)i} - e_{c)h}e_{c)i}\} e_{b)j}e_{c)k} \\
& \qquad + \{h_{hj} - e_{b)h}e_{b)j} - e_{c)h}e_{c)j}\} e_{b)i}e_{c)k} \\
& \qquad + \{h_{hk} - e_{b)h}e_{b)k} - e_{c)h}e_{c)k}\} e_{b)i}e_{c)j}] \\
& + \sum_{b\neq c} 2T_{bc}e_{b)h}\{e_{b)i}e_{b)j}e_{c)k} + e_{b)i}e_{c)j}e_{b)k} + e_{c)i}e_{b)j}e_{b)k}\} \\
& + \sum_b 3T_{bb}e_{b)h}e_{b)i}e_{b)j}e_{b)k} \\
= & \sum_{b,c} T_{bc}\{h_{hi}e_{b)j}e_{c)k} + h_{hj}e_{b)i}e_{c)k} + h_{hk}e_{b)i}e_{c)j}\}.
\end{aligned}
$$

Thus, if we put $\mathring{T}_{jk} = \sum_{b,c} T_{bc}e_{b)j}e_{c)k}$, this is indicatory and symmetric, and then we obtain the conclusion $T_{hijk} = h_{hi}\mathring{T}_{jk} + h_{hj}\mathring{T}_{ik} + h_{hk}\mathring{T}_{ij}$.

It is clear that the assumption (2) of Lemma 9.1 becomes nonsense for T^i_j of (1, 1)-type. Consequently we immediately get

Lemma 9.2. *If T^i_j is indicatory and $T^i_jB_iB^j_\alpha = 0$ for each (B^i_α, B^i), then we have a scalar $\mathring{T}$ satisfying $T^i_j = \mathring{T}h^i_j$.*

Next, we shall deal with $T_{ij\cdots k}$ which is symmetric in $i, j, \cdots, k$ instead of the assumption (2) of Lemma 9.1:

Lemma 9.3. (1) *If a tensor T_{ijk} is indicatory, symmetric and $T_{ijk}B^iB^{jk}_{\alpha\beta} = 0$ for each (B^i_α, B^i), we have $T_{ijk} = 0$.*

(2) *If a tensor T_{hijk} is indicatory, symmetric and $T_{hijk}B^hB^{ijk}_{\alpha\beta\gamma} = 0$ for each (B^i_α, B^i), we have a scalar $\mathring{T}$ such that $T_{hijk} = \mathring{T}(h_{hi}h_{jk} + h_{hj}h_{ki} + h_{hk}h_{ij})$.*

Proof. (1) From Lemma 9.1 we first get $T_{ijk} = h_{ij}\mathring{T}_k + h_{ik}\mathring{T}_j$. Owing to symmetry, from (1.6) we have

$$0 = (h_{ij}\mathring{T}_k + h_{ik}\mathring{T}_j)B^i_\alpha B^j B^k_\beta = h_{\alpha\beta}\mathring{T}_jB^j,$$

which implies $\mathring{T}_jB^j = 0$. Thus $\mathring{T}_j$ satisfies $\mathring{T}_jl^j = \mathring{T}_jB^j = 0$ for each B^j orthogonal to l^j, and so we have $\mathring{T}_j = 0$.

(2) Similarly we have

$$
\begin{aligned}
0 = T_{hijk}B^h_\alpha B^iB^{jk}_{\beta\gamma} &= (h_{hi}\mathring{T}_{jk} + h_{hj}\mathring{T}_{ik} + h_{hk}\mathring{T}_{ij})B^h_\alpha B^iB^{jk}_{\beta\gamma} \\
&= h_{\alpha\beta}\mathring{T}_{ik}B^iB^k_\gamma + h_{\alpha\gamma}\mathring{T}_{ij}B^iB^j_\beta,
\end{aligned}
$$

and contraction by $g^{\alpha\beta}$ gives $\mathring{T}_{ik}B^iB^k_\gamma = 0$. Applying Lemma 9.2 to $\mathring{T}_{ik}$, we get $\mathring{T}_{ik} = \mathring{T}h_{ik}$. Thus the proof has been completed.

Lemma 9.4. *Suppose a tensor* $T^i_{j\cdots k}$ *of* (1, r)*-type be* (1) *indicatory with respect to* $j, \cdots, k$, (2) *symmetric in* $j, \cdots, k$, *and* (3) *for each hypersurface element* (B^i_α, B^i) *we have* $B_i T^i_{j\cdots k} B^{j\cdots k}_{\alpha\cdots\beta}=0$. *Then we have an indicatory and symmetric tensor* $\mathring{T}_{j\cdots k}$ *of* (0, $r-1$)*-type such that*

$$(4') \quad T^i_{j_1\cdots j_r}=l^i T^0_{j_1\cdots j_r}/L+h^i_{j_1}\mathring{T}_{j_2\cdots j_r}+\cdots+h^i_{j_r}\mathring{T}_{j_1\cdots j_{r-1}}.$$

Proof. We indicatorize $T^i_{j\cdots k}$ with respect to the index i (*Definition 31.3):

$$'T^i_{j\cdots k}=T^i_{j\cdots k}-l^i T^0_{j\cdots k}/L.$$

This $'T^i_{j\cdots k}$ satisfies three assumptions of Lemma 9.1, hence we can apply Lemma 9.1 to $'T^i_{j\cdots k}$. Q.E.D.

Now we return to the considerations of three theorems mentioned at the beginning of the present section. The equation (9.1), contracting by v^β, gives

$$(9.1') \qquad R_{i0k}B^i B^k_\gamma=0.$$

It is well-known (*§ 17) that R_{i0k} is indicatory, and application of Lemma 9.2 to R_{i0k} shows that R_{i0k} is proportional to h_{ik}. Therefore *Theorem 26.1 shows $R_{i0k}=KL^2 h_{ik}$; F^n is of scalar curvature K.

Next we shall consider (9.2). Since the hv-curvature tensor G^i_{jkh} of the Berwald connection $B\Gamma$ is indicatory and symmetric in the subscripts (*§ 18, [30]), Lemma 9.4 immediately gives

$$(9.5) \qquad G^i_{jkh}=l^i G^0_{jkh}/L+h^i_j\mathring{G}_{kh}+h^i_k\mathring{G}_{jh}+h^i_h\mathring{G}_{jk},$$

where $\mathring{G}_{kh}$ is indicatory and symmetric. (9.5) leads to $G_{jk}(=G^i_{jki})=(n+1)\mathring{G}_{jk}$, and so we have

$$(9.5') \qquad G^i_{jkh}=l^i G^0_{jkh}/L+(h^i_j G_{kh}+h^i_k G_{jh}+h^i_h G_{jk})/(n+1).$$

Differentiating this by y^l, we get

$$G^i_{jkh\cdot l}=l^i G^0_{jkh\cdot l}/L+G^0_{jkh}(h^i_l-l^i l_l)/L^2$$
$$+\mathfrak{S}_{(jkh)}\{-(h^i_l l_j+h_{lj}l^i)G_{kh}/L+h^i_j G_{kh\cdot l}\}/(n+1),$$

where $(\,.\,)=\dot{\partial}$. Contraction with respect to $i=l$ yields

$$G^0_{jkh}=L^2 G_{jk\cdot h}/(n+1)+L(l_j G_{kh}+l_k G_{hj}+l_h G_{jk})/(n+1),$$

because of $n>2$. Therefore (9.5′) may be rewritten as

$$(9.5'') \qquad G^i_{jkh}=(y^i G_{jk\cdot h}+\delta^i_j G_{kh}+\delta^i_k G_{hj}+\delta^i_h G_{jk})/(n+1),$$

which just shows vanishing of the Douglas tensor D^i_{jkh} of F^n [29].

On the other hand, Z. Szabó ([34], [29]) showed that $F^n(n>2)$ is of scalar curvature if and only if its Weyl tensor W^i_{jkh} vanishes identically. Therefore F^n is projectively flat [29] (and necessarily of scalar curvature).

Next it is well-known that $P_{ijk}(=g_{ijk|0})$ of $C\Gamma$ and g_{ijk} are indicatory and symmetric in i, j, k. Therefore (9.3) and (9.4), applying Lemma 9.3 (1), immediately enable us to recognize that the condition for F^n in (II) and (III) is necessary.

Now we are in a position to show the sufficiency of the conditions for F^n in (I). Suppose that $F^n(n>2)$ be projectively flat. Then we have a rectilinear coordinate system (x^i) [31] such that $G^i(x, y)$ appearing in (7.1) is written as

$$G^i(x, y) = -P(x, y)y^i, \tag{9.6}$$

where $P(x, y)$ is a (1) p-homogeneous function defined on the domain of the (x^i). Then we have

$$G^i_j = -P_j y^i - P\delta^i_j, \qquad G^i_{jk} = -P_j\delta^i_k - P_k\delta^i_j - y^i P_{jk}, \tag{9.7}$$

where $P_j = \dot{\partial}_j P$ and $P_{jk} = \dot{\partial}_k P_j$. Thus (5.4) ($\Gamma^{*i}_{0j} = G^i_j$) and (6.6) lead to

$$H_\beta = B_i B^i_{0\beta}, \qquad \overset{b}{H}_{\beta\gamma} = B_i B^i_{\beta\gamma}. \tag{9.8}$$

We consider an arbitrary hyperplane P^{n-1} in this coordinates which is given by a linear equation $\Phi(x) = b_i x^i + b = 0$ with constant b's. Differentiating this by parameters u^α and u^β, we have $b_i B^i_\alpha = 0$ and $b_i B^i_{\alpha\beta} = 0$; the former shows that b_i are proportional to B_i and then the latter implies $B_i B^i_{\alpha\beta} = 0$. Therefore (9.8) shows that H_α and $\overset{b}{H}_{\alpha\beta}$ of the P^{n-1} vanish. Consequently each hyperplane P^{n-1} is a hyperplane of the 1st kind by means of Theorem 7.1 and consequently we get

Theorem 9.1. (Rapcsák) *There exists a hyperplane of the* 1*st kind* (*with respect to* $C\Gamma$) *at each hypersurface element of a Finsler space* $F^n(n>2)$, *if and only if* F^n *is projectively flat. The hyperplanes are represented by a linear equation in a rectilinear coordinate system.*

Remark. As mentioned above, this F^n is necessarily of scalar curvature by Szabó's theorem, and $C\Gamma$ may be changed for $B\Gamma$ or $H\Gamma$ or $R\Gamma$.

Varga [15] shows that such an F^n as in Theorem 9.1 should be of constant curvature, but we can hardly understand Varga's discussions and especially his differential equations (29).

Next, if a Finsler space $F^n(n>2)$ is projectively flat and Landsberg ($P_{ijk} = 0$), then F^n admits the above hyperplane P^{n-1} and further $Q_{\alpha\beta} = P_{ijk} B^i B^{jk}_{\alpha\beta} = 0$, so that P^{n-1} is of the 2nd kind by means of Theorem 7.2. (6.7) gives $\overset{c}{H}_{\alpha\beta} = 0$. Consequently.

Theorem 9.2. (Rapcsák) *There exists a hyperplane of the* 2*nd kind* (*with respect to* $C\Gamma$) *at each hypersurface element of a Finsler space* $F^n(n>2)$, *if and only if* F^n *is projectively flat and Landsberg.*

From this necessary condition we recall Numata's theorem ([32], *Theorem 30.6): If $F^n(n>2)$ is of nonzero scalar curvature K and Landsberg, then F^n is a Riemannian space of constant curvature K. Therefore we have

Corollary 9.1. *A Finsler space $F^n(n>2)$ of Theorem 9.2 is of scalar curvature K. If $K\neq 0$, F^n is a Riemannian space of constant curvature K.*

Finally (III), proved first by Kikuchi [9], is stated as follows:

Theorem 9.3. (Kikuchi) *If a Finsler space $F^n(n>2)$ admits a hyperplane of the 3rd kind at each hypersurface element of F^n, then F^n is a Riemannian space of constant curvature. The converse is also true.*

Remark. See §6 of [31] about rectilinear coordinate systems in Riemannian spaces of constant curvature.

Finslerian projective geometry. Theorem 9.1, together with the author's previous paper [31], reminds us of this word! In fact, the concept of Finsler space *with rectilinear extremals*, originally suggested by Hilbert, had caused various interesting theories already before Finsler's thesis in 1918. A Finsler space is said to be with rectilinear extremals, if there exists a covering by coordinate systems (x^i) in which each extremal (geodesic) curve is represented by a system of $n-1$ linear equations in x^i, and such a coordinate system is called *rectilinear* by the present author [31]. Now a Finsler space with rectilinear extremals is really *projectively flat* (projective to a locally Minkowski space), and the rectilinear coordinate systems obey projective transformations [31].

Then it is natural to recall the dual geometrical figures: What is a figure represented by a linear equation in a rectilinear coordinate system? Theorem 9.1 answers this question: It is a hyperplane of the first kind, just the same circumstances as in a projective space! It will be obvious that the similar facts hold for subspaces of arbitrary dimensions in a projectively flat Finsler space. Further it is easy to observe that Theorem 9.1 concerns a nonlinear connection $(N^i_j(x, y))$ alone; if we are concerned with a Finsler metric, Theorem 9.1 is to assert the existence of totally geodesic hypersurfaces instead of hyperplanes of the first kind.

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