J. Math. Kyoto Univ. (JMKYAZ)
25-4 (1985) 789-806

Differentiability of generalized Fourier transforms associated with Schrödinger operators

By

Hiroshi ISOZAKI

(Received Oct. 12, 1984)

Introduction

In our previous work [1], we developed a theory of eigenfunction expansion or generalized Fourier transformation associated with the Schrödinger operator $H=-\Delta+V(x)$ in $L^2(\boldsymbol{R}^n)$ $(n\geq 2)$ with a long-range potential $V(x)$ satisfying the following assumption

$$\text{(A)}\quad \begin{cases} V(x) \text{ is a real smooth function on } \boldsymbol{R}^n \text{ such that for some constant } \varepsilon_0>0 \\ \qquad D_x^\alpha V(x)=O(|x|^{-|\alpha|-\varepsilon_0}) \text{ as } |x|\longrightarrow\infty \\ \text{for all multi-index } \alpha. \end{cases}$$

More precisely, we constructed a partially isometric operator $\mathscr{F}$ with initial set $L^2_{ac}(H)$ (the absolutely continuous subspace for H) and final set $L^2(\boldsymbol{R}^n)$ satisfying

$$(\mathscr{F}\alpha(H)f)(\xi)=\alpha(|\xi|^2)(\mathscr{F}f)(\xi)$$

for any bounded Borel function $\alpha(\lambda)$ on $\boldsymbol{R}$ and $f\in L^2(\boldsymbol{R}^n)$. The main idea was as follows: First we construct a real function $\phi(x, \xi)$ which behaves like $x\cdot\xi$ as $|x|\to\infty$ and solves the eikonal equation

$$|\nabla_x\phi(x, \xi)|^2+V(x)=|\xi|^2$$

in an appropriate region of the phase space $\boldsymbol{R}^n\times\boldsymbol{R}^n$. We set $G_0(x, \xi)=e^{-i(x,\xi)}\cdot(-\Delta+V(x)-|\xi|^2)e^{i\phi(x,\xi)}$ and $R(z)=(H-z)^{-1}$. We then define $\mathscr{F}$ formally by

$$(0.1)\qquad (\mathscr{F}f)(\xi)=(2\pi)^{-n/2}\int_{\boldsymbol{R}^n} e^{-i\phi(x,\xi)}f(x)dx - (2\pi)^{-n/2}\int_{\boldsymbol{R}^n} e^{-i\phi(x,\xi)}\,\overline{G_0(x, \xi)}R(|\xi|^2+i0)f(x)dx.$$

If $V(x)$ is short-range, i.e. $V(x)=O(|x|^{-1-\varepsilon_0})$, one can take $x\cdot\xi$ as $\phi(x, \xi)$. Then the above formula (0.1) takes the following form

$$(0.2)\qquad (\mathscr{F}f)(\xi)=(2\pi)^{-n/2}\int e^{-ix\cdot\xi}f(x)dx - (2\pi)^{-n/2}\int e^{-ix\cdot\xi}V(x)R(|\xi|^2+i0)f(x)dx.$$

As is clear from the above definition, the operator $\mathscr{F}$ is a generalization of the ordinary Fouier transformation

$$(0.3)\qquad (\mathscr{F}_0 f)(\xi)=\hat{f}(\xi)=(2\pi)^{-n/2}\int e^{-ix\cdot\xi}f(x)dx.$$

One knows many interesting properties of $\mathscr{F}_0$: Various Paley-Winer type theorems, transforming rapidly decreasing functions into smooth ones, etc. It may be of interest to consider to what extent these properties extend to $\mathscr{F}$. The main purpose of this paper is to prove the following differentiability property for $\mathscr{F}$. For a real number s, let $L^{2,s}$ denote the space of measurable functions $f(x)$ on $\boldsymbol{R}^n$ such that

$$\|f\|_s^2=\int(1+|x|)^{2s}|f(x)|^2dx<\infty.$$

Theorem 0.1. *Let $\gamma>1/2$ be arbitrarily fixed and N a non-negative integer. If $f\in L^{2,N+\gamma}$, $(\mathscr{F}f)(\xi)$ is N times differentiable with respect to $\xi\neq 0$, and for any $\varepsilon>0$*

$$\sum_{|\alpha|\leq N}\int_{|\xi|>\varepsilon}\langle\xi\rangle^{-2\gamma}|D_\xi^\alpha(\mathscr{F}f)\,(\xi)|^2d\xi\leq C\|f\|_{N+\gamma}^2.$$

The differentiability of $\mathscr{F}$ is closely connected with the decay rates for scattering states. Using the above result, we can prove the following

Theorem 0.2. *Let $\chi(\lambda)$ be a smooth function on $\boldsymbol{R}^1$ such that for some $\varepsilon>0$, $\chi(\lambda)=1$ for $\lambda>2\varepsilon$, $\chi(\lambda)=0$ for $\lambda<\varepsilon$. Then for any $s\geq 0$ and $\delta>0$*

$$\|\chi(H)e^{-itH}f\|_{-s}\leq C(1+|t|)^{-s}\|f\|_{s+\delta}.$$

One can make use of the above result as an intermediate step to prove the best possible decay rate

$$(0.4)\qquad \|\chi(H)e^{-itH}f\|_{-s}\leq C(1+|t|)^{-s}\|f\|_s \qquad (s\geq 0),$$

whose proof will be given in a forthcoming paper.

As can be seen from (0.1) and (0.2), in order to prove the differentiability of $\mathscr{F}$, one should consider that of the resolvent $R(\lambda+i0)$, which occupies the major part of this work and is studied in §1 (Theorem 1.9) utilizing the recent results of Isozaki–Kitada [2], [3] concerning the micro-local estimates for the resolvent. The differentiability with respect to λ of $R(\lambda+i0)$ is also discussed by Jensen–Mourre–Perry [8], where they employ the commutator method due to Mourre [9].

Let us list the notations used in this paper. For a vector $x\in\boldsymbol{R}^n$, $\hat{x}=x/|x|$ and $\langle x\rangle=(1+|x|^2)^{1/2}$. $\mathscr{B}(\boldsymbol{R}^n)$ denotes the space of smooth functions on $\boldsymbol{R}^n$ with bounded derivatives. $C_0^\infty(\boldsymbol{R}^n)$ is the space of smooth functions on $\boldsymbol{R}^n$ with compact support. For two Banach spaces X and Y, $\boldsymbol{B}(X;Y)$ denotes the totality of bounded linear operators from X to Y. For a multi-index α, $D_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$, $|\alpha|=\alpha_1+\cdots+\alpha_n$. Throughout the paper, C's denote various constants independent of the parameters in question.

§ 1. Differentiability of the resolvent

Let $H=-\Delta+V(x)$, where V satisfies the assumption (A) in the introduction, and $R(z)=(H-z)^{-1}$. Our starting point is the following limiting absorption principle (see e.g. [2], Theorem 1.2).

Lemma 1.1. *For any* $\lambda>0$ *and* $\gamma>1/2$, *there exists a strong limit* $\operatorname{s-lim}_{\varepsilon\downarrow 0} R(\lambda\pm i\varepsilon)=R(\lambda\pm i0)$ *in* $\boldsymbol{B}(L^{2,\gamma}; L^{2,-\gamma})$. *Moreover for any* $\varepsilon>0$, *there exists a constant* $C>0$ *such that*

$$\|R(\lambda\pm i0)f\|_{-\gamma}\leq C/\sqrt{\lambda}\|f\|_{\gamma}$$

for $\lambda>\varepsilon$.

Our aim of this section is to discuss the differentiability with respect to λ of the resolvent $R(\lambda\pm i0)$. It leads us to consider the powers of $R(\lambda\pm i0)$, since by the formal calculus

$$\left(\frac{d}{d\lambda}\right)^{N} R(\lambda\pm i0)=N!\, R(\lambda\pm i0)^{N+1}.$$

Needless to say, one cannot use Lemma 1.1 directly to treat $R(\lambda\pm i0)^{N+1}$. If one inserts some pseudo-differential operators (Ps. D. Op.'s), however, one can give a definite meaning to $R(\lambda\pm i0)^{N+1}$. The estimates of resolvents multiplied by Ps. D. Op.'s, which we call the micro-local resolvent estimates, have been intensively studied in [2] and [3]. Let us begin with recalling the results.

Definition 1.2. Let $a_0>0$ be arbitrarily fixed and $\mu\geq 0$. A smooth function $p(x, \xi; \lambda)$ belongs to $W(\mu)$ if for any α, β

$$\sup_{x,\xi\in\boldsymbol{R}^n,\lambda>a_0} \langle x\rangle^{\mu+|\alpha|}\langle\xi\rangle^{|\beta|}|D_x^{\alpha}D_{\xi}^{\beta}\, p(x, \xi; \lambda)|<\infty.$$

Definition 1.3. $p(x, \xi; \lambda)\in S_{\infty}$ if

(1) $p(x, \xi; \lambda)\in W(0)$,

(2) there exists a constant $\varepsilon>0$ such that

$$p(x, \xi; \lambda)=0 \quad \text{if} \quad ||\xi|/\sqrt{\lambda}-1|<\varepsilon,\ \lambda>a_0,$$

(ε may depend on $p(x, \xi; \lambda)$).

Definition 1.4. $p_{\pm}(x, \xi; \lambda)\in S_{\pm}$ if

(1) $p_{\pm}(x, \xi; \lambda)\in W(0)$,

(2) there exists a constant $\varepsilon>0$ such that

$$p_{\pm}(x, \xi; \lambda)=0 \quad \text{if} \quad ||\xi|/\sqrt{\lambda}-1|>\varepsilon,\ \lambda>a_0,$$

(ε may depend on $p_{\pm}(x, \xi; \lambda)$),

(3) there exists a constant $\mu_{\pm}$ such that $-1<\mu_{\pm}<1$ and

$$p_+(x, \xi; \lambda)=0 \quad \text{if} \quad \hat{x}\cdot\hat{\xi}<\mu_+ ,$$

$$p_-(x, \xi; \lambda)=0 \quad \text{if} \quad \hat{x}\cdot\hat{\xi}>\mu_- ,$$

($\mu_\pm$ may depend on $p_\pm(x, \xi; \lambda)$).

For a Ps. D. Op. $P(\lambda)$, $P(\lambda)\in S_\infty$ (or $S_\pm$) means that its symbol belongs to S_∞ (or $S_\pm$). Then we have shown in [2], Theorems 3.3 and 3.5 the following

Lemma 1.5.
(1) *Let* $P(\lambda)\in S_\infty$. *Then for any* $s>1/2$ *and* $\lambda>a_0$

$$\|P(\lambda)R(\lambda\pm i0)f\|_s\leq C/\lambda\|f\|_s .$$

(2) *Let* $P_\pm(\lambda)\in S_\pm$. *Then for any* $s>1/2$ *and* $\lambda>a_0$

$$\|P_\mp(\lambda)R(\lambda\pm i0)f\|_{s-1}\leq C/\sqrt{\lambda}\|f\|_s .$$

A remark should be added here concerning the limits $\lim_{\varepsilon\downarrow 0} P(\lambda)R(\lambda\pm i\varepsilon)$ and $\lim_{\varepsilon\downarrow 0} P_\mp(\lambda)R(\lambda\pm i\varepsilon)$. What we have shown in [2] actually is that for any $s>1/2$

$$\sup_{0<\varepsilon<1}\|P_\mp(\lambda)R(\lambda\pm i\varepsilon)f\|_{s-1}\leq C/\sqrt{\lambda}\|f\|_s \tag{1.1}$$

(see [2], Theorem 3.7), which does not necessarily imply the existence of the strong limit $s\text{-}\lim_{\varepsilon\downarrow 0} P_\mp(\lambda)R(\lambda\pm i\varepsilon)$ in $\boldsymbol{B}(L^{2,s}; L^{2,s-1})$. As can be checked easily, however, (1.1) implies the existence of the strong limit $s\text{-}\lim_{\varepsilon\downarrow 0} P_\mp(\lambda)R(\lambda\pm i\varepsilon)$ in $\boldsymbol{B}(L^{2,s}; L^{2,s-1-\delta})$ for any $\delta>0$. In the same way, one can show the existence of the strong limit $s\text{-}\lim_{\varepsilon\downarrow 0} P(\lambda)R(\lambda\pm i\varepsilon)$ in $\boldsymbol{B}(L^{2,s}; L^{2,s-\delta})$ for any $\delta>0$.

Definition 1.6. (1) Let $P(\lambda)$, $Q(\lambda)$ be Ps. D. Op.'s $\in W(0)$ with symbols $p(x, \xi; \lambda)$, $q(x, \xi; \lambda)$, respectively. $\{P(\lambda), Q(\lambda)\}$ is said to be a disjoint pair of type I if

$$\inf_{x\in p^n, \lambda<a_0} \operatorname{dis}(\operatorname{supp}_\xi p(x, \xi; \lambda), \operatorname{supp}_\xi q(x, \xi; \lambda))>0,$$

where $\operatorname{dis}(A, B)$ denotes the distance of two sets $A, B\subset \boldsymbol{R}^n$, and $\operatorname{supp}_\xi p(x, \xi; \lambda)$ means the support of $p(x, \xi; \lambda)$ as a function of ξ.
(2) Let $P_\pm(\lambda)$ be Ps. D. Op.'s $\in S_\pm$ with symbols $p_\pm(x, \xi; \lambda)$. $\{P_+(\lambda), P_-(\lambda)\}$ is said to be a disjoint pair of type II if there exist constants $\mu_\pm$ such that $-1<\mu_-<\mu_+<1$ and

$$p_+(x, \xi; \lambda)=0 \quad \text{if} \quad \hat{x}\cdot\hat{\xi}<\mu_+ ,$$

$$p_-(x, \xi; \lambda)=0 \quad \text{if} \quad \hat{x}\cdot\hat{\xi}>\mu_- .$$

Lemma 1.7. (1) *Let* $P(\lambda), Q(\lambda)\in S_\infty$. *Suppose* $\{P(\lambda), Q(\lambda)\}$ *is a disjoint pair of type I. Then for any* $s>0$ *and* $\lambda>a_0$, *there exists a strong limit* $s\text{-}\lim_{\varepsilon\downarrow 0} P(\lambda)R(\lambda\pm i\varepsilon)Q(\lambda)$ *in* $\boldsymbol{B}(L^{2,-s}; L^{2,s})$ *and*

$$\|P(\lambda)R(\lambda\pm i0)Q(\lambda)f\|_s\leq C/\sqrt{\lambda}\|f\|_{-s} .$$

(2) *Let $P_{\pm}(\lambda)\in S_{\pm}$ and $Q(\lambda)\in S_{\infty}$. Suppose that $\{P_{\pm}(\lambda), Q(\lambda)\}$ are disjoint pairs of type I. Then for any $s>0$ and $\lambda>a_0$, there exist strong limits*

$$\operatorname*{s-lim}_{\varepsilon\downarrow 0} Q(\lambda)R(\lambda\pm i\varepsilon)P_{\pm}(\lambda), \quad \operatorname*{s-lim}_{\varepsilon\downarrow 0} P_{\mp}(\lambda)R(\lambda\pm i\varepsilon)Q(\lambda)$$

in $B(L^{2,-s}; L^{2,s})$. Moreover,

$$\|Q(\lambda)R(\lambda\pm i0)P_{\pm}(\lambda)f\|_s \le C/\sqrt{\lambda}\|f\|_{-s},$$

$$\|P_{\mp}(\lambda)R(\lambda\pm i0)Q(\lambda)f\|_s \le C/\sqrt{\lambda}\|f\|_{-s}.$$

(3) *Let $P_{\pm}(\lambda)\in S_{\pm}$. Suppose $\{P_{+}(\lambda), P_{-}(\lambda)\}$ is a disjoint pair of type II. Then for any $s>0$ and $\lambda>a_0$, there exists a strong limit $\operatorname*{s-lim}_{\varepsilon\downarrow 0} P_{\pm}(\lambda)R(\lambda\pm i\varepsilon)P_{\pm}(\lambda)$ in $B(L^{2,-s}; L^{2,s})$ and*

$$\|P_{\mp}(\lambda)R(\lambda_{\pm} i0)P_{\pm}(\lambda)f\|_s \le C/\sqrt{\lambda}\|f\|_{-s}.$$

For the proof, see [2] Theorems 4.2, 4.3, 4.4 and [3] Theorem 2.

We now study the limit $\operatorname*{s-lim}_{\varepsilon\downarrow 0} R(\lambda\pm i\varepsilon)^N$.

Theorem 1.8. *Let $\gamma>1/2$ be arbitrarily fixed and N an integer ≥ 1. Let $\lambda>a_0$. Then we have:*

(1) *There exists a strong limit* $\operatorname*{s-lim}_{\varepsilon\downarrow 0} R(\lambda\pm i\varepsilon)^N \equiv R(\lambda\pm i0)^N$ in $B(L^{2,\gamma+N-1}; L^{2,-\gamma-N+1})$ *and*

$$\|R(\lambda\pm i0)^N f\|_{-\gamma-N+1} \le C\lambda^{-N/2}\|f\|_{\gamma+N-1}.$$

(2) *Let $P_{\pm}(\lambda)\in S_{\pm}$. For any $s\ge N+\gamma$ and $\delta>0$, there exists a strong limit $\operatorname*{s-lim}_{\varepsilon\downarrow 0} P_{\mp}(\lambda)R(\lambda\pm i\varepsilon)^N$ in $B(L^{2,s}; L^{2,s-N-\delta})$ and*

$$\|P_{\mp}(\lambda)R(\lambda\pm i0)^N f\|_{s-N} \le C\lambda^{-N/2}\|f\|_s.$$

(3) *Let $P_{\pm}(\lambda)\in S_{\pm}$. For any $s\ge N+\gamma$ and $\delta>0$, there exists a strong limit $\operatorname*{s-lim}_{\varepsilon\downarrow 0} R(\lambda\pm i\varepsilon)^N P_{\pm}(\lambda)$ in $B(L^{2,-s+N}; L^{2,-s-\delta})$ and*

$$\|R(\lambda\pm i0)^N P_{\pm}(\lambda)f\|_{-s} \le C\lambda^{-N/2}\|f\|_{-s+N}.$$

(4) *Let $P_{\pm}(\lambda)\in S_{\pm}$. Suppose $\{P_{+}(\lambda), P_{-}(\lambda)\}$ is a disjoint pair of type II. Then for any $s>0$, there exists a strong limit $\operatorname*{s-lim}_{\varepsilon\downarrow 0} P_{\mp}(\lambda)R(\lambda\pm i\varepsilon)^N P_{\pm}(\lambda)$ in $B(L^{2,-s}; L^{2,s})$ and*

$$\|P_{\pm}(\lambda)R(\lambda\pm i0)^N P_{\pm}(\lambda)f\|_s \le C\lambda^{-N/2}\|f\|_{-s}.$$

(5) *Let $Q(\lambda)\in S_{\infty}$ and $s\ge\gamma+N-1$. For any $\delta>0$ there exists a strong limit $\operatorname*{s-lim}_{\varepsilon\downarrow 0} Q(\lambda)R(\lambda\pm i\varepsilon)^N$ in $B(L^{2,s}: L^{2,s-N+1-\delta})$ and*

$$\|Q(\lambda)R(\lambda\pm i0)^N f\|_{s-N+1} \le C\lambda^{-N/2}\|f\|_s.$$

Proof (by induction on N). The assertions of the theorem have already been proved for $N=1$ (see Lemmas 1.5, 1.7). Assume the theorem for N.

Choose $\phi_0(\xi)\in C^\infty(\boldsymbol{R}^n)$ such that

$$\phi_0(\xi)=\begin{cases}1 & ||\xi|-1|<\varepsilon\\ 0 & ||\xi|-1|>2\varepsilon,\end{cases}$$

where $0<\varepsilon<1/2$. We set

$$\phi_\infty(\xi)=1-\phi_0(\xi).$$

Let $\chi_0(x)$, $\chi_\infty(x)\in C^\infty(\boldsymbol{R}^n)$ be such that $\chi_0(x)+\chi_\infty(x)=1$,

$$\chi_0(x)=0 \quad \text{for} \quad |x|>2,$$

$$\chi_0(x)=1 \quad \text{for} \quad |x|<1.$$

Choose constants $-1<\tilde{\mu}_\pm<1$ and C^∞-functions $\rho_\pm(t)$ so that $\tilde{\mu}_-<\tilde{\mu}_+$, $\rho_+(t)+\rho_-(t)=1$ and

$$\rho_+(t)=0 \quad \text{for} \quad t<\tilde{\mu}_+,$$

$$\rho_-(t)=0 \quad \text{for} \quad t>\tilde{\mu}_-.$$

Let $A(\lambda)$, $B(\lambda)$, $\tilde{P}_\pm(\lambda)$ be the Ps. D. Op.'s with symbols $\phi_\infty(\xi/\sqrt{\lambda})$, $\chi_0(x)\phi_0(\xi/\sqrt{\lambda})$, $\chi_\infty(x)\rho_\pm(\hat{x}\cdot\hat{\xi})\phi_0(\xi/\sqrt{\lambda})$, respectively. By definition $A(\lambda)\in S_\infty$, $\tilde{P}_\pm(\lambda)\in S_\pm$, the symbol of $B(\lambda)$ is compactly supported for x and

$$A(\lambda)+B(\lambda)+\tilde{P}_+(\lambda)+\tilde{P}_-(\lambda)=1.$$

We further introduce the following notations. Let

$$I=\{z\in \boldsymbol{C};\ \mathrm{Re}\, z>a_0,\ \mathrm{Im}\, z>0\}.$$

For an operator $T(z)$ defined for $z\in I$, $T(z)\in C(\bar{I};\ L^{2,s},\ L^{2,r};\ k)$ means that there exists a strong limit $\underset{\varepsilon\downarrow 0}{s\text{-lim}}\ T(\lambda+i\varepsilon)$ in $\boldsymbol{B}(L^{2,s};\ L^{2,r})$ for $\lambda>a_0$ and

$$\|T(\lambda+i0)f\|_r\le C\lambda^{-k/2}\|f\|_s.$$

Pfroof of (1) *for* $N+1$. We split $R(\lambda+i\varepsilon)^{N+1}$ into four parts:

$$(1.2)\qquad R(\lambda+i\varepsilon)^{N+1}=R(\lambda+i\varepsilon)^N A(\lambda)R(\lambda+i\varepsilon)+R(\lambda+i\varepsilon)^N B(\lambda)R(\lambda+i\varepsilon)$$
$$+R(\lambda+i\varepsilon)^N\tilde{P}_+(\lambda)R(\lambda+i\varepsilon)+R(\lambda+i\varepsilon)^N\tilde{P}_-(\lambda)R(\lambda+i\varepsilon).$$

Since $A(\lambda)\in S_\infty$, Lemma 1.5 (1) shows that $A(\lambda)R(\lambda+i\varepsilon)\in C(\bar{I};\ L^{2,\gamma+N-1},\ L^{2,\gamma+N-1};\ 1)$. By our induction hypothesis (1), $R(\lambda+i\varepsilon)^N\in C(\bar{I};\ L^{2,\gamma+N-1},\ L^{2,-\gamma-N+1};\ N)$. Thus the first term belongs to $C(\bar{I};\ L^{2,\gamma+N-1},\ L^{2,-\gamma-N+1};\ N+1)$.

Since the symbol of $B(\lambda)$ is compactly supported for x, $R(\lambda+i\varepsilon)^N B(\lambda)\in C(\bar{I};\ L^{2,-\gamma},\ L^{2,-\gamma-N+1};\ N)$ by our induction hypothesis (1). This, combined with Lemma 1.1, shows that the second term belongs to $C(\bar{I};\ L^{2,\gamma},\ L^{2,-\gamma-N+1};\ N+1)$.

In view of Lemma 1.1 and our induction hypothesis (3), we have $R(\lambda+i\varepsilon)\in C(\bar{I};\ L^{2,\gamma},\ L^{2,-\gamma};\ 1)$ and $R(\lambda+i\varepsilon)^N\tilde{P}_+(\lambda)\in C(\bar{I};\ L^{2,-\gamma},\ L^{2,-\gamma-N};\ N)$, which shows that the third term belongs to $C(\bar{I};\ L^{2,\gamma},\ L^{2,-\gamma-N};\ N+1)$.

Making use of our induction hypotheses (1) and (2) for N, we have for small $\delta>0$, $\tilde{P}_-(\lambda)R(\lambda+i\varepsilon)\in C(\bar{I}; L^{2,\gamma+N}, L^{2,\gamma+N-1-\delta}; 1)$ and $R(\lambda+i\varepsilon)^N\in C(\bar{I}; L^{2,\gamma-\delta+N-1}, L^{2,-\gamma+\delta-N+1}; N)$. Thus the fourth term belongs to $C(\bar{I}; L^{2,\gamma+N}, L^{2,-\gamma-N+1}; N+1)$.

Proof of (2) *for* $N+1$. We multiply (1.2) by $P_-(\lambda)$. By methods similar to the above, one can show that $P_-(\lambda)R(\lambda+i\varepsilon)^N A(\lambda)R(\lambda+i\varepsilon)$, $P_-(\lambda)R(\lambda+i\varepsilon)^N B(\lambda)R(\lambda+i\varepsilon)$ and $P_-(\lambda)R(\lambda+i\varepsilon)^N\tilde{P}_-(\lambda)R(\lambda+i\varepsilon)$ belong to $C(\bar{I}; L^{2,s}, L^{2,s-N-1-\delta}; N+1)$ for $s\geq N+1+\gamma$ and $\delta>0$. In order to treat the term $P_-(\lambda)R(\lambda+i\varepsilon)^N\tilde{P}_+(\lambda)R(\lambda+i\varepsilon)$, we note that $\{P_-(\lambda), \tilde{P}_+(\lambda)\}$ becomes a disjoint pair of type II if $\tilde{\mu}_+>\mu_-$. Thus by our induction hypothesis (4) for N, we have $P_-(\lambda)R(\lambda+i\varepsilon)^N\tilde{P}_+(\lambda)\in C(\bar{I}; L^{2,-\gamma}, L^{2,s}; N)$ for any $s>0$, which, combined with Lemma 1.1, shows that $P_-R(\lambda+i\varepsilon)\tilde{P}_+(\lambda)R(\lambda+i\varepsilon)\in C(\bar{I}; L^{2,\gamma}, L^{2,s}; N+1)$ for any $s>0$.

Proof of (3). By the asymptotic expansion of the symbol of $P_\pm(\lambda)^*$, we have for any $m>0$,

$$P_\pm(\lambda)^*=P_\mp^{(m)}(\lambda)+Q_m(\lambda),$$

where $P_\mp^{(m)}(\lambda)\in S_\pm$ and the symbol $q_m(x, \xi; \lambda)$ of $Q_m(\lambda)$ verifies

$$|D_x^\alpha D_\xi^\beta q_m(x, \xi; \lambda)|\leq C_{\alpha\beta}\langle x\rangle^{-m-|\alpha|}\langle\xi\rangle^{-|\beta|}$$

(see [2], Theorem 2.4). Thus if $s\geq N+\gamma$ and $m\geq\gamma+s-1$

$$\begin{aligned}&\|P_\mp(\lambda)^*R(\lambda\pm i\varepsilon)^N f\|_{s-N}\\ &\leq\|P_\mp^{(m)}\|(\lambda)R(\lambda\pm i\varepsilon)^N f\|_{s-N}+\|Q_m(\lambda)R(\lambda\pm i\varepsilon)^N f\|_{s-N}\\ &\leq C\lambda^{-N/2}\|f\|_s,\end{aligned}$$

where we have used (1) and (2). Taking the adjoint, we have $\|R(\lambda\pm i\varepsilon)^N P_\pm(\lambda)f\|_{-s}\leq C\lambda^{-N/2}\|f\|_{-s+N}$, which proves that (3) for N follows from (1) and (2) for N.

Proof of (4) *for* $N+1$. First we choose $\tilde{\mu}_\pm$ in such a way that $-1<\mu_-<\tilde{\mu}_-<\tilde{\mu}_+<\mu_+<1$ so that $\{P_-(\lambda), \tilde{P}_+(\lambda)\}$ and $\{\tilde{P}_-(\lambda), P_+(\lambda)\}$ form disjoint pairs of type II. Next we recall that the support with respect to ξ of the symbol of $P_+(\lambda)$ lies in a small neighborhood of the sphere $\{\xi; |\xi|=\sqrt{\lambda}\}$. Thus for a suitable choice of ε for $A(\lambda)$, $\{A(\lambda), P_+(\lambda)\}$ becomes a disjoint pair of type I.

We multiply (1.2) by $P_\pm(\lambda)$ from both sides. Consider the resulting first term. By Lemma 1.7 (2), $A(\lambda)R(\lambda+i\varepsilon)P_+(\lambda)\in C(\bar{I}; L^{2,-s}, L^{2,s}; 1)$ for any $s>0$. We also have by our induction hypothesis (2), $P_-(\lambda)R(\lambda+i\varepsilon)^N\in C(\bar{I}; L^{2,s}, L^{2,s-N-1}; N)$. Thus the first term belongs to $C(\bar{I}; L^{2,-s}, L^{2,s}; N+1)$ for $s>0$.

The treatment of the second term is easy, hence is omitted.

Taking the adjoint in Lemma 1.5 (2), one can show using Lemma 1.1 that $R(\lambda+i\varepsilon)P_+(\lambda)\in C(\bar{I}; L^{2,-s}, L^{2,-s-2}; 1)$ for $s>0$. Since $\{P_-(\lambda), \tilde{P}_+(\lambda)\}$ is a disjoint pair of type II, we have $P_-(\lambda)R(\lambda+i\varepsilon)^N\tilde{P}_+(\lambda)\in C(\bar{I}; L^{2,-s-2}, L^{2,s}; N)$ for $s>0$. Thus the third term has the desired property.

Since $\{\tilde{P}_-(\lambda), P_+(\lambda)\}$ is a disjoint pair of type II, $\tilde{P}_-(\lambda)R(\lambda+i\varepsilon)P_+(\lambda)\in C(\bar{I}; L^{2,-s}, L^{2,s}; 1)$ for $s>0$. This, combined with (2) proves that the fourth term has the desired property.

Proof of (5) *for* $N+1$. We shall estimate

$$Q(\lambda)R(\lambda+i\varepsilon)^{N+1}=Q(\lambda)R(\lambda+i\varepsilon)A(\lambda)R(\lambda+i\varepsilon)^{N}+Q(\lambda)R(\lambda+i\varepsilon)B(\lambda)R(\lambda+i\varepsilon)^{N}$$
$$+Q(\lambda)R(\lambda+i\varepsilon)\tilde{P}_{+}(\lambda)R(\lambda+i\varepsilon)^{N}+Q(\lambda)R(\lambda+i\varepsilon)\tilde{P}_{-}(\lambda)R(\lambda+i\varepsilon)^{N}.$$

The treatment of the first two terms is easy. We have only to use (1), (5) for N and Lemma 1.5 (1).

Since $Q(\lambda)\in S_{\infty}$, one can assume that $\{Q(\lambda), \tilde{P}_{+}(\lambda)\}$ is a disjoint pair of type I by an appropriate choice of ε. Therefore by Lemma 1.7 (2), $Q(\lambda)R(\lambda+i\varepsilon)\tilde{P}_{+}(\lambda)\in C(\bar{I}; L^{2,-s}, L^{2,s}; 1)$ for $s>0$. This, combined with (1) for N, shows that the third term belongs to $C(\bar{I}; L^{2,\gamma+N}, L^{2,s}; N+1)$ for $s>0$.

In order to treat the fourth term, we have only to take note of (2) for N and Lemma 1.5 (1). □

In view of Theorem 1.8 and the formula $\left(\frac{d}{d\lambda}\right)^{N}R(\lambda\pm i\varepsilon)=N!R(\lambda\pm i\varepsilon)^{N+1}$, one can conclude the strong differentiability of the resolvent $R(\lambda\pm i0)$.

Theorem 1.9. *Let* $\gamma>1/2$ *and* N *be an integer* ≥ 0.
(1) *As an operator* $\in \boldsymbol{B}(L^{2,\gamma+N}, L^{2,-\gamma-N})$, $R(\lambda\pm i0)$ *is* N*-times strongly differentiable and for* $\lambda>a_0>0$,

$$\left\|\left(\frac{d}{d\lambda}\right)^{N}R(\lambda\pm i0)f\right\|_{-\gamma-N}\leq C\lambda^{-(N+1)/2}\|f\|_{\gamma+N}.$$

(2) *Let* $P_{\pm}(\lambda)\in S_{\pm}$. *For any* $s\geq N+1+\gamma$ *and* $\lambda>a_0>0$

$$\left\|P_{\mp}(\lambda)\left(\frac{d}{d\lambda}\right)^{N}R(\lambda\pm i0)f\right\|_{s-N-1}\leq C\lambda^{-(N+1)/2}\|f\|_{s},$$

$$\left\|\left[\left(\frac{d}{d\lambda}\right)^{N}R(\lambda\pm i0)\right]P_{\pm}(\lambda)f\right\|_{-s}\leq C\lambda^{-(N+1)/2}\|f\|_{-s+N+1}.$$

(3) *Let* $P_{\pm}(\lambda)\in S_{\pm}$. *Suppose that* $\{P_{+}(\lambda), P_{-}(\lambda)\}$ *is a disjoint pair of type II. Then for* $s>0$ *and* $\lambda>a_0>0$

$$\left\|P_{\mp}(\lambda)\left[\left(\frac{d}{d\lambda}\right)^{N}R(\lambda\pm i0)\right]P_{\pm}(\lambda)f\right\|_{s}\leq C\lambda^{-(N+1)/2}\|f\|_{-s}.$$

(4) *Let* $Q(\lambda)\in S_{\infty}$. *For any* $s>N+\gamma$ *and* $\lambda>a_0>0$

$$\left\|Q(\lambda)\left(\frac{d}{d\lambda}\right)^{N}R(\lambda\pm i0)f\right\|_{s-N}\leq C\lambda^{-(N+1)}\|f\|_{s}.$$

For later use, it is convienient to rewrite the above theorem in the following form.

Theorem 1.10. *In addition to the assumptions of Theorem 1.9, suppose that the symbols* $p_{\pm}(x, \xi; \lambda)$, $q(x, \xi; \lambda)$ *of* $P_{\pm}(\lambda)$ *and* $Q(\lambda)$ *have the following properties*

$$|D_x^\alpha D_\xi^\beta D_k^m\, p_\pm(x, \xi; k^2)| \le C_{\alpha\beta m} \langle x\rangle^{-|\alpha|} \langle \xi\rangle^{-|\beta|},$$

$$|D_x^\alpha D_\xi^\beta D_k^m\, q(x, \xi; k^2)| \le C_{\alpha\beta m} \langle x\rangle^{-|\alpha|} \langle \xi\rangle^{-|\beta|},$$

where the constant $C_{\alpha\beta m}$ *is independent of* $k > k_0 = \sqrt{a_0} > 0$. *Then we have for* $k > k_0 > 0$

(1) $$\left\| \left(\frac{d}{dk}\right)^N R(k^2 \pm i0) f \right\|_{-\gamma-N} \le C k^{-1} \|f\|_{\gamma+N},$$

(2) *for* $s \ge N+1+\gamma$

$$\left\| \left(\frac{d}{dk}\right)^N [P_\mp(k^2) R(k^2 \pm i0)] f \right\|_{s-N-1} \le C k^{-1} \|f\|_s,$$

$$\left\| \left(\frac{d}{dk}\right)^N [R(k^2 \pm i0) P_\pm(k^2)] f \right\|_{-s} \le C k^{-1} \|f\|_{-s+N+1},$$

(3) *for* $s > 0$

$$\left\| \left(\frac{d}{dk}\right)^N [P_\mp(k^2) R(k^2 \pm i0) P_\pm(k^2)] f \right\|_s \le C k^{-1} \|f\|_{-s},$$

(4) *for* $s \ge N+\gamma$

$$\left\| \left(\frac{d}{dk}\right)^N [Q(k^2) R(k^2 \pm i0)] f \right\|_{s-N} \le C k^{-1} \|f\|_s.$$

§ 2. Differentiability of generalized Fourier transforms

In [1], we constructed a solution to the eikonal equation

(2.1) $$|\nabla_x \phi(x, \xi)|^2 + V(x) = |\xi|^2$$

and used it to develop an eigenfunction expansion theory for the Schrödinger operator H. In [4], we gave a slightly different method of construction. First we recall the results of [1] and [4] (see [1], Theorem 1.16 and [4], Theorem 2.5).

Lemma 2.1. *Let* $\varepsilon > 0$ *be arbitrarily fixed. Choose* $d > 0$ *arbitrarily. Then there exists a real function* $\phi(x, \xi) \in C^\infty(\boldsymbol{R}^n \times (\boldsymbol{R}^n - \{0\}))$ *having the following properties*:
(1) *For any* $\delta > 0$

$$|D_x^\alpha D_\xi^\beta (\phi(x, \xi) - x\cdot\xi)| \le C_{\alpha\beta} \langle x\rangle^{1-\varepsilon_0-|\alpha|} \langle \xi\rangle^{-1}$$

for $x \in \boldsymbol{R}^n$ *and* $|\xi| > \delta$.

(2) $$\sup_{x\in \boldsymbol{R}^n, |\xi|>d} \left| \left(\frac{\partial^2}{\partial x_i \partial \xi_j}\phi\right)(x, \xi) - I \right| < 1/2,$$

where I *is the* $n \times n$ *identity matrix.*
(3) *For any* $\delta > 0$, *there exists a constant* $R > 0$ *such that for* $|x| > R$, $|\xi| > \delta$ *and* $\hat{x}\cdot\hat{\xi} > -1 + \varepsilon/2$, $\phi(x, \xi)$ *solves the eikonal equation* (2.1).

Lemma 2.2 ([3], *Theorem* 2.3). *Choose* ε, $d>0$ *arbitrarily. Let* $\phi(x, \xi)$ *be as in Lemma* 2.1. *Then there exists a smooth function* $a(x, \xi)\in\mathscr{B}(\boldsymbol{R}^n)$ *having the following properties*:

(1) $$|D_x^\alpha D_\xi^\beta(a(x, \xi)-1)|\leq C_{\alpha\beta}\langle x\rangle^{-\varepsilon_0-|\alpha|}\langle\xi\rangle^{-1},$$

if $|\xi|>d$, $\hat{x}\cdot\hat{\xi}>-1+\varepsilon$, $|x|>2R$, R *being the constant specified in Lemma* 2.1 *for* $\delta=d/2$. $a(x, \xi)=0$ *if* $|\xi|<d/2$ *or* $\hat{x}\cdot\hat{\xi}<-1+\varepsilon/2$ *or* $|x|<R$.
(2) *Let* $G(x, \xi)=e^{-i\phi(x,\xi)}(-\Delta+V-|\xi|^2)e^{i\phi(x,\xi)}a(x, \xi)$. *Then for* $\hat{x}\cdot\hat{\xi}>-1+\varepsilon$, *we have for any* $N>0$,

$$|D_x^\alpha D_\xi^\beta G(x, \xi)|\leq C_{\alpha\beta N}\langle x\rangle^{-N}\langle\xi\rangle.$$

If $\hat{x}\cdot\hat{\xi}<-1+\varepsilon$,

$$|D_x^\alpha D_\xi^\beta G(x, \xi)|\leq C_{\alpha\beta}\langle x\rangle^{-1-|\alpha|}\langle\xi\rangle.$$

Our generalized Fourier transformation in [1] is constructed by the following method.

Lemma 2.3 ([1], *Theorem* 5.5). *Let* $\phi(x, \xi)$ *be as in Lemma* 2.1. *Choose* $0<\mu<1$ *arbitrarily and let* $\rho(t)\in C^\infty(\boldsymbol{R}^1)$ *be such that* $\rho(t)=1$ *for* $t>1-\mu/2$, $\rho(t)=0$ *for* $t<1-\mu$. *Let* $\psi(t)\in C^\infty(\boldsymbol{R}^1)$ *be such that* $\psi(t)=1$ *for* $t<1$, $\psi(t)=0$ *for* $t>2$. *We set* $\psi_R(x)=\psi(|x|/R)$. *Then for* $f\in L^{2,\gamma}$ *and* $k>0$, *there exists the following strong limit*

$$\underset{R\to\infty}{s\text{-lim}}\, 2ik(2\pi)^{-n/2}\int e^{-i\phi(x,k\omega)}\Big(\frac{\partial}{\partial r}\psi_R(x)\Big)\rho(\hat{x}\cdot\omega)R(k^2+i0)f(x)dx=\mathscr{F}(k)f$$

in $L^2(S^{n-1})$. *This* $\mathscr{F}(k)$ *is independent of* μ *and for any* $\delta>0$

(2.2) $$\|\mathscr{F}(k)f\|_{L^2(S^{n-1})}\leq Ck^{-(n-1)/2}\|f\|_\gamma, \qquad (k>\delta).$$

Let us take notice that (2.2) follows from the formulae (8.1), (9.4) in [1] and Lemma 1.1 in the present paper. The fact that $\mathscr{F}(k)$ is independent of μ follows from the proof of [1], §5.

For $f\in L^{2,\gamma}$, we define $(\mathscr{F}f)(\xi)$ by

(2.3) $$(\mathscr{F}f)(\xi)=(\mathscr{F}(|\xi|)f)(\xi/|\xi|).$$

Then $\mathscr{F}$ is uniquely extended to a partial isometry with initial set $L^2_{ac}(H)$ and final set $L^2(\boldsymbol{R}^n)$, and plays the role of a generalization to the Fourier transformation ([1], Theore 7.1). Moreover, the above Lemma 2.3 shows that $\mathscr{F}$ depends only on the behavior of the phase function $\phi(x, \xi)$ in a neighborhood of $\hat{x}=\hat{\xi}$. As has been noted in the introduction, $\mathscr{F}f(\xi)$ can be written formally as in (0.1). We now rewrite (0.1) by using $a(x, \xi)$.

Definition 2.4. Let $\phi(x, \xi)$ and $a(x, \xi)$ be as in Lemmas 2.1 and 2.2. Let $\psi_R(x)$ be as in Lemma 2.3. We define for $f\in L^{2,\gamma}$ and $k>0$

$$\mathscr{F}(k,R)f(\omega)=(2\pi)^{-n/2}\int\psi_R(x)e^{-i\phi(x,k\omega)}\,\overline{a(x,k\omega)}\,f(x)dx$$

$$-(2\pi)^{-n/2}\int\psi_R(x)e^{-i\phi(x,k\omega)}\,\overline{G(x,k\omega)}\,R(k^2+i0)f(x)\,dx.$$

Lemma 2.5. *For $f\in L^{2,\gamma}$ and $k>d$,*

$$\operatorname*{s-lim}_{R\to\infty}\mathscr{F}(k,R)f=\mathscr{F}(k)f \quad in \quad L^2(S^{n-1}).$$

Proof. We proceed as in [1], § 5. Let $u=R(k^2+i0)f$. Since

$$(2\pi)^{n/2}\mathscr{F}(k,R)f=\int\psi_R\{\Delta(e^{-i\phi}\bar{a})\}u\,dx-\int\psi_R e^{-i\phi}\bar{a}\Delta u\,dx,$$

we have by integration by parts

$$(2\pi)^{n/2}\mathscr{F}(k,R)f=\int e^{-i\phi}(\Delta\psi_R)\,\bar{a}u\,dx+2\int e^{-i\phi}\left(\frac{\partial}{\partial r}\psi_R\right)\bar{a}\left(\frac{\partial u}{\partial r}-iku\right)dx$$

$$+2ik\int e^{-i\phi}\left(\frac{\partial}{\partial r}\psi_R\right)\bar{a}u\,dx$$

$$=I_1(R)+I_2(R)+I_3(R).$$

One can argue as in the proof of [1], Lemma 5.2 to see that $I_1(R)\to 0$, $I_2(R)\to 0$ as $R\to\infty$. Let $\rho(t)$ be as in Lemma 2.3. Then as in the proof of [1], Lemma 5.3, we have

$$\int e^{-i\phi(x,k\omega)}\left(\frac{\partial}{\partial r}\psi_R(x)\right)\overline{a(x,k\omega)}(1-\rho(\hat{x}\cdot\omega))u(x)dx\longrightarrow 0$$

as $R\to\infty$. Thus we have only to consider

$$2ik\int e^{-i\phi}\left(\frac{\partial}{\partial r}\psi_R\right)\bar{a}\rho(\hat{x}\cdot\omega)u(x)dx.$$

Since $|(a(x,k\omega)-1)\rho(\hat{x}\cdot\omega)|\le C\langle x\rangle^{-\varepsilon_0}$ by Lemma 2.2, we have as in the proof of Lemma 5.2 in [1],

$$\int e^{-i\phi}\left(\frac{\partial}{\partial r}\psi_R\right)(\bar{a}-1)\rho(\hat{x}\cdot\omega)u(x)dx\longrightarrow 0.$$

Therefore by Lemma 2.3, we have

$$\operatorname*{s-lim}_{R\to\infty}(2\pi)^{n/2}\mathscr{F}(k,R)f=\operatorname*{s-lim}_{R\to\infty}2ik\int e^{-i\phi}\left(\frac{\partial}{\partial r}\psi_R\right)\rho(\hat{x}\cdot\omega)u(x)dx$$

$$=(2\pi)^{n/2}\mathscr{F}(k)f. \qquad \square$$

It follows formally from Lemma 2.5 that

$$(2\pi)^{n/2}\mathscr{F}(k)f=\int e^{-i\phi(x,k\omega)}\,\overline{a(x,k\omega)}\,f(x)dx \tag{2.4}$$

$$-\int e^{-i\phi(x,k\omega)}\,\overline{G(x,k\omega)}R(k^2+i0)f(x)dx.$$

The rest of this section is devoted to showing that the right-hand side is a well-defined bounded operator on $L^{2,\gamma}$ and that it is differentiable with repect to $\xi=k\omega$ if f decays rapidly.

Lemma 2.6. *Let $b(x, \xi)\in\mathscr{B}(\boldsymbol{R}^{2n})$ be such that for some $\varepsilon>0$ $b(x, \xi)=0$ if $|\xi|<\varepsilon$. Then the integral transformation*

$$Tf(\xi)=\int e^{-i\phi(x,\xi)}b(x, \xi)f(x)dx$$

has the following properties:

(1) *If $f\in L^{2,N}$, $Tf(\xi)$ is N times differentiable and*

$$\sum_{|\alpha|\leq N}\|D_\xi^\alpha Tf(\xi)\|_{L^2}\leq C_N\|f\|_N\,.$$

(2) *Let $\gamma>1/2$ and $\varepsilon>0$ be arbitrarily fixed. Then for any $k>\varepsilon$,*

$$\|(Tf)\,(k\cdot)\|_{L^2(S^{n-1})}\leq Ck^{-(n-1)/2}\|f\|_\gamma\,.$$

Proof. (1) follows from [1], Theorem 3.2. Arguing in the same way as in [1], Theorem 3.4, we have

$$\int_{|\theta|=k}|Tf(\theta)|^2dS_\theta\leq C\|f\|_\gamma^2,$$

where the constant C is independent of $k>\varepsilon$. The assertion (2) directly follows from this inequality. □

Lemma 2.7. *Let $S(k)$ be defined by*

$$S(k)f(\omega)=\int^{-i\phi(x,k\omega)}b(x, k\omega)f(x)dx \qquad (k>0),$$

where $b(x, \xi)\in C^\infty(\boldsymbol{R}^{2n})$ and

$$|D_x^\alpha D_\xi^\beta b(x, \xi)|\leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\,.$$

Let $P(k)$ be the Ps. D. Op. with symbol $p(x, \xi; k)$ such that

$$|D_x^\alpha D_\xi^\beta p(x, \xi; k)|\leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}$$

for a constant $C_{\alpha\beta}$ independent of $k>k_0$, k_0 being as in Theorem 1.10. *Suppose that $b(x, \xi)$ and $p(x, \xi; k)$ satisfy either of the following assumptions* (1), (2):

(1) *There exists a constant $\varepsilon>0$ such that*

$$p(x, \xi; k)=0 \qquad \text{if} \quad ||\xi|/k-1|<\varepsilon,\ k>k_0\,.$$

(2) *There exist constants $\mu_\pm$ such that $-1<\mu_-<\mu_+<1$ and*

$$b(x, \xi)=0 \qquad \text{if} \quad \hat{x}\cdot\hat{\xi}>\mu_-\,,$$

$$p(x, \xi; k)=0 \qquad \text{if} \quad \hat{x}\cdot\hat{\xi}<\mu_+,\ k>k_0\,.$$

Then for any $s\geq 0$, $k>k_0$ and $N>0$

$$\|S(k)\langle x\rangle^N P(k)f\|_{L^2(S^{n-1})} \le C_s k^{-s}\|f\|_{-s}.$$

Proof. Choose $\chi_1(x)$, $\chi_2(x) \in C^\infty(\boldsymbol{R}^n)$ *such that* $\chi_1(x)+\chi_2(x)=1$, $\chi_1(x)=1$ for $|x|<1$, $\chi_1(x)=0$ for $|x|>2$. We split $S(k)\langle x\rangle^N P(k)$ into two parts: $S(k)\langle x\rangle^N P(k) = A_1(k)+A_2(k)$, where

$$A_j(k)f(\omega)=\iint e^{-i(\phi(x,k\omega)-x\xi)}b(x,k\omega)\langle x\rangle^N p(x,\xi;k)\chi_j\left(\frac{|x|}{R}\right)\hat{f}(\xi)d\xi dx,$$

$R>0$ being a constant yet to be determined. By Lemma 2.1 (1),

$$|\nabla_x(\phi(x,k\omega)-x\cdot\xi))-(k\omega-\xi)| \le Ck^{-1}\langle x\rangle^{-\varepsilon_0}.$$

In view of the assumptions (1) or (2), one can find a constant $C>0$ such that on the support of the integrand

$$|k\omega-\xi| \ge Ck \qquad \text{for} \quad k>k_0.$$

Therefore, there is a constant $R>0$ such that

$$|\nabla_x(\phi(x,k\omega)-x\cdot\xi)| \ge Ck \qquad \text{for} \quad k>k_0 \quad \text{and} \quad |x|>R.$$

Letting $\psi(\omega,x,\xi;k)=\phi(x,k\omega)-x\cdot\xi$ and using the relation $e^{-i\psi}=|\nabla_x\psi|^{-2}i\nabla_x\psi\cdot\nabla_x e^{-i\psi}$, we have by integrating by parts in x $3N$ times

$$A_2(k)f(\omega)=\iint e^{-i\psi(\omega,x,\xi;k)}b_N(\omega,x,\xi;k)\hat{f}(\xi)d\xi dx,$$

where

$$|D_x^\alpha D_\xi^\beta b_N(\omega,x,\xi;k)| \le C_{\alpha\beta}\langle x\rangle^{-2N}k^{-2N}. \tag{2.5}$$

Let $B_2(k,\omega)$ be the Ps. D. Op. with symbol $\langle x\rangle^N b_N(\omega,x,\xi;k)$. Then we have

$$A_2(k)f(\omega)=\int e^{-i\phi(x,k\omega)}\langle x\rangle^{-N}(B_2(k,\omega)f)(x)dx.$$

Thus for large N

$$\begin{aligned}\|A_2(k)f\|_{L^2(S^{n-1})} &\le C\sup_{\omega\in S^{n-1}}|A_2(k)f(\omega)| \\ &\le C\sup_{\omega\in S^{n-1}}\|B_2(k,\omega)f\|_{L^2(S^{n-1})}.\end{aligned}$$

(2.5) implies that $\|B_2(k,\omega)f\|_{L^2} \le Ck^{-2N}\|f\|_{-N}$, which shows that $\|A_2(k)f\|_{L^2(S^{n-1})} \le Ck^{-2N}\|f\|_{-N}$.

Next we consider $A_1(k)$. Since in this case the symbol $\langle x\rangle^N p(x,\xi;k)\chi_1(x/R)$ is compactly supported for x, one can easily show for any $s\ge 0$

$$\|A_1(k)f\|_{L^2(S^{n-1})} \le C\|f\|_{-s},$$

with a constant C independent of $k>k_0$. In order to derive the decay with respect to k, we have only to note that for large k, $|\nabla_x(\phi(x,k\omega)-x\cdot\xi)| \ge Ck$ for a constant $C>0$ and integrate by parts. □

We turn to the estimate of the right-hand side of (2.4). The first term is treated by Lemma 2.6 (1). In order to treat the second term, we set

$$T(k)f(\omega)=\int e^{-i\phi(x,k\omega)}\,\overline{G(x,\,k\omega)}R(k^2+i0)f(x)dx$$

and

$$(Tf)(\xi)=T(|\xi|)f(\xi/|\xi|).$$

Lemma 2.8. *Let $d>0$ be the constant specified in Lemma 2.2. Choose $\gamma>1/2$ arbitrarily. Then we have for any $N\geq 0$ and $k>d$*

$$\sum_{|\alpha|\leq N}\|D_\xi^\alpha Tf(\xi)|_{\xi=k\omega}\|_{L^2(S^{n-1})}\leq Ck^{-(n-1)/2}\|f\|_{N+\gamma}.$$

Proof. We make use of the localizations used in the proof of Theorem 1.8. Let $\phi_0(\xi)$, $\phi_\infty(\xi)$, $\chi_0(x)$ and $\chi_\infty(x)$ be as in the proof of Theorem 1.8. Choose $\rho_\pm(t)\in C^\infty(\boldsymbol{R}^1)$ such that $\rho_+(t)+\rho_-(t)=1$ and $\rho_+(t)=0$ if $t<1/2$, $\rho_-(t)=0$ if $t>3/4$. Let $A(k)$, $B(k)$, $P_\pm(k)$ be the Ps. D. Op.'s with symbols $\phi_\infty(\xi/k)$, $\chi_0(x)\phi_0(\xi/k)$, $\chi_\infty(x)\rho_\pm(\hat{x}\cdot\hat{\xi})\phi_0(\xi/k)$, respectively. Then $T(k)=\sum_{j=1}^4 T_j(k)$, where

$$T_1(k)f(\omega)=\int e^{-i\phi(x,k\omega)}\overline{G(x,\,k\omega)}A(k)R(k^2+i0)f(x)fx,$$

$$T_2(k)f(\omega)=\int e^{-i\phi(x,k\omega)}\overline{G(x,\,k\omega)}B(k)R(k^2+i0)f(x)dx,$$

$$T_3(k)f(\omega)=\int e^{-i\phi(x,k\omega)}\overline{G(x,\,k\omega)}P_+(k)R(k^2+i0)f(x)dx,$$

$$T_4(k)f(\omega)=\int e^{-i\phi(x,k\omega)}\overline{G(x,\,k\omega)}P_-(k)R(k^2+i0)f(x)dx.$$

We set

$$T_jf(\xi)=T_j(|\xi|)f(\xi/|\xi|).$$

First we consider T_1. By a straightforward calculation we have

$$\sum_{|\alpha|\leq N}D_\xi^\alpha T_1f(\xi)$$
$$=\sum_{\substack{|\alpha|\leq N\\ \beta\leq\alpha}}\int e^{-i\phi(x,\xi)}\langle\xi\rangle a_\beta(x,\,\xi)\langle x\rangle^{|\beta|}A_\beta(|\xi|)D_\xi^{\alpha-\beta}R(|\xi|^2+i0)\,f(x)\,dx,$$

where $\langle\xi\rangle a_\beta(x,\,\xi)$ and $A_\beta(|\xi|)$ arise from the derivatives of $\overline{G(x,\,\xi)}$ and $A(|\xi|)$, respectively. In particular, $a_\beta(x,\,\xi)\in\mathscr{B}(\boldsymbol{R}^{2n})$ by Lemma 2.2. In view of Lemma 2.7, we have for any $s>0$

$$\sum_{|\alpha|\leq N}\|D_\xi^\alpha T_1f(\xi)|_{\xi=k\omega}\|_{L^2(S^{n-1})}$$
$$\leq Ck^{-s}\sum_{m\leq N}\left\|\left(\frac{d}{dk}\right)^m R(k^2+i0)f\right\|_{-s}$$
$$\leq Ck^{-s}\|f\|_{N+\gamma},$$

where we have used Theorem 1.10 (1).

Since the symbol of $B(k)$ is compactly supported for x, we have by using Lemma 2.6 (2),

$$\sum_{|\alpha|\leq N} \|D_\xi^\alpha T_2 f(\xi)|_{\xi=k\omega}\|_{L^2(S^{n-1})}$$

$$\leq Ck^{-(n-1)/2} \sum_{m\leq N} \left\| \left(\frac{d}{dk}\right)^m R(k^2+i0)f \right\|_{-N-\gamma}$$

$$\leq Ck^{-(n-1)/2} \|f\|_{N+\gamma}.$$

Since $G(x, \xi)=O(\langle\xi\rangle\langle x\rangle^{-1})$, we have, similarly,

$$\sum_{|\alpha|\leq N} \|D_\xi^\alpha T_4 f(\xi)|_{\xi=k\omega}\|_{L^2(S^{n-1})}$$

$$\leq Ck^{-(n-1)/2} \sum_{m\leq N} \left\| \langle x\rangle^{N-m-1+\gamma} \left(\frac{d}{dk}\right)^m P_-(k)R(k^2+i0)f \right\|_{L^2}$$

$$\leq Ck^{-(n-1)/2} \|f\|_{N+\gamma}.$$

The treatment of T_3 is slightly different. Choose $\chi_\pm(t)\in C^\infty(\boldsymbol{R}^1)$ such that $\chi_+(t)+\chi_-(t)=1$ and $\chi_+(t)=0$ if $t<-1/4$, $\chi_-(t)=0$ if $t>1/4$. We split T_3 into two parts: $T=T_3^{(+)}+T_3^{(-)}$, where

$$T_3^{(\pm)}(k)f(\omega)=\int e^{-i\phi(x,k\omega)}\overline{G(x, k\omega)}\chi_\pm(\hat{x}\cdot\omega)P_+(k)R(k^2+i0)f(x)dx.$$

Since $\overline{G(x, k\omega)}\chi_+(\hat{x}\cdot\omega)$ is rapidly decreasing in x, we have as for T_2

$$\sum_{|\alpha|\leq N} \|D_\xi^\alpha T_3^{(+)} f(\xi)|_{\xi=k\omega}\|_{L^2(S^{n-1})}$$

$$\leq Ck^{-(n-1)/2} \|f\|_{N+\gamma}.$$

Using Lemma 2.7, one can treat $T_3^{(-)}$ in the same way as T_1:

$$\sum_{|\alpha|\leq N} \|D_\xi^\alpha T_3^{(-)} f(\xi)|_{\xi=k\omega}\|_{L^2(S^{n-1})}$$

$$\leq Ck^{-s} \|f\|_{N+\gamma}. \qquad \square$$

Theorem 0.1 in the introduction now ready follows from Lemma 2.8 by integrating in k.

§3. Decay rates for scattering states

As an application of the differentiability property of $\mathscr{F}$, we derive in this section a decay rate for scattering states of the Schrödinger operator H.

Theorem 3.1. *Let $\gamma>1/2$. Let $\chi(\lambda)\in C^\infty(\boldsymbol{R}^1)$ be such that for some $\varepsilon>0$ and an integer $N\geq 1$, $\chi(\lambda)=0$ for $\lambda<\varepsilon$, and $\left|\left(\frac{d}{d\lambda}\right)^m \chi(\lambda)\right| \leq C_m\lambda^{(N-2\gamma-m)/2}$ for $\lambda>\varepsilon$, $m=0, 1, 2,\dots$. Then we have for $t>0$*

$$\|\chi(H)e^{-itH}f\|_{-N}\leq Ct^{-N}\|f\|_{N+1+\gamma}.$$

Proof. Let $\psi(\xi)=(\mathscr{F}\chi(H)f)(\xi)$. By [1], Theorem 7.1,

$$e^{-itH}\chi(H)f=\int_0^\infty \mathscr{F}(k)^*e^{-itk^2}\psi(k\cdot)k^{n-1}dk.$$

From (2.4) it follows that

$$\mathscr{F}(k)^*\psi(k\cdot)=(2\pi)^{-n/2}\int_{S^{n-1}} e^{i\phi(x,k\omega)}a(x,k\omega)\psi(k\omega)d\omega-R(k^2-i0)v(k),$$

where

$$v(k)(x)=(2\pi)^{-n/2}\int_{S^{n-1}} e^{i\phi(x,k\omega)}G(x,k\omega)\psi(k\omega)d\omega. \tag{3.1}$$

Therefore,

$$e^{-itH}\chi(H)f=(2\pi)^{-n/2}\int e^{i(\phi(x,\xi)-t|\xi|^2)}a(x,\xi)\psi(\xi)d\xi$$
$$-\int_0^\infty R(k^2-i0)v(k)e^{-itk^2}k^{n-1}dk. \tag{3.2}$$

First we consider the first term of (3.2). Using the relation $(-2it|\xi|^2)^{-1}\xi\cdot\nabla_\xi e^{-it|\xi|^2}=e^{-it|\xi|^2}$, we have by integration by parts

$$\langle x\rangle^{-N}\int e^{i(\phi(x,\xi)-t|\xi|^2)}a(x,\xi)\psi(\xi)d\xi$$
$$=\sum_{|\alpha|\leq N}\int e^{i(\phi(x,\xi)-t|\xi|^2)}a_\alpha(x,\xi;t)D_\xi^\alpha\psi(\xi)d\xi,$$

where $|D_x^\beta D_\xi^\gamma a_\alpha(x,\xi;t)|\leq C_{\beta\gamma}(t|\xi|)^{-N}$. Thus by an L^2-boundedenss theorem of Fourier integral operators ([1], Theorem 3.2), we have

$$\left\|\langle x\rangle^{-N}\int e^{i(\phi(x,\xi)-t|\xi|^2)}a(x,\xi)\psi(\xi)d\xi\right\|_{L^2}$$
$$\leq Ct^{-N}\sum_{|\alpha|\leq N}\|\langle\xi\rangle^{-N}D_\xi^\alpha\psi\|_{L^2}. \tag{3.3}$$

The sceond term of (3.2) is treated by the technique employed in [5], Lemma 5.1. For $\varepsilon>0$

$$R(k^2-i\varepsilon)=-ie^{-it(H-(k^2-i\varepsilon))}\int_t^\infty e^{is(H-(k^2-i\varepsilon))}ds.$$

Therefore letting $g(k)=v(k)k^{n-1}$, we have

$$\int_0^\infty R(k^2-i\varepsilon)e^{-itk^2}g(k)dk=-i\int_t^\infty e^{-(s-t)\varepsilon}e^{-i(t-s)}\hat{g}(s)ds,$$

$$\hat{g}(s)=\int_0^\infty e^{-isk^2}g(k)dk.$$

In the following, we show for finy $s>0$

(3.4) $$\|\hat{g}(s)\|_{L^2}\le Cs^{-N-1}\sum_{|\alpha|\le N+1}\|\langle\xi\rangle^{-N}D_\xi^\alpha\psi\|_{L^2}.$$

If (3.4) is established, we have by the dominated convergence theorem

$$\left\|\int_0^\infty R(k^2-i0)e^{-itk^2}g(k)\,dk\right\|_{L^2}$$
$$\le Ct^{-N}\sum_{|\alpha|\le N+1}\|\langle\xi\rangle^{-N}D_\xi^\alpha\psi\|_{L^2}.$$

Therefore for $t>0$

$$\|\chi(H)e^{-itH}f\|_{-N}\le Ct^{-N}\sum_{|\alpha|\le N+1}\|\langle\xi\rangle^{-N}D_\xi^\alpha\psi\|_{L^2}.$$

Since $\psi(\xi)=\chi(|\xi|^2)(\mathscr{F}f)(\xi)$ and $|D_\xi^\alpha\chi(|\xi|^2)|\le C\langle\xi\rangle^{N-2\gamma}$,

$$\sum_{|\alpha|\le N+1}\|\langle\xi\rangle^{-N}D_\xi^\alpha\psi\|_{L^2}\le C\sum_{|\alpha|\le N+1}\|\langle\xi\rangle^{-\delta/2}D_\xi^\alpha(\mathscr{F}f)(\xi)\|_{L^2}$$
$$\le C\|f\|_{N+1+\gamma},$$

by Theorem 0.1. This proves the theorem.

Now we prove (3.4). By (3.1),

$$\hat{g}(t)=(2\pi)^{-n/2}\int e^{i(\phi(x,\xi)-t|\xi|^2)}G(x,\xi)\psi(\xi)d\xi.$$

Choose $\chi_\pm(t)\in C^\infty(\boldsymbol{R}^1)$ such that $\chi_+(t)+\chi_-(t)=1$, $\chi_+(t)=0$ for $t<1/2$, $\chi_-(t)=0$ for $t>3/4$. Split $\hat{g}(t)$ into two parts: $\hat{g}(t)=g_+(t)+g_-(t)$, where

$$g_\pm(t)=(2\pi)^{-n/2}\int e^{i(\phi(x,\xi)-t|\xi|^2)}G(x,\xi)\chi_\pm(\hat{x}\cdot\hat{\xi})\psi(\xi)d\xi.$$

Since $G(x,\xi)\chi_+(\hat{x}\cdot\hat{\xi})$ is rapidly decreasing in x, we have by integration by parts as we have derived (3.3)

$$\|g_+(t)\|_{L^2}\le Ct^{-N-1}\sum_{|\alpha|\le N+1}\|\langle\xi\rangle^{-N}D_\xi^\alpha\psi\|_{L^2}.$$

(Take notice of the estimates for $G(x,\xi)$ in Lemma 2.2).

Choose $\rho_1(t)$, $\rho_2(t)\in C^\infty(\boldsymbol{R}^1)$ such that $\rho_1(t)+\rho_2(t)=1$, $\rho_1(t)=0$ for $t>2$, $\rho_2(t)=0$ for $t<1$. Split $g_-(t)$ into two parts: $g_-(t)=g_-^{(1)}(t)+g_-^{(2)}(t)$, where

$$g_-^{(j)}(t)=(2\pi)^{-n/2}\int e^{i(\phi(x,\xi)-t|\xi|^2}\rho_j(|x|/R)G(x,\xi)\chi_-(\hat{x}\cdot\hat{\xi})\psi(\xi)d\xi,$$

$R>0$ being a constant yet to be determined. Since $\rho_1(|x|/R)$ is compactly supported, we have as above

$$\|g_-^{(1)}(t)\|_{L^2}\le Ct^{-N-1}\sum_{|\alpha|\le N+1}\|\langle\xi\rangle^{-N}D_\xi^\alpha\psi\|_{L^2}.$$

On the support of the integrand of $g_-^{(2)}$, we have for large $R>0$

$$|\nabla_\xi(\phi(x, \xi)-t|\xi|^2)|\geq C(|x|+t|\xi|) \qquad (|x|>R).$$

Thus by integration by parts

$$g_-^{(2)}(t)=\int e^{i(\phi(x,\xi)-t|\xi|^2)} \sum_{|\alpha|\leq N+1} b_\alpha(x, \xi; t)D_\xi^\alpha\psi(\xi)d\xi,$$

where $|D_x^\beta D_\xi^\gamma b_\alpha(x, \xi; t)|\leq Ct^{-N-1}|\xi|^{-N}$. Therefore again using the L^2-boundedness theorem of Fourier integral operators

$$\|g_-^{(2)}(t)\|_{L^2}\leq Ct^{-N-1} \sum_{|\alpha|\leq N+1} \|\langle\xi\rangle^{-N}D_\xi^\alpha\psi\|_{L^2}. \qquad \square$$

In order to prove Theorem 0.2, we have only to interpolate the estimate in Theorem 3.1 with the obvious one

$$\|\chi(H)e^{-itH}f\|_{L^2}\leq f\|_{L^2}.$$

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] H. Isozaki, On the generalized Fourier transforms associated with Schrödinger operators with long-range perturbations, J. reine angew. Math., **337** (1982), 18–67.

[2] H. Isozaki and H. Kitada, Micro-local resolvent estimates for two-body Schrödinger operators, (preprint).

[3] H. Isozaki and H. Kitada, A remark on the micro-local resolvent estimates for two-body Schrödinger operators, (preprint).

[4] Isozaki and H. Kitada, Modified wave operators with time-independent modifiers, (preprint).

[5] T. Ikebe and H. Isozaki, A stationary approach to the existence and completeness of long-range wave operators, Integr. Equat. and Oper. Theory, **5** (1982), 18–49.

[6] T. Ikebe, Spectral representations for Schrödinger operators with long-range potentials, J. Functional Anal., **20** (1975), 158–176.

[7] Y. Saitō, Spectral representation for the Schrödinger operator with long-range potentials, Springer Lecture Notes in Math. 727, Berlin-Heidelberg-New York 1979.

[8] A. Jensen, E. Mourre and P. Perry, Multiple commutator estimates and resolvent smoothness in quantum scattering theory, (preprint).

[9] E. Mourre, Absence of singular continuous spectrum for certain self-adjoint operators, Comm. Math. Phys., **78** (1981), 391–408.