J. Math. Kyoto Univ. (JMKYAZ)
25-4 (1985) 597–617

Square integrable harmonic differentials on arbitrary Riemann surfaces with a finite number of nodes

Dedicated to Professor Yukio Kusunoki on his sixtieth birthday

By

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(Received November 21, 1983)

Introduction

A quasiconformal mapping between two Riemann surfaces induces an isomorphism between the Hilbert spaces of square integrable harmonic differentials on those surfaces, and it is known (cf. [10]) that such an isomorphism preserves several important subspaces. The first purpose of this paper is to generalize such an isomorphism for the case of a deformation from a Riemann surface with a finite number of nodes to another (cf. §1–1°)). Let $(f; R, R_0)$ be an allowable deformation (cf. §2–1°)) from such a surface R to another R_0, then we will show in §2 (Theorem 1) that the mapping H_f naturally induced from f is a bounded linear injection from the Hilbert space $\Gamma_h(R_0)$ of square integrable harmonic differentials on R_0 into $\Gamma_h(R)$, and has similar properties as in the case of quasiconformal mappings. We also give an estimate of the norm of H_f which is coincident with the known one when f is a quasiconformal mapping.

Now there are several investigations concerning on continuity properties of the above Marden–Minda's isomorphisms on the Teichmüller space (cf., for example, [7] and [12]). The second purpose of this paper is to show certain continuity property of H_f on the finitely augmented Teichmüller space $\hat{T}(R^*)$ of arbitrary Riemann surface R^* (cf. §1–1°)). Actually, we will show in §3 (Theorem 4) that $H_{f_k}(\omega)$ converges to ω strongly metrically for every $\omega \in \Gamma_h(R_0)$ when (R_k corresponding to f_k converges to R_0 and) $\{f_k\}_{k=1}$ is an admissible sequence.

Also we state related results on Dirichlet finite harmonic functions. See Theorem 2 in §2–2°) and Theorems 5 and 6 in §3–2°).

§1 is preliminaries, where we give definitions of notions concerning on the finitely augmented Teichmüller space and the spaces of differentials and functions. Theorem 1 is proved in §2–3°), and a general sufficient condition with which a given sequence of differentials converges strongly metrically is given in §3–1°)

(Theorem 3), and proved in §3-3°). As applications we show several results including Theorem 4.

§ 1. Preliminaries

1°) Finitely augmented Teichmüller spaces. A Riemann surface with nodes is a connected complex space R, on which three are k points $N(R)=\{p_j\}_{j=1}^{k}$ $(0\leq k\leq +\infty)$, called *nodes* of R, such that (i) p_j has a neighbourhood homeomorphic to the analytic set $\{z_1\cdot z_2=0, |z_1|<1, |z_2|<1\}$, with p_j corresponding to $(0, 0)$; (ii) each component of $R-N(R)$, called a *part* of R, is an ordinary Riemann surface whose universal covering surface is conformally equivalent to $\{|z|<1\}$ (cf. [4]). A continuous surjection f from a Riemann surface R with nodes onto another R' is called *a deformation* if f^{-1} restricted on $R'-\bar{U}$ is quasiconformal for every neighbourhood U of $N(R')$ and $f^{-1}(p)$ is either a node of R or a simple closed curve on $R-N(R)$ for every p in $N(R')$.

Now let a Riemann surface R^* without nodes be given and consider the set of all deformations from R^* onto another surfaces with or without nodes. Two deformations $(f_1; R^*, R_1)$ and $(f_2; R^*, R_2)$ are called equivalent if there are homeomorphisms g from R_1 onto R_2 and h from R^* onto itself which are homotopic to a conformal mapping from R_1 onto R_2 (namely, a homeomorphism from R_1 onto R_2 which is conformal on $R_1-N(R_1)$) and to the identical mapping of R^*, respectively, such that $g\circ f_1=f_2\circ h$ on R^*. *The augmented Teichmüller space of* R^* is the set of all equivalence classes of deformations from R^*.

In this paper, we consider only surfaces with a finite number of nodes. The subset of the augmented Teichmüller space of R^* consisting of all equivalence classes of deformations from R^* to such surfaces (with a finite number of nodes) is called *the finitely augmented Teichmüller space of* R^* and is denoted by $\hat{T}(R^*)$. A point of $\hat{T}(R^*)$, an equivalence class of a deformation $(f; R^*, R)$ is called *a marked Riemann surface with nodes*, and is denoted simply by a representative $(f; R^*, R)$, or even by R when the marking f is clear from the context. A deformation $f=(f; R_1, R_2)$ from a marked Riemann surface $R_1=(f_1; R^*, R_1)$ with nodes to another $R_2=(f_2; R^*, R_2)$ is called *a marking-preserving deformation from* R_1 *to* R_2 if there are homeomorphisms g and h from R_2 and R^*, respectively, onto themselves which are homotopic to the identical mappings such that $g\circ f\circ f_1=f_2\circ h$. Recall that the subset $\{R\in\hat{T}(R^*)\colon N(R)$ is empty$\}$ is the usual Teichmüller space of R^* (without topology). For the basic results on Teichmüller spaces, see for example, [2].

Next, following Abikoff [1], we define a Hausdorff topology on $\hat{T}(R^*)$. Let $R_0\in\hat{T}(R^*)$ be fixed. For every positive ε and every neighbourhood U of $N(R_0)$ on R_0 we set

$$W(\varepsilon, U)=\{R\in\hat{T}(R^*)\colon \text{There is a marking-preserving deformation } (f; R, R_0) \text{ such that } f^{-1} \text{ is } (1+\varepsilon)\text{-quasiconformal on } R_0-U\}.$$

Taking these sets $\{W(\varepsilon, U)\colon \varepsilon$ is positive and U is a neighoburhood of $N(R_0)\}$ as a fundamental neighborhood system at R_0, we have a Hausdorff topology on $\hat{T}(R^*)$, which is called *the conformal topology*. It is clear that $\hat{T}(R^*)$ equipped with this topology satisfies the first countability axiom, hence we need to consider only sequential convergence. Also note that on the Teichmüller space $T(R^*)$ the conformal topology is coincident with the usual Teichmüller topology.

Convergence of R_k to R_0 on $\hat{T}(R^*)$ is equivalent to the existence of a sequence $\{(f_k;\ R_k,\ R_0)\}_{k=1}^{\infty}$ of marking-preserving deformations f_k from R_k to R_0 such that for every neighbourhood U of $N(R_0)$, (f_k^{-1} is quasiconformal on $R_0-\overline{U}$ for every k from the definifion and) it holds that $\lim_{k\to\infty} K(f_k^{-1},\ R_0-\overline{U})=1$, where and in the sequel, we denote by $K(f,\ E)$ the maximal dilatation of a quasiconformal mapping f on a borel set E. We call such a sequence *an admissible sequence for* $\{R_k\}_{k=1}^{\infty}$ (converging to R_0 on $\hat{T}(R^*)$).

Finally for every $R\in\hat{T}(R^*)$, we call an open neighborhood U of $N(R)$ such that $\overline{U}$ is compact in R, and every component of U contains exactly one node of R and is homeomorphic to $\{z_1\cdot z_2=0\colon |z_1|<1,\ |z_2|<1\}$ *a standard neighbourhood* of $N(R)$. Fix such a U, and map each component of $U-N(R)$ conformally onto $\{0<|z|<2\}$. Then the union $U(\varepsilon)$ of $N(R)$ and all preimages of the part $\{0<|z|<\varepsilon\}$ is a standard neighbourhood of $N(R)$ for every $\varepsilon\in(0,\ 1]$ and $\{U(\varepsilon)\colon \varepsilon\in(0,\ 1]\}$ is a fundamental neighbourhood system of $N(R)$, for $N(R)$ is a finite set. We call this system *the standard system of neighbourhoods* of $N(R)$ with respect to U. Also for every $\varepsilon\in(0,\ 1]$, we denote by $R(\varepsilon,\ U)$ the set $R-N(R)\cup\overline{U(\varepsilon)}$.

Here we give a special example of a standard neighbourhood of $N(R)$, though we use such a neighbourhood only in the proof of Theorem 3 in §3. For every part S of R, let $\{p_j=p_j(S)\}_{j=1}^{n(S)}$ be punctures of S corresponding to $N(R)$. If S admits Green's functions, then letting $g(p,\ p_j)$ be Green's function on $S\cup\{p_j\}$ with the pole p_j for every j, set

$$\text{(i)}\quad b_S(p)=\sum_{j=1}^{n(S)} g(p,\ p_j).$$

If not, then for every pair $\{p_j,\ p_{j'}\}$ in $\{p_j\}_{j=1}^{n(S)}$, fix a harmonic function $g(p;\ p_j,\ p_{j'})$ on S uniquely determined up to constants by the following conditions; $g(p;\ p_j,\ p_{j'})$ is bounded outside any neighbourhood of $\{p_j,\ p_{j'}\}$, and $g(z_j;\ p_j,\ p_{j'})+\log|z_j|$ and $g(z_{j'};\ p_j,\ p_{j'})-\log|z_{j'}|$, respectively, are harmonic in a neighbourhood of $z_j=0$ and $z_{j'}=0$, where z_j and $z_{j'}$ are local parameters near p_j and $p_{j'}$ such that p_j and $p_{j'}$ correspond to $z_j=0$ and $z_{j'}=0$, respectively. Here we may assume that $\{\{p_{2j-1},\ p_{2j}\}\}_{j=1}^{m}$ are all pairs in $\{p_j\}_{j=1}^{n(S)}$ such that p_{2j-1} and p_{2j} correspond to the same node of R_0 (which may be empty). Now set

$$\text{(ii)}\quad b_S(p)=\sum_{j=1}^{m} g(p;\ p_{2j-1},\ p_{2j})+\sum_{j=2m+2}^{n(S)} g(p;\ p_{2m+1},\ p_j),$$

where the second term of the right hand side is empty if $n(S)=2m$ or $2m+1$. In each case, we call $b_S(p)$ *the indicator function* on S. Then for every sufficiently large M, the set $U_{S,M}=\{|b_S(p)|>M\}$, added a suitable doubly connected deleted neighbourhood of $p_{n(S)}$ if $n(S)=2m+1$, is a deleted neighbourhood of $\{p_j\}_{j=1}^{n(S)}$ whose components are doubly connected and relatively compact in R for every part S. The union U_M of $N(R)$ and $\{U_{S,M}\colon S \text{ is a part of } R\}$ is a standard neighbourhood of $N(R)$.

2°) Spaces of differentials and functions. For every R in $\hat{T}(R^*)$, we denote by $\Gamma(R)$ the Hilbert space of square integrable differentials on the finite union $R-N(R)$ of Riemann surfaces with the inner product $(\omega, \omega')_R=\iint_{R-N(R)} \omega \wedge {}^*\bar{\omega}'$, where ${}^*\omega'$ is the conjugate differential of ω'. A differential ω belongs to $\Gamma(R)$ if and only if $\omega|_S \in \Gamma(S)$ for every part S of R, where $\omega|_S$ is ω restricted on S. Various subspaces of $\Gamma(R)$ are defined similarly as in the case of ordinary Riemann surfaces (cf. for example, [3] and [6]). In this paper, we treat only real differentials and consider $\Gamma(R)$ as the real Hilbert space. The only one exception is the space $\Gamma_a(R)$ of all analytic square integrable differentials on $R-N(R)$, which is used in §3.

For the sake of convenience, we recall some definitions. Let $\Gamma_h(R)=\{\omega \in \Gamma(R)$: ω is harmonic$\}$, $\Gamma_{e0}(R)$ be the closure of the space $\{df: f$ is a C^∞-function with compact support on $R-N(R)\}$ in $\Gamma(R)$, and ${}^*\Gamma_{e0}(R)=\{{}^*\omega\colon \omega \in \Gamma_{e0}(R)\}$. Then we have the orthogonal decomposition; $\Gamma(R)=\Gamma_h(R)+\Gamma_{e0}(R)+{}^*\Gamma_{e0}(R)$, and $\Gamma_h(R)+\Gamma_{e0}(R)$ is equal to the space $\Gamma_c(R)$ of all square integrable closed differentials on $R-N(R)$. Next let $\Gamma_{he}(R)=\{\omega \in \Gamma_h(R)$: ω is exact$\}$ and $\Gamma_{hse}(R)=\{\omega \in \Gamma_h(R)$: ω is semi-exact, namely $\int_c \omega=0$ for every dividing curve on every part of $R\}$. Then the subspaces $\Gamma_{h0}(R)$ and $\Gamma_{hm}(R)$, respectively, are characterized by the orthogonal decompositions; $\Gamma_h(R)=\Gamma_{h0}(R)+{}^*\Gamma_{he}(R)=\Gamma_{hm}(R)+{}^*\Gamma_{hse}(R)$. Here we recall other characterizations of these spaces. For every 1-cycle c on $R-N(R)$, there is the uniquely determined harmonic differential $\sigma(c)=\sigma(c, R)$ in $\Gamma_h(R)$ such that $(\omega, \sigma(c))_R=\int_c \omega$ for every ω in $\Gamma_h(R)$, which is called the period reproducer for c. Then $\Gamma_{h0}(R)$ and $\Gamma_{hm}(R)$ are the closure in $\Gamma_h(R)$ of the spaces spaned by

$$\{{}^*\sigma(c)\colon c \text{ is a cycle on } R-N(R)\}, \quad \text{and}$$

$$\{{}^*\sigma(c)\colon c \text{ is a dividing curve on a part of } R\},$$

respectively.

Next for every R in $\hat{T}(R^*)$, let R_G be the union of all parts of R admitting Green's functions, and set

$$HD(R)=\{u\colon u \text{ is a harmonic function on } R_G \text{ such that } du \in \Gamma_h(R)\},$$

where and in the sequel, u is considered to be zero on $R-N(R) \cup R_G$ for every $u \in HD(R)$. Clearly, $\Gamma_{he}(R)=\{du\colon u \in HD(R)\}$. Also corresponding to $\Gamma_{e0}(R)$ and $\Gamma_e(R)=\Gamma_{e0}(R)+\Gamma_{he}(R)$, we consider the spaces $D_0(R)$ and $D(R)$ consisting of all Dirichlet potentials and functions, respectively, on $R-N(R)$, where we set $D_0(S)=D(S)$ on a parabolic surface S. For definitions see [5] Ch. 7. It is known ([5] Satz 7.5 and 7.6) that $D(R)=HD(R)+D_0(R)$ and $\Gamma_{e0}(R)=\{df\colon f \in D_0(R)\}$.

Finally take a pair of points R and R_0 in $\hat{T}(R^*)$ such that there is a marking-preserving deformation $(f; R, R_0)$. Let $L(R, R_0)$ be the set of cycles on R whose representatives are simple closed curves freely homotopic to some $f^{-1}(p)$ with $p \in N(R_0)$ (and with suitable orientations), and set

$$\Gamma_N(R, R_0)=\{\sum_{c_j \in L(R,R_0)} a_j \cdot {}^*\sigma(c_j) \text{ with real } a_j\}, \quad \text{and}$$

$$\Gamma_x(R, R_0)=\{\omega\in\Gamma_x(R): \int_c \omega=0 \text{ for every } c\in L(R, R_0)\},$$

where x stands for h, he, hm, hse or $h0$. It is clear that $\Gamma_N(R, R_0)$ is a finite dimensional (closed) subspace of $\Gamma_h(R, R_0)$ and that $\Gamma_{he}(R, R_0)=\Gamma_{he}(R)$ and $\Gamma_{hm}(R, R_0)=\Gamma_{hm}(R)$. We also note the following

Proposition 1. (i) $\Gamma_{hse}(R, R_0)=\{\omega\in\Gamma_h(R): \int_c \omega=0$ *for every dividing curve* c' *on every component of* $R'\}$, *where* $R'=R-\{f^{-1}(p): p\in N(R_0)\}$ *(which is an open subset of* $R-N(R)$*).*

(ii) $\Gamma_{h0}(R, R_0)$ *and* $\Gamma_{hm}(R, R_0)=\Gamma_{hm}(R)$ *are the closure in* $\Gamma_h(R)$ *of the spaces spaned by* $\{*\sigma(c', R)$: c' *is a cycle on* $R'\}$ *and* $\{*\sigma(c', R)$: c' *is a cycle on* R' *dividing both on every component of* R' *and on every part of* $R\}$, *respectively.*

Proof. (i) Let $\omega\in\Gamma_{hse}(R, R_0)$ and a dividing curve c' on a component S' of R' be given arbitrarily. Then clearly, c' is homologous to a dividing curve c on a part of R (containing S') modulo cycles generated by $L(R, R_0)$ on S'. Hence from the assumption on ω, it holds that $\int_{c'}\omega=\int_c\omega=0$.

Next let $\omega\in\Gamma_h(R)$ satisfy the condition that $\int_{c'}\omega=0$ for every dividing curve c' on every component of R', and c be any dividing curve on a part S of R. Take a boundary component $c_0=f^{-1}(p)$ (with $p\in N(R_0)$) of a component of R' contained in S arbitrarily. Then the algebraic intersection number $c_0\times c$ between c_0 and c is zero, for c is dividing on S. Since c_0 is a simple closed curve, we can find by a standard argument a cycle c_1 homologous to c on S such that $c_0\cap c_1=\emptyset$. Considering the representation by a canonical homology basis on $S-c_0$, we can see that c_1 is a dividing cycle on each (or the one) component of $S-c_0$. Repeat this argument, and we can find a cycle c' homologous to c on S, contained in R', and dividing on every component of R'. Hence from the assumption, it holds that $\int_c\omega=\int_{c'}\omega=0$, which implies that $\omega\in\Gamma_{hse}(R)$. And since every $f^{-1}(p)$ with $p\in N(R_0)$ is a boundary component of some component of R', $\int_c\omega=0$ for every $c\in L(R, R_0)$, namely, $\omega\in\Gamma_{hse}(R, R_0)$.

(ii) Denote by Γ' and Γ'' the closure in $\Gamma_h(R)$ of the spaces spanned by $\{*\sigma(c', R)$: c' is a cycle on $R'\}$ and $\{*\sigma(c', R)$: c' is a cycle on R' dividing both on every component of R' and on every part of $R\}$, respectively. Then it is clear that $\Gamma_{h0}(R, R_0)\supset\Gamma'$ and $\Gamma_{hm}(R)\supset\Gamma''$. First since we have shown in (i) that every dividing curve c on $R-N(R)$ is homologous to a dividing cycle on R', we have that $\Gamma_{hm}(R, R_0)=\Gamma''$.

Next suppose that Γ' is a proper subspace of $\Gamma_{h0}(R, R_0)$. Then from the second characterization of $\Gamma_{h0}(R)$, we can find a curve c' on a part S of R such that $*\sigma(c', R)\in\Gamma_{h0}(R, R_0)-\Gamma'$. Let c_0 be as in (i), then $c'\times c_0=\int_{c_0}*\sigma(c', R)=0$ (, where c_0 is considered as a cycle on S), for $c_0\in L(R, R_0)$. Hence by the same argument as in (i), we can show that c' is homologous to a cycle on R', which implies that $*\sigma(c', R)\in\Gamma'$. This contradiction shows that $\Gamma_{h0}(R, R_0)=\Gamma'$. q.e.d.

§ 2. Isomorphisms induced from allowable deformations

1°) In this section, a marking-preserving deformation $(f;\, R,\, R_0)$ is specially called *an allowable deformation*, to stress the property that f^{-1} is quasiconformal on $R_0-\bar{U}$ for every neighbourhood U of $N(R_0)$.

Now, let an allowable deformation $(f;\, R,\, R_0)$ and a standard neighbourhood U of $N(R_0)$ be given arbitrarily. Take $\omega\in\Gamma_h(R_0)$, then $\omega|_{U-N(R_0)}$ is exact, hence there is a harmonic function $h(p)$ on $U-N(R_0)$ such that $dh=\omega$ on $U-N(R_0)$. Here we can assume that $h(p)$ is continuous on the whole U. We call a continuous function on U, harmonic on $U-N(R_0)$, *a harmonic function on U*. Next take a continuous $e(p)\in D(R_0)$ such that $0\leq e(p)\leq 1$, $e(p)\equiv 0$ on $U(\varepsilon_0)$ and $e(p)\equiv 1$ outside $U(1)$, where $\{U(\varepsilon)\colon 0<\varepsilon\leq 1\}$ is the standard system of neighbourhoods of $N(R_0)$ with respect to U and ε_0 is an arbitrarily given value in $(0,\,1)$. In the sequel, such a function $e(p)$ is called *a U-function*. Now set

$$E(\omega)=e(p)\cdot\omega+h(p)\cdot de$$

with a convention that $h(p)\cdot de=0$ outside of $U(1)$. Then it is easy to see that $E(\omega)\equiv 0$ on $U(\varepsilon_0)$, $E(\omega)\equiv\omega$ outside $U(1)$ and $E(\omega)\in\Gamma_c(R_0)$ (cf. [10] Proposition 4). Hence the pull-back $(E(\omega))\circ f$ by f belongs to $\Gamma_c(R)$, for f^{-1} is quasiconformal on $R_0-U(\varepsilon_0)$ which contains the support of $E(\omega)$. Thus we can define a linear operator H_f from $\Gamma_h(R_0)$ into $\Gamma_h(R)$ by setting $H_f(\omega)$ be the projection of $(E(\omega))\circ f$ into $\Gamma_h(R)$ for every $\omega\in\Gamma_h(R_0)$.

Lemma 1. *H_f is well-defined, namely, $H_f(\omega)$ does not depend on the choice of U, $h(p)$ and $e(p)$. And $H_f(\Gamma_h(R_0))$ is contained in $\Gamma_h(R,\, R_0)$.*

Proof. Take another triple of a standard neighbourhood U', a harmonic function $h'(p)$ on U' such that $dh'=\omega$, and a U' function $e'(p)$. Setting $E'(\omega)=e'(p)\cdot\omega+h'(p)\cdot de'$, let $H'_f(\omega)$ be the projection of $(E'(\omega))\circ f$ into $\Gamma_h(R)$. Here considering the third neighbourhood, it suffices to show that $H_f(\omega)=H'_f(\omega)$ in case that $U(1)\supset U'$. Then it holds that

$$E(\omega)-E'(\omega)=d((e-e')\cdot h)+(h-h')\cdot de'.$$

And since $h-h'$ is constant on each component of U', $g=(e-e')\cdot h-(h-h')(1-e')$ is a continuous bounded Dirichlet potential and $E(\omega)-E'(\omega)=dg$. Because $g(p)$ is constant on each component of a neighbourhood of $N(R_0)$ and has a compact support in $R_0-N(R_0)$, we can see (cf. Proposition 2 below) that $g\circ f\in D_0(R)$, or equivalently, $dg\circ f\in\Gamma_{e0}(R)$, which implies that $H_f(\omega)=H'_f(\omega)$.

Next since $E(\omega)\circ f\equiv 0$ on $f^{-1}(U(\varepsilon_0))$, $H_f(\omega)$ is exact on $f^{-1}(U(\varepsilon_0))$, hence $\int_c H_f(\omega)=0$ for every $c\in L(R,\, R_0)$. Thus $H_f(\Gamma_h(R_0))\subset\Gamma_h(R,\, R_0)$. q.e.d.

Remark 1. (i) If an allowable deformation $f=(f;\, R,\, R_0)$ is a homeomorphism, then for every neighbourhood U of $N(R_0)$ we can construct a (marking-preserving) quasiconformal mapping $\tilde{f}$ from $R_0-N(R_0)$ onto $R-N(R)$ such that $f\circ\tilde{f}$ is the identical mapping outside U(cf. the proof of [14] Lemma 1). And we can easily see that H_f is coincident with the linear operator $(\tilde{f}^{-1})_h^\sharp$ induced by $\tilde{f}^{-1}$ in the sense of Marden–Minda (cf. [10]).

(ii) Allowability is inessential for the definition of H_f. In fact, we have defined H_g for every marking-preserving surjection $(g; R, R_0)$ such that g^{-1} is quasiconformal outside a standard neighbourhood $U_0(=U(\varepsilon_0))$ of $N(R_0)$. Also note that for such a g we can construct an allowable deformation $\tilde{f}$ such that $\tilde{f}^{-1}=g^{-1}$ outside an arbitrarily taken neighbourhood of $N(R_0)$ containing U_0, and then H_g is coincident with $H_{\tilde{f}}$. On the other hand, allowability gives a reasonable estimate of the norm of H_f (Theorem 1 (i)).

Now the main theorem of this section is the following one, which generalizes the Marden–Minda's results (cf. [10] Theorems 6 and 7).

Theorem 1. (i) *Let $(f; R, R_0)$ be an allowable deformation, then the operator H_f induced from f is a bounded linear injection from $\Gamma_h(R_0)$ into $\Gamma_h(R, R_0)$, and $\pi\circ H_f$ is an isomorphism from $\Gamma_h(R_0)$ onto $\Gamma_h(R, R_0)/\Gamma_N(R, R_0)$, where and in the sequel, π is the natural projection from $\Gamma_h(R, R_0)$ onto $\Gamma_h(R, R_0)/\Gamma_N(R, R_0)$. Moreover, it holds that*

$$\|H_f\|^2 \leq \inf_{0<\varepsilon<1}\left(1+\left(\frac{2}{\log(1/\varepsilon)}\right)^{1/2}\right)^2 \cdot K(f^{-1}, R_0(\varepsilon, U)),$$

where U is any given standard neighbourhood of $N(R_0)$.

(ii) *It holds that $\pi\circ H_f(\Gamma_x(R_0))=\pi(\Gamma_x(R, R_0))$, where x stands for he, hse, hm or h0.*

The proof will be given in §2–3°).

Remark 2. In case that f is quasiconformal homeomorphism, that estimate in Theorem 1 (i) reduces to the known one (i.e. $\|H_f\|^2 \leq K(f^{-1}, R_0)$). Also $\pi(\Gamma_x(R, R_0))=\Gamma_x(R)$ for any x as in Theorem 1 (ii), and H_f itself is an isomrophism from $\Gamma_x(R_0)$ onto $\Gamma_x(R)$ as is shown by Marden–Minda.

2°) Corresponding to the operator H_f, we can define a mapping H^f from $HD(R_0)$ into $HD(R)$. Let $(f; R, R_0)$ be an allowable deformation (cf. Reamrk 1 (ii)), U be a standard neighbourhood of $N(R_0)$ and $e(p)$ be a U-function, then $e(p)u(p) \in D(R_0)$ (with a convention that $u(p)\equiv 0$ on $R_0-(R_0)_G \cup N(R_0)$ for every $u \in HD(R_0)$). Since $e\cdot u\equiv 0$ in a neighbourhood of $N(R_0)$, $(e\cdot u)\circ f$ belongs to $D(R)$. We denote by $H^f(u)$ the projection of $(e\cdot u)\circ f$ into $HD(R)$ for every $u \in HD(R_0)$. This mapping H^f is well-defined (, for letting U' and e' be another pair of a standard neighbourhood of $N(R_0)$ and a U'-function, $(e\cdot u)\circ f-(e'\cdot u)\circ f$ is clearly a Dirichlet potential), and can be considered as the correspondance between the Dirichlet solutions. To state this more precisely, let $\bar{S}^D$ be the Royden's compactification of S for every Riemann surface S, and $\Delta(S)$ be the harmonic boundary of $\bar{S}^D$. (For the definition and basic facts on the Royden's compactification, see [5] and [11].) The compactification $\bar{R}^D$ of a Riemann surface R with a finite number of nodes is the union of R and the boundary $\{\bar{S}^D - S \cup \overline{(U \cap S)}: S$ is a part of $R\}$ with the natural topology, where $\overline{(U \cap S)}$ is the closure of $U \cap S$ in $\bar{S}^D$ with a standard neighbourhood U of $N(R_0)$. The harmonic boundary $\varDelta(R)$ of $\bar{R}^D$ is the union of $\varDelta(S)$ of all part S of R. A continuous Dirichlet function g on $R - N(R)$ belongs to $D_0(R)$ if and only if $g \equiv 0$ on $\varDelta(R)$. (Here recall that every continuous Dirichlet function can be extended continuously to $\bar{R}^D - R$). The following fact is essentially well-known.

Proposition 2. *Let $(f; R, R_0)$ be an allowable deformation, then f can be extended to a continuous surjection from $\bar{R}^D$ onto $\overline{(R_0)}^D$. Moreover, f gives a homeomorphism from $\bar{R}^D - R$ and $\varDelta(R)$ onto $\overline{(R_0)}^D - R_0$ and $\varDelta(R_0)$, respectively.*

Proof. Fix a standard neighbourhood U of $N(R_0)$, then by the localization theorem ([11] III. 5C or [5] Satz 9.11) and invariance under quasiconformal mappings ([11] III. 7C), we can see that f gives a homeomorphism from $\bar{R}^D - f^{-1}(U)$ onto $\overline{(R_0)}^D - U$ such that $f(\varDelta(R)) = \varDelta(R_0)$. q. e. d.

Now every $u \in HD(R_0)$ have the continuous boundary function on $\varDelta(R_0)$, which is denoted also by u, and the Dirichlet solution $H^R_{u \circ f}$ for the continuous boundary function $u \circ f$ is coincident with $H^f(u)$, for $(e \cdot u) \circ f$ has the same boundary function $u \circ f$ on $\varDelta(R)$, hence $(e \cdot u) \circ f - H^R_{u \circ f} \in D_0(R)$. Here concerning Theorem 1, we note the following

Theorem 2. *Let $(f; R, R_0)$ be an allowable deformation, then the linear mapping H^f is a bijection from $HD(R_0)$ onto $HD(R)$, and it holds that*

$$\pi \circ d \circ H^f(u) = \pi \circ H_f(du) \qquad \text{for every } u \in HD(R_0).$$

Proof. Let U and $e(p)$ be as above and take $\varepsilon_0 \in (0, 1)$ so that $e(p) \equiv 0$ on $U(\varepsilon_0)$. For every $v \in HD(R)$, $v \circ f^{-1}$ restricted on $R_0(\varepsilon_0, U)$ is a continuous Dirichlet function, bounded in a neighbourhood of the support of de, hence $e \cdot (v \circ f^{-1})$ can be considered as an element of $D(R_0)$. We denote by $I^f(v)$ the projection of $e \cdot (v \circ f^{-1})$ into $HD(R_0)$. Then similarly as before, I^f is well-defined and $I^f(H^f(u))$ is the projection of $e(e \cdot u - g \circ f^{-1})$ for every $u \in HD(R_0)$, where $g = (e \cdot u) \circ f - H^f(u)$ is a continuous Dirichlet potential on $R - N(R)$. Because of Proposition 2, $e \cdot g \circ f^{-1} \in D_0(R_0)$ and it is clear that $e \cdot e \cdot u - u \in D_0(R_0)$. Hence it holds that $I^f(H^f(u)) = u$ for every $u \in HD(R_0)$. Similarly we can show that $H^f(I^f(v)) = v$ for every $v \in HD(R)$. Thus $(H^f)^{-1}$exists and equal to I^f, which show the first assertion.

Next to prove the second assertion, we need the following

Lemma 2. *Let $(f; R, R_0)$ and U be as above and $u(p)$ be the characteristic function on $U - N(R_0)$ of a fixed component of $U - N(R_0)$ (namely, $u \equiv 0$ on $U - N(R_0)$ except for the fixed one component of $U - N(R_0)$, where $u \equiv 1$). Then $d(e \cdot u) \circ f$ with a U-function $e(p)$ can be considered as an element of $\Gamma_c(R)$ and the projection of $d(e \cdot u) \circ f$ into $\Gamma_h(R)$ belongs to $\Gamma_N(R, R_0)$.*

Proof. It is clear that $(e \cdot u) \circ f$ is a continuous bounded Dirichlet function on $f^{-1}(U)$ and is constant on a neighourhood of every boundary component of $f^{-1}(U - N(R_0))$. Hence $\omega = d(e \cdot u) \circ f \in \Gamma_c(f^{-1}(U))$ and has a compact support in

$f^{-1}(U)$, so we can consider ω as an element of $\Gamma_c(R)$.

Next we can easily see that the projection of ω into $\Gamma_h(f^{-1}(U))$ is equal to $^*\sigma(c, f^{-1}(U))$ with some $c \in L(R, R_0)$ (oriented suitably), hence it holds that

$$\int_c \tilde{\omega} = (\tilde{\omega}, \sigma(c, f^{-1}(U)))_{f^{-1}(U)} = (\tilde{\omega}, -{}^*d(e \cdot u)\circ f)_{f^{-1}(U)}$$

$$= (\tilde{\omega}, -{}^*\omega)_R \qquad \text{for every} \quad \tilde{\omega} \in \Gamma_h(R).$$

Thus the projection of ω into $\Gamma_h(R)$ is equal to $^*\sigma(c, R)$ with the same $c \in L(R, R_0)$ as above, which shows the assertion. q.e.d.

Now returning to the proof of Theorem 2, let $u \in HD(R_0)$ and a harmonic function $h(p)$ on U such that $dh = du$ on U be fixed. Then we have that

$$dH^f(u) - H_f(du) = d(e \cdot u)\circ f - (e \cdot du + h \cdot de)\circ f + dg = ((u-h) \cdot de)\circ f + dg$$

with a suitable $g \in D_0(R)$. Since $u - h$ is constant on every component of $U - N(R_0)$, we conclude by Lemma 2 that $dH^f(u) - H_f(du) \in \Gamma_N(R, R_0)$, which implies that $\pi \circ dH^f(u) = \pi \circ H_f(du)$. q.e.d.

3°) **The proof of Theorem 1.** First we show the following

Lemma 3. *Let $(f; R, R_0)$ be as in Theorem 1, then H_f is bounded linear, and for any standard neighbourhood U of $N(R_0)$, it holds that*

$$\|H_f\|^2 \leq \inf_{0<\varepsilon<1} \left(1 + \left(\frac{2}{\log(1/\varepsilon)}\right)^{1/2}\right)^2 \cdot K(f^{-1}, R_0(\varepsilon, U)).$$

Proof. Fix a standard neighbourhood U of $N(R_0)$ arbitrarily, and for every $\varepsilon \in (0, 1)$, take as a U-function the function $e_\varepsilon(p)$ corresponding to

$$e_\varepsilon(z) = \log(|z|/\varepsilon)/\log(1/\varepsilon) \quad \text{on} \quad \{\varepsilon < |z| < 1\},$$

$$\equiv 0 \quad \text{on} \quad \{|z| \leq \varepsilon\}, \quad \text{and} \quad \equiv 1 \quad \text{on} \quad \{1 \leq |z| < 2\}$$

under a mapping from every component of $U - N(R_0)$ onto $\{0 < |z| < 2\}$. Then by a simple computation, we have that $\|de_\varepsilon\|_{R_0}^2 = 2\pi/\log(1/\varepsilon)$, where and in the sequel, $\|\omega\|_R$ implies the Dirichlet norm of a differential ω on R (i.e. $\|\omega\|_R^2 = (\omega, \omega)_R$).

Next fix $\omega \in \Gamma_h(R_0)$, and take a harmonic function $h(p)$ on U with $dh = \omega$ so that $h(p) = 0$ for every $p \in N(R_0)$. Then we can show that

$$|h(p)|^2 \leq \frac{1}{\pi} \|dh\|_U^2 \qquad \text{for every} \quad p \in U(1).$$

Hence we have that

$$\|E_\varepsilon(\omega)\|_{R_0} (= \|e_\varepsilon \cdot \omega + h de_\varepsilon\|_{R_0})$$

$$\leq (1 + \|de_\varepsilon\|_{R_0}/\sqrt{\pi}) \cdot \|\omega\|_{R_0} \leq \left(1 + \left(\frac{2}{\log(1/\varepsilon)}\right)^{1/2}\right) \cdot \|\omega\|_{R_0}.$$

Also since the support of $E_\varepsilon(\omega)$ is contained in $\overline{R_0(\varepsilon, U)}$, we can show that

$$\|E_\varepsilon(\omega)\circ f\|_R^2 \leq K(f^{-1}, R_0(\varepsilon, U)) \cdot \|E_\varepsilon(\omega)\|_{R_0}^2.$$

Thus noting that $\|H_f(\omega)\|_R \leq \|E_\varepsilon(\omega)\circ f\|_R$ and ω is arbitrary, we have the desired estimate on the norm of H_f. q.e.d.

Next to prove Theorem 1, the following lemmas are crucial, which generalize [10] Theorem 4. (Also see [7].)

Lemma 4. *For every cycle c on $R_0 - N(R_0)$, it holds that*

$$H_f(*\sigma(c)) - *\sigma(f^{-1}(c)) \in \Gamma_N(R, R_0).$$

Proof. We may assume that c is a simple closed curve. Take a relatively compact doubly connected region W in $R_0 - N(R_0)$ which contains c. Then we can find continuous $g_c \in D(W)$ and $g \in D_0(R_0)$ such that $0 \leq g_c \leq 1$, $g_c \equiv 0$ and $\equiv 1$, respectively, in a neighbourhoods of one and the other boundary components of W and $*\sigma(c) - dg_c = dg$ with a convention that $dg_c \equiv 0$ on $R_0 - W$.

Now take a standard neighbourhood U so that $W \cap U = \emptyset$ and let $h(p)$ be a harmonic function on U such that $dh = *\sigma(c)$, then we have that

$$E(*\sigma(c))\circ f = (e(dg_c + dg) + hde)\circ f = dg_c\circ f + d(e \cdot g)\circ f + ((h-g)de)\circ f.$$

Because $(e \cdot g)\circ f \in D_0(R)$ and $h - g$ is constant on every component of $U - N(R_0)$, we have by Lemma 2 that $H_f(*\sigma(c)) - dg_c\circ f \in \Gamma_N(R, R_0) + \Gamma_{e0}(R)$.

On the other hand, similarly as in the proof of Lemma 2 we have that

$$(\omega', *(dg_c\circ f))_R = (\omega', *(dg_c\circ f))_{f^{-1}(W)} = (\omega', -\sigma(f^{-1}(c), f^{-1}(W)))_{f^{-1}(W)}$$

$$= -\int_{f^{-1}(c)} \omega' \qquad \text{for every } \omega' \in \Gamma_h(R).$$

Thus we conclude that $H_f(*\sigma(c)) - *\sigma(f^{-1}(c)) \in \Gamma_N(R, R_0)$. q.e.d.

Lemma 5. *For every ω and ω' in $\Gamma_h(R_0)$, it holds that*

$$(H_f(\omega), *H_f(\omega'))_R = (\omega, *\omega')_{R_0}.$$

In particular, for every $\omega \in \Gamma_h(R_0)$ and every cycle c on $R_0 - N(R_0)$ we have

$$\int_{f^{-1}(c)} H_f(\omega) = \int_c \omega.$$

Proof. Let E be an operator as before, then it holds that

$$(H_f(\omega), *H_f(\omega'))_R = (E(\omega)\circ f, *(E(\omega')\circ f))_R$$

$$= \iint_{R-N(R)} -E(\omega)\circ f \wedge E(\omega')\circ f = \iint_{R_0 - N(R_0)} -E(\omega) \wedge E(\omega')$$

$$= (E(\omega), *E(\omega'))_{R_0} = (\omega, *\omega')_{R_0}$$

for every ω and ω' in $\Gamma_h(R_0)$. In particular, for every $\omega \in \Gamma_h(R_0)$ and every cycle c on $R_0 - N(R_0)$, we have from Lemma 4 that

$$\int_{f^{-1}(c)} H_f(\omega)=(H_f(\omega),\ \sigma(f^{-1}(c)))_R$$

$$=(H_f(\omega),\ -{}^*H_f({}^*\sigma(c)))_R=(\omega,\ \sigma(c))_{R_0}=\int_c \omega,$$

for $H_f(\omega)\in\Gamma_h(R, R_0)$ by Lemma 1 and ${}^*\Gamma_N(R, R_0)$ is orthogonal to $\Gamma_h(R, R_0)$ from the definition. q.e.d.

Finally we construct the inverse mapping of $\pi\circ H_f$ explicitly. Again fix U and $e(p)$ as before. Let $V=f^{-1}(U)$, then $\omega|_V$ is exact for every $\omega\in\Gamma_h(R, R_0)$, hence there is a bounded harmonic function v on V such that $dv=\omega$ on V. Then $v\circ f^{-1}$ is also a continuous bounded Dirichlet function on $U-U(\varepsilon)$ for every $\varepsilon\in(0, 1)$, and we can consider the differential

$$F(\omega)=e\cdot(\omega\circ f^{-1})+v\circ f^{-1}\cdot de$$

as an element of $\Gamma_c(R_0)$. We denote by $I_f(\omega)$ the projection of $F(\omega)$ into $\Gamma_h(R_0)$. Then we have the following

Lemma 6. *The mapping I_f is well-defined. And it holds that $I_f(\omega)=0$ for every $\omega\in\Gamma_N(R, R_0)$, hence I_f can be considered as a mapping from $\Gamma_h(R, R_0)/\Gamma_N(R, R_0)$ into $\Gamma_h(R_0)$.*

Proof. Let U', e' and v' be another triple such as used in the definition of F, then noting that $(v'-v)\circ f^{-1}$ is constant on each component of U' (, where we assume that $U'\subset U(1)$ as in the proof of Lemma 1), we can show similarly as in the proof of Lemma 1 that I_f is well-defined.

Next let U, e and V be as above. Then for every $c\in L(R, R_0)$, we can find a component V_c of V, a continuous $g_c\in D(V_c)$ and a continuous $g\in D_0(R)$ such that $0\le g_c\le 1$, $g_c\equiv 0$ and $\equiv 1$, respectively, in a neighbourhood of one and the other boundary components of V_c, and ${}^*\sigma(c)-dg_c=dg$ with a convention that $dg_c\equiv 0$ on $R-V_c$. Take ε_0 so that $e(p)\equiv 0$ on $U(\varepsilon_0)$, and let U_1 be the component of $U-U(\varepsilon_0)$ on which $e(g_c\circ f^{-1})$ is not a Dirichlet potential. Then similarly as in the proof of Lemma 2 (using U_1 instead of $f^{-1}(U)$), we can show that $\int_{c_1}\omega=(\omega,\ -{}^*d(e\cdot g_c\circ f^{-1}))_{R_0}$ for every $\omega\in\Gamma_h(R_0)$, and hence $d(e\cdot g_c\circ f^{-1})-{}^*\sigma(c_1, R_0)\in\Gamma_{e0}(R_0)$, where c_1 is the cycle in U_1 corresponding to c. Since ${}^*\sigma(c_1)=0$ and $e\cdot(g\circ f^{-1})\in D_0(R_0)$, we conclude that $F({}^*\sigma(c))\in D_0(R_0)$, or equivalently, $I_f({}^*\sigma(c))=0$. q.e.d.

Lemma 7. (i) $I_f(H_f(\omega))=\omega$ *for every* $\omega\in\Gamma_h(R_0)$.

(ii) $H_f(I_f(\omega))-\omega\in\Gamma_N(R, R_0)$ *for every* $\omega\in\Gamma_h(R, R_0)$.

Proof. Fix $\omega\in\Gamma_h(R_0)$, U, $e(p)$ and h as before. Then there is a continuous $g\in D_0(R)$ such that $dg=H_f(\omega)-E(\omega)\circ f$, where $E(\omega)=e\cdot\omega+hde$. Take another standard neighbourhood U' of $N(R_0)$ such that $e(p)\equiv 0$ on U', and set $V=f^{-1}(U')$. Since $E(\omega)\circ f\equiv 0$ on V, we can take g as v for $H_f(\omega)$ in the definition of F. Hence letting e' be a U'-function, we have that

$$F'(H_f(\omega)) = e'H_f(\omega)\circ f^{-1} + g\circ f^{-1}de' = d(e'\cdot g\circ f^{-1}) + E(\omega).$$

Since $e'\cdot g\circ f^{-1}\in D_0(R_0)$ and $E(\omega)-\omega\in\Gamma_{e0}(R_0)$, we have the assertion (i).

Next fix $\omega\in\Gamma_h(R, R_0)$, and let g be a continuous Dirichlet potential on R_0 such that $I_f(\omega)=F(\omega)+dg$ $(=e\cdot\omega\circ f^{-1}+v\circ f^{-1}de+dg)$. Then $dg=I_f(\omega)$ on U', for $e(p)\equiv 0$ on U'. Hence for every harmonic function $h'(p)$ on U' such that $dh'=I_f(\omega)$, it holds that $g-h'$ is a linear combination of the characteristic functions of components of $U'-N(R_0)$. Since $e'(p)\equiv 1$ on the support of $e(p)$, we have that

$$\begin{aligned} E'(I_f(\omega))-\omega\ (&=(e'\cdot I_f(\omega)+h'\cdot de')\circ f-\omega) \\ &= e\circ f\cdot\omega+v\cdot de\circ f+(e'\cdot dg)\circ f+(h'\cdot de')\circ f-\omega \\ &= -d((1-e\circ f)\cdot v)+d(e'\cdot g)\circ f+((h'-g)\cdot de')\circ f. \end{aligned}$$

Here it is clear that $-(1-e\circ f)\cdot v$ and $(e'\cdot g)\circ f$ belong to $D_0(R)$. Also from above we know that $((h'-g)de')\circ f=d((h'-g)e')\circ f$, and by Lemma 2 we conclude that $H_f(I_f(\omega))-\omega\in\Gamma_N(R, R_0)$. q.e.d.

Now we give the proof of Theorem 1. First $H_f(\Gamma_h(R_0))\subset\Gamma_h(R, R_0)$ by Lemma 1. Next, if $\pi\circ H_f(\omega)=\pi\circ H_f(\omega')$, then $I_f(H_f(\omega))=I_f(H_f(\omega'))$ by Lemma 6. Hence $\omega=\omega'$ by Lemma 7 (i). which shows that $\pi\circ H_f$ is injective, hence so is also H_f. Finally by Lemma 7 (ii) we have that $\pi\circ H_f(\Gamma_h(R_0))=\Gamma_h(R, R_0)/\Gamma_N(R, R_0)$. Thus by Lemma 3 $\pi\circ H_f$ is a bounded linear bijection from $\Gamma_h(R_0)$ onto $\Gamma_h(R, R_0)/\Gamma_N(R, R_0)$, and the closed graph theorem implies that $\pi\circ H_f$ is also an isomorphism. Thus we have the assertion (i).

The assertion (ii) for Γ_{he} has been already shown in Theorem 2, for $\pi\circ H_f(\Gamma_{he}(R_0))=\pi\circ H_f(d(HD(R_0)))=\pi\circ d(H^f(HD(R_0)))=\pi\circ d(HD(R))=\pi(\Gamma_{he}(R))$. For Γ_{h0}, note that c is a cycle on $R_0-N(R_0)$ if and only if $f^{-1}(c)$ is one on R' $(=R-\{f^{-1}(p)\colon p\in N(R_0)\})$. Since $\pi\circ H_f({}^*\sigma(c))=\pi({}^*\sigma(f^{-1}(c)))$, hence ${}^*\sigma(c)=I_f({}^*\sigma(f^{-1}(c)))$ for every cycle c on $R_0-N(R_0)$ by Lemmas 4 and 7 (i), and H_f and I_f are continuous as are shown above, we see from Proposition 1 (ii) that $\pi\circ H_f(\Gamma_{h0}(R_0))\subset\pi(\Gamma')=\pi(\Gamma_{h0}(R, R_0))$, where Γ' is as in the proof of Proposition 1, and that $I_f(\Gamma_{h0}(R, R_0))=I_f(\Gamma')\subset\Gamma_{h0}(R_0)$. Here Lemma 7 (ii) implies that $\pi(\Gamma_{h0}(R, R_0))=\pi\circ H_f(I_f(\Gamma_{h0}(R, R_0)))\subset\pi\circ H_f(\Gamma_{h0}(R_0))$, hence we conclude that $\pi\circ H_f(\Gamma_{h0}(R_0))=\pi(\Gamma_{h0}(R, R_0))$.

For the case of Γ_{hm}, note that every dividing curve c on a part S of R_0, there is a dividing curve c_1 on S such that $\sigma(c, R_0)=\sigma(c_1, R_0)$ and one of component of $S-c_1$ contains no punctures of S corresponding to nodes of R_0. Clearly, $f^{-1}(c_1)$ is a dividing curve both on the component of R' and on the part of R containing it. Hence similarly as above, we see that $\pi\circ H_f(\Gamma_{hm}(R_0))\subset\pi(\Gamma'')=\pi(\Gamma_{hm}(R, R_0))$, where Γ'' is as in the proof of Proposition 1. On the other hand, for every dividing cycle c' on a component of R', $f(c')$ is a dividing curve on the corresponding part of R_0, hence it is clear that $I_f(\Gamma_{hm}(R, R_0))=I_f(\Gamma'')\subset\Gamma_{hm}(R_0)$. Thus the assertion for Γ_{hm} follows from Lemma 7 (ii) as above.

Finally, Lemma 5 it holds that $\int_{c'}H_f(\omega)=\int_{f(c')}\omega=0$ for every $\omega\in\Gamma_{hse}(R_0)$ and every dividing curve c' on every component of R', hence $H_f(\Gamma_{hse}(R_0))\subset\Gamma_{hse}(R, R_0)$ by Proposition 1 (i). Also by Lemmas 5 and 7 (ii) we see that

$$\int_c I_f(\omega)=\int_{f^{-1}(c)} H_f(I_f(\omega))=\int_{f^{-1}(c)} \omega=0$$

for every $\omega\in\Gamma_{hse}(R, R_0)$ and every dividing curve c on every part of R_0, for $\int_{f^{-1}(c)} {}^*\sigma(c'')=0$ for every $c''\in L(R, R_0)$. Hence $I_f(\Gamma_{hse}(R, R_0))\subset\Gamma_{hse}(R_0)$, and again by Lemma 7 (ii) we have the assertion for Γ_{hse}.

Thus we have proved Theorem 1.

§3. A continuity property of H_f on $\hat{T}(R^*)$.

1°) In this section, we investigate a continuity property of the operator H_f on the finitely augmented Teichmüller space $\hat{T}(R^*)$. For this purpose, first we state such a property for holomorphic abelian differentials. We consider the Hilbert space $\Gamma_a(R)$ of square integrable holomorphic abelian differentials on $R-N(R)$ for every $R\in\hat{T}(R^*)$, and for every marking-preserving deformation $(f; R, R_0)$, set $\Gamma_a(R, R_0)=\{\omega\in\Gamma_a(R)$: $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega$ belong to $\Gamma_h(R, R_0)\}$. Also we denote by d_zh the holomorphic differential $dh+i^*dh$ for every real harmonic function h on $R-N(R)$.

Fix a sequence $\{R_k\}_{k=1}^{\infty}$ converging to R_0 in $\hat{T}(R^*)$, and an admissible sequence $\{(f_k; R_k, R_0)\}_{k=1}^{\infty}$ of marking-preserving deformations (cf. §1–1°)) once for all. Let $\theta_k\in\Gamma_a(R_k)$ be given for every k, then we deform $\theta_k\circ f_k^{-1}$ as follows. For every part S of R_0, let $\{p_j\}_{j=1}^{n(S)}$ be punctures corresponding to $N(R_0)$, and $g(p, p_j)$ and $g(p; p_j, p_{j'})$ be as in §1–1°). Set $a_{j,k}^S=\int_{f_k^{-1}(c_j)}\theta_k$, where c_j is the cycle corresponding to a simple closed curve which surrounds only p_j on S. Now consider the indicator function $b_S(p)$ on S defined in §1–1°) for every part S, and define the holomorphic differential $\theta_{S,k}$ by

(i) $$\theta_{S,k}=\frac{i}{2\pi}\sum_{j=1}^{n(S)} a_{j,k}^S\cdot d_zg(\cdot, p_j),\ \text{or}$$

(ii) $$\theta_{S,k}=\frac{i}{2\pi}\sum_{j=1}^{m} a_{2j-1,k}^S\cdot d_zg(\cdot; p_{2j-1}, p_{2j})$$
$$-\frac{i}{2\pi}\sum_{j=2m+2}^{n(S)} a_{j,k}^S\cdot d_zg(\cdot; p_{2m+1}, p_j),$$

according as $b_S(p)$ is defined by (i) or (ii) in §1–1°), for every k and S. Then we note the following

Lemma 8. *For every k, j and S, it holds that*

$$\int_{c_j}\theta_{S,k}=a_{k,j}^S.$$

Proof. Because $\int_{c_j} d_zg(\cdot, p_j)=\int_{c_j} d_zg(\cdot; p_j, p_{j'})=-\int_{c_{j'}} d_zg(\cdot; p_j, p_{j'})=-2\pi i$, we need only to show that *) $a_{2j,k}^S=-a_{2j-1,k}^S$ and **) $a_{2m+1,k}^S=-\sum_{j=2m+2}^{n(S)} a_{j,k}^S$ (or $=0$) if $n\geq 2m+2$ (or $n=2m+1$) in the case (ii).

First *) is clear, for c_{2j-1} and c_{2j} bound a component of a standard neighbourhood of $N(R_0)$, hence $f_k^{-1}(c_{2j-1})$ is freely homotopic to $-f_k^{-1}(c_{2j})$ on R_k for every k.

Next because S admits no Green's functions in the case (ii), $\sum_{n=1}^{n(S)}(-c_j)$ bounds a parabolic end in S, hence so is $\sum_{j=1}^{n(S)} -f_k^{-1}(c_j)$ no R_k. Then it is well-known (cf. [6] Corollary 8.9) that $\sum_{j=1}^{n(S)}\int_{f_k^{-1}(c_j)}\omega=0$ for every $\omega\in\Gamma_a(R_k)$, which, togather with *), implies **). q.e.d.

Now fix a standard neighbourhood U of $N(R_0)$ and $\varepsilon_0\in(0, 1)$ arbitrarily. Then from the assumption f_k^{-1} is quasiconformal on $R_0(\varepsilon_0, U)$ for every k. We define a differential ω_k on R_0 by setting

$$\omega_k|_S=\theta_k\circ f_k^{-1}|_S-\theta_{S,k}$$

on every part S of R_0. Then by Lemma 8, we can see that ω_k is exact on each component of $U(1)\cap R_0(\varepsilon_0, U)$. Hence we can find a continuous bounded Dirichlet function g_k on $U(1)\cap R_0(\varepsilon_0, U)$ such that $dg_k=\omega_k$. So for every k, we can consider the closed differential

$$\tilde{F}\theta_k=e(p)\cdot\omega_k+g_k\cdot de,$$

where $e(p)$ is a U-function such that $e(p)\equiv 0$ on $U(\varepsilon_0)$. It is clear that $\tilde{F}\theta_k$ is square integrable and $\tilde{F}\theta_k\equiv\omega_k$ on $R_0(1, U)$. We denote by $R\theta_k$ and $I\theta_k$ the projections of $\mathrm{Re}\,\tilde{F}\theta_k$ and $\mathrm{Im}\,\tilde{F}\theta_k$, respectively, into $\Gamma_h(R_0)$. (We can show similarly as in the proof of Lemma 1, that R and I are well-defined.) Note that if $\theta_k\in\Gamma_a(R_k, R_0)$, then it is clear that $R\theta_k=I_{f_k}(\mathrm{Re}\,\theta_k)$ and $I\theta_k=I_{f_k}(\mathrm{Im}\,\theta_k)$, where I_f is as in §2. (Also note that H_f and I_f can be defined and bounded linear for any surjection $(g; R, R_0)$ such that g^{-1} is quasiconformal outside some standard neighbourhood of $N(R_0)$, cf. Remark 1 (ii).) Now we can prove the following

Theorem 3. *Let R_k converge to R_0 on $\hat{T}(R^*)$, $\{(f_k; R_k, R_0)\}_{k=1}^{\infty}$ be an admissible sequence of marking-preserving deformations, and $\theta_k\in\Gamma_a(R_k)$ be given for every k. Suppose that*

1) *$\{\|\theta_k\|_{R_k}\}_{k=1}^{\infty}$ is a bounded sequence, and*
2) *$(R\theta_k-\mathrm{Re}\,\theta_0, *(I\theta_k-\mathrm{Im}\,\theta_0))_{R_0}=0$ for every k.*

Then θ_k converges to θ_0 strongly metrically (cf. [14] §2), namely, for every neighbourhood U of $N(R_0)$ it holds that

$$\lim_{k\to\infty}\|\theta_k\circ f_k^{-1}-\theta_0\|_{R_0-U}=0.$$

The proof will be given in 3°) of this section. Here we note the following

Corollary 1. *Let $\{R_k\}_{k=1}^{\infty}$ and $\{(f_k; R_k, R_0)\}_{k=1}^{\infty}$ be as in Theorem 3, and $\theta_k\in\Gamma_a(R_k, R_0)$ be given for every k. Suppose that*

1) *$\{\|\theta_k\|_{R_k}\}_{k=1}^{\infty}$ is a bounded sequence, and*
2) $(\mathrm{Re}\,\theta_k-H_{f_k}(\mathrm{Re}\,\theta_0), *(\mathrm{Im}\,\theta_k-H_{f_k}(\mathrm{Im}\,\theta_0))_{R_k}=0$

for every k, then θ_k converges to θ_0 strongly metrically.

Proof. First for every k, let $\omega_1^k = \mathrm{Re}\,\theta_k - H_{f_k}(I_{f_k}(\mathrm{Re}\,\theta_k))$ and $\omega_2^k = \mathrm{Im}\,\theta_k - H_{f_k}(I_{f_k}(\mathrm{Im}\,\theta_k))$. Then by Lemma 7 (ii), ω_1^k and ω_2^k belong to $\Gamma_N(R_k, R_0)$. Because $H_{f_k}(\Gamma_h(R_0)) \subset \Gamma_h(R_k, R_0)$ by Lemma 1, $\Gamma_N(R_k, R_0) \subset \Gamma_h(R_k, R_0)$ from the definition, $^*\Gamma_N(R_k, R_0)$ is orthogonal to $\Gamma_h(R_k, R_0)$, and $R\theta_k = I_{f_k}(\mathrm{Re}\,\theta_k)$ and $I\theta_k = I_{f_k}(\mathrm{Im}\,\theta_k)$ in this case, we see from Lemma 5 that

$$
\begin{aligned}
&(\mathrm{Re}\,\theta_k - H_{f_k}(\mathrm{Re}\,\theta_0),\ {}^*(\mathrm{Im}\,\theta_k - H_{f_k}(\mathrm{Im}\,\theta_0)))_{R_k} \\
&\quad = (\omega_1^k + H_{f_k}(I_{f_k}(\mathrm{Re}\,\theta_k) - \mathrm{Re}\,\theta_0),\ {}^*\omega_2^k + {}^*H_{f_k}(I_{f_k}(\mathrm{Im}\,\theta_k) - \mathrm{Im}\,\theta_0))_{R_k} \\
&\quad = (\omega_1^k, {}^*\omega_2^k)_{R_k} + (H_{f_k}(I_{f_k}(\mathrm{Re}\,\theta_k) - \mathrm{Re}\,\theta_0),\ {}^*H_{f_k}(I_{f_k}(\mathrm{Im}\,\theta_k) - \mathrm{Im}\,\theta_0))_{R_k} \\
&\quad = (I_{f_k}(\mathrm{Re}\,\theta_k) - \mathrm{Re}\,\theta_0,\ {}^*(I_{f_k}(\mathrm{Im}\,\theta_k) - \mathrm{Im}\,\theta_0))_{R_0} \\
&\quad = (R\theta_k - \mathrm{Re}\,\theta_0,\ {}^*(I\theta_k - \mathrm{Im}\,\theta_0))_{R_0}.
\end{aligned}
$$

Thus the condition 2′) implies the condition 2) in Theorem 3, and the assertion follows from Theorem 3. q.e.d.

Remark 3. (i) Several kinds of continuity of (, and more quantitative results on) differentials under quasiconformal mappings have been investigated by several authors. See, for example, [7], [9], [13] and references of them.

(ii) We can also show the strongly metrical convergence of certain differentials of the third kind. And in general, the strongly metrical convergence with some reasonable conditions implies the geometrical convergence even on the finitely augmented Teichmüller spaces, as in the case of the Teichmüller spaces treated in [14]. These investigations will be appeared elsewhere. (See also [15].)

2°) Applications. Returning to real harmonic differentials, we have the following

Theorem 4. *Let $\{R_k\}_{k=0}^\infty$ and $\{f_k\}_{k=1}^\infty$ be as in Theorem 3. For arbitrarily given $\omega_0 \in \Gamma_h(R_0)$, $H_{f_k}(\omega_0)$ converges to ω_0 strongly metrically.*

Proof. Let θ_k be the element of $\Gamma_a(R_k)$ such that $\mathrm{Re}\,\theta_k = H_{f_k}(\omega_0)$ for every k. Then by Theorem 1 (i), $\{\|\mathrm{Re}\,\theta_k\|_{R_k}\}_{k=1}^\infty$ is a bounded sequence, hence so is $\{\|\theta_k\|_{R_k}\}_{k=1}^\infty$. Next we show that $R\theta_k = I_{f_k}(\mathrm{Re}\,\theta_k) = \omega_0$. Because $\mathrm{Re}\,\theta_k \in \Gamma_h(R_k, R_0)$, every $a_{j,k}^S$ is purely imaginary, hence $\mathrm{Re}\,\theta_{S,k}$ is a real linear combination of $dg(\cdot, p_j)$ or $dg(\cdot\,; p_j, p_{j'})$. In each case, we can see that $\mathrm{Re}\,\theta_{S,k}$ is exact and there is a harmonic function $u_{S,k}$ on S such that $du_{S,k} = \mathrm{Re}\,\theta_{S,k}$ and $e(p)\cdot u_{S,k}(p) \in D_0(S)$ for every S and k, where $e(p)$ is any U-function with a standard neighbourhood U of $N(R_0)$. Hence $\mathrm{Re}\,\tilde{F}\theta_k - I_{f_k}(\mathrm{Re}\,\theta_k) \in D_0(R_0)$, or equivalently, $R\theta_k = I_{f_k}(\mathrm{Re}\,\theta_k) = \omega_0$.

Thus we have that $(R\theta_k - \mathrm{Re}\,\theta_0, {}^*(I\theta_k - \mathrm{Im}\,\theta_0))_{R_0} = (0, {}^*(I\theta_k - \mathrm{Im}\,\theta_0))_{R_0} = 0$ for every k, and from Theorem 3 we have that θ_k converges to θ_0 strongly metrically, hence so does $H_{f_k}(\omega_0)$ to ω_0. q.e.d.

Here we note the following

Proposition 3. *Let $\{R_k\}_{k=0}^\infty$ and $\{f_k\}_{k=1}^\infty$ be as in Theorem 3. Fix $p \in N(R_0)$*

and let c_k be the cycle on R_k corresponding to $f_k^{-1}(p)$ for every k. Then $\sigma(c_k)$ converges to zero strongly metrically.

Proof. Set $\theta_0\equiv 0$ and $\theta_k=\sigma(c_k)+i^*\sigma(c_k)$ for every k. Since $\mathrm{Im}\,\theta_k\in\Gamma_N(R_k, R_0)$ $\subset\Gamma_h(R_k, R_0)$, we can show similarly as in the proof of Theorem 4 that $I\theta_k=I_{f_k}(\mathrm{Im}\,\theta_k)$, which is equal to zero by Lemma 6. Hence we have that $(R\theta_k-\mathrm{Re}\,\theta_0, {}^*(I\theta_k-\mathrm{Im}\,\theta_0))_{R_0}=(R\theta_k, {}^*I_{f_k}(\mathrm{Im}\,\theta_k))_{R_0}=(R\theta_k, 0)_{R_0}=0$ for every k. On the other hand, it is well-known that $\|\sigma(c_k)\|_{R_k}^2$ is the extremal length of the homology class of c_k on R_k, which tends to 0 as k tends to $+\infty$. Thus the assertion follows from Theorem 3. q.e.d.

Proposition 4. *Let $\{R_k\}_{k=0}^{\infty}$ and $\{f_k\}_{k=1}^{\infty}$ be as in Theorem 3, and a cycle c on $R_0-N(R_0)$ be given arbitrarily. Then $\sigma(f_k^{-1}(c))$ converges to $\sigma(c)$ strongly metrically.*

Proof. Set $\theta_k=\sigma(f_k^{-1}(c))+i^*\sigma(f_k^{-1}(c))$ for every k. Since we can easily see that $\{\|\sigma(f_k^{-1}(c))\|_{R_k}\}_{k=1}^{\infty}$ is a bounded sequence, so is $\{\|\theta_k\|_{R_k}\}_{k=1}^{\infty}$. Next Lemmas 4 and 7 (i) implies that $I_{f_k}({}^*\sigma(f_k^{-1}(c)))={}^*\sigma(c)$, and as in the proof of Theorem 4, we can show that $I\theta_k=I_{f_k}({}^*\sigma(f_k^{-1}(c)))$, for ${}^*\sigma(f_k^{-1}(c))\in\Gamma_h(R_k, R_0)$. Hence it holds that

$$(R\theta_k-\mathrm{Re}\,\theta_0, {}^*(I\theta_k-\mathrm{Im}\,\theta_0))_{R_0}=(R\theta_k-\mathrm{Re}\,\theta_0, 0)_{R_0}=0$$

for every k. Thus the assertion follows from Theorem 3. q.e.d.

Now for harmonic functions, as a variant of Theorem 3, we have the following generalization of Shiga's result ([12] Theorem 1).

Theorem 5. *Let $\{R_k\}_{k=0}^{\infty}$ and $\{f_k\}_{k=1}^{\infty}$ be as in Theorem 3, and $u_0\in HD(R_0)$ be given arbitrarily. Then $dH^{f_k}(u_0)$ converges to du_0 strongly metrically, and $H^{f_k}(u_0)\circ f_k^{-1}$ converges to u_0 locally uniformly on $(R_0)_G$.*

Corollary 2 (cf. [12] Theorem 4). *Let $\{R_k\}_{k=0}^{\infty}$ and $\{f_k\}_{k=1}^{\infty}$ be as in Theorem 3, and a continuous bounded function h on $\Delta(R_0)$ be given. Let u_k be the Dirichlet solution of the boundary function $h\circ f_k$ on $\Delta(R_0)$ for every k. Then $u_k\circ f_k^{-1}$ converges to u_0 locally uniformly on $(R_0)_G$.*

Proof. Using Theorem 5, we have the assertion by the same argument as in the proof of [12] Corollary 1. q.e.d.

Finally we note that by the same argument as in the proof of Theorems 3 and 5, we can show the following

Theorem 6. *Let $\{R_k\}_{k=0}^{\infty}$ and $\{f_k\}_{k=1}^{\infty}$ be as in Theorem 3, and $u_k\in HD(R_k)$ be given for every k. Suppose that*

(1′) *$\{u_k\}_{k=0}^{\infty}$ are uniformly bounded, and*

(2) *$(R\theta_k-du_0, {}^*(I\theta_k-{}^*du_0))_{R_0}=0$ for every k,*

where $\theta_k=du_k+i^*du_k$ for every k. Then θ_k converges to θ_0 strongly metrically, and $u_k\circ f_k^{-1}$ converges to u_0 locally uniformly on $(R_0)_G$.

The proofs of Theorems 5 and 6 will be given at the end of 3°).

3°) **The proofs of Theorems.** Let $\{R_k\}_{k=0}^{\infty}$ and $\{f_k\}_{k=1}^{\infty}$ be as in Theorem 3. Let S be a part of R_0, and consider the mapping $f_k^S = f_k^{-1}|_S$ from S onto $f_k^{-1}(S)$. Attaching punctured disks along each borders of $f_k^{-1}(S)$ in R_k, we have a surface S_k homeomorphic to S for every k. Then we can easily construct a sequence of homeomorphisms $\tilde{f}_k^S$ from S onto S_k such that for every neighbourhood U of $N(R_0)$ we can find a $k(U)$ satisfying that $\tilde{f}_k^S = f_k^S$ on $S-U$ for every $k \geq k(U)$. Because $\{f_k\}_{k=1}^{\infty}$ is an admissible sequence, $\{\tilde{f}_k^S\}_{k=1}^{\infty}$ is a weakly admissible sequence (in the sense defined in [14] §1). So, letting G^S be a Fuchsian group acting on the unit disk U_1 associated with S, G_k^S be the point in the reduced Teichmüller space $T^{\#}(G^S)$ corresponding to $(\tilde{f}_k^S;\ S,\ S_k)$ for every k, and F_k^S be the lift of $\tilde{f}_k^S$ on U_1 with respect to G_0^S $(=G^S)$, we know the following

Lemma 9 ([14] Lemma 3). *F_k^S converges to the identical mapping locally uniformly on U_1 and G_k^S converges to G_0^S elementwise for every S.*

Next for given $\theta_k \in \Gamma_a(R_k)$, we consider $\theta_k^S = \theta_k|_{f_k^{-1}(S)}$ as a differential on $f_k^{-1}(S) \subset S_k$. Then, though θ_k^S is defined not on the whole S_k, we can show by the same argument as in the proof of [14] Proposition 1 and Corollary 3 the following

Lemma 10. *Let $a_k^S(z)dz$ be the lift of θ_k^S in U_1 with respect to G_k^S.*

(i) *If $\{\|\theta_k\|_{R_k}\}_{k=1}^{\infty}$ is a bounded sequence, then $\{a_k^S(z)\}_{k=1}^{\infty}$ are locally uniformly bounded, hence is a normal family, for every S.*

(ii) *If θ_k converges to θ_0 strongly metrically (with $\{f_k\}_{k=1}^{\infty}$), then $a_k^S(z)$ (and also $a_k^S(F_k^S(z))$) converges to $a_0^S(z)$ locally uniformly on U_1 for every S.*

Lemma 11. *If $a_k^S(z)$ converges to $a_0^S(z)$ locally uniformly on U_1 for every S, then θ_k converges to θ_0 metrically (namely, $\lim_{k\to\infty} \|\theta_k \circ f_k^{-1} - \theta_0\|_E = 0$ for every compact set E in $R_0 - N(R_0)$), and $\int_{f_k^{-1}(c)} \theta_k$ converges to $\int_c \theta_0$ for every cycle c on $R_0 - N(R_0)$.*

Next we note the following

Lemma 12. *Fix a sufficiently large M so that the set U_M defined in §1–1°) is a standard neighbourhood of $N(R_0)$, and let $e_\varepsilon(p)$ be the U_M-function defined as in the proof of Lemma 3 with $U = U_M$. Then it holds that*

(i) $\|(e_\varepsilon)_z dz\|_{R_0}^2 = \|(e_\varepsilon)_{\bar{z}} d\bar{z}\|_{R_0}^2 = \pi/\log(1/\varepsilon)$, *and*

(ii) $\|e_\varepsilon(\sum_S \theta_{S,k})\|_{R_0}^2 \leq (2/\pi) \cdot A_k^2 \cdot (M + \log(1/\varepsilon) + B)$,

where $A_k = \sum_S \left(\sum_{j=1}^{n(S)} |a_{j,k}^S| \right)$, *and B is a constant depending only on R_0* and U.

Proof. The equality (i) follows by a simple computation. To show (ii), fix a standard neighbourhood U of $N(R_0)$ arbitrarily. Then from the definition of $g(p, p_j)$ and $g(p;\ p_j,\ p_{j'})$, we can find a constant B_0 depending only on R_0 and U such that $|g| \leq B_0$ on $S - U_g$ for every S and $g = g(p, p_j)$ or $g(p;\ p_j,\ p_{j'})$ according as S admits Green's functions or not, where U_g is (the union of) the component(s) of $U - N(R_0)$ containing p_j (or p_j and $p_{j'}$). Also for every g appeared in b_S, it holds that $|g| \leq |b_S - g| + |b_S| \leq |b_S| + NB_0$ on $U_g \cap S$ except for the component of $U_g - N(R_0)$ corresponding to p_{2m+1}, (which exists only if $g = g(p;\ p_{2m+1},\ p_j)$ and) where we also have that $|g|\leq(n(S)-2m-1)|g|\leq|b_S-(n(S)-2m-1)\cdot g|+|b_S|\leq|b_S|+NB_0$, for $g(p;\, p_{2m+1},\, p_{j'})=g(p;\, p_{2m+1},\, p_j)+g(p;\, p_j,\, p_{j'})$. Here N is, at most, twice the number of nodes of R_0. Thus we conclude that, for every S and g appeared in b_S, it holds that $|g|\leq|b_S|+NB_0$ on S.

On the other hand, it is well-known that $\|d_z g\|^2_{\{|g|\leq M_1\}}=4\pi M_1$ or $8\pi M_1$ for every positive M_1, according as $g=g(p,\, p_j)$ or $g(p;\, p_j,\, p_{j'})$. Hence we have that $\|e_\varepsilon\cdot(\sum_S \theta_{S,k})\|^2_{R_0}\leq\|\sum_S \theta_{S,k}\|^2_{R_0-U_M(\varepsilon)}\leq(2/\pi)\cdot(M+\log(2/\varepsilon)+NB_0)\cdot A_k^2$, for $|b_S|\leq M+\log(2/\varepsilon)$ on $R_0-U_M(\varepsilon)$. q.e.d.

Now suppose that the condition (i) in Theorem 3 holds. Then by Lemma 10 we can find a subsequence, say $\{k'\}$, such that $a^S_{k'}(z)$ converges to a holomorphic function $a^S(z)$ locally uniformly on U_1 for every S. And we can see that $a^S(z)dz$ is G^S_0-invariant for every S, hence defines a holomorphic differential θ'_0 on $R_0-N(R_0)$. Here by Lemma 11, it holds that $\|\theta'_0\|^2_E=\lim_{k'\to\infty}\|\theta_{k'}\circ f^S_{k'}\|^2_E\leq\lim_{k'\to\infty}K(f^{-1}_{k'},\, E)\cdot\|\theta_{k'}\|^2_{f^{-1}_{k'}(E)}\leq\sup_k\|\theta_k\|^2_{R_k}(=C)<+\infty$ for every compact set E in $R_0-N(R_0)$. Since E is arbitrary, we have that $\|\theta'_0\|_{R_0}\leq C^{1/2}$, and hence $\theta'_0\in\Gamma_a(R_0)$. Also we can show the following

Lemma 13. (i) $\lim_{k'\to\infty}A_{k'}=0$,

(ii) *For every* $\varepsilon\in(0,\, 1)$, *we can find a continuous* $h_{k'}(p)\in D(U_M-\overline{U_M(\varepsilon/2)})$ *such that* $dh_{k'}=\omega_{k'}$ *and* $|h_{k'}(p)|^2\leq\frac{1}{\pi}(\|\theta'_0\|^2_{\bar{U}_M}+1)$ *on* $U_M(1)-\overline{U_M(\varepsilon)}$ *for every* $k'\geq k'(\varepsilon)$ *with a sufficiently large* $k'(\varepsilon)$ *depending on* ε.

Proof. The assertion (i) follows from Lemma 11, for every $a^S_{j,k'}$ converges to $\int_{c_j}\theta'_0=0$. To show (ii), recall that, for the holomorphic function g_0 on U_M such that $dg_0=\theta'_0$ and $g_0(p)=0$ for every $p\in N(R_0)$, it holds that $|g_0(p)|^2\leq\frac{1}{2\pi}\|\theta'_0\|^2_{\bar{U}_M}$ on $U_M(1)$. Fix S and a component W of $U_M\cap S$, and let $\tilde{W}$ be a component of the lift of W on U_1 with respect to G^S_0 and $\tilde{g}_0(z)$ be the lift of $g_0(p)$ on $\tilde{W}$ (i.e. $\frac{d\tilde{g}_0(z)}{dz}=a^S(z)$). Then since $a^S_{k'}(z)$ converges to $a^S(z)$ locally uniformly on U_1, we can find, for any given $\varepsilon\in(0,\, 1)$ and compact set E in $\tilde{W}$ covering $(U_M(1)-\overline{U_M(\varepsilon)})\cap W$, a holomorphic function $\tilde{g}_{k'}(z)$ on the lift of $f^{-1}_{k'}(U_M-\overline{U_M(\varepsilon/2)})$ such that $\frac{d\tilde{g}_{k'}(z)}{dz}=a^S_{k'}(z)$ for every sufficiently large k' and that $\tilde{g}_{k'}(z)$, hence $\tilde{g}_{k'}(F_{k'}(z))$ converges to $\tilde{g}_0(z)$ uniformly on E. Also, for every k', letting $b^S_{k'}(z)dz$ be the lift of $\theta_{S,k'}$ with respect to G^S_0, we can find a holomorphic function $\tilde{g}_{S,k'}(z)$ with $\frac{d\tilde{g}_{S,k'}(z)}{dz}=b^S_{k'}(z)$ converging to zero locally uniformly on U_1, for so does $b^S_{k'}(z)$ by the above (i). Hence we can find an $k'(\varepsilon)$ such that $|g_{k'}(F_{k'}(z))|^2\leq\frac{1}{2\pi}\left(\|\theta'_0\|^2_{\bar{U}_M}+\frac{1}{2}\right)$ and $|\tilde{g}_{S,k'}(z)|^2\leq\frac{1}{4\pi}$ on the above E for every $k'\geq k'(\varepsilon)$.

Finally set $h_{k'}(z)=\tilde{g}_{k'}(F_{k'}(z))-\tilde{g}_{S,k'}(z)$ on E for every $k'\geq k'(\varepsilon)$. From the construction we can show that $\tilde{h}_{k'}(z)$ can be projected to a continuous Dirichlet function $h_{k'}(p)$ on $(U_M-\overline{U_M(\varepsilon/2)})\cap W$, and that $dh_{k'}=\omega_{k'}$ for every k'. Also from above, $h_{k'}$ satisfies the desired estimate, hence we have the assertion (ii). q.e.d.

Lemma 14. *Let $\{k'\}$ be as above, and suppose that the condition* (ii) *in Theorem* 3 *holds. Then $\theta_{k'}$ converges to θ_0 strongly metrically* (*, hence $\theta'_0 \equiv \theta_0$*).

Proof. Using $h_{k'}(p)$ in Lemma 13 (ii), set $F_\varepsilon(\theta_{k'}) = e_\varepsilon \cdot \omega_{k'} + h_{k'} \cdot de_\varepsilon$. Then we see from the condition (ii) that

$$0 = 2i(\mathrm{Re}\,(F_\varepsilon(\theta_{k'}) - \theta_0),\ {}^*\mathrm{Im}\,(F_\varepsilon(\theta_{k'}) - \theta_0))_{R_0}$$

$$= -(F_\varepsilon(\theta_{k'}) - \theta_0,\ {}^*(F_\varepsilon(\theta_{k'}) - \theta_0))_{R_0} = \iint_{R_0} (F_\varepsilon(\theta_{k'}) - \theta_0) \wedge \overline{(F_\varepsilon(\theta_{k'}) - \theta_0)},$$

which implies, by writing $\theta_{k'} \circ f_{k'}^{-1} = a_{k'} \circ f_{k'}^{-1} \cdot ((f_{k'}^{-1})_z dz + (f_{k'}^{-1})_{\bar z}\, d\bar z)$, that

$$\| e_\varepsilon \cdot (a_{k'} \circ f_{k'}^{-1} \cdot (f_{k'}^{-1})_z dz - \sum_S \theta_{S,k'}) - \theta_0 + h_{k'} \cdot (e_\varepsilon)_z dz \|_{R_0}^2$$

$$= \| e_\varepsilon \cdot a_{k'} \circ f_{k'}^{-1} \cdot (f_{k'}^{-1})_{\bar z}\, d\bar z + h_{k'} \cdot (e_\varepsilon)_{\bar z}\, d\bar z \|_{R_0}^2.$$

Hence by Lemmas 12 and 13 (ii) we have that

$$\| e_\varepsilon \cdot a_{k'} \circ f_{k'}^{-1} \cdot (f_{k'}^{-1})_z dz - \theta_0 \|_{R_0} \le \| e_\varepsilon \cdot a_{k'} \circ f_{k'}^{-1} \cdot (f_{k'}^{-1})_{\bar z}\, d\bar z \|_{R_0} + A(k', \varepsilon, M)$$

for every $k' \ge k'(\varepsilon)$, where $A(k', \varepsilon, M) = ((2/\pi)(M + \log\ (1/\varepsilon) + B))^{1/2} \cdot A_{k'} + 2\,(\|\theta'_0\|_{U_M} + 1)\ (\log\ (1/\varepsilon))^{-1/2}$. Then by a standard argument (cf. the proof of [8] Theorem 1), for every $k' \ge k'(\varepsilon)$, we can show that

$$(*) \qquad \| e_\varepsilon \cdot \theta_{k'} \circ f_{k'}^{-1} - \theta_0 \|_{R_0}$$

$$\le (K(f_{k'}^{-1}, R_0(\varepsilon, U_M)) - 1) \cdot \|\theta_0\|_{R_0} + K(f_{k'}^{-1}, R_0(\varepsilon, U_M)) \cdot A(k', \varepsilon, M)$$

Now let a neighbourhood U of $N(R_0)$ be given arbitrarily. Then take M so that $U_M \subset U$, and fix $\varepsilon \in (0, 1)$ arbitrarily. Since $\{f_{k'}\}$ is an admissible sequence and $\lim_{k' \to \infty} A_{k'} = 0$ by Lemma 13 (i), we have from $(*)$ that

$$\lim_{k' \to \infty} \| \theta_{k'} \circ f_{k'}^{-1} - \theta_0 \|_{R_0 - U}$$

$$\le \lim_{k' \to \infty} \| e_\varepsilon \cdot \theta_{k'} \circ f_{k'}^{-1} - \theta_0 \|_{R_0} \le 2(\|\theta'_0\|_{R_0} + 1) \cdot (\log\,(1/\varepsilon))^{-1/2}.$$

Since ε is arbitrary, we conclude that $\lim_{k' \to \infty} \| \theta_{k'} \circ f_{k'}^{-1} - \theta_0 \|_{R_0 - U} = 0$. And since U is also arbitrary, we have the assertion. q. e. d.

Proof of Theorem 3. Under the assumptions in Theorem 3, every subsequence of θ_k also contains a subsequence such as $\{\theta_{k'}\}$ defined before Lemma 13, which converges to θ_0 strongly metrically by Lemma 14. Thus θ_k itself converges to θ_0 strongly metrically. q. e. d.

In the course of the above proof, we have shown the following estimate, which seems to be interesting in itself.

Proposition 5. *Under the same assumptions as in Theorem* 3, *fix a sufficiently large M and $\varepsilon \in (0, 1)$, then for every sufficiently large k it holds that*

$$\|e_\varepsilon\cdot\theta_k\circ f_k^{-1}-\theta_0\|_{R_0}$$
$$\leq(K(f_k^{-1},\ R_0(\varepsilon,\ U_M))-1)\cdot\|\theta_0\|_{R_0}+K(f_k^{-1},\ R_0(\varepsilon,\ U_M))\cdot[((2/\pi)(M+\log(1/\varepsilon)$$
$$+B))^{1/2}\cdot A_k+2(\|\theta_0\|_{U_M}+1)(\log(1/\varepsilon))^{-1/2}]$$

where e_ε *and* A_k *are as in Lemma* 12.

Proof of Theorem 5. First from Theorem 2, $dH^{f_k}(u_0)-H_{f_k}(du_0)\in\Gamma_N(R_k,\ R_0)$, and hence $I_{f_k}(dH^{f_k}(u_0))=I_{f_k}(H_{f_k}(du_0))=du_0$ by Lemmas 6 and 7 (i). Also since $dH^{f_k}(u_0)\in\Gamma_h(R_k,\ R_0)$, we can show as in the proof of Theorem 4 that $R\theta_k=I_{f_k}(dH^{f_k}(u_0))=du_0$ for every k, where $\theta_k=d_zH^{f_k}(u_0)$.

Now from the definition, it is clear that $\{\|\theta_k\|_{R_k}\}_{k=1}^\infty$ is a bounded sequence and it holds that $(R\theta_k-du_0,\ {}^*(I\theta_k-{}^*du_0))_{R_0}=(0,\ {}^*(I\theta_k-{}^*du_0))_{R_0}=0$ for every k. Thus we have the first assertion from Theorem 3.

Next fix a sufficiently large M and $\varepsilon\in(0,\ 1)$ arbitrarily, then as in the proof of Lemma 13 (ii), we can find a continuous (real) $u_k\in D(U_M-\overline{U_M(\varepsilon/2)})$ such that $du_k=dH^{f_k}(u_0)\circ f_k^{-1}$ and $|u_k(p)|^2\leq\frac{1}{\pi}(\|\theta_0\|_{R_0}^2+1)$ on $U_M(1)-\overline{U_M(\varepsilon)}$ for every k. Let e_ε be as in Lemma 12, $F_\varepsilon(dH^{f_k}(u_0))=e_\varepsilon\cdot dH^{f_k}(u_0)\circ f_k^{-1}+u_k\cdot de_\varepsilon$, and $\alpha_{k,\varepsilon}=F_\varepsilon(dH^{f_k}(u_0))-du_0$, then from above we have that $\alpha_{k,\varepsilon}\in\Gamma_{e0}(R_0)$, hence $\alpha_{k,\varepsilon}=dg_{k,\varepsilon}$ with some $g_{k,\varepsilon}\in D_0(R_0)$ for every k. Also since $dg_{k,\varepsilon}=d(e_\varepsilon\cdot H^{f_k}(u_0)\circ f_k^{-1}-u_0)$ on R_0-U_M, we have that $g_{k,\varepsilon}=H^{f_k}(u_0)\circ f_k^{-1}-u_0$ on $(R_0)_G-U_M$. Moreover, Proposition 5 with Lemmas 12 (i) and 13 (i) implies that

$$\lim_{k\to\infty}\|dg_{k,\varepsilon}\|_{R_0}\leq\lim_{k\to\infty}(\|e_\varepsilon\cdot\theta_k\circ f_k^{-1}-\theta_0\|_{R_0}+\|u_k\cdot de_\varepsilon\|_{R_0})$$
$$\leq 2(\|\theta_0\|_{R_0}+1)(\log(1/\varepsilon))^{-1/2}+(\|\theta_0\|_{R_0}+1)(\log(1/\varepsilon))^{-1/2}.$$

Hence setting $\varepsilon_n=\exp(-2^{2n+4}(\|\theta_0\|_{R_0}+1)^2)$ for every n, we can find a subsequence $\{k(n)\}_{n=1}^\infty$ such that $\|dg_{k(n),\varepsilon_n}\|_{R_0}\leq 2^{-n}$ for every n. Then by [5] Hilfsatz 7.8, we conclude that $\lim_{n\to\infty} g_{k(n),\varepsilon_n}=0$ for almost every p on $(R_0)_G$, which implies that $\lim_{n\to\infty} H^{f_{k(n)}}(u_0)\circ f_{k(n)}^{-1}(p)=u_0(p)$ for almost every p on $(R_0)_G-U_M$. Thus (, since $\theta_{k(n)}=d_zH^{f_{k(n)}}$ converges to $\theta_0=d_zu_0$ strongly metrically), we can show, by using Lemmas 9 and 10 (ii), that $H^{f_{k(n)}}(u_0)\circ f_{k(n)}^{-1}$ converges to u_0 locally uniformly not only on $(R_0)_G-U_M$, but also on $(R_0)_G$.

Now we have shown that every subsequence of $\{H^{f_k}(u_0)\circ f_k^{-1}\}_{k=1}^\infty$ contains a subsequence converging to u_0 locally uniformly on $(R_0)_G$, which shows the second assertion of Theorem 5. q.e.d.

Proof of Theorem 6. First because $|u_k|\leq M_1$ for every k with some M_1 by the condition (i'), the lifts $\tilde{u}_k^S(z)$ of u_k on U_1 with respect to G_k^S form a normal family for every S. Also $a_k^S(z)=\frac{\partial\tilde{u}_k^S}{\partial z}(z)$ are locally uniformly bounded on U_1, and hence make a normal family. Thus the first conclusion of Lemma 10 still holds in this case.

Next for any sequence $\{k'\}$ such as defined before Lemma 13, we may assume that $\tilde{u}_{k'}^S(z)$ also converges to a harmonic function $\tilde{u}^S(z)$ such that $|\tilde{u}^S(z)|\leq M_1$ locally uniformly on U_1. Then using the same notation as in the proof of Lemma 13, it holds that $|\mathrm{Re}\,\tilde{g}_0(z)| \leq |\tilde{u}^S(z)| + M_1 \leq 2M_1$ on $\tilde{W}$ for every $\tilde{W}$. Hence we can find a constant $C(M_1)$ depending only on M_1 such that $|g_0(p)| \leq C(M_1)$ on $U_M(1)$. From this estimate of g_0, we can have a similar assertion as in Lemma 13 (ii) (using $C(M_1)^2$ instead of $\frac{1}{\pi} \|\theta_0'\|^2_{U_M}$).

Because the other parts of the proof of Theorem 3 are available without change also to this case, we have the assertion. q.e.d.

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