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The Cauchy problem for effectively hyperbolic equations (general cases)

Dedicated to Professor Sigeru Mizohata
on the occasion of his sixtieth birthday

By

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§0. Introduction

In two previous papers by the author [4] and [5], it was shown for second order operators by the factorization method that effectively hyperbolic operators are strongly hyperbolic. This fact will easily be extended to the general cases of single operators of higher order, because the essential and difficult points of the proofs have been settled through the above papers. This paper aims the proof of this extension by the method which we call the recombination of characteristics. And also it will be shown that it is able to apply to the Cauchy problem for non-linear equations by virtue of the Nash-Moser implicit function theorem. The conclusion of our clarifications and of the results by V. Ya Ivriĭ and V. Petkov [3] is that the effective hyperbolicity is equivalent to the strong hyperbolicity with respect to the non characteristic Cauchy problem for single partial differential equations. We leave the commentary of the related fields to V. Ya Ivriĭ and V. Petkov [3] and to the author [6].

§1. Notations, Assumptions and Results

We consider a single partial differential operator p of order m on an open set Ω of $\boldsymbol{R}^{n+1}$. The non characteristic Cauchy problem is to find a solution of the equation $pu=f$ on Ω satisfying the initial data on a hypersurface of the derivatives up to $m-1$ of u to the conormal direction of the surface. We here use the notation "well posed" defined by following in this connection.

Definition 1.1. 1) The Cauchy problem for p is said to be well posed at a point $x^{\sim}$ with respect to a non characteristic direction $\theta \neq 0$ if there exists a neighborhood Ω of $x^{\sim}$ for any infinitely differentiable function ϕ satisfying $\phi(x^{\sim})=0$ and $d\phi(x^{\sim})=\theta$ such that the following statements $(E)_t$ and $(U)_t$ hold for any small t.

$(E)_t$ For every f belonging to $C_0^{+\infty}(\Omega)$ there is a distribution u belonging to $\mathscr{E}'(\Omega)$ and satisfying the equation $pu=f$ on Ω_t, where Ω_t is the set of x in Ω such that $\phi(x)<t$.
$(U)_t$ If u belonging to $\mathscr{E}'(\Omega)$ satisfies $pu=0$ on Ω_t, then u vanishes identically on Ω_t.

2) The Cauchy problem for P is said to be $\mathscr{E}$-well posed if the above distribution solution u is always infinitely differentiable.

V. Ya Ivriĭ and V. Petkov showed us that the strongly hyperbolic operators should be effectively hyperbolic. The notations, appearing in this statement and the converse one we are going to complete, stand for the following ones defined to the principal parts p_m.

Definition 1.2. 1) Let p_m be hyperbolic with respect to the direction $\theta\neq 0$. p_m is said effectively hyperbolic (with respect to the direction $\theta\neq 0$) if the fundamental matrix at any singular point of $\{p_m=0\}$ has non zero real eigenvalues. (Refer to [6].)

2) p_m is said strongly hyperbolic (with respect to the direction $\theta\neq 0$) if the Cauchy problem for p with any arbitrary lower order term is well posed (with respect to the direction $\theta\neq 0$).

We quote the results for the operators of second order from the author [4] and [5], and assume them. We consider the operator p, which is a classical pseudodifferential operator in the variables x and a differential operator in the variable x_0 of second order. Namely, we suppose that p is written as

$$p=-\partial_0^2+a^\sim(x_0, x, \partial)\partial_0+b^\sim(x_0, x, \partial)$$

with pseudodifferential operators $a^\sim$ and $b^\sim$ of order 1 and 2 in x, respectively. So we denote the principal symbol of p by p_2.

$$p_2=-\xi_0^2+a_1^\sim(x_0, x, \xi)\xi_0+b_2^\sim(x_0, x, \xi).$$

Moreover we assume that p depends smoothly on another parameter t varying on the interval [0, 1]. We obtain Theorem 1.1 from [5] and Theorem 1.2 from [4].

Theorem 1.1. *Any effectively hyperbolic p_2 with infinitely differentiable coefficients is written locally as*

$$p_2=-(\xi_0-\Lambda_1)(\xi_0-\Lambda_0)+b_2,$$

where Λ_0, Λ_1 and b_2 are infinitely differentiable functions in (x_0, x, ξ, t) of homogeneous order 1, 1 *and* 2 *in ξ, respectively, such that*

$$b_2\geq 0,$$

$$\{\xi_0-\Lambda_0, \xi_0-\Lambda_1\}>0 \quad \text{at} \quad \Lambda_0-\Lambda_1=b_2=0$$

and

$$\{\xi_0-\Lambda_0, b_2\}+cb_2=0$$

with an infinitely differentiable function c in (x_0, x, ξ, t).

Theorem 1.2. *If p_2 forms like the conclusions of the above Theorem 1.1 on a neighborhood of $0 \leq x_0 \leq T$, then the Cauchy problem for p with the principal part p_2 with respect to dx_0,*

$$pu = f \quad \text{on} \quad 0 \leq x_0 < T$$

and

$$u = 0 \quad \text{on} \quad x_0 < 0,$$

has the unique infinitely differentiable solutions u for infinitely differentiable and compactly supported data f vanishing on $x_0 < 0$. The solution u satisfies the estimates, by the Sobolev norms on $x_0 < T$, that $\|u\|_s \leq C_s \|f\|_{s+l}$ with some constant l independent of s and the parameter t.

Remarks. 1) Theorems 1.1–2 have been proved at [5] and [4] in the case of p not varied by the parameter t. However their proofs assure the results in this article by treating the parameter t as well as the variables x in the first case and by verifying all processes of the proof are uniform on the parameter t in the second case. This uniformity of the results in the parameter t makes it easy to verify the uniformity with respect to small perturbations of coefficients.

2) The constant l naturally depends on the first order terms. However, if it is described as

$$p_1 = q_1 + \textstyle\sum_{j=1}^{2} f_j(\xi_0 - \Lambda_j) + g \cdot \nabla b_2,$$

with pseudodifferential operators q_1, f_j and g, then the constant l depends only on the bound of q_1 in the space of pseudodifferential operators and is independent of ones of f_j and g.

Under assuming Theorems 1.1–2, we shall prove Theorem 1.3, being the main theorem for single linear equations, and Theorem 1.4, being the extension to non-linear equations. Theorem 1.3 will be proved by a somewhat different method from the usual one. And the Nash–Moser implicit function theorem will apply to showing Theorem 1.4 because all requisites for linear parts will be assured by Theorem 1.3.

Theorem 1.3. *Let p be a single partial differential operator of order m. If p is effectively hyperbolic with respect to dx_0 at a neighborhood of the origin, then p is $\mathscr{E}$-well posed at the origin with respect to dx_0. Moreover there exists a lens neighborhood Ω of the origin contained in a neighborhood given beforehand such that for any infinitely differentiable function u on Ω supported on $x_0 \geq 0$, it holds the estimates*

$$\|u\|_s \leq C_s(\|pu\|_{s+l} + \|a\|_{s+l}\|pu\|_l)$$

for any real positive s and for a fixed l, where a is coefficients of p and where $\|\cdot\|_s$ stand for the Sobolev norms on the lens domain

$$\Omega = \{(x_0, x) : -\varepsilon < x_0 < \varepsilon - |x|^2\}.$$

Ω, the constants C_s and l are uniform as far as the hyperbolic principal parts p_m

and coefficients a of p belong to a small neighborhood of a fixed effectively hyperbolic one and a bounded set in the space of infinitely differentiable functions, respectively.

Let us consider a non-linear equation of order m

$$pu = p(x_0, x, \partial^\alpha u) = 0$$

where $(\partial^\alpha u)_{|\alpha|=l}$ stand for l-th derivatives of the real valued function u and p is a real valued infinitely differentiable function in the variables $(x_0, x, \eta_\alpha)_{|\alpha|\leq m}$ with multi-indices $\alpha = (\alpha_0, \ldots, \alpha_n)$. We define the principal symbol p_m of p by

$$p_m(x_0, x, \partial^\alpha u, \xi_0, \xi) = \sum_{|\beta|=m} (\partial/\partial\eta_\beta) p(x_0, x, \partial^\alpha u)(\xi_0, \xi)^\beta.$$

Definition 1.3. p is said to be effectively hyperbolic at $(x_0^{\sim}, x^{\sim}, u^{\sim})$ with respect to dx_0 if $u^{\sim}$ is a formal solution of $p(x_0^{\sim}, x^{\sim}, \partial^\alpha u^{\sim}) = 0$ at $(x_0, x) = (x_0^{\sim}, x^{\sim})$ and if the principal symbol $p_m(x_0, x, \partial^\alpha u, \xi_0, \xi)$ is an effectively hyperbolic symbol with respect to dx_0 at a neighborhood of $(x_0^{\sim}, x^{\sim}, u^{\sim})$.

Theorem 1.4. *Let a non-linear operator p be effectively hyperbolic at $(x_0, x, u) = (0, 0, 0)$ with respect to dx_0. Then at a small neighborhood of the origin there exists uniquely an infinitely differentiable solution u satisfying that*

$$pu = p(x_0, x, \partial^\alpha u) = 0$$

and

$$(\partial/\partial x_0)^j u|_{x_0=0} = 0 \qquad (j = 0, \ldots, m-1).$$

We show an example. Let us consider the following equation on $\boldsymbol{R}^2$.

$$pu = \partial_x^2 u - \partial_y(a(x, y, u)\partial_y u) - b(x, y, u, \partial_x u, \partial_y u) = 0,$$

$$u|_{x=0} = u_0 \quad \text{and} \quad \partial_x u|_{x=0} = u_1,$$

where $a(x, y, u)$ and $b(x, y, u, v, w)$ are infinitely differentiable functions and a is non negative. If the function

$$\partial_x^2 a(x, y, u) + 2\partial_x\partial_u a(x, y, u)v + \partial_u^2 a(x, y, u)v^2$$

is positive at $(x, y, u, v) = (0, 0, u_0(0), u_1(0))$ when $a(0, 0, u_0(0))$ vanishes, then p is effectively hyperbolic so that the non-linear equation has a unique solution at a neighborhood of $(x, y) = (0, 0)$.

§ 2. Recombination of Characteristics

The characteristics of hyperbolic operators with the multiplicity of at most two is micro-locally equivalent to one of hyperbolic operators of second order. Therefore the results for equations of second order are extended to equations of general higher order. However, the usual method needs the micro-local estimates for the equations of second order, which are apparently moreprecise and more intricate than the global estimates. We have to improve the previous results in spite of the repetition of the same technics if we wish to take the usual method. In order to avoid this repetition we shall introduce another method which we call the recombination of characteristics.

A basic lemma is the following variation of well known and simple ones with respect to commutations of two involutive operators of first order.

Lemma 2.1. *Let a and b be two pseudodifferential operators of order* 1 *in x (with the parameter x_0). And put two operators p and q as $p=\xi_0-a$ and $q=\xi_0-b$. We assume that the principal parts p_1 and q_1 of p and q are real and involutive to each other, that is,*

$$\{p_1, q_1\}=c(p_1-q_1)=c(b_1-a_1)$$

with c a pseudodifferential operator of order 0. *Then there exists another pseudodifferential operator d of order* 0 *in x such that*

$$(p+c)(q+d-c)-(q+d)p \tag{2.1}$$

is a pseudodifferential operator of order $-l$ $(0\le l\le +\infty)$ in x. Particularly, if $p=q$, therefore $c=0$, on a conic domain Γ which is invariant with respect to the Hamilton flow defined by p_1, then d may be taken as $d=0$ on Γ.

Proof. Solve asymptotically the equation

$$pd-dp=qp-pq+pc-cq\ (=f).$$

The principal part is

$$\{p_1, d_0\}=f_0. \qquad \text{q.e.d.}$$

We prepare another same type of lemma.

Definition 2.1. 1) We say p a Kovalevskian d-ps-d operator of order m if p is a differential operator in x_0 with coefficients of pseudodifferential operators in x such that

$$p=\partial_0^m+\sum_{j=1}^m a_j(x_0, x, \partial)\partial_0^{m-j},$$

where the order of $a_j\le j$.

2) We say r a (general) d-ps-d operator of order k and differential order m for convenience' sake if

$$r=\sum_{j=0}^m b_j(x_0, x, \partial)\partial_0^{m-j}$$

and

$$k=\mathrm{Max}_j\,\{m-j+\text{order of } b_j\}$$

with pseudodifferential operators b_j in x.

Lemma 2.2. *p and q are two Kovalevskian (hyperbolic) d-ps-d operators* *defined micro-locally of order m and n. Let us assume that the principal parts p_m and q_n have no common root. Then for given r a $d-ps-d$ operator of order and differential order $m+n-1$, there exist a and b operators of order and differential order $m-1$ and $n-1$, respectively, such that*

$$(p+a)(q+b)-(pq+r)=\sum_{j=0}^{\min(m,n)} c_j\partial_0^j, \tag{2.2}$$

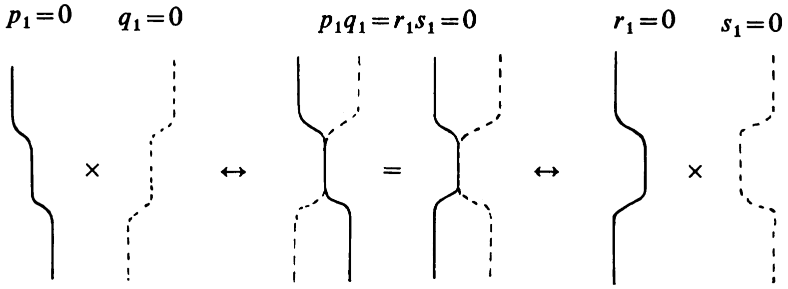
where c_j are pseudodifferential operators of order $-l-j$ $(0\leq l\leq +\infty)$. a and b are unique modulo $d-ps-d$ operators of order $-l-n$ and $-l-m$, respectively.

The same statement holds for

$$(p+a)q-(q+b)p \tag{2.3}$$

with respect to given p and q.

Lemma 2.3. *The last statement of Lemma 2.1 assures a partial commutation of p and q on a conic domain Γ_0 surrounded by another conic domain Γ, invariant with respect to the Hamilton flow defined by p_1, on which $p=q$. In another point of view, it means that the characteristics of p and q are able to exchange one of each other in part. (Refer to the figure.)*



We apply this techinic to hyperbolic operators with double characteristics, namely, whose multiplicities of characteristics are at most two. Then we obtain the next proposition.

Proposition 2.1. *Let p be a hyperbolic Kovalevskian $d-ps-d$ operator of order $m\geq 2$ with double characteristics. If the domain in x_0 is small, then there exists a strictly hyperbolic Kovalevskian $d-ps-d$ operator q of order $m-2$ such that*

$$qp-r_{(1)}r_{(2)}\cdots r_{(m-1)}=\sum_{j=0}^{\mathrm{order}(qp)-1} a_j\partial_0^j \tag{2.4}$$

where the principal part of qp has the double characteristics, $r_{(j)}$ $(j=1,\dots, m-1)$ are hyperbolic Kovalevskian $d-ps-d$ operators of order 2, and a_j are pseudodifferential operators of order $-l-j$ in x $(0\leq l\leq +\infty)$. The double characteristic points of $r_{(j)}$ coinside exactly with ones of p so that, if p is effectively hyperbolic, then all $r_{(j)}$ are also effectively hyperbolic.

Remark. If p is varied in a small neighborhood of hyperbolic operators with infinitely differentiable coefficients, then the width of the interval in x_0 and the bound of coefficients of q, $r_{(j)}$ and the ramainder term are fixed commonly. Particularly the coefficients satisfy the estimates as

$$|a|_s \leq C_{s,l}(|b|_{s+l'}+1)$$

with a uniform number l' depending on the finite number l fixing the class of the remainder term, where a stands for the coefficients of q, $r_{(j)}$ and a_j, and b for all ones of p, and where $|\cdot|_s$ stands for some weighted Hölder norms of pseudo-differential operators.

Proof. We may assume $m \geq 3$ because the case $m=2$ is trivial. Let the principal part of p be p_m. The ordered roots of p_m are denoted by $\lambda_j(x_0, x, \xi)$ $(j=1,\dots, m)$ such that

$$\lambda_j \leq \lambda_{j+1} \qquad (j=1,\dots, m-1).$$

Since the multiplicity of the roots is at most two by definition, so there exist $m-2$ distinct smooth functions ψ_j in (x_0, x, ξ) (homogeneous in ξ) such that

$$\lambda_j < \psi_j < \lambda_{j+2} \qquad (j=1,\dots, m-2).$$

$$\Omega_j = \{(x_0, x, \xi) \colon \lambda_{j+1} = \psi_j\}$$

are closed conic domains ("closed domain" stands for the closure of an open set) and

$$\{\xi_0 - \lambda_{j+1}, \xi_0 - \psi_j\} = 0$$

at a neighborhood of Ω_j. If we put the principal part q_{m-2} of q as

$$q_{m-2} = \textstyle\prod_{j=1}^{m-2}(\xi_0 - \psi_j),$$

then we can find the lower order terms of q such as the lemma holds. In fact we define the principal parts $r_{(j)2}$ of $r_{(j)}$ by

$$r_{(j)2} = (\xi_0 - \lambda_{\tilde{j}})(\xi_0 - \psi_{\tilde{j}}),$$

where

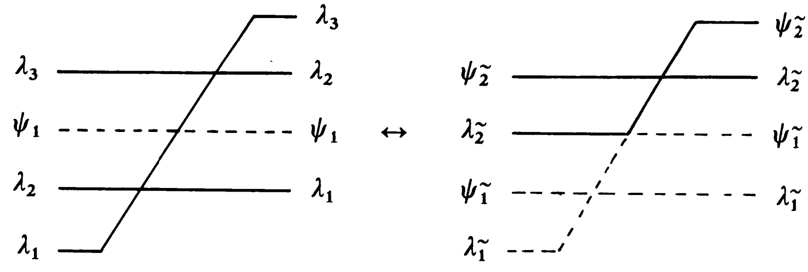
$$\lambda_{\tilde{1}} = \lambda_1, \quad \psi_{\tilde{m}} = \lambda_m,$$

$$\psi_{\tilde{j}} = \psi_j, \quad \lambda_{\tilde{j}+1} = \lambda_{j+1} \qquad \text{if} \quad \psi_j \leq \lambda_{j+1}$$

and

$$\psi_{\tilde{j}} = \lambda_{j+1}, \quad \lambda_{\tilde{j}+1} = \psi_j, \qquad \text{otherwise}$$

for $j=1,\dots, m-2$. It is clear that $r_{(j)2}$ are smooth. (Refer to the figure.)



Now we will find the lower order terms of $r_{(j)}$ and q with given principal parts as at the above. It is proved by induction in m. To do so we consider $r_{(1)}$, $q_{(1)}$ and $q_{(2)}$ with the principal parts

$$r^{\sim}_{(1)1}=r_{(1)2},$$

$$q_{(1)1}=\xi_0-\psi_1$$

and

$$q_{(2)m-3}=\prod_{j=2}^{m-3}(\xi_0-\psi_j) \qquad \text{(if necessary)},$$

respectively. We show that the lower order terms of $r^{\sim}_{(1)}$ and $q_{(1)}$ are found such that

$$q_{(1)}p\equiv r^{\sim}_{(1)}p^{\sim}, \tag{2.5}$$

where $p^{\sim}$ is an operator of order $m-1$ with the principal part

$$p^{\sim}_{m-1}=(\xi_0-\lambda^{\sim}_2)\prod_{j=3}^{m}(\xi_0-\lambda_j),$$

which satisfies the conditions in this lemma. If $m=3$, then the proof is complete, and if $m>3$, then the assumption for induction yields that there exist $r_{(j)}$ $(j=2,\dots, m-2)$ and $q_{(2)}$ with the principal parts $r_{(j)2}$ and $q_{(2)m-3}$, respectively, such that

$$q_{(2)}p^{\sim}=r_{(2)}\cdots r_{(m-1)}.$$

On the other hand we take $q^{\sim}_{(2)}$ and $r_{(1)}$ such that

$$q^{\sim}_{(2)}r^{\sim}_{(1)}\equiv r_{(1)}q_{(2)},$$

by (2.2) for (2.3) in Lemma 2.2 because $q_{(2)m-3}$ and $r_{(1)2}$ have no common root to each other. Therefore we get

$$\begin{aligned}q^{\sim}_{(2)}q^{\sim}_{(1)}p&\equiv q^{\sim}_{(2)}r^{\sim}_{(1)}p^{\sim}\\&\equiv r_{(1)}q_{(2)}p^{\sim}\\&\equiv r_{(1)}r_{(2)}\cdots r_{(m-1)}.\end{aligned}$$

We have to prove (2.5) in order to complete the proof. It is done by applying Lemma 2.3 at a neighborhood of Ω_1 and Lemma 2.2 outside of it.

We can pick out $p_{(1)}$ and $p_{(2)}$ from p at a neighborhood of Ω_1 by Lemma 2.2 such that the principal parts of $p_{(j)}$ are

$$p_{(j)1}=(\xi_0-\lambda_j) \qquad \text{for} \quad j=1,\,2,$$

and

$$p_{(3)m-2}=\prod_{j=3}^{m}(\xi_0-\lambda_j)$$

and that

$$p\equiv p_{(1)}p_{(2)}p_{(3)}, \tag{2.6}$$

because their characteristic are separated there. Now Lemma 2.3 applies to $q^{\sim}_{(1)}$,

$p_{(2)}$, $s_{(1)}$ and $s_{(2)}$, where the principal parts of $q_{(1)}^{\sim}$, $s_{(1)}$ and $s_{(2)}$ are $(\xi_0-\psi_1)$, $(\xi_0-\psi_1^{\sim})$ and $(\xi_0-\lambda_2^{\sim})$, respectively. Then there exist the lower order terms of $q_{(1)}^{\sim}$, $s_{(1)}$ and $s_{(2)}$ such that

$$q_{(1)}^{\sim}p_{(2)}\equiv s_{(1)}s_{(2)}\,.$$

Moreover $q_{(1)}$ is defined as

$$q_{(1)}p_{(1)}\equiv p_{(1)}^{\sim}q_{(1)}^{\sim}$$

by finding the lower order terms of $q_{(1)}$, and of $p_{(1)}^{\sim}$ with the same principal part as one of $p_{(1)}$, because their roots are separted. Then we get on a neighborhood of Ω_1

$$\begin{aligned} q_{(1)}p &\equiv q_{(1)}p_{(1)}p_{(2)}p_{(3)} \\ &\equiv p_{(1)}^{\sim}q_{(1)}^{\sim}p_{(2)}p_{(3)} \\ &\equiv p_{(1)}^{\sim}s_{(1)}s_{(2)}p_{(3)} \\ &\equiv r_{(1)}^{\sim}p^{\sim} \end{aligned}$$

if we put

$$r_{(1)}^{\sim}\equiv p_{(1)}^{\sim}s_{(1)}$$

and

$$p^{\sim}\equiv s_{(2)}p_{(3)}\,.$$

On the domain that $\psi_1<\lambda_2$, we extend the lower order terms of $q_{(1)}$ from on a neighborhood of Ω_1 and find one of $p_{(1)}$ such that

$$p\equiv p_{(1)}p^{\sim}.$$

Then we put

$$r_{(1)}^{\sim}=q_{(1)}p_{(1)}$$

to get

$$q_{(1)}p\equiv r_{(1)}^{\sim}p^{\sim}.$$

$r_{(1)}^{\sim}$ and $p^{\sim}$ are smoothly connected from on a neighborhood of Ω_1 because the decomposition is unique.

On the domain that $\psi_1>\lambda_2$, we also pick out $r'_{(1)}$ from p as (2.6) such that the principal part of $r'_{(1)}$ is

$$(\xi_0-\lambda_1)(\xi_0-\lambda_2)$$

and

$$p\equiv r'_{(1)}p_{(2)}\,.$$

We use again Lemma 2.2 to get the lower order terms of $q_{(1)}$, $q_{(1)}^{\sim}$ and $r_{(1)}^{\sim}$ such that

$$q_{(1)}r'_{(1)}\equiv r_{(1)}^{\sim}q_{(1)}^{\sim}.$$

Putting

$$p^{\sim}=q^{\sim}_{(1)}p_{(3)},$$

we get

$$q_{(1)}p\equiv r^{\sim}_{(1)}p^{\sim}.$$

$q_{(1)}$, $r_{(1)}$ and $p^{\sim}$ are also smoothly connected from a neighborhood of Ω_1. Therefore we have proved (2.5). q.e.d.

§3. Stability for Smooth Perturbations

A well known fact is mentioned here about the stability of solvability with respect to perturbations of lower order terms.

Let us consider two Kovalevskian d−ps−d operators p and q of order m, which are differential operators in x_0 with coefficients of pseudodifferential operators in x such as

$$\partial_0^m+\sum_{j=1}^m a_j(x_0, x, \partial_x)\partial_0^{m-j}, \tag{3.1}$$

where the order of $a_j\leq j$. We suppose that they have the same principal part, that is,

$$\begin{aligned} r&\equiv p-q \\ &=\sum_{j=1}^m b_j(x_0, x, \partial_x)\partial_0^{m-j}, \end{aligned} \tag{3.2}$$

where the order of $b_j\leq j-1$, namely, the order and the differential order of $r\leq m-1$.

We give a sufficient condition for holding the proposition that the Cauchy problem for p is well posed if the order of the perturbed term r is small with respect to fixed q. Let $\boldsymbol{H}^s(t, T)$ for $t<T$ stand for subspaces of the Sobolev space $\boldsymbol{H}^s$ on $\{(x_0, x): x_0<T\}$ consisting of all elements supported on $\{(x_0, x): x_0\geq t\}$. Then, p is an operator from $\boldsymbol{H}^s(t, T)$ to $\boldsymbol{H}^{s-m}(t, T)$ for sufficiently many s and for all (t, T): $0\leq t<T\leq 1$.

Definition 3.1. We say here the Cauchy problem for p strongly well posed on the level l if there exists an operator r from $\boldsymbol{H}^s(0, 1)$ to $\boldsymbol{H}^{s+l}(0, 1)$ with a constant l such that r is the inverse of p for sufficiently many s with respect to l and also satisfies the estimate

$$\|rf\|_{s+l,t}\leq C_s\int_0^t \|f\|_{s,\tau}d\tau \tag{3.3}$$

with respect to the norm $\|\cdot\|_{s,T}$ of $\boldsymbol{H}^s(0, T)$.

Remark. 1) Naturally, the level l should be less than m as far as the non characteristic Cauchy problem is considered.

2) Let r be an operator from $\boldsymbol{H}^s(0, 1)$ to $\boldsymbol{H}^{s+l}(0, 1)$ satisfying

$$\|rf\|_{s+l,t}\leq C_s\|f\|_{s,t} \tag{3.4}$$

for all t $(0<t\leq 1)$. We are able to obtain the estimate (3.3) for r making a concession of the level l to $l-1$.

There is no result known up to the present except for ones yielding the wellposedness for the infinitely differentiable class of the Cauchy problem in the sense of Definition 3.1, if they are modified to the global case in x. Therefore we shall show the stability of such strong wellposedness.

Proposition 3.1. *Let p, q and r be defined as* (3.1) *and* (3.2). *If q is strongly well posed on the level l and if the order of r is less than or equal to l, then $p=q+r$ is strongly well posed on the level l.*

We prepare a lemma before proving the proposition.

Lemma 3.1. 1) *Let h be an operator on $\boldsymbol{H}^s(0, 1)$ satisfying* (3.3) *with $l=0$. Then $I+h$ has the inverse on $\boldsymbol{H}^s(0, 1)$, which is also the identity plus an operator* on $\boldsymbol{H}^s(0, 1)$ satisfying (3.3) with $l=0$.

2) *Let q be strongly well posed on the level l. If r is an operator from $\boldsymbol{H}^s(0, 1)$ to $\boldsymbol{H}^{s-l}(0, 1)$ satisfying* (3.4), *then $p=q+r$ is strongly well posed on the level l.*

Proof. 1) $(I+h)u=f$ is a Volterra's equation. Therefore the Neumann series converges.

2) Let us put $h=q^{-1}r$ or rq^{-1}. They satisfy the conditions of 1). Therefore

$$p^{-1}=(I+q^{-1}r)^{-1}q^{-1}=q^{-1}(I+rq^{-1})^{-1}. \qquad \text{q.e.d.}$$

Proof of Proposition 3.1. Let us consider

$$\Lambda=\partial_0+\langle D\rangle,$$

which is an operator from $\boldsymbol{H}^s(0, 1)$ to $\boldsymbol{H}^{s-1}(0, 1)$ and invertible. Λ and Λ^{-1} satisfy (3.4) with $l=-1$ and 1, respectively. Therefore Λ may seem strongly well posed on the level 0. $q+r$ is divided as

$$q+r=(I+r_0)q+r_1, \tag{3.5}$$

where r_0 is an operator on $\boldsymbol{H}^s(0, 1)$ satisfying (3.3) with $l=0$, and r_1 is an operator from $\boldsymbol{H}^s(0, 1)$ to $\boldsymbol{H}^{s-l}(0, 1)$ satisfying (3.4). In fact we consider

$$r=\Lambda^{l-m}\Lambda^{m-l}r.$$

Then $\Lambda^{m-l}r$ is a general d$-$ps$-$d operator of order m and the order of differential operators in x_0 included in it are at most $2m-1-l$. Therefore the division by q yields

$$\Lambda^{m-l}r=r_0^{\sim}q+r_1^{\sim},$$

where $r_0^{\sim}$ and $r_1^{\sim}$ are general d$-$ps$-$d operators of order 0 and m including differential operators in x_0 of order at most $m-l-1$ and $m-1$, respectively. Thus, $\Lambda^{l-m+1}r_0^{\sim}$ is an operator on $\boldsymbol{H}^s(0, 1)$ satisfying (3.4) so that $\Lambda^{l-m}r_0^{\sim}=r_0$ is one from $\boldsymbol{H}^s(0, 1)$ to $\boldsymbol{H}^{s+1}(0, 1)$ satisfying (3.4), namely, (3.3) on $\boldsymbol{H}^s(0, 1)$. It is also clear that $\Lambda^{l-m}r_1^{\sim}=r_1$ is an operator from $H^s(0, 1)$ to $H^{s-l}(0, 1)$ satisfying (3.4).

Lemma 3.1 applies to (3.5) to imply the conclusion as follows. 1) Lemma 3.1 applies to $(I+r_0)$ to show the existence of $(I+r_0)^{-1}$. 2) applies to $q+(I+r_0)^{-1}r_1$ to yield

$$(q+r)^{-1}=(q+(I+r_0)^{-1}r_1)^{-1}(I+r_0)^{-1}. \qquad \text{q. e. d.}$$

§4. Proof of Theorem 1.3.

We use the decomposition at Proposition 2.1 and show that p is strongly well posed. It implies immediately Theorem 1.3 because the effective hyperbolicity is invariant under the Holmgren transformation. So, we may assume that $q\cdot p$ is a Kovalevskian d$-$ps$-$d operator of order $2m-2$ which is a product of Kovalevskian d$-$ps$-$d operators of second order $r_{(1)}\cdots r_{(m-1)}$ modulo smooth perturbations. We have assumed Theorem 1.1–2 in this paper. We may suppose that each $r_{(j)}$ is strongly well posed with a fairly low level l in the sense of Definition 3.1, since all $r_{(j)}$ $(j=1,\dots, m-1)$ are effectively hyperbolic if p is effectively hyperbolic. In fact, if r is a Kovalevskian d$-$ps$-$d operator of second order and effectively hyperbolic, then r is decomposable as Theorem 1.1 and so applicable to Theorem 1.2. Here the constants C_s at (3.3) and the level l are uniform if the principal part of r lies on a small neighborhood of a fixed effectively hyperbolic operator and if other lower order coefficients belong to a bounded set of the space of pseudodifferential operators. If otherwise, we can choose a convergent sequence of operators r_n to an effectively hyperbolic r_0 such that they could not holds the uniformity of C_s and l. Choosing the subsequence, we can construct an operator with parameter t such that

$$r_0^{\sim}=r_0,$$
$$r_{1/n}^{\sim}=r_n$$

and $r_t^{\sim}$ is infinitely differentiable in t, by virtue of the convexity of hyperbolic operators of second order noted in Lemma 4.1. It contradicts the results of Theorem 1.1–2. However, the level l never depend on terms of subprincipal parts expanded by the first derivatives ∇p_m of the principal symbol p_m, as noting at Remark after Theorem 1.2, because they are expanded by ones of $r_{(j)}$ even in the subprincipal parts of $r_{(j)}$ after the decomposition.

Lemma 4.1. *Let p_j $(j=0, 1)$ be two quadratic polynomial in τ with only real roots such that*

$$p_j=(\tau-a_j)^2-b_j.$$

then

$$p_\varepsilon=(\tau-\varepsilon a_0-(1-\varepsilon)a_1)^2-\varepsilon b_0-(1-\varepsilon)b_1$$

has also only real roots for $0\leq\varepsilon\leq 1$.

The results for $r_{(j)}$ imply that $q\cdot p$ is strongly well posed with some level l by

Proposition 3.1. q is strongly well posed on the level $m-3$, if it exists, since q is strictly hyperbolic. Theorefore p is also strongly well posed with another level l because

$$p^{-1}=(q\cdot p)^{-1}q.$$

We have finished the proof of Theorem 1.3 except for the precise dependence of the estimate on the coefficients of p, though it is already proved that the constant C_s at (3.3) is independent of the bound of derivatives of coefficients over fixed order if s is fixed.

Now we suppose p a partial differential operator and start from holding the estimate (3.3) for fixed $s_0=s+l$ ($\geq m$) with the level l. The constant C_s at (3.3) is uniform on the coefficients a of p bounded as

$$|a|_\gamma \leq C$$

with a Hölder norm $|\cdot|_\gamma$, where γ and the level l are fixed as far as the principal part of p and the essential part of the subprincipal part are fixed. (The fact is that they may vary slightly.)

We operate $\langle D\rangle^{2\alpha}$ (α is an integer) to p from the left hand side and commute them.

$$\langle D\rangle^{2\alpha}p=p^{\sim}\langle D\rangle^{2\alpha}+r,$$

where

$$p^{\sim}=\sum_{j=l'+1}^{m} p_j^{\sim},$$

$$p_m^{\sim}=p_m,$$

$$p_{m-1}^{\sim}=p_{m-1}-(2i)^{-1}\{\langle\xi\rangle^{2\alpha},\ p_m\}\langle\xi\rangle^{-2\alpha},$$

$$p_j^{\sim}=-\sum_{k=-j}^{m-j} C_k\sigma_k(\langle\xi\rangle^{2\alpha},\ p_{j+k})\langle\xi\rangle^{-2\alpha}$$

for $l'=l-m-1<j<m$, and r is the remainder terms.

$$(\sigma_k(a,\ b)=\sum_{|\alpha+\beta|=k}(-1)^{|\beta|}C_{\alpha\beta}^{k}a_{(\beta)}^{(\alpha)}b_{(\alpha)}^{(\beta)}.)$$

$p^{\sim}$ is also strongly well posed with the same level l as p because they have the same principal and same essential subprincipal part. Therefore for the solution u of $pu=f$, we get the estimate

$$\|\langle D\rangle^{2\alpha}u\|_{\alpha_0}\leq C_\alpha(\|\langle D\rangle^{2\alpha}f\|_{\alpha_0-l}+\|ru\|_{\alpha_0-l})$$

with a fixed constant C_α as far as $|a|_{2m-l+\gamma}$ has a fixed bound, which may depend on α. On the other hand, ru has a bilinear relation with the derivatives of the coefficients a of p and the derivatives of the solution u. The order of the derivatives of a is 2α or less, the order of the derivatives of u is $2\alpha+m+l'$ or less, and the total order of each term is $2\alpha+m$ or less. Therefore,

$$\|ru\|_{\alpha_0-l}\leq C\sum_{j,k}|a|_j\|u\|_k,$$

where the right hand side is summed up for all (j, k) such that

$$0\leq j\leq 2\alpha+\alpha_0-l,$$

$$0\leq k\leq 2\alpha+\alpha_0-1$$

and

$$j+k\leq 2\alpha+\alpha_0+m-l.$$

By the interpolation theory, it holds for any small positive ε that

$$\|ru\|_{\alpha_0-l}\leq\varepsilon\|u\|_{2\alpha+\alpha_0}+C_\varepsilon(1+|a|_{2\alpha+\alpha_0-l})\,\|u\|_{\alpha_0}$$

With another small positive ε, we get

$$\begin{aligned}&\|\langle D\rangle^{2\alpha}u\|_{\alpha_0}\\&\leq\varepsilon\|u\|_{2\alpha+\alpha_0}+C_\varepsilon(\|f\|_{2\alpha+\alpha_0-l}+(1+|a|_{2\alpha+\alpha_0-1})\|f\|_{\alpha_0-l}).\end{aligned}$$

Moreover, we can yield the estimate for $\partial_0^{2\alpha+\alpha_0}u$ by taking the derivatives in x_0 of the equation and by applying the interpolation theory as

$$\|\partial_0^{2\alpha+\alpha_0}u\|_0\leq\varepsilon\|\partial_0^{2\alpha+\alpha_0}u\|_0+C_\varepsilon(\|\langle D\rangle^{2\alpha}u\|_{\alpha_0}+\|f\|_{2\alpha+\alpha_0-m}+(1+|a|_{2\alpha+\alpha_0-m})\|u\|_{\alpha_0}).$$

Therefore we conclude

$$\|u\|_{2\alpha+\alpha_0}\leq C(\|f\|_{2\alpha+\alpha_0-l}+(1+|a|_{2\alpha+\alpha_0-l})\|f\|_{\alpha_0-l}).$$

q.e.d.

§5. Non Linear Equations

Proving Theorem 1.4 is nothing but stating the Nash–Moser implicit function theorem. We quote it from L. Hörmander [2]. (We also refered to S. Klainerman [7] and R. S. Hamilton [1].) We consider it on the Hölder class. Let the domain Ω be a lens domain

$$\{(x_0, x)\colon 0<x_0\leq\varepsilon(1-|x|^2)\}$$

with a small positive ε. We define that the Hölder class $H_0^s(\Omega)$ $(s\geq 0)$ stands for the subspaces of the Hölder class $H^s(\Omega^\sim)$ on

$$\Omega^\sim=\{(x_0, x)\colon x_0\leq\varepsilon(1-|x|^2)\}$$

consisting of the functions whose supports are included in the closure of Ω.

$$(H_0^{+\infty}(\Omega)=\cap_{s\geq 0}H_0^s(\Omega).)$$

It is clear that the Banach scales $H_0^s(\Omega)$ have the following smoothing operator.

Lemma 5.1. *There exists the smoothing operators S_θ having the following properties for $\theta>1$ and u belonging to $H_0^a(\Omega)$ if a and b are non-negative and bounded numbers.*

i) $|S_\theta u|_b \leq C|u|_a$, *if* $b \leq a$.
ii) $|S_\theta u|_b \leq C\theta^{b-a}|u|_a$, *if* $a \leq b$.
iii) $|u - S_\theta u|_b \leq C\theta^{b-a}|u|_a$, *if* $b \leq a$.
iv) $|(\partial/\partial\theta)S_\theta u|_b \leq C\theta^{b-a-1}|u|_a$.

Proof. Find a function $\psi(t)$ belonging to $\mathscr{S}$ such that the support of ψ lies on the negative axis,

$$\int_{-\infty}^{0} \psi dt = 1$$

and

$$\int_{-\infty}^{0} t^k \psi dt = 0$$

for all integer $k \geq 1$. q. e. d.

Therefore holds the Nash–Moser implicit function theorem, that is stated as follows.

Proposition 5.1 (The Nash–Moser Implicit Function Theorem).
Let Φ be a map on $H_0^{+\infty}(\Omega)$ and be defined on a $H_0^{\mu}(\Omega)$ neighborhood V of an element u_0 of $H_0^{+\infty}(\Omega)$, $(\mu \geq 1)$, which has a second differential and satisfies (5.3) *for u belonging to $V \cap H_0^{+\infty}(\Omega)$.*

1) (*Existence*) *Assume that the first differential has a right inverse Ψ satisfying* (5.4) *for u belonging to $V \cap H_0^{+\infty}(\Omega)$. If $\alpha \geq \max(\mu, \mu')$ and $\alpha > \max(m+n+\lambda, 2(\mu-1))$ and if α is not integer, then there exists a neighborhood W of* 0 *in $H_0^{\alpha+\lambda}(\Omega)$ such that the equation*

$$\Phi(u) = \Phi(u_0) + f \tag{5.1}$$

has a solution $u(f)$ belonging to $H_0^{\alpha}(\Omega)$ for each f belonging to W. Moreover, it holds the following i) *and* ii).

i) $|u(f) - u_0|_\alpha \to 0$ *if* $|f|_{\alpha+\lambda} \to 0$.

ii) *If f belongs to $H_0^{\beta+\lambda}(\Omega)$ for β being not integer$>\alpha$, then $u(f)$ belongs to $H_0^{\beta}(\Omega)$ and satisfies*

$$|u(f)|_\beta \leq C(|f|_{\beta+\lambda} + 1),$$

so that $u(f)$ belongs to $H_0^{+\infty}(\Omega)$ if f does so.

2) (*Uniqueness*) *Assume that the first differential has a left inverse Ψ satisfying* (5.4) *for u belonging to $V \cap H_0^{+\infty}(\Omega)$ and that*

$$\alpha > \max(\mu + \lambda + m, \lambda + m + 2, \mu + \mu' - 1).$$

Let W be a bounded set of $H_0^{\alpha}(\Omega)$. Then there exists a positive constant δ such that

$$|u - v|_0 \geq \delta \tag{5.2}$$

if u and v are two different solutions of (5.1) *belonging to $V \cap W$, where the constant δ depends only on the constants α, μ, μ', λ, m, C_a's in the estimates* (5.3–4), *the* *bounded set W and the spaces $H_0^s(\Omega)$.*

(5.3) $$|\Phi(u)|_a \le C_a(|u|_{m+a}+1),$$

$$|\Phi'(u: v)|_a \le C_a(|u|_{a+m+1}|v|_{\mu-1}+|v|_{a+m})$$

and

$$|\Phi''(u: v, w)|_a \le C_a(|u|_{a+m+2}|v|_{\mu-1}|w|_{\mu-1}+|v|_{a+m+1}|w|_{\mu-1}+|v|_{\mu-1}|w|_{a+m+1})$$

for some m, $\mu-1\ge 0$, for all $a\ge 0$ and for all v, w belonging to $H_0^{+\infty}(\Omega)$.

(5.4) $$|\Psi(u)g|_a \le C_a(|g|_{a+\lambda}+|g|_\lambda |u|_{a+\mu'})$$

for some λ, $\mu'\ge 0$, for all $a\ge 0$ and for all g belonging to $H_0^{+\infty}(\Omega)$.

Proof of Theorem 1.4. We may assume that $p|_{u=0}=g$ is supported in $\{x_0\ge 0\}$, and find a solution in $H_0^{+\infty}(\Omega)$, because p is Kovalevskian so that an asymptotic solution at $x_0=0$ is constructed. By the assumption, p is effectively hyperbolic for sufficiently small u. By virtue of the Sobolev's lemma, Theorem 1.3 shows that the linearized equations are solvable and satisfies the estimate (5.4) with respect to the Hölder class $H_0^s(\Omega)$. On the other hand, p satisfies (5.3) because p is a non-linear partial differential operator. By Proposition 5.1, for each f belonging to a neighborhood W of $u_0=0$ in $H_0^s(\Omega)$, there exists a solution u of the equation

$$p(x_0, x, \partial^\alpha u)=pu=g+f$$

such that u belongs to $H_0^{+\infty}(\Omega)$ if f does so. Since g belongs to $H_0^{+\infty}(\Omega)$, so we can choose f belonging to W as f coincides with $-g$ at $\{x_0\le\varepsilon\}$ for a sufficiently small positive ε. Then solution u for f satisfies

$$pu=0$$

at $\{x_0\le\varepsilon\}$. We denote by $H_t^s(\Omega)$ the spaces of functions f such that $f(x_0-t, x)$ belongs to $H_0^s(\Omega)$. If we take ε defining Ω small, then the linearized equations are solved on $H_t^s(\Omega)$ for $|t|\le t_0>\varepsilon$ and the inverses satisfy the common estimates (5.4). Therefore two different solutions u and v for $pu=0$ has the difference

(5.5) $$|u-v|_0\ge\delta>0$$

with a common positive constant δ. On the other hand, the norm of $u-v$ in $H_t^s(\Omega)$ tends to zero if $t>-\varepsilon$ tends to $-\varepsilon$. This implies that holding the inequality (5.5) is impossible, so that u should coincide with v near the origin $(x_0, x)=(0, 0)$. Sweeping away from $t=-\varepsilon$ to $t=0$, we conclude the solution u in $H_0^{+\infty}(\Omega)$ unique.

q. e. d.

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