J. Math. Kyoto Univ. (JMKYAZ)
25-4 (1985) 619-626

On Teichmüller spaces and modular transformations

Dedicated to Professor Yukio Kusunoki on his 60th birthday

By

Hiroshige SHIGA

(Received February 4, 1984, Revised April 9, 1984)

§1. Introduction

Let G be a finitely generated Fuchsian group of the first kind acting on the upper half plane U such that the Riemann surface U/G is of type (p, n). The Teichmüller space $T(G)$ of G is identified with a bounded domain in C^{3p-3+n}, which is called the Bers embedding of $T(G)$ (Bers [5]). Recently, Bers investigated the action of modular transformations on the boundary $\partial T(G)$ of $T(G)$ in the Bers embedding (cf. [8], [9]). He showed that all modular transformations can be extended to some set of boundary points which is dense in $\partial T(G)$ and that the infinite iterations of a hyperbolic modular transformation accumulate to boundary points corresponding to totally degenerate groups.

In this paper, we shall investigate the infinite iterations of parabolic and pseudo-hyperbolic modular transformations. Furthermore, we shall give a new characterization of the Thurston-Bers classifications of modular transformations in terms of their infinite iterations. Roughly speaking, accumulation points of elliptic, parabolic, pseudo-hyperbolic and hyperbolic modular transformations correspond to quasi-Fuchsian groups, regular b-groups, degenerate cusps and totally degenerate groups, respectively (Theorem 3.3). And we shall give some examples about the infinite iterations of pseudo-hyperbolic modular transformations (§4).

The author wishes to express his deepest gratitude to Prof. Y. Kusunoki for his encouragement and comments. The author also thanks to Mr. H. Ohtake and Mr. M. Masumoto for their useful and stimulating conversations with him.

§2. Preliminaries

In this section, we shall introduce some notations and recall some known results (for details see Bers [5], [7] and Kra [10]).

Throughout this paper, we denote by G a finitely generated Fuchsian group of the first kind acting on the upper half plane U such that U/G is a Riemann surface of type (p, n) with $2p+n-2>0$, and denote by π a canonical projection of U onto U/G.

Let $B_2(L, G)$ be the Banach space consisting of all holomorphic functions ϕ defined on the lower half plane L such that

$$\phi(g(z))g'(z)^2=\phi(z), \quad z \in L, \ g \in G,$$

with the norm

$$\|\phi\| = \sup_{z \in L} (\operatorname{Im} z)^2 |\phi(z)| < +\infty.$$

Let $M(G)$ be the set of all measurable functions μ defined on U such that $\|\mu\|_\infty < 1$ and

$$\mu(g(z))\overline{g'(z)}/g'(z)=\mu(z), \quad z \in U, \ g \in G.$$

For each $\mu \in M(G)$ we denote by w^μ the quasiconformal homeomorphism of $\hat{C}$ satisfying $(w^\mu)_{\bar z}(z)=\mu(z)(w^\mu)_z(z)$ on U, $(w^\mu)_{\bar z}=0$ on L, $w^\mu(0)=0$, $w^\mu(1)=1$ and $w^\mu(\infty)=\infty$. Then w^μ is compatible with G, that is, it induces an isomorphism; $g \mapsto w^\mu \circ g \circ (w^\mu)^{-1}$ of G onto $G_\mu = w^\mu G (w^\mu)^{-1}$ with two simply connected invariant components (a quasi-Fuchsian group). If w^μ and $w^{\mu'}$ ($\mu, \mu' \in M(G)$) induces the same isomorphism, they are called to be equivalent. The equivalence class of w^μ is denoted by $[w^\mu]$. The Teichmüller space $T(G)$ is the set of Schwarzian derivatives $\{w^\mu, z\}$ of w^μ in L for all μ in $M(G)$. If $[w^\mu]=[w^{\mu'}]$, then $\{w^\mu, z\}=\{w^{\mu'}, z\}$. If $[w^\mu]\neq[w^{\mu'}]$, then $\{w^\mu, z\}\neq\{w^{\mu'}, z\}$. It is known that $\phi_\mu=\{w^\mu, z\}$ belongs to $B_2(L, G)$ and $T(G)$ is a bounded domain in $B_2(L, G)$.

For each $\phi \in T(G) \cup \partial T(G) \subset B_2(L, G)$ we denote by W_ϕ the univalent function on L satisfying $\{W_\phi, z\}=\phi(z)$ and $W_\phi(z)=(z+i)^{-1}+o(1)$ as $z \to -i$. Then for $\phi \in T(G)$ $G^\phi = W_\phi G (W_\phi)^{-1}$ is a quasi-Fuchsian group and for $\phi \in \partial T(G)$ G^ϕ is a Kleinian group called *b-group*, which has only one simply connected invariant component. All b-groups are classified into *regular b-groups*, *partially degenerate groups* and *totally degenerate groups* ([1], [5], [11]). We set $h_\phi(g)=W_\phi \circ g \circ (W_\phi)^{-1}$ ($g \in G$), the isomorphism of G onto G^ϕ induced by W_ϕ.

It is also well known that the Teichmüller space $T(G)$ has the holomorphic automorphism group $\operatorname{Mod}(G)$ called the *modular group* of G. Each modular transformation $\chi \in \operatorname{Mod}(G)$ is defined as follows (cf. [6], [8]).

Let w_θ be a quasiconformal self-mapping of U with the Beltrami differential θ such that $w_\theta G (w_\theta)^{-1}=G$. Then for each μ in $M(G)$ we can define $\theta_*(\mu)$ as the Beltrami differential of $A \circ w^\mu \circ w_\theta^{-1}$ where A is a Möbius transformation chosen so that $A \circ w^\mu \circ w_\theta^{-1}$ fixes 0, 1, and ∞. It is known that the mapping; $[w^\mu] \mapsto [w^{\theta_*(\mu)}]$ is well defined on $T(G)$ and isometry with respect to the Teichmüller metric t_G. Furthermore, the mapping depends only on $w_\theta | \boldsymbol{R}$, and the homotopy class determined by a quasiconformal self mapping of $S_0=U/G$ induces the above mapping uniquely.

Now, we shall introduce the *Thurston-Bers classification* of modular transformations.

A finite non-empty set of disjoint closed Jordan curves, $C=\{C_1, \dots, C_s\}$ on S_0 will be called admissible if no C_j can be deformed to a point, a puncture or into C_k with $k \neq j$. An orientation preserving homeomorphism f; $S_0 \mapsto S_0$ is calles to be *reduced* by C if $f(C)=C$. A self-mapping f is called *reducible* if f is isotopic to a reduced mapping, *irreducible* otherwise. When f is reduced by C, each component $S_0^{(k)}$ of $S_0 - C$ ($k=1,\dots, m$) is called a part of $S_0 - C$.

Let $J(k)$ be the smallest positive integer so that $f^{J(k)}|S_0^{(k)}$ fixed $S_0^{(k)}$. Bers [7] showed that if f is reduced by C, then for each part $S_0^{(k)}$ $f^{J(k)}|S_0^{(k)}$ is irreducible as a self-mapping of $S_0^{(k)}$.

Let χ be in $\mathrm{Mod}(G)$ and not the identity. We shall say that χ is *elliptic* if χ is periodic, and a non-periodic χ is *hyperbolic* if χ is induced by an irreducible mapping, *parabolic* if there exists a reduced mapping f by some C such that f induces χ and for each part $S_0^{(k)}$ of $S_0 - C$ $f^{J(k)}|S_0^{(k)}$ is isotopic to a periodic mapping of $S_0^{(k)}$, and *pseudo-hyperbolic* otherwise. These definitions are equivalent to those defined by the Teichmüller distance, and it is also known that a $\chi \in \mathrm{Mod}(G)$ is a hyperbolic modular transformation if and only if it leaves a Teichmüller geodesic in $T(G)$ (cf. [7], [10]). But we do not need these results in this paper.

§3. Infinte iterations of parabolic and pseudo-hyperbolic modular transformations

For each $\chi \in \mathrm{Mod}(G)$ and for $x \in T(G)$ we denote by $A(\chi; x)$ the set of all accumulation points of $\{\chi^n(x)\}_{n=-\infty}^{+\infty}$ if χ is not elliptic, and we set $A(\chi; x) = \bigcup_{n=0}^{+\infty} \chi^n(x)$ if χ is elliptic. From the discontinuity of $\mathrm{Mod}(G)$, if χ is not elliptic, then $A(\chi; x)$ is contained in $\partial T(G)$.

Bers [9] showed that for a hyperbolic modular transformation χ every ϕ of $A(X; x)$ is (corresponding to) a totally degenerate group with no accidental parabolic transformation (APT).

In this section, we shall study $A(\chi; x)$ when χ is parabolic or pseudo-hyperbolic.

Theorem 3.1. *Let $\chi \in \mathrm{Mod}(G)$ be parabolic. Then $A(\chi; x)$ consists of regular b-groups for each $x \in T(G)$.*

To prove Theorem 3.1 we note the following lemmas given in Abikoff[1] and Bers [9].

Lemma 3.1. *Let G_0 be a Kleinian group. Assume that there exist a domain D and homeomorphisms w_n of $\hat{C}$ ($n=1, 2,\dots$) satisfying the following conditions; (a) for each $g \in G_0$, $w_n g w_n^{-1}$ and $\lim_{n\to\infty} w_n g w_n^{-1} = q(g)$ are Möbius transformations and q is an isomorphism of G_0 onto $H = \lim_{n\to\infty} w_n G_0 w_n^{-1}$, (b) w_n ($n=1, 2,\dots$) are uniformly quasiconformal mappings on D, that is, their maximal dilatations are uniformly bounded on D, (c) there exist loxodromic transformations g_1 and g_2 in G_0 and a point z in D such that $g_1 \circ g_2 \neq g_2 \circ g_1$ and $g_1(z)$, $g_2(z)$ belong to D. Then H is either discontinuous or non-discrete. Furthermore, if H is discontinuous, then $H_{w(D)} = q(G_{0,D})$, where w is the limit function of $\{w_n|D\}_{n=1}^{+\infty}$ and $H_{w(D)} (\subset H)$, $G_{0,D} (\subset G_0)$ are the stabilized subgroups of $w(D)$ and D, respectively.*

Lemma 3.2. *Let $\{x_j\}_{j=1}^{+\infty}$ and $\{y_j\}_{j=1}^{+\infty}$ be sequences in $T(G)$, with $\lim_{j\to+\infty} x_j = \phi$ and $\lim_{j\to+\infty} y_j = \psi$. Assume that $t_G(x_j, y_j) < a < +\infty$ ($j=1, 2,\dots$) and ϕ, $\psi \in \partial T(G)$.*

Then boundary groups G^{ψ} and G^{ϕ} are quasiconformally equivalent, that is, there is a quasiconformal self-mapping F of $\hat{C}$ such that $FG^{\phi}F^{-1}=G^{\psi}$. In particular, if G^{ϕ} is a regular b-group, partially degenerate group, cusp, and totally degenerate group, then G^{ψ} is so respectively.

Proof of Theorem 3.1. Let f be a self-mapping of $S_0=U/G$ inducing χ. Then we may assume that f is reduced by certain $C=\{C_1, C_2,\dots, C_s\}$ and each C_j $(j=1, 2,\dots, s)$ is a simple geodesic curve on S_0. For each part $S_0^{(k)}$ of S_0-C, there is an integer n_k such that $f^{n_k}|S_0^{(k)}$ is a self-mapping and isotopic to the identity on $S_0^{(k)}$ because χ is parabolic. Therefore, there exists an integer N such that for all parts $S_0^{(k)}$ $f^N|S_0^{(k)}$ is a self mapping and isotopic to the identity on $S_0^{(k)}$. Then, we can easily construct a quasiconformal mapping $\tilde{f}_N$; $S_0\mapsto S_0$ which is isotopic to f^N, equal to the identity on $S_0-\bigcup_{j=1}^{s}U_j$ where U_j $(j=1, 2,\dots, s)$ are annular neighbourhoods of C_j. Hence $\tilde{f}_N$ induces χ^N. Denote by F_N the self-mapping of U being a lift of $\tilde{f}_N$, i.e. $F_N=\tilde{f}_N\circ\pi$, and by μ_m the Beltrami differential of F_N^{-m} $(m\in Z)$. Then $\{w^{\mu_m}, z\}$ on L is $\chi^{mN}(0)$ from the definitions. Obviously, w^{μ_m} $(m=0, \pm 1, \pm 2,\dots)$ are conformal on $V=\pi^{-1}(S_0-\bigcup_{j=1}^{s}U_j)$. Thus, the conditions of Lemma 3.1 are satisfied for $\{w^{\mu_m}\}_{m=-\infty}^{+\infty}$ and for each component V_k of V corresponding to $S_0^{(k)}\cap(S_0-\bigcup_{j=1}^{s}U_j)$ via π. By using the classification of b-groups in Maskit [11], we verify that every $\phi\in A(\chi^N; 0)$ must be a regular b-group.

Let $\{\chi^{n_p}(0)\}$ be a subsequence of $\{\chi^n(0)\}_{n=-\infty}^{+\infty}$ which converges to a boundary group ϕ'. Set $n_p=Nk_p+\ell_p$ with $0\leqslant \ell_p<N$ where N is the integer defined as above, then $t_G(\chi^{n_p}(0), \chi^{Nk_p}(0))=t_G(\chi^{\ell_p}(0), 0)\leqslant t_G(0, \chi(0))+t_G(\chi(0), \chi^2(0))+\cdots+t_G(\chi^{\ell_p-1}(0), \chi^{\ell_p}(0))=\ell_p t_G(\chi(0), 0)<Nt_G(\chi(0), 0)$. Hence from Lemma 3.2 and the above argument we conclude that ϕ' is a regular b-group.

Since $t_G(\chi^n(x), \chi^n(0))=t_G(x, 0)$ for $x\in T(G)$, by using Lemma 3.2 again, we can show the statement for every $x\in T(G)$. q.e.d.

As for the infinite iterations of a pseudo-hyperbolic modular transformation, we have

Theorem 3.2. *Let $\chi\in\mathrm{Mod}(G)$ be pseudo-hyperbolic. Then for each $x\in T(G)$ $A(\chi; x)$ consists of cusps and contains a degenerate cusp.*

Remarks. 1) We can construct examples so that (a) $A(\chi; x)$ contains a *totally degenerate cusp* and (b) $A(\chi; x)$ contains a *partially degenerate group* (see §4).

2) The author does not know whether or not $A(\chi; x)$ consists of degenerate cusps.

Proof. Let f be a homeomorphism of S_0 inducing χ. We may assume that f is reduced by certain $C=\{C_1, C_2,\dots, C_s\}$ and each C_j $(j=1, 2,\dots, s)$ is a simple geodesic curve on S_0.

Assume that $\phi\in A(\chi; 0)$ is the limit of $\{\chi^{n_p}(0)\}$ and G^{ϕ} is a totally degenerate group without an accidental parabolic transformation. Let $g_j\in G$ $(j=1, 2,\dots, s)$ be hyperbolic transformations determined by C_j, and set $g_{j,n}=W_{\chi^n(0)}\circ g_j\circ(W_{\chi^n(0)})^{-1}$.

Then each $g_{j,n}$ is associated with $f^n(C_j)$, and the axis of $g_{j,n}$ in $\hat{C}-W_{\chi^n(0)}(L)$ is a lift of $f^n(C_j)$.

Now, we take a Fuchsian equivalent $H_n=\varphi_n^{-1}G^{\chi^n(0)}\varphi_n$ of $G^{\chi^n(0)}$ where φ_n; $U\mapsto\hat{C}-\overline{W_{\chi^n(0)}(L)}$ is a conformal mapping, and let $\lambda_{j,n}$ (>1) be the multiplier of $\eta_n=\varphi_n^{-1}\circ g_{j,n}\circ\varphi_n$. Since $\log\lambda_{j,n}$ is equal to Poincaré length of $f^n(C_j)$ on S_0, we have

$$(3.1)\qquad 1<\lambda_{j,n}<M<+\infty,\quad j=1, 2,\dots,s,\quad n=\pm 1, \pm 2,\dots$$

Set $g'_j=\lim_{n_p\to\infty} g_{j,n_p}\in G^{\phi}$, then g'_j must be loxodromic. Therefore from Abikoff [2] Theorem 2, we have $\lim_{n_p\to\infty}\lambda_{j,n_p}=+\infty$. This contradicts with (3.1). Hence G^{ϕ} is a cusp. By using Lemma 3.2 as in the proof of Theorem 3.1, we verify that for every $x\in T(G)$ $A(\chi; x)$ consists of cusps.

Next, we take an integer N as in the proof of Theorem 3.1. For simplicity, we set $\tilde{\chi}=\chi^N$ and $\tilde{f}=f^N$.

We assume that $\phi=\lim_{n_p\to\infty}\tilde{\chi}^{n_p}(0)$ and $\psi_1=\lim_{n_p\to\infty}\tilde{\chi}^{n_p+1}(0)$ exist for a subsequence $\{n_p\}$ and both are regular b-groups. Let η_j $(j=1, 2,\dots, s)$ be accidental parabolic transformations in G^{ϕ} such that $\eta_i\circ\eta_j\neq\eta_j\circ\eta_i$ $(i\neq j)$ and every accidental parabolic transformation in G^{ϕ} is conjugate to some η_j, and let C_j $(j=1, 2,\dots, s)$ be geodesic closed curves on S_0 associated with axes of η_j in the invariant component of G^{ϕ}. Then $C=\{C_1, C_2,\dots, C_s\}$ is admissible (cf. [1], [11]) and furthermore we can take as f a self-mapping of S_0 reduced by C (cf. [9] Lemma 5).

Since $\psi_1=\lim_{n_p\to\infty}\tilde{\chi}^{n_p+1}(0)=\lim_{n_p\to\infty}\tilde{\chi}(\tilde{\chi}^{n_p}(0))$, from [8] Propositions 3.2 and 3.8 there exists a quasiconformal self-mapping F of $\hat{C}$ such that

$$(3.2)\qquad \begin{aligned} &F\circ\Omega_{\phi}\circ w_{\theta}^{-1}=\Omega_{\psi_1}\quad\text{on}\quad L,\quad\text{and}\\ &F\circ\Omega_{\phi}\circ g\circ\Omega_{\phi}^{-1}\circ F^{-1}=\Omega_{\psi_1}\circ w_{\theta}\circ g\circ w_{\theta}^{-1}\circ\Omega_{\psi_1}^{-1}\qquad(g\in G)\quad\text{on}\ \Omega_{\psi_1}(L),\end{aligned}$$

where w_{θ} is a quasiconformal self mapping of L which corresponds to χ and has the continuous Beltrami differential θ, and Ω_{ϕ}, Ω_{ψ_1} are univalent functions on L taking $(-i, -2i, -3i)$ into $(0, 1, \infty)$ with $\{\Omega_{\phi}, z\}=\phi(z)$, $\{\Omega_{\psi_1}, z\}=\psi_1(z)$ respectively. Furthermore, F is conformal on $\Omega(G^{\phi})-\Delta(G^{\phi})$ where $\Omega(G^{\phi})$ and $\Delta(G^{\phi})$ are the region of discontinuity and the invariant component of G^{ϕ}, respectively.

For each part $S_0^{(k)}$ of S_0-C $(k=1, 2,\dots, m)$, denote by N_k a component of $\pi^{-1}(S_0^{(k)})$, then N_k is a Nielsen convex region on U bounded by some geodesics in $\pi^{-1}(C)$. Since in the equation (3.2) θ need not be continuous any longer, we may assume that $N_{k,\theta}=w_{\theta}(N_k)$ is a component of $\pi^{-1}(S_0^{(k)})$.

Since χ is pseudo-hyperbolic, there exists a part of S_0-C, say $S_0^{(1)}$, such that $\tilde{f}|S_0^{(1)}$ is an irreducible and non-periodic self-mapping of $S_0^{(1)}$. We may assume that the stabilized subgroup G_{N_k} of N_k in G is corresponding to a non-invariant component subgroup of G^{ϕ} ([1] Theorem 1), that is, there exists a non-invariant component Δ_k such that $(G^{\phi})_{\Delta_k}=h_{\phi}(G_{N_k})$. Since $w_{\theta}G_{N_k}w_{\theta}^{-1}=G_{N_{k,\theta}}$ and for some Möbius transformation A_{ϕ}, $\Omega_{\phi}=A\circ W_{\phi}$ $(\phi\in T(G)\cup\partial T(G))$, we have from (3.2) $W_{\psi_1}=\tilde{F}\circ W_{\phi}\circ w_{\theta}^{-1}$ and

$$(3.3)\qquad \tilde{F}\circ h_{\phi}(G_{N_k})\circ\tilde{F}^{-1}=h_{\psi_1}(G_{N_{k,\theta}})=h_{\psi_1}(w_{\theta}G_{N_k}w_{\theta}^{-1}),$$

where $\tilde{F}=A_{\psi_1}^{-1}\circ F\circ A_\phi$.

A quasiconformal self-mapping $\tilde{f}|S_0^{(k)}$ is naturally deformed to a quasiconformal self-mapping $\tilde{f}_k$ of $\Delta_k/(G^\Phi)_{\Delta_k}$ because $S_0^{(k)}$ is homeomorphic to $\Delta_k/(G^\Phi)_{\Delta_k}$. Each $\tilde{f}_k$ $(k=1, 2,\dots, m)$ is lifted to a self-mapping $\hat{f}_k$ of $G^\Phi(\Delta_k)$ satisfying $\hat{f}_k\circ h_\phi(g)\circ\hat{f}_k^{-1}=h_\phi(w_\theta^{-1}gw_\theta)$ for all $g\in G$ as follows.

Let $\Delta_{k,\theta}$ be a non-invariant component of G^Φ which is invariant by $h_\phi(G_{N_k,\theta})$. From the definition of $\tilde{f}_k$, we can construct a mapping $\hat{f}_k$: $\Delta_k\to\Delta_{k,\theta}$ satisfying

$$\hat{f}_k\circ h_\phi(g)\circ\hat{f}_k^{-1}(z)=h_\phi(w_\theta^{-1}gw_\theta)(z)$$

for every $h_\phi(g)\in(G^\Phi)_{\Delta_k}=h_\phi(G_{N_k})$ and for every $z\in\Delta_{k,\theta}$.

Next, for each representative g' of G/G_{N_k}, we define

$$\hat{f}_k\colon h_\phi(g')\,(\Delta_k)\longrightarrow h_\phi(w_\theta^{-1}g'w_\theta)\,(\Delta_{k,\theta})\ \text{as}$$

$$\hat{f}_k(z)=h_\phi(w_\theta^{-1}g'w_\theta)\circ(\hat{f}_k|\Delta_k)\circ h_\phi(g')^{-1}(z)$$

for every $z\in h_\phi(g')(\Delta_k)$.

If $g'_mG_{N_k}\neq g'_nG_{N_k}$, then $h_\phi(g'_m)(\Delta_k)\cap h_\phi(g'_n)(\Delta_k)=\emptyset$ and $h_\phi(w_\theta^{-1}g'_mw_\theta)(\Delta_{k,\theta})\cap h_\phi(w_\theta^{-1}g'_nw_\theta)\,(\Delta_{k,\theta})=\emptyset$. Furthermore, $G^\Phi(\Delta_k)=\bigcup_{g'\in G^\Phi/G_{N_k}} h_\phi(g')\,(\Delta_k)=\bigcup_{g'\in G^\Phi/G_{N_k}} h_\phi(w_\theta^{-1}g'_nw_\theta)(\Delta_{k,\theta})$. Hence $\hat{f}_k$ is a self mapping of $G^\Phi(\Delta_k)$ for a fixed coset representation $G=\sum_{n=0}^{\infty}g'_nG_{N_k}$.

For each $z\in G^\Phi(\Delta_k)$ and for each $g\in G$ there exist g'_1, g'_2, g'_3, representatives of G/G_{N_k} as above such that z is in $h_\phi(w_\theta^{-1}g'_1w_\theta)(\Delta_{k,\theta})$, $g=g'_2\circ g_1$ and $g'_2\circ g_1\circ g'_1=g'_3\circ g_2$ for some g_1, $g_2\in G$. Then we have

$$\begin{aligned}
\hat{f}_k\circ h_\phi(g)\circ\hat{f}_k^{-1}(z)&=\hat{f}_k\circ h_\phi(g'_2\circ g_1)\circ h_\phi(g'_1)\circ(\hat{f}_k|\Delta_k)^{-1}\circ h_\phi(w_\theta^{-1}g'_1w_\theta)^{-1}(z)\\
&=\hat{f}_k\circ h_\phi(g'_2\circ g_1\circ g'_1)\circ(\hat{f}_k|\Delta_k)^{-1}\circ h_\phi(w_\theta^{-1}g'_1w_\theta)^{-1}(z)\\
&=\hat{f}_k\circ h_\phi(g'_3\circ g_2)\circ(\hat{f}_k|\Delta_k)^{-1}\circ h_\phi(w_\theta^{-1}g'_1w_\theta)^{-1}(z)\\
&=h_\phi(w_\theta^{-1}g'_3w_\theta)\circ(\hat{f}_k|\Delta_k)\circ h_\phi(g_2)\circ(\hat{f}_k|\Delta_k)^{-1}\circ h_\phi(w_\theta^{-1}g'_1w_\theta)^{-1}(z)\\
&=h_\phi(w_\theta^{-1}g'_3w_\theta)\circ h_\phi(w_\theta^{-1}g_2w_\theta)\circ h_\phi(w_\theta^{-1}g'_1w_\theta)^{-1}\\
&=h_\phi(w_\theta^{-1}(g'_3\circ g'_2\circ g_1'^{-1})w_\theta)=h_\phi(w_\theta^{-1}(g'_2\circ g_1)w_\theta)\\
&=h_\phi(w_\theta^{-1}gw_\theta).
\end{aligned}$$

We define

$$\tilde{W}^{(1)}(z)=\begin{cases}\tilde{F}\circ\hat{f}_k(z) & z\in G^\Phi(\Delta_k),\quad k=1,\dots,m.\\ \tilde{F}\circ W_\phi\circ w_\theta^{-1}\circ(W_\phi)^{-1}(z) & z\in\Delta(G^\Phi).\end{cases}\tag{3.4}$$

Then $\tilde{W}^{(1)}$ is a quasiconformal mapping of $\Omega(G^\Phi)$ onto $\Omega(G^{\psi_1})$. Therefore, from [1] §4 Corollary, $\tilde{W}^{(1)}$ can be extended to a quasi-conformal self-mapping $W^{(1)}$ of $\hat{\boldsymbol{C}}$ with the Beltrami differential $\mu^{(1)}$. Furthermore, $W^{(1)}h_\phi(g)(W^{(1)})^{-1}=h_{\psi_1}(g)$ for all

$g \in G$ by (3.3) and (3.4). In particular, $W^{(1)}|\Delta_1$ is a lift of a quasiconformal self-mapping $\tilde{f}_1$ of $\Delta_1/(G^\Phi)_{\Delta_1}$.

By using the argument as above for (a convergent subsequence of) $\{X^{n_p+j}(0)\}_{n_p=1}^{+\infty}$ ($j=2, 3,\dots$), we can construct a quasiconformal self-mapping $W^{(j)}$ of $\hat{C}$ with the Beltrami differential $\mu^{(j)}$ so that $W^{(j)}|\Delta_1$ is a lift of $(\tilde{f}_1)^j$ of $\Delta_1/(G^\Phi)_{\Delta_1}$.

On the other hand, there exists a canonical injection i; $\prod_{k=1}^{m} T(G_k) \mapsto \partial T(G)$ where G_k ($k=1, 2,\dots, m$) are Fuchsian equivalents of $(G^\Phi)_{\Delta_k}$ ([1] §5 Corollary 1). Furthermore, if $i((x_1^{(n)}, x_2^{(n)},\dots, x_m^{(n)})) \to \phi \in \partial T(G)$ as $n \to \infty$ for a sequence $\{(x_1^{(n)}, x_2^{(n)},\dots, x_m^{(n)})\} \subset \prod_{k=1}^{m} T(G_k)$ and $\{x_1^{(n)}\}$ converges to a totally degenerate group on $\partial T(G_1)$, then G^Φ is a degenerate group ([1] §6 Theorem 7). From the above consideration, we have $\psi_j = i((x_1^{(j)}, x_2^{(j)},\dots, x_m^{(j)}))$ for each j and may assume that there exists a hyperbolic modular transformation $\chi_1 \in \mathrm{Mod}(G_1)$ such that $x_1^{(j)} = \chi_1(x_1^{(j-1)})$. Therefore, $\{x_1^{(j)}\}_{j=1}^{\infty}$ converges to a totally degenerate group on $\partial T(G_1)$ by [9] Theorem 1, and $\psi_j = \lim_{n_p \to \infty} \tilde{\chi}^{n_p+j}(0)$ converges to a degenerate group $\psi \in \partial T(G)$ as $j \to \infty$. This implies $\psi \in A(\chi; 0)$. By using Lemma 3.2 again, we verify that $A(\chi; x)$ contains a degenerate group for each $x \in T(G)$. Thus we have completely proved the theorem.

Theorem 3.3. *Let* χ *be in* Mod(G). *Then*

χ *is elliptic* $\Leftrightarrow A(\chi; x)$ *consists of quasi-Fuchsian groups for all (or some)* x *in* $T(G)$,

χ *is parabolic* $\Leftrightarrow A(\chi; x)$ *consists of regular b-groups for all (or some)* x *in* $T(G)$,

χ *is pseudo-hyperbolic* $\Leftrightarrow A(\chi; x)$ *consists of cusps and contains a degenerate cusp for all (or some)* x *in* $T(G)$,

χ *is hyperbolic* $\Leftrightarrow A(\chi; x)$ *consists of totally degenerate groups with no accidental parabolic transformations for all (or some)* x *in* $T(G)$.

Proof. If χ is elliptic, then $A(\chi; x)$ is contained in $T(G)$ because χ is of finite order. Hence the statements are easily obtained by Theorem 3.1, Theorem 3.2 and [9] Theorem 1.

§4. Examples

In this section, we shall give examples mentioned at §3 Remark 1).

Let $S_0 = U/G_0$ be a Riemann surface of type $(p, 1)$ and let P_0 be the puncture of S_0. It is known that there exists a quasi-conformal self-mapping f' of S_0 such that $f'(P_0) = P_0$ and f' induces a hyperbolic transformation of Mod(G_0), say χ' (cf. [10] Theorem 2). Furthermore, we may assume that f' fixes each point in a small disk D with the center P_0 (cf. [10] Proof of Proposition 1). Set $S' = S_0 - \bar{D}$, then $f'|S'$ is also an irreducible self-mapping of S'. Let S be the *Schottky double* of S' with respect to $\partial S' = \partial D$ and let φ be the indirectly conformal mapping of S. We define $f = f'$ on $S' \cup \partial S'$ and $= \varphi \circ f' \circ \varphi$ on $S - S' \cup \partial S'$. Then f is a reduced mapping by $C = \{\partial S'\}$ and induces a pseudo-hyperbolic modular transformation χ in Mod(G), where G is a Fuchsian group with $U/G=S$. Considering the proof of Theorem 3.2 and the symmetricity of f, we conclude that $A(\chi;\, x)$ contains a totally degenerate cusp.

In the above case, we set $f_0=f'$ on $S' \cup \partial S'$ and $=id.$ on $S-S' \cup \partial S'$, then f_0 is also a reduced self-mapping of S snd induces a pseudo-hyperbolic transformation χ_0 in Mod (G). By using Lemma 3.1 as in the proof of Theorem 3.1, we verify that $A(\chi_0;\, x)$ contains no totally degenerate group, and contains a partially degenerate group by Theorem 3.2.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] W. Abikoff, On boundaries of Teichmüller spaces and on Kleinian groups. III, Acta Math., **134** (1975), 217–247.

[2] W. Abikoff, Two theorems on totally degenerate Kleinian groups, Amer. J. Math., **98** (1976), 109–118.

[3] W. Abikoff, *Topics in the real analytic theory of Theichmüller space*, Lecture Notes in Math., vol. 820, Springer–Verlag, Berlin and New York, 1980.

[4] L. V. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand, New York, 1966.

[5] L. Bers, On boundaries of Teichmüller spaces and on Kleinian groups. I. Ann. of Math., (2) **91** (1970), 570–600.

[6] L. Bers, Fiber spaces over Teichmüller spaces, Acta Math., **130** (1973), 89–126.

[7] L. Bers, An extremal problem for quasiconformal mappings and a theorem by Thurston, Acta Math., **141** (1978), 73–98.

[8] L. Bers, The action of the modular group on the complex boundary, *Riemann Surfaces and Related Topics*; *Proceeding of the* 1978 *Stony Brook Conference*, Ann. of Math. Studies, No. 97, 1981, pp. 33–52.

[9] L. Bers, On iterates of hyperbolic transformations of Teichmüller space, Amer. J. Math., **105** (1983), 1–11.

[10] I. Kra, On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces, Acta Math., **146** (1981), 231–270.

[11] B. Maskit, On boundaries of Teichmüller spaces and on Kleinian groups. II, Ann. of Math., **91** (1970), 607–639.

[12] H. Shiga, On analytic and geometric properties of Teichmüller spaces, J. Math. Kyoto Univ., **24** (1984), 441–452.