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Singular hydrodynamical continuations of finite Riemann surfaces[†)]

Dedicated to Professor Yukio Kusunoki on his sixtieth birthday

By

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Introduction

The present study arose, in close relationships to a series of our investigations [16], [17] and [18], from an attempt to embed an arbitrary open Riemann surface of finite genus into another closed Riemann surface of the same genus, so that *the prolongation of the surface accompanies that of some global meromorphic uniformizer of the given surface*[1)].

It is a well known fact that an open Riemann surface of finite genus is prolongable anyway up to some closed Riemann surface of the same genus (see Bochner [2]). It belongs, however, to comparatively recent questions, what variety of like prolongations occur, or whether we are always able to find among them the one endowed with somewhat distinguished features such as extremalities. As for the first problem the readers are referred to, e.g. Mori [10], Heins [5], Grunsky [3], [4], Oikawa [11], and for the second to Ioffe [6], Timmann [21], Shiba [16], [17], Shiba-Shibata [18], and others. They are also intimately connected with the problems of conformal mappings and of realizations of Riemann surfaces, to which significant contributions have been made above all by the mathematicians in Hannover: Tietz [19], [20], Köditz-Timmann [7] and Schmieder [14], [15].

In our preceding papers [16] and [18] we have shown the following: Given an open Riemann surface R of finite genus together with a special kind of single-valued meromorphic function f (called an S-function — a generalization of the classical Strömungsfunkion) on it, there is a closed Riemann surface $R^* \supset R$ of the same genus as R such that f is continued holomorphically up to $R^* \setminus R$ beyond ∂R. More precisely, for the given R and f there exists a closed Riemann surface R^*, a conformal

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1) We are concerned here with no more than the extensions with the property 'non-increase of genus'.

injection ι^* of R into R^* and a single-valued meromorphic function f^* on R^* satisfying the conditions:

(1) $f^* \circ \iota^* = f$ on R;

(2) $\operatorname{Im} f^* = \text{const.}$ on each component of $R^* \setminus \iota^*(R)$;

(3) $R^* \setminus \iota^*(R)$ is a set of measure zero, on which f^* is holomorphic.

We called the continuation (R^*, ι^*) of R a *hydrodynamische Abschließung* (*hydrodynamical closing-up*) of R with respect to the S-function f after a phenomenal interpretation it reveals.

In this paper we will construct another continuation of R, so that it preserves the properties (1), (2) and that the S-function is extended *meromorphically* off R.

§1. Preliminaries

1.1. Every open Riemann surface R of finite genus is realized by means of an S-function f onto a covering surface over $\hat{C}$ with horizontal slits, which originates from the ideal boundary ∂R and are schlicht except at most finitely many ones (see [16]). Therefore, via a natural sewing process of those schlicht slits, R comes to be embedded conformally in another bordered Riemann surface of the same genus.

So far as our continuation theory is concerned, we may thus concentrate our attention exclusively upon a compact bordered Riemann surface without any restriction of generality.

In the following we shall commit ourselves to employ the notations below without particular mention:

R always denotes the interior of a compact bordered Riemann surface $\bar{R}$ of finite genus g and β_j $(j=1, 2,\dots, h)$ are the contours of $\bar{R}$ oriented so that $\partial\bar{R} = \beta_1+\beta_2+\cdots+\beta_h$. Then there exists another Riemann surface R' with the same genus g comprising $\bar{R}$ as its closed subdomain, on which every β_j $(j=1, 2,\dots, h)$ is realized as an analytic Jordan curve. Next let f be a non-constant (single- or multiple-valued) S-function on R[2]; namely, f is a non-constant meromorphic function on $\bar{R}$ with constant imaginary part on β_j $(j=1, 2,\dots, h)$, whose possible multiple-valuedness comes only from the non-planarity of R.

1.2. It is known to us that the S-function f can be characterized by any one of the following four equivalent propositions:

(A) $\operatorname{Im}\{df\}$ is a real *distinguished* harmonic differential in the sense of Ahlfors (Ahlfors-Sario [1], p. 313);

(K) idf is a *canonical semi-exact* differential of Kusunoki (Kusunoki [8]);

(S_0) $\operatorname{Re} f$ is an L_0-*principal* function of Sario;

or

(S_1) $\operatorname{Im} f$ is a $(Q)L_1$-*principal* function of Sario, where Q stands for a canonical partition of ∂R (Sario-Oikawa [13], pp. 23–24).

One of the results we have obtained in the previous investigations is as follows:

2) Although we discussed in [18] mainly single-valued S-functions, the multiple-valuedness like this is no doubt allowed to f.

Theorem 1. *For the above introduced pair* (R, f) *there exists a triple* (R^*, ι^*, f^*) *satisfying the conditions*:

1°) R^* *is a closed Riemann surface of genus* g;

2°) $\iota^*: R \to R^*$ *is a conformal injection*; $\operatorname{meas}(R^* \setminus \iota^*(R)) = 0$;

3°) f^* *is a function meromorphic on* R^*, *which is single-valued or multiple-valued according as* f *is*;

4°) f^* *is single-valued holomorphic in a neighbourhood of* $R^* \setminus \iota^*(R)$;

5°) $\operatorname{Im} f^* = \text{const.}$ *on each component of* $R^* \setminus \iota^*(R)$;

6°) $f^* \circ \iota^* = f$ *on* R.

In general, the closing-up (R^*, ι^*, f^*) was not uniquely determined by the prescribed pair (R, f), but depends on a finite number of real parameters except the case $g=0$ with a dipole f and the case $g=1$ with an f holomorphic everywhere on R, in which the closing-up (R^*, f^*) is unique. Several significant extremum properties that the slit torus enjoys were detailed in [17].

§ 2. Heuristic observation

2.1. The crux of our ideas in [18] can be sketched as follows. In terms of physics, the stream f on R never overflows ∂R, which amounts to saying mathematically that ∂R is an impenetrable boundary for f (see Rodin-Sario [12]; p. 260). Fix an index j at will. We showed in [18] that, for every value $u \in \boldsymbol{R}$ taken by $\operatorname{Re} f$ on β_j, it is possible to identify an appropriate n-tuple of points of β_j —$n \in \boldsymbol{Z}^+$ depending on u— such that $\operatorname{Re} f = u$. We extend f holomorphically up to β_j with the aid of Painlevé's theorem. Letting j run from 1 to h, we obtained f^* as an extension of f and simultaneously R^* as the existence domain for f^*. Thus each component of $R^* \setminus \iota^*(R)$ takes part in the totality of streamilines of the complex velocity potential (S-function) f^* on R^*.

The above mentioned fact supplies a reason why we could obtain the closing-up R^* without attaching any 'big point set' to R, while one might try in vain to continue f holomorphically up to some closed surface R' such that $R' \setminus R$ contains an interior point (on account of Maximum Principle).

2.2. On the other hand, the state of a steady flow in the presence of some obstructive solid body are unaltered and invariant outside the solid after removal of the solid and replacement by an appropriate streaming with source and sink (and vice versa), as is familiar in Milne-Thomson's Circle Theorem, Rankine's Solid (Ovoid) Theorem etc. in hydrodynamics (see Milne-Thomson [9], pp. 154–155, pp. 461–462). Here the border of the body agrees to part of streamlines for the obstructed flow. Translated into our terms, it suggests the possibility of cutting out the closed surface R^* again along ∂R —which is realized on R^* as part of the streamlines of f^*— and of inserting a 'big point set' into the lacuna. This implies that the original complex potential f on R is prolonged across the ideal boundary ∂R analytically but now with some isolated singular points. In the subsequent argument we concern ourselves with such continuations of the pair (R, f) with singularities, which we wish to name *singular hydrodynamical* continuation of R in regard to f. See Theorem 2 below.

§ 3. Singular hydrodynamical continuations

3.1. Now we are prepared to state and prove our Main Theorem:

Theorem 2. *There exists a proper continuation $\check{R}$ of R and a meromorphic function $\check{f}$ on $\check{R}$ which coincides with f on R. To be precise, to a prescribed pair (R, f) it is possible to associate a triplet $(\check{R}, \check{\iota}, \check{f})$ possessing the properties:*

1°) *$\check{R}$ is a closed Riemann surface of genus g;*

2°) *$\check{\iota}: R \to \check{R}$ is a conformal injection; $\check{R} \setminus \check{\iota}(R)$ has interior points;*

3°) *$\check{f}$ is a meromorphic function on $\check{R}$, which has at least one pole in each component of $\check{R} \setminus \check{\iota}(R)$;*

4°) *$\check{f} \circ \check{\iota} = f$ on R.*

We propose here to name the above $\check{R}$ a singular hydrodynamical continuation of R. In proving this assertion it suffices to observe a suitable planar neighbourhood of ∂R, where f is necessarily single-valued. Furthermore, a recursive argument permits us to confine ourselves to the case $h=1$. So we write $\beta=\beta_1=\partial R$ for shortness' sake. Under the circumstances thus specified, the matter will be rather simple.

Recall that the non-negative integer

$$N: = -\frac{1}{2\pi}\int_{\beta} d \arg df - 1$$

could favourably describe the covering property of the map $f: R \to \hat{C}$ near β (see [16]). Only for the time being let us agree to use the wording as follows: a subset of R is called a neighbourhood[3)] of β, if it is a doubly connected subdomain bounded by the 1-cycle β itself and by an analytic Jordan curve β' such that $-\beta'$ is homologous to β. We have shown in [16]

f is univalent in a neighbourhood U_0 of β if and only if $N=0$.

More generally:

f is at most $(N+1)$-valent in a neighbourhood U_N of β.

3.2. We will now expose how to construct $\check{R}$ in comparison with the manner in which R^* has already been achieved. The notation $w=f(p)$ $(p \in R)$ for the S-function on R shall be consistently employed. To begin with, we consider the simpler case $N=0$, which will be typical for later line of argument.

Based on the univalency of f in U_0, we work with our construction procedure in reference to the conformal image $V_0=f(U_0)$ on $\boldsymbol{C}$ equivalent to U_0 rather than on the abstract subsurface U_0 itself. By the definition of S-function, $\operatorname{Im} f$ takes on a constant value on β, i.e. the image set $f(\beta)$ is a horizontal segment σ. When a point

3) Often referred to as an 'end of R'.

$p \in \beta$ makes a round along β, its image point $f(p)$ goes and returns the way σ exactly once. Hence the image set $V_0 = f(U_0)$ of the neighbourhood U_0 is a so-called horizontal slit subdomain of $\boldsymbol{C}$. Then we have attempted in [18] to identify a pair of points which are located on the upper and lower edges of the slit and are of an equal w-coordinate. In this way the range of values of f have been extended up to the Jordan domain V'_0 bounded by a single Jordan curve $f(\beta')$. The inverse conformal mapping f^{-1} clearly extend up to V'_0, which implies that we have sewn β via the S-function f to get the closed surface R^*.

Now instead, we keep the above-mentioned slit-sides not identified, but take another copy $\hat{\boldsymbol{C}}'$ of the extended complex w-plane $\hat{\boldsymbol{C}}$ slit along a segment congruent to σ. Then we connect $\hat{\boldsymbol{C}}'$ with V'_0 cross-wise along each pair of the upper and lower edges of their slits. The range of $w = f(p)$ is thus enlarged towards the w-sphere just attached across σ. Back to the closed surface R^* by f^{-1}, we come to have welded a bordered Riemann surface of genus zero to R^* along β, which implies the birth of a new abstract closed Riemann surface $\check{R}$ ($\neq R^*$) of the same genus as R^*. In addition, the S-function f on R is prolonged up to a meromorphic S-function $\check{f}$ on $\check{R}$, whose range has been ready in advance as the covering surface over $\hat{\boldsymbol{C}}$, i.e.

$$\check{f}(p) = \begin{cases} f(p), & p \in R \\ w, & p \in \check{R} \setminus R. \end{cases}$$

More precisely, $\check{f}$ associates some value $w \in \hat{\boldsymbol{C}}'$ with every point of $\check{R} \setminus \bar{R}$ bijectively; in particular, $\check{f}$ has a simple pole at a single point p_∞ corresponding to ∞ on $\hat{\boldsymbol{C}}$.

§4. Proof of Theorem 2

4.1. Our line of argument to demonstrate the Main Theorem in full generality lies, as preceded by the one in the particular case $N = 0$, in identifying adequate part of the boundary streamlines of the S-function by its equal values and inserting some bordered surfaces of vanishing genus into the slits consisting of the remainder part of those boundary streamlines.

Before commencing the main body of proof we want to bring to mind something about the stagnation points of S-functions quoting from [18].

A point q on β was called a *boundary stagnation point* if $df(q) = 0$. All the boundary stagnation point was classified into one of the two, an *even stagnation point* or an *odd stagnation point*, according as the number of the interior streamlines meeting it. Curiously enough, the even stagnation points were found of hardly any use in building the hydrodynamical closing-up R^* of R by f, whereas the odd stagnation points have filled a significant rôle; the existence proof for R^* has been done inductively with respect to the number of the odd stagnation points on β. By the way we have seen through a simple observation on the orientation of streams that β necessarily carries more than one and an even number of odd stagnation points (see Fig. 1).

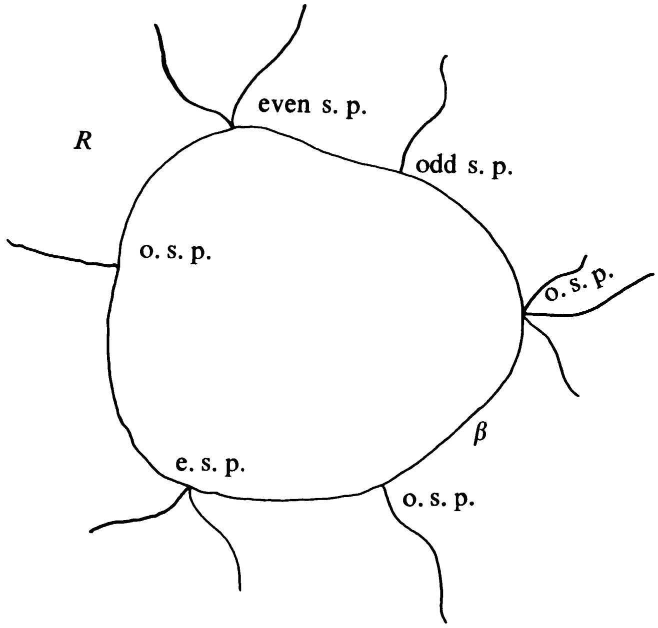

Fig. 1.

4.2. Roughly speaking, R^* had grown out of R via an identification of points on β according to the equal values taken by f, which resulted the elimination of the old boundary curve β of R. In the simplest case ($N=0$ in our terms) it reminds us nearly of the usual conformal sewings in the classical sense. To be precise, every point of $f(\beta)$ should have been identified, by its equal complex w-coordinate, with a unique point with the sole exception of the both extremities, which were odd stagnation points (of the first order) and were identified with no other points. The exceptional points reveal no longer any singularities after the identification.

In case $N>0$ too, R^* was obtained in working on some partially similar sewing processes. But unlike the first case $N=0$, the closing-up here was not unique on account of the multivalency of f. One of the measures to achieve it was as follows (see [18]). Analogously to the above, (1) even number of (more than two) odd stagnation points on β are identified with no other points; (2) all point $p\in\beta$, except finitely many ones, is identified with a point p' $(\neq p)\in\beta$ such that $f(p)=f(p')$; (3) as for a finite number of those exceptional points, more than three of them are simultaneously identified according to an equal value taken by f there to form a single point q^* of R^*. To be precise, q^* comes from an even number of odd stagnation points or an even number of even stagnation points of f. It would of course be easy to prove these facts, but the situations are more directly convinced of through a hydrodynamic intuition. See Figure 2.

At any rate, β is realized on R^* as a finite union B of simple closed analytic arcs, which makes a compact connected subset of R^*. With the aid of Painlevé's theorem f was prolongable analytically up to an S-function[4] f^* on R^*. Hence R can

4) This simply means that f is an ordinary meromorphic function on the closed surface R^*.

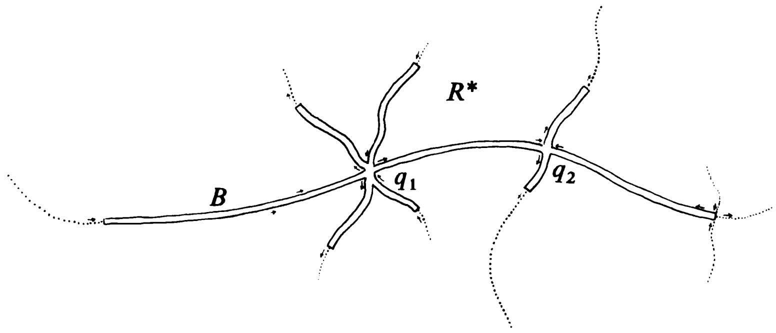

Fig. 2.

be identified with the restriction $R^* \setminus B$ of R^* as the existence domain of f^*.

The differential df^* vanishes at a finite number of points of multiplicity q_1, $q_2, \ldots, q_m$ of B. Here, the extremities of B shall be excluded. Hence f^* is not univalent in some neighbourhood of each q_j, i.e. the value $c_j = f^*(q_j)$ is taken by f^* ν_j times there $(2 \leq \nu_j \in \boldsymbol{Z})$; $2\nu_j$ streamlines of f^* meet the single point q_j $(j=1, 2, \ldots, m)$. Note that the equality

$$\sum_{j=1}^{m} (\nu_j - 1) = N \tag{1}$$

holds.

Remark 1. Now that B takes part in the whole streamlines of the S-function f^* on R^*, it also constitutes a subset of the trajectories for the analytic quadratic differential $-(df^*)^2$.

4.3. It is possible to find a neighbourhood W_j of q_j $(j=1, 2, \ldots, m)$, which contains no interior stagnation points originating from f, and satisfies $W_j \cap W_k = \emptyset$ $(j \neq k)$. The analytic subarcs of B contained in W_j shoots radially out of the core q_j, some of which may reach ∂W_j (see Fig. 3). We need the following two propositions.

Proposition 1. *There exists a closed subset $\tilde{B}$ of B with the following properties*:

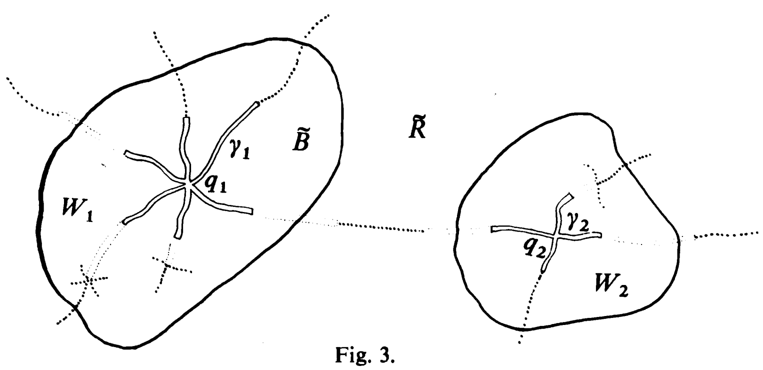

Fig. 3.

1°) *$\tilde{B}$ consists of exactly m connected components γ_j comprised entirely in the interior of W_j;*

2°) *df^* does not vanish at every extremity of γ_j $(j=1, 2,\dots, m)$.*

Proof. Along all the subarcs of B lying outside W_j, R can be sewn up according to the equal values of f^* in a similar manner as before. The sewing is unique and may invade the interiors of all W_j as close to q_j as desired $(j=1, 2,\dots, m)$. Hence $\tilde{B}$ sought after is readily made of B. q. e. d.

Proposition 2. *There exists a triplet $(\tilde{R}, \tilde{\iota}, \tilde{f})$ satisfying the conditions:*

1°) *$\tilde{R}$ is a finite Riemann surface of genus g;*

2°) *$\tilde{\iota}: R \to \tilde{R}$ is a conformal injection;*

3°) *$\tilde{f}$ is an S-function on $\tilde{R}$;*

4°) *the whole boundary streamlines of $\tilde{f}$ coincide with $\bar{\tilde{R}} \setminus \tilde{\iota}(R)$;*

5°) *$\tilde{f} \circ \tilde{\iota} = f$ on R.*

Proof. We have only to set $\tilde{R} = R^* \setminus \tilde{B}$, while $\tilde{f}$ is the restriction of f^* to $\tilde{R}$. q. e. d.

By this Proposition we may consider $\tilde{B}$ as the realization of the boundary of $\tilde{R}$ on R^*.

Remark 2. In accordance with the alteration from (R, f) to $(\tilde{R}, \tilde{f})$, the integers m and N in the identity (1) change their values in general, which shall be denoted by the same letters, however.

For every index $j=1, 2,\dots, m$ an appropriate choice of W_j and a local coordinate z_j in W_j permits us to assume

$$(2) \qquad z_j(q_j)=0, \qquad w=f(p)=z_j^{\nu_j}, \qquad (2 \leq \nu_j \in \boldsymbol{Z}).$$

In terms of z_j, $\gamma_j = \tilde{B} \cap W_j$ is realized as a degenerate concave polygon A_j in $\boldsymbol{C}$, which

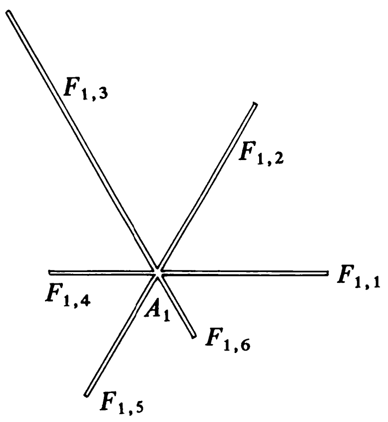

Fig. 4.

consists of $2\nu_j$ rectilinear segments $F_{j,1},\dots, F_{j,2\nu_j}$: every $F_{j,i}$ ($i=1, 2,\dots, 2\nu_j$) starts at the origin and the angle $\widehat{F_{j,i}, F_{j,i+1}}$ composed by the adjacent pairs $F_{j,i}$, $F_{j,i+1}$ ($i=1, 2,\dots, 2\nu_j$, mod $2\nu_j$) are equal to π/ν_j (see Fig. 4).

We fix an index j at will ($1\leqq j\leqq m$). Then the shorter notations γ, ν, z, and A are preferred to γ_j, ν_j, z_j, and A_j, respectively. The exterior of A (i.e. the complement of A with respect to $\hat{C}$) can be mapped onto $|\zeta|>1$ conformally and the mapping function is written down in closed form e.g. by means of Schwarz-Christoffel's transformation. In fact, there are 2ν points $\zeta^{(1)}$, $\zeta^{(2)},\dots$, $\zeta^{(2\nu)}$ located on $|\zeta|=1$ in this order and a constant C such that

$$z=\frac{C}{\zeta}\prod_{n=1}^{2\nu}(\zeta-\zeta^{(n)})^{1/\nu}, \tag{3}$$

Substitution of (3) into

$$w=z^{\nu} \tag{2'}$$

yields the expression

$$w=\frac{C^{\nu}}{\zeta^{\nu}}\prod_{n=1}^{2\nu}(\zeta-\zeta^{(n)})$$

of $\tilde{f}(p)$ in terms of a new local coordinate ζ near γ ($\subset\tilde{B}$), which admits the S-function $w=\tilde{f}(p)$ to be continued meromorphically up to $|\zeta|<1$ in a unique manner. If one attaches a surface portion to $\tilde{R}$ such as to correspond to $|\zeta|<1$, one obtains a meromorphic prolongation of the S-function $\tilde{f}(p)$ onto the new surface. The S-function $\tilde{f}(p)$ on $\tilde{R}$ thus turns out to have been prolonged meromorphically beyond γ, so that it may have a pole of order ν outside $\tilde{R}$.

Working on the indices $j=1, 2,\dots, m$ in the same way, we have finally a continuation $\check{R}$ of $\tilde{R}$ and $\check{f}$ of $\tilde{f}$ looked for. q.e.d.

§5. Concluding remarks

5.1. At the j-th step in the course of proving Theorem 2, $\check{f}$ has arisen from $\tilde{f}$ by getting a new pole of order ν_j ($j=1, 2,\dots, m$). Hence $\check{f}$ has more poles than $\tilde{f}$ by $\sum_{j=1}^{m}\nu_j$, namely $d\check{f}$ is greater than $d\tilde{f}$ by $\sum_{j=1}^{m}(\nu_j+1)$ in the degree of their polar divisors. On the other hand $d\check{f}$ is greater than $d\tilde{f}$ by $2\sum_{j=1}^{m}\nu_j$ in the degree of their zero divisors, since $d\check{f}$ has a simple zero at each endpoint of γ_j. Consequently we have the relation

$$\begin{aligned}&\deg(d\check{f})_{\check{R}}-\deg(d\tilde{f})_{\tilde{R}}\\&=2\sum_{j=1}^{m}\nu_j-\sum_{j=1}^{m}(\nu_j+1)=\sum_{j=1}^{m}(\nu_j-1)=N,\end{aligned}$$

which agrees well with the statement "$d\tilde{f}$ possesses the ramification of total order N on $\tilde{B}$" in [16], p. 372. In other words, the classical Riemann-Hurwitz relation for the covering $\check{f}$: $\check{R}\to\hat{C}$ is reproduced in quite a natural manner from the same formula for $\tilde{f}$ in our generalized sense (see [16]).

5.2. Non-uniqueness. The singular hydrodynamical continuation $\check{R}$ is no more determined uniquely by the prescribed pair (R, f) than R^* was. There would subsist some opportunities, on which the non-uniqueness might have entered. Even at a glance of the simplest case $N=0$ in the proof of Theorem 2 we shall be aware of the following circumstances: In section 3.2 $\hat{C}$ has been joined to V_0' so that the resultant covering surface may possess the branch points of the first order over the two endpoints of σ. They need not, however, be located on the extremities of σ, but might be any two points on σ. In this way one could obtain many closed Riemann surfaces which are conformally equivalent neither to $\check{R}$ nor to each other, if $g>0$. Much more in case $N>0$, we notice that an arbitrary choice of the neighbourhoods W_j $(j=1, 2,..., m)$ has already caused a similar non-uniqueness. This circle of idea may go further, for we presume that R is at first an open Riemann surface of finite genus but possibly *with infinitely many boundary components.* The S-function f on R still lends itself to the construction of the hydrodynamical closing-up R^* of R (see [18]). Then take a finite number of connected components $B_1,..., B_h$ out of $R^* \setminus R$ at will. Application of Theorem 2 to the finite Riemann surface $R^* \setminus \bigcup_{j=1}^{h} B_j$ yields a singular hydrodynamical continuation $\check{R}_h$ of R, such that f extends to a meromorphic function $\check{f}_h$ on $\check{R}_h$ whose order is at least h. Since h was arbitrary, it enables us to obtain a singular hydrodynamical continuation of R of as high total order as we please.

5.3. Our continuation processes of R to $\check{R}$ was based upon a specific meromorphic function on R in the sense that it was only performed through the S-function f on R. Note that a similar continuation problem is dealt with in Grunsky [3]. Meromorphic continuations of Riemann surfaces referring to more general functions are found in Mori [10] and Grunsky [4].

5.4. Just as we remarked earlier there are widely known results such as Milne-Thomson's Circle Theorem, Rankine's Solid Theorem (a kind of generalization of the former) and others in the classical hydrodynamics, which are very akin to ours in the treatment of subjects. Those theorems assert that under the existence of a parallel uniform flow on the whole complex plane $\boldsymbol{C}$ the placement of an obstacle has an effect equivalent to the composition of the original flow with another flow generated by some adequate singularities. What is more, in Circle Theorem the composite flow has a dipole —its sole singularity— at the centre of the circle, which seems to make no essential difference from Reflexion Principle. So an attempt of its direct (but obvious) translation to a Riemann surface R would lead us to nothing but the Schottky double, which is of genus $2g$.

In our Theorem 2, however, the hypothesis on the existence domains of the given regular flows is weakened to a more general Riemann surface R than $\boldsymbol{C}$, and the continuation $\check{R}$ of R shall be always of genus g. Thus our Theorem 2 is regarded as a non-trivial generalization of both Circle Theorem and Ovoid Theorem to arbitrary open Riemann surfaces of finite genera.

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