J. Math. Kyoto Univ. (JMKYAZ)
26-2 (1986) 177-190

On factorization of certain entire and meromorphic functions

Dedicated to Professor Yukio Kusunoki on his 60th birthday

By

Hironobu URABE

Introduction.

In the previous paper [20], we investigated the problem concerning the uniqueness of the factorization (under composition) of certain entire functions, as well as the primeness etc. And in [22], we introduced the notion of primeness in divisor sense (for entire functions) and studied about it.

In this paper, using results obtained in [20] and [22], we shall deal with some related problems on factorization of certain entire and meromorphic functions (which are closely related to periodic functions). Among others, we are mainly concerned about entire functions which belong to $\boldsymbol{J}(b)$ or $\boldsymbol{L}(b)$ (cf. § 1). For instance, an entire function $f(z)$ belongs to $\boldsymbol{J}(b)$ $(b\neq 0)$, if $f(z)$ can be expressed as $f(z)=cz+H(z)$, where c is a non-zero constant and $H(z+b)\equiv H(z)$ is, periodic, entire.

In § 1, we recall some definitions, terminologies and several known facts, needed later. In § 2, we will generalize and complement the former results on $\boldsymbol{J}(b)$ and $\boldsymbol{L}(b)$. § 3 contains a result on the deficiency and factorizability, relating to the work due to Gross-Osgood-Yang [9] (originally Goldstein [4]). In § 4, we shall consider a factorization problem concerning the (iterative) functional equation: $f\circ f=g\circ g$, and prove that certain entire functions f and g satisfying this equation are identical (see Theorem 6). In § 5, applying a result on primeness in divisor sense, we shall exhibit certain meromorphic functions which are prime.

§ 1. Preliminaries.

1.1. Definitions and terminologies. For a meromorphic function $F(z)$, the factorization under composition operation such as

$$F(z)=f\circ g(z)=f(g(z)) \qquad (1)$$

has been considered, where f and g are meromorphic functions. Of course, when f is transcendental, then g should be entire. Then, by definition ([5]), F is called to be *prime* (*pseude-prime*; *right-prime*; *left-prime*), if, for every factorization as above, we can always deduce the following assertion: f or g

Communicated by Prof. Kusunoki, Nov. 2, 1984.

is linear (f or g is rational; g is linear whenever f is transcendental; f is linear whenever g is transcendental, respectively).

When F is entire and both factors f (left-factor) and g (right-factor) of F under (1) are restricted to entire functions, then it is called that the factorization is to be in entire sense. Thus, we may use the phrase "prime in entire sense" instead of "prime" etc. It is well-known that any non-periodic entire function is prime if it is prime in entire sense (cf. [6]).

Typically, the functions $z\cdot e^z$ and $z+e^z$ are prime. And it might be noteworthy that, especially, the latter function has occupied the significant position in factorization theory (see [20] etc.).

Suppose that a non-constant entire function $F(z)$ has two factorizations:

$$F(z)=f_1\circ f_2\circ\cdots\circ f_m(z)=g_1\circ g_2\circ\cdots\circ g_n(z)$$

into non-linear entire factors f_j and g_k. If $m=n$ and if with suitable linear polynomials T_j $(1\leqq j\leqq n-1)$ the relations

$$f_1(z)=g_1\circ T_1^{-1}(z),\quad f_2(z)=T_1\circ g_2\circ T_2^{-1}(z),\ \cdots,\ f_n(z)=T_{n-1}\circ g_n(z)$$

hold identically, then the two factorizations are called to be *equivalent* (in entire sense). If every factorization of $F(z)$ into non-linear, prime, entire factors is equivalent, then we say that $F(z)$ is *uniquely factorizable.*

Of course, prime functions are considered to be uniquely factorizable. In [20], it is proved, for example, that the function $F(z)=(z+e^z)\circ(z+e^z)$ is uniquely factorizable.

Now, an entire function $F(z)$ ($\not\equiv$ const.), with zeros, is called to be *prime in divisor sense* (*pseudo-prime in divisor sense*; *right-prime in divisor sense*; *left-prime in divisor sense*), if, for every identical relation such as

$$F(z)=f(g(z))\cdot e^{A(z)}, \tag{2}$$

where f, g ($\not\equiv$ const.) and A are entire functions, we can deduce the following assertion: f has just one simple zero or g is a linear polynomial (f has only a finite number of zeros or g is a polynomial; g is a linear polynomial whenever f has an infinite number of zeros; f has just one simple zero whenever g is transcendental, respectively). As is shown in [22], the function $z+e^z$ is also prime in divisor sense.

1.2. Two fundamental Theorems on $J(b)$ and $L(b)$. Following Koont [13], we have considered (in [20]) the two classes of entire functions, defined by

$$J(b)=\{cz+H(z);\ H \text{ is entire, periodic, with period } b\ (H(z+b)\equiv H(z)) \text{ and } c \text{ is a non-zero constant.}\}$$

and

$$L(b)=\{z\cdot e^{H_1(z)}+H_2(z);\ e^{H_1} \text{ and } H_2 \text{ are entire, periodic, with period } b.\}$$

Evidently, $J(b)\subset L(b)$. For example, $z+e^{e^z}$ and $(z+e^z)\circ(z+e^z)$ belong to $J(2\pi i)$, and $(z\cdot e^z)\circ(z+e^z)$ belongs to $L(2\pi i)$. Concerning the right and left factors into which the functions in $\boldsymbol{J}(b)$ or $\boldsymbol{L}(b)$ are factorized, the following theorems will be used later.

Theorem A. *Let $F(z)\in \boldsymbol{L}(b)$ $(b\neq 0)$ and $F(z)=f(g(z))$ with non-linear entire functions f and g, then we have $f\in \boldsymbol{L}(b')$ for some $b'\neq 0$ and $g\in \boldsymbol{J}(b)$.*

Theorem B. *Let $F(z)\in \boldsymbol{J}(b)$ $(b\neq 0)$ and $F(z)=f(g(z))$ with non-linear entire functions f and g, then we have $f\in \boldsymbol{J}(b')$ for some $b'\neq 0$ and $g\in \boldsymbol{J}(b)$.*

Remark. In the above Theorems, if $g\in \boldsymbol{J}(b)$ has the expression $g(z)=cz+H(z)$ with $H(z+b)\equiv H(z)$, then we know that $b'=bc$ (cf. proof in [20], around p. 102).

1.3. Concerning the primeness in divisor sense, we obtained in [22] the following results, which are used in § 5.

Theorem C. *Let $F(z)$ be one of the following three types of functions*; (1) $z+P(e^z)$, *where P is a polynomial,* (2) $P(z)+Q(e^z)$, *where P and Q are two non-constant polynomials such that, for any natural number k and constant c, the function $e^{-kz}\cdot[Q(e^z)+c]$ is non-constant,* (3) $P(z)\cdot Q(e^z)$, *where P and Q are two non-constant polynomials such that $[P(z+c)]^2$ is not an even function for any constant c and $Q(z)$, with $Q(0)\neq 0$, has only simple zeros. Then $F(z)$ is prime in divisor sense.*

§ 2. Supplement of former results.

2.1. At first, we recall the following fact.

Theorem D (cf. [20] Cor. 8). *The function $z+e_m(z)$ is prime, where $e_m(z)$ is the m-th iterate of e^z ($e_m(z)=\exp[e_{m-1}(z)]$, $e_0(z)=z$).*

As a generalization of this fact, we shall show the following result, by using Theorem B.

Theorem 1. *Let $F(z)=z+h(e^z)$, where h is an entire function. Assume that* $\exp[h(z)]$ *is periodic, with some period b $(\neq 0)$. Then $F(z)$ is prime.*

Indeed, this Theorem is a direct consequence of the Propositions 1 and 2, below. We wish to note here that we can show also that, if h is an entire function with the order $\rho(h)<1$, then the function $z+h(e^z)$ is prime.

Proposition 1. *Let h be an entire function. Then the function $F(z)=z+h(e^z)$ is prime if and only if $G(z)=z\cdot e^{h(z)}$ is so.*

Proof. Let $F(z)=f\circ g(z)$, with non-linear entire functions f and g. Then, by Theorem B in § 1, setting $b=2\pi i$, we have

$$f(z)=cz+H_1(z)\,,\qquad \in J(b') \tag{3}$$

and

$$g(z)=c'(z+H_2(z))\,,\qquad \in J(b) \tag{4}$$

with $b'=bc'$. By the assumption, we have

$$\begin{aligned} f\circ g(z)&=cc'z+cc'H_2(z)+H_1[c'(z+H_2(z))]\\ &=z+h(e^z)\,. \end{aligned} \tag{5}$$

Here we can write

$$H_1(z)=h_1(e^{z/c'})\,,\quad H_2(z)=h_2(e^z)\,, \tag{6}$$

where h_1 and h_2 are holomorphic in the punctured plane $0<|z|<\infty$ (since $b'=bc'$ with $b=2\pi i$). In the identity (5), we see at first $cc'=1$. Hence, under (5), cancelling z and then, in view of (6), replacing the variable (e^z by z), we obtain the identical relation

$$h_2(z)+h_1(z\cdot e^{h_2(z)})=h(z)\,, \tag{7}$$

for $z\neq 0$. Now, noting Lemma 6 in [20], we know from (7) that h_1 and h_2 are entire functions. Hence the equation (7) must be satisfied for any z. Then, by (7), we have the following factorization of the function $z\cdot e^{h(z)}$;

$$\begin{aligned} z\cdot e^{h(z)}&=z\cdot e^{h_2(z)}\cdot\exp[h_1(z\cdot e^{h_2(z)})]\\ &=(z\cdot e^{h_1(z)})\circ(z\cdot e^{h_2(z)})\,, \end{aligned} \tag{8}$$

where h_j $(j=1, 2)$ are entire, as shown above. Hence, if $G(z)=z\cdot e^{h(z)}$ is prime, then h_1 or h_2 is constant, so that f or g is linear, that is, F is prime. Note that, since F and G are non-periodic, we need only consider the entire factors w. r. t. the primeness.

Conversely, if $G(z)$ is not prime, then, under (8), we may assume that both h_j $(j=1, 2)$ are non-constant and entire. In this case, using the relations (3)-(8), we see at once, under $F=f\circ g$, that f and g are both transcendental, so that F is not (pseudo-)prime.

2.2. Relating to the uniqueness of the factorization, we note the following two facts. As a supplement of Theorem 3 in [20], we have

Theorem 2. *Let*

$$F(z)=(z+e^{-kz}\cdot P(e^z))\circ(z+Q(e^z))\,,$$

where P and Q are polynomials with $P(0)\neq 0$ and k is a natural number. Then $F(z)$ is uniquely factorizable.

The proof of this result can be done quite similarly as that of Theorem 3 in [20] (even somewhat simpler), hence omitted. As a generalization of Theorem 7 in [20], we obtain

Theorem 3. *Let*

$$F(z)=(z+H_1(z))\cdot\exp[H_2(z)],$$

where H_1 and $\exp[H_2]$ *are periodic entire functions with the same period b ($\neq 0$) (hence $F\in \boldsymbol{L}(b)$). And assume that $F(z)=f(g(z))$, with non-linear entire functions f and g. Then, two functions $z+H_1(z)$ and $H_2(z)$ must have $g(z)$ as the common right-factor, that is,*

$$z+H_1(z)=K_1(g(z)) \quad \textit{and} \quad H_2(z)=K_2(g(z))$$

for some entire functions K_j ($j=1, 2$).

Remark. In the above Theorem, F is prime, if two functions $z+H_1(z)$ and $H_2(z)$ have no common, non-linear, entire function as the right-factor. Further, if $z+H_1(z)$ is prime, then $F(z)$ is uniquely factorizable such as

$$F(z)=(z\cdot\exp[K_2(K_1^{-1}(z))]\circ(z+H_1(z)).$$

Here note that, in this case, K_1 is a linear polynomial and so the function $z\cdot\exp[K_2(K_1^{-1}(z))]\in \boldsymbol{L}(b)$ (in the above expression) is prime.

As noted above, in particular, we have

Proposition 2 (cf. [20] Cor. 3). *The function $z\cdot e^{H(z)}$, which is contained in $\boldsymbol{L}(b)$ ($b\neq 0$), is prime.*

Proof of Theorem 3. Since $F\in \boldsymbol{L}(b)$, by Theorem A and Remark there, we may write

$$f(z)=H_3(z)+z\cdot e^{H_4(z)}, \quad g(z)=z+H_5(z), \tag{9}$$

where H_3, $\exp[H_4]$ and H_5 are periodic, entire, with period b. Hence

$$f(g(z))=H_6(z)+z\cdot\exp[H_4(z+H_5(z))] \tag{10}$$

with $H_6(z)=H_3(z+H_5(z))+H_5(z)\cdot\exp[H_4(z+H_5(z))]$. Clearly $H_6(z+b)\equiv H_6(z)$. Now, $F=f\circ g$ and

$$F(z)=H_1(z)\cdot\exp[H_2(z)]+z\cdot\exp[H_2(z)]. \tag{11}$$

By considering $F(z+b)-F(z)$ and cancelling the periodic parts from (10) and (11), we have $\exp[H_2(z)]=\exp[H_4(z+H_5(z))]=\exp[H_4(g(z))]$. Hence

$$\begin{aligned} z+H_1(z)&=f(g(z))\cdot\exp[-H_2(z)] \\ &=(f(z)\cdot\exp[-H_4(z)])\circ(g(z)), \end{aligned}$$

and $H_2(z)=H_4(g(z))+2m\pi i$ for some integer m. Thus, in order to obtain the assertion, we may take

$$K_1(z)=f(z)\cdot\exp[-H_4(z)], \quad K_2(z)=H_4(z)+2m\pi i.$$

Clearly, these K_j ($j=1, 2$) are entire functions.

2.3. Here we show the following result, which is needed in § 3.

Theorem 4. *Let*

$$F(z)=z\cdot\exp[\lambda\cdot e_m(h(z))]+H(z),$$

where λ is a non-zero constant, $e_m(w)$ is the m-th iterate of e^w and both $h(z)$ ($\not\equiv$const.) and $H(z)$ are entire, periodic, functions with the same period (b, $\neq 0$, say). Further assume that $h(z)$ is prime. Then $F(z)$ is prime.

Remark. We know that there exist prime, periodic, entire functions (cf. [2], [10], [17], [21]).

Proof of Theorem 4. Let $F(z)=f(g(z))$, with non-linear entire functions f and g. Since $F\in \boldsymbol{L}(b)$, as before, we may write

$$f(z)=H_1(z)+z\cdot e^{H_2(z)} \quad \text{and} \quad g(z)=z+H_3(z),$$

where $H_j(z+b)=H_j(z)$ ($j=1, 3$) and $\exp[H_2(z+b)]=\exp[H_2(z)]$ are entire functions. Note that $H_2(z)\not\equiv$const., since $F\notin \boldsymbol{J}(b)$, and also $H_3(z)\not\equiv$const., since g is non-linear by assumption. Under $F=f\circ g$, by considering $F(z+b)-F(z)$ and cancelling the periodic parts, we have $\exp[H_2(z+H_3(z))]=\exp[\lambda\cdot e_m(h(z))]$. Hence

$$H_2(z+H_3(z))=\lambda\cdot e_m(h(z))+2k_1\pi i, \tag{12}$$

for some integer k_1. Because $z+H_3(z)$ takes every values (cf. [20], footnote at p. 102), from (12), it follows that

$$H_2(z)=2k_1\pi i+\lambda\cdot\exp[U_1(z)], \tag{13}$$

for some entire function $U_1(z)$ ($\not\equiv$const., since H_2 is so). In view of (12) and (13), we have $\exp[U_1(z+H_3(z))]=e_m(h(z))$, and so

$$U_1(z+H_3(z))=e_{m-1}(h(z))+2k_2\pi i,$$

for some integer k_2. Then, $U_1(z)=2k_2\pi i+\exp[U_2(z)]$ for some entire function $U_2(z)$. Hence $\exp[U_2(z+H_3(z))]=e_{m-1}(h(z))$ so that $\cdots$. Repeating this argument, we have that

$$U_{m-1}(z+H_3(z))=\exp[h(z)]+2k_m\pi i,$$

$$U_{m-1}(z)=2k_m\pi i+\exp[U_m(z)],$$

and

$$U_m(z+H_3(z))=2k_{m+1}\pi i+h(z), \tag{14}$$

for some entire functions U_j ($\not\equiv$const.) and integers k_j.

Now, $h(z)$ is prime by assumption. Since $g(z)=z+H_3(z)$ is non-linear, then by (14), $U_m(z)$ must be linear ;$=az+b$ (say). In this case, the identity (14) can be rewritten as

$$a\cdot H_3(z)-h(z)=-az-b+2k_{m+1}\pi i. \tag{15}$$

By the way, the left hand side of (15) is periodic with period b. While $a\neq 0$, since $U_m(z)$ is non-constant. Under (15), these clearly lead us to a contradiction. Thus, under $F=f\circ g$, f or g is linear. Hence F is prime, which is to be proved.

Remark. Also, as a generalization of Theorem 12 in [20], we can prove the following result: The function $p(e^z)+z\cdot\exp[z+q(e^z)]$ is prime, where p and q are two entire functions with the order $\rho(p)$, $\rho(q)<1$ (both).

§ 3. Deficiency and factorizability.

Here, in connection with the work due to Gross-Osgood-Yang [9], we will exhibit prime entire functions of infinite order (arbitrary rapid growth) with prescribed Nevanlinna deficiency. For this purpose, using their line of argument in [9], we note the following

Proposition 3. *For constant* $\lambda_j>0$ $(j=1, 2)$ *and an entire function* $g(z)$ $(\not\equiv const.)$, *we consider the function* $F(z)$ *defined by*

$$F(z)=\exp[\lambda_1 g(z)]\cdot\{\exp[\lambda_2 g(z)]+z+c\},$$

where c *is a constant. Then, for a suitable value of* c *(precisely, if* $c\in \boldsymbol{C}-E$, *where* E *is a set with inner* $\operatorname{Cap} E=0$), *we have*

$$\delta(0, F)=\lambda_1/(\lambda_1+\lambda_2),$$

where $\delta(*, *)$ *denotes the Nevanlinna deficiency.*

Proof. We wish to show, firstly, that

$$T(r, F)\sim(\lambda_1+\lambda_2)\cdot T(r, e^{g(z)}). \tag{16}$$

To do so, we shall take certain $n\in\boldsymbol{N}$ and $\lambda>0$ such that $\lambda+\lambda_1=n\lambda_2$ and we consider the function

$$G(z)=e^{\lambda g(z)}\cdot F(z), \tag{17}$$

which is rewritten as

$$\begin{aligned} G(z)&=\exp[n\lambda_2 g(z)]\cdot\{\exp[\lambda_2 g(z)]+z+c\} \\ &=h(z)^n\cdot\{h(z)+z+c\}, \end{aligned} \tag{18}$$

where $h(z)=\exp[\lambda_2 g(z)]$. Here we note the following fact: For a positive constant μ and an entire function $g(z)$, $T(r, e^{\mu g(z)})=\mu T(r, e^{g(z)})$, where $T(*, *)$ denotes the Nevanlinna characteristic function. Then, using this fact, we obtain that

$$T(r, G)\sim(n+1)T(r, h)=(n+1)\lambda_2 T(r, e^{g(z)}),$$

by (18). Note also that the first assertion above can be shown by considering $m(r, G)$, in which the set where $|h(re^{i\theta})|$ is large should only be considered. Hence, using (17), we have

$$\begin{aligned}T(r, F)&=T(r, e^{-\lambda g(z)}\cdot G(z))\\&\geqq T(r, G)-T(r, e^{\lambda g(z)})\\&\sim\{(n+1)\lambda_2-\lambda\}\cdot T(r, e^{g(z)})\\&=(\lambda_1+\lambda_2)\cdot T(r, e^{g(z)}).\end{aligned}$$

While,

$$\begin{aligned}T(r, F)&\leqq T(r, e^{\lambda_1 g(z)})+T(r, e^{\lambda_2 g(z)}+z+c)\\&\sim T(r, e^{\lambda_1 g(z)})+T(r, e^{\lambda_2 g(z)})\\&=(\lambda_1+\lambda_2)\cdot T(r, e^{g(z)}).\end{aligned}$$

From above, we know that the assertion (16) is valid.

By the way, noting a well-known fact ([15], p. 276), for $c\in \boldsymbol{C}-E$ where E is a set with inner $\operatorname{Cap} E=0$, we have

$$\begin{aligned}N(r, 0, F)&=N(r, -c, e^{\lambda_2 g(z)}+z)\\&\sim T(r, e^{\lambda_2 g(z)}+z)\sim T(r, e^{\lambda_2 g(z)})\\&=\lambda_2\cdot T(r, e^{g(z)}).\end{aligned}$$

Hence $\delta(0, F)=1-\limsup_{r\to\infty} N(r, 0, F)/T(r, F)=\lambda_1/(\lambda_1+\lambda_2)$ (for a suitable value of c), which is to be shown.

Theorem 5. *Let α be a given constant with $0<\alpha<1$. Suppose that $h(z)$ is a prime, periodic, entire function and λ_j $(j=1, 2)$ are positive constants such that $\lambda_1/(\lambda_1+\lambda_2)=\alpha$. We consider the function $F(z)$ given by*

$$F(z)=\exp[\lambda_1 e_m(h(z))]\cdot\{\exp[\lambda_2 e_m(h(z))]+z+c\},$$

where $c\in \boldsymbol{C}-E$ for a suitable subset of $\boldsymbol{C}$ with inner $\operatorname{Cap} E=0$. Then $F(z)$ is a prime entire function with Nevanlinna deficiency $\delta(0, F)=\alpha$.

This is valid by Theorem 4 and Proposition 3.

Remark. 1) Prime entire functions (of infinite order) with Nevanlinna deficiency 0 or 1 can easily be given by using Theorem 1 or Theorem 4 in § 2, respectively. 2). As a prime entire function with prescribed Nevanlinna deficiency, by the results due to Y. Noda ([16], Theorem 3, etc.), we may also take the function of the form: $(z+c)\cdot f(z)$, where f is a suitable transcendental entire function and c is a certain constant.

§ 4. Equation $f\circ f=g\circ g$.

Relating to the problem on the uniqueness of the factorization, or specializing this, we shall consider the following problem: When f and g are entire and they satisfy the identity $f(f(z))=g(g(z))$, then what can be said about f and g? Generally, it seems very difficult to get a positive answer for the above question. However, if we put a certain specific condition on $f(z)$, we can prove the following result.

Theorem 6. *Let f be an entire function of the form:*

$$f(z)=z+p(e^z),$$

where $p(z)$ is a non-constant entire function such that $\exp[k\cdot p(0)]\neq -1$ for any natural number k. Suppose that g is entire and satisfies the following identical relation

$$g(g(z))=f(f(z)). \tag{19}$$

Then we have necessarily $g(z)\equiv f(z)$.

Corollary. *Let $f(z)=z+e_m(z)$, where m is a natural number ($e_m(z)$ denotes the m-th iterate of e^z). And, if g is entire and satisfies the identity $g(g(z))=f(f(z))$, then we have necessarily $g(z)=z+e_m(z)$.*

Remark. In the case where $m=1$ (for the above f), we know further that the composite function $(z+e^z)\circ(z+e^z)$ is uniquely factorizable. But, so far, we have not been able to prove that the function $(z+e_k(z))\circ(z+e_m(z))$ is uniquely factorizable (or not), in the case where one of (natural numbers) k and m is larger than one.

Also, note that, if $f(z)=z-\sin z$ and $g(z)=-(z-\sin z)+2k\pi$ with an integer k, then we have $f(f(z))\equiv g(g(z))$ (cf. [8], p. 242).

For the proof of Theorem 6, we shall need the following fact.

Lemma A. *Let $G(z)$ be entire and $q(z)$ be holomorphic in $0<|z|<\infty$. Suppose that they satisfy the equation*

$$q(z)-q\Big(\frac{1}{z}\cdot e^{-q(z)}\Big)=G(z) \tag{20}$$

for any $z\neq 0$. Then $q(z)$ must be entire.

This is proved similarly as Lemma 6 in [20]. Indeed, if the Laurent expansion of $q(z)$ at $z=0$ contains terms of negative powers of z, then there exists a sequence $\{z_j\}$ ($z_j\neq 0$), $z_j\to 0$ ($j\to\infty$) such that

$$\left|\frac{1}{z_j}\cdot e^{-q(z_j)}\right|=1, \tag{21}$$

because the function $e^{-q(z)}/z$ has the origin ($z=0$) as the essential singularity. Then, for the sequence $\{z_j\}$ satisfying (21), we have $\mathrm{Re}\,q(z_j)\to\infty$ ($j\to\infty$). Hence, putting $z=z_j$ in (20), we know that the real part of the left hand side of (20) tends to infinity, while the right hand side remains bounded, as $j\to\infty$. This is a contradiction. Thus $q(z)$ has no terms of negative powers of z, so that q is to be entire.

Proof of Theorem 6. Since $f(f(z))=z+p(e^z)+p(\exp[z+p(e^z)])\in \boldsymbol{J}(2\pi i)$, by Theorem B, it follows that, under (19), $g\in \boldsymbol{J}(2\pi i)$. Writing $g(z)=c(z+H(z))$, with $H(z+2\pi i)\equiv H(z)$, entire, we see firstly $c=\pm 1$. Now, we have $H(z)=q(e^z)$, for some holomorphic function $q(w)$ in $0<|w|<\infty$.

Then, from the identity (19), cancelling z and replacing the variable (from e^z to z, as before), we have two identical relation for $z\neq 0$, according as $c=1$ or $c=-1$. In both cases, we see that q is entire. Indeed, when $c=1$, then we know it by Lemma 6 in [20], and when $c=-1$, then we know it by the above Lemma A.

By the way, we can rule out the case where $c=-1$. In fact, in this case, the identical relation (valid for $z\neq 0$) mentioned above can be written as

$$q(z)-q\left(\frac{1}{z}\cdot e^{-q(z)}\right)=p(z)+p(z\cdot e^{p(z)}).$$

Here, we know that q is entire, as noted above, by Lemma A. Further, by considering the behavior of the above identity near $z=0$, we have that the right hand side is naturally holomorphic at the origin, but the left hand side has singularity there (cannot be extended holomorphically). Thus, we have a contradiction.

Hence, in the following, we need only consider the equation

$$q(z)+q(z\cdot e^{q(z)})=p(z)+p(z\cdot e^{p(z)}), \tag{22}$$

which is valid for any z in $\boldsymbol{C}$. In this case, note that $g(z)=z+q(e^z)$, since $c=1$.

Under (22), setting $z=0$, it follows that $q(0)=p(0)$. Here we put $s=e^{q(0)}=e^{p(0)}$ (say). Then it is assumed that $1+s^m\neq 0$ for any natural number m.

In the following, we wish to show that

$$q^{(m)}(0)=p^{(m)}(0) \qquad (m\in \boldsymbol{N}), \tag{23}$$

by induction. We note that, differentiating the both sides of (22) and setting $z=0$, we have $(1+s)\cdot q'(0)=(1+s)\cdot p'(0)$ so that $q'(0)=p'(0)$. Assume now that we have

$$q(0)=p(0),\ \cdots,\ q^{(m-1)}(0)=p^{(m-1)}(0). \tag{24}$$

Then by differentiating (22) m-times, we have

$$q^{(m)}(z)+\frac{\mathrm{d}^m}{\mathrm{d}z^m}\{q(z\cdot e^{q(z)})\}=p^{(m)}(z)+\frac{\mathrm{d}^m}{\mathrm{d}z^m}\{p(z\cdot e^{p(z)})\}. \tag{25}$$

Now, for instance,

$$\begin{aligned}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\{p(z\cdot e^{p(z)})\}=&\,p^{(m)}(z\cdot e^{p(z)})\cdot\{(z\cdot e^{p(z)})'\}^m\\&+\sum_{k=1}^{m-1}p^{(k)}(z\cdot e^{p(z)})\cdot D_{mk}[z\cdot e^{p(z)}],\end{aligned} \tag{26}$$

where $D_{mk}[***]$ is a homogeneous differential polynomial of degree k (cf. [19]). In particular,

$$D_{m1}[z\cdot e^{p(z)}]=m\cdot\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\{e^{p(z)}\}+z\cdot\frac{\mathrm{d}^{m}}{\mathrm{d}z^{m}}\{e^{p(z)}\}.$$

Here note that the second term of the above equation is zero at $z=0$, and so that we have $D_{m1}[z\cdot e^{q(z)}]=D_{m1}[z\cdot e^{p(z)}]$ at the origin. Also we obtain

$$D_{mk}[z\cdot e^{q(z)}]|_{z=0}=D_{mk}[z\cdot e^{p(z)}]|_{z=0}$$

for $k=1, \cdots, m-1$ under (24). Hence, in view of (24) and (26), by setting $z=0$ in the identity (25), we see that $(1+s^m)\cdot q^{(m)}(0)=(1+s^m)\cdot p^{(m)}(0)$ so that $q^{(m)}(0)=p^{(m)}(0)$, since $1+s^m\neq 0$ by assumption. Therefore the assertion (23) is checked.

Hence, by the unicity theorem, we conclude that q and p are identical. This means that two functions f and g are identical; $g(z)\equiv f(z)$, which is to be proved.

§ 5. Certain prime meromorphic functions.

In this section, by using the results on primeness in divisor sense shown in [22] (cf. Theorem C, § 1), we shall exhibit certain prime meromorphic functions. For instance, we can prove the following results.

5.1. Theorem 7. *Let*

$$F(z)=\frac{P(z)+Q(e^z)}{S(z)+R(z)\cdot H(z)},$$

where P ($\not\equiv$const.), Q ($\not\equiv$const.), R ($\not\equiv 0$) and S are polynomials such that, for any $k\in\boldsymbol{N}$ and $c\in\boldsymbol{C}$, the function $e^{-kz}\cdot[Q(e^z)+c]$ is non-constant, and $H(z)$ is an entire function which is periodic with the period $2\pi i$ such that the order and lower order of H satisfy $1<\underline{\rho}(H)\leqq\rho(H)<2$. Assume that the numerator and denominator have no common zeros. Then $F(z)$ is prime.

Theorem 8. *Let*

$$F(z)=\frac{P(z)+Q(e^z)}{S(z)+R(z)\cdot\alpha(e^z)},$$

where P ($\not\equiv$const.), Q ($\not\equiv$const.), R ($\not\equiv 0$) and S are polynomials such that, for any $k\in\boldsymbol{N}$ and $c\in\boldsymbol{C}$, the function $e^{-kz}\cdot[Q(e^z)+c]$ is non-constant, and α is an entire function such that the order of $\alpha(e^z)$ is finite and not integer and the lower order of it is larger than one. Assume that the numerator and denominator have no common zeros. Then $F(z)$ is prime.

The proof of Theorem 7 is quite similar to that of Theorem 8 (even somewhat simpler). Hence, we shall only prove Theorem 8.

Proof of Theorem 8. Suppose that

$$F(z)=f(g(z)), \tag{27}$$

where f and g are non-linear meromorphic functions. Here, g cannot be a polynomial, as easily seen by noting the distribution of the zeros of $F(z)$ (they are distributed mainly in the right half plane). In the following, we'll consider the two cases separately.

(I) The case where f is transcendental meromorphic (not entire) and g is transcendental entire. In this case, by a Theorem of Edrei-Fuchs ([3], Cor. 1.2), we have $\rho(f)=0$ and $\rho(g)<\infty$, so that we can write

$$f(z)=\beta(z)/\gamma(z)\,,$$

where β and γ ($\not\equiv$const.) are entire functions, without common zeros, both of order zero. Then, by (27) and (28), we have the identities

$$P(z)+Q(e^z)=\beta(g(z))\cdot e^{A(z)} \tag{29}$$

and

$$S(z)+R(z)\cdot\alpha(e^z)=\gamma(g(z))\cdot e^{A(z)}\,, \tag{30}$$

for some entire function $A(z)$, since by the assumption the numerator and denominator of $F(z)$ have no common zeros. By Theorem C, the left hand side of (29) is prime in divisor sense (by the assumption in Theorem 8). Hence, by definition, $\beta(z)$ must be linear. Here, without loss of generality, we may assume that $\beta(z)\equiv z$. Then, from (29), it follows that

$$g(z)=(P(z)+Q(e^z))\cdot e^{-A(z)}\,. \tag{31}$$

Since the order $\rho(g)<\infty$, we know that $A(z)$ is a polynomial. Now, from (30) and (31), we have the following identical relation

$$\{S(z)+R(z)\cdot\alpha(e^z)\}\cdot e^{-A(z)}=\gamma([P(z)+Q(e^z)]\cdot e^{-A(z)})\,. \tag{32}$$

Here, when $\deg A=0$ or 1, then we compare the growth of the both sides of the above identity (32) along the upper or lower half of the imaginary axis (If $e^{A(z)}$ is periodically bounded on the imaginary axis, then any selection will do. If it is not the case, then the selection must be made so as to $e^{-A(z)}\to\infty$ as $z\to\infty$ on the considered part.). When $\deg A\geqq 2$, there exists a straight half line L issued from the origin and contained in the left half plane such that, if $z\to\infty$ on L, $e^{-A(z)}\to\infty$. Then we compare the growth of the both sides of (32) along this half line L. In either case, by applying the minimum modulus theorem (if necessary), we have a contradiction (cf. argument around p. 108 in [20]). Note that γ is transcendental entire with $\rho(\gamma)=0$, and that both $\alpha(e^z)$ and $Q(e^z)$ are bounded on the considered half line.

(II) The case where f is rational and g is transcendental meromorphic (not entire). Write

$$f(z)=U(z)/V(z)\,, \tag{33}$$

where U and V are polynomials without common zeros. Note that, because the order $\rho(F)$ is finite and not integer, in this case, the same is true for $\rho(g)$. Further note that the lower order $\underline{\rho}(g)>1$ (since F is so). If $\deg U=0$, $\deg V\geqq 2$

as f is non-linear. Now, in this case, the poles of g are the multiple zeros of F. And they are infinite in number. But the zeros of F are all simple except at most a finite number of them (as easily seen). This is a contradiction.

If $(1\leqq)\ \deg U<\deg V$ under (33), then, letting b be a zero-point of $U(z)$, we have

$$N(r, \infty, g)+N(r, b, g)\leqq N(r, 0, F). \tag{34}$$

Here, $N(r, 0, F)$ is clearly of order 1 (cf. [11], around p. 16), and hence the left hand side of (34) is also. But this is not valid, when $\rho(g)$ is finite and not integer with $\rho(g)>1$, by a Theorem due to Nevanlinna ([14], p. 51). This is a contradiction.

If $\deg U\geqq\deg V$, then $\deg U\geqq 2$ (f is non-linear). Since the zeros of $F(z)$ are almost simple, $U(z)$ must have at least two distinct zero-points. Hence, letting b and c $(b\neq c)$ be two zeros of $U(z)$, we have

$$N(r, b, g)+N(r, c, g)\leqq N(r, 0, F).$$

Using this inequality, we obtain a contradiction similarly as above.

Thus, under the factorization such as (27), f or g must be linear. Hence F is prime. This completes the proof of Theorem 8.

5.2. Remark. For the functions of the form:

$$F(z)=\frac{P_1(z)+Q_1(e^z)}{P_2(z)+Q_2(e^z)}, \quad \text{or} \quad \frac{P_1(z)+Q_1(e^z)}{P_2(z)+Q_2(e^{-z})},$$

where P_j and Q_j are some non-constant polynomials, generally, we can only prove that they are right-prime.

Note, for example, that the function $([P(z)]^k+e^z)/([P(z)]^k-e^z)=(\{w^k+1\}/\{w^k-1\})\circ(P(z)\cdot e^{-z/k})$ is not prime for any polynomial $P(z)$ $(\not\equiv 0)$ if $k\geqq 2$.

By the way, concretely speaking, we do not know whether the functions such as $[z+e^{2z}]/[z+e^z]$ and $[z+e^z]/[z+e^{-z}]$ are prime or not, though we have the following assertion.

Proposition 4. *For a meromorphic function* $h(z)$, *the function* $[h(z)+e^z]/[h(z)-e^z]$ *is prime if and only if* $h(z)\cdot e^{-z}$ *is so.*

In fact, this is trivially valid in view of the relation such as $[h(z)+e^z]/[h(z)-e^z]-1=2/[h(z)\cdot e^{-z}-1]$. Hence, in particular, the function $[z+e^z]/[z-e^z]$ is prime, by Proposition 4, because $z\cdot e^{-z}$ is prime (cf. Proposition 2).

DEPARTMENT OF MATHEMATICS,
KYOTO UNIVERSITY OF EDUCATION

References

[1] I.N. Baker, Zusammensetzungen ganzer Funktionen, Math. Z., **69** (1958), 121-163.

[2] I.N. Baker and C.-C. Yang, An infinite order periodic entire function which is prime, Lecture Notes in Math., Springer, 599 (Complex Analysis, Kentucky, 1976), 7-10.

[3] A. Edrei and W.H.J. Fuchs, On the zeros of $f(g(z))$ where f and g are entire functions, J. d'analyse Math., **12** (1964), 243-255.

[4] R. Goldstein, On factorization of certain entire functions, J. London Math. Soc., (2), **2** (1970), 221-224.

[5] F. Gross, On factorization of meromorphic functions, Trans. Amer. Math. Soc., **131** (1968), 215-222.

[6] F. Gross, Factorization of entire functions which are periodic mod g, Indian J. Pure and Appl. Math., **2** (1971), 561-571.

[7] F. Gross, Prime entire functions, Trans. Amer. Math. Soc., **161** (1971), 219-233.

[8] F. Gross, Factorization of meromorphic functions, U.S. Government Printing Office, 1972.

[9] F. Gross, Ch. Osgood and C.-C. Yang, Prime entire functions with prescribed Nevanlinna deficiency, Nagoya Math. J., **47** (1972), 91-99.

[10] F. Gross and C.-C. Yang, On prime periodic entire functions, Math. Z., **174** (1980), 43-48.

[11] W.K. Hayman, Meromorphic functions, Clarendon Press, Oxford, 1964.

[12] T. Kobayashi, Permutability and unique factorizability of certain entire functions, Kodai Math. J., **3** (1980), 8-25.

[13] S. Koont, On factorization in certain classes of entire functions, Math. Research Report No. 76-5, Univ. Maryland (1976), 19pp.

[14] R. Nevanlinna, Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Gauthier Villars, 1929 (Chelsea, 1974).

[15] R. Nevanlinna, Analytic functions, Springer, 1970.

[16] Y. Noda, On factorization of entire functions, Kodai Math. J., **4** (1981), 480-494.

[17] M. Ozawa, On the existence of prime periodic entire functions, Kodai Math. Sem. Rep., **29** (1978), 308-321.

[18] G. Pólya, On an integral function of an integral function, J. London Math. Soc., **1** (1926), 12-15.

[19] N. Steinmetz, Über die factorisierbaren Lösungen gewöhnlicher Differentialgleichungen, Math. Z., **170** (1980), 169-180.

[20] H. Urabe, Uniqueness of the factorization under composition of certain entire functions, J. Math. Kyoto Univ., **18** (1978), 95-120.

[21] H. Urabe, Infinite order, periodic, entire functions which together with their derivatives are prime, Bull. Kyoto Univ. of Education, Ser. B, No. 55 (1979), 21-32.

[22] H. Urabe, Primeness in divisor sense for certain entire functions, J. Math. Kyoto Univ., **24** (1984), 127-140.

[23] C.-C. Yang and H. Urabe, On permutability of certain entire functions, J. London Math. Soc., (2), **14** (1976), 153-159.