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Mixed problems for pluriparabolic equations

By

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Petrowski considered the well-posedness of Cauchy problems for evolution equations with t-dependent coefficients, and introduced two typical subclasses —strictly hyperbolic and p-parabolic ([1]). Volevich-Gindikin considered Cauchy problems for pluriparabolic equation—the third subclass of evolution equations ([2]). Finally, Volevich proved the well-posedness of Cauchy problems for $\mathscr{H}$-correct evolution equations with (t, x)-dependent coefficients, where the class of $\mathscr{H}$-correct evolution equations is a subclass of evolution equations containing the above three classes ([3]). On the other hand, there are little works on mixed problems for evolution equations other than hyperbolic or parabolic equations ([4], [5]).

In this paper, the author considers the mixed problems for pluriparabolic equations. She uses the energy method, where the main tools are the pseudo-differential operators with weight functions ([6]). She uses two types of weight functions and pays attentions to the separation of two types of symbols. To get the energy inequalities, the choice of energy forms is based on the technique used in [7].

A typical example of pluriparabolic mixed problems is given by

$$\begin{cases} \partial_t u = -\partial_x u + \partial_y^2 u + f & (t>0,\ x>0,\ -\infty<y<+\infty)\,, \\ u|_{x=0} = g & (t>0,\ -\infty<y<+\infty)\,, \\ u|_{t=0} = h & (x>0,\ -\infty<y<+\infty)\,. \end{cases}$$

More general pluriparabolic equations of order 1 with respect to ∂_t are investigated under the name of ultraparabolic equations ([8], [9]).

§1. Pseudo-differential operators with weight functions.

1.1. For $\rho=(\rho_1, \rho_2, \cdots, \rho_n)$ $(\rho_i>0)$, we say that $\lambda(\xi)$ $(\geqq 1)$ is a weight function if

$$|\partial_\xi^\alpha \lambda(\xi)| \leqq C_\alpha \lambda(\xi)^{1-\rho\cdot\alpha}\,.$$

Moreover, we say that $a(x, \xi) \in S_{\lambda,\rho}^m$ if

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$$|\partial_\xi^\alpha D_x^\beta a(x, \xi)| \leqq C_{\alpha\beta}\, \lambda(\xi)^{m-\rho\cdot\alpha},$$

Then we can define the pseudo-differential operator $a(x, D_x)$ by

$$a(x, D_x)\, u(x) = (2\pi i)^{-n} \int e^{ix\cdot\xi}\, a(x, \xi)\, \hat{u}(\xi)\, d\xi$$

(see [6]). Moreover, we say that $a(x, \xi) \in S^m_{\lambda,\rho}(\Omega)$ if

$$\chi(x, \xi)\, a(x, \xi) \in S^m_{\lambda,\rho}$$

for any $\chi(x, \xi) \in S^0_{\lambda,\rho}$ satisfying supp $[\chi] \subset \Omega$.

Lemma 1.1. *Let us assume that $a(x, \xi) \in S^m_{\lambda,\rho}(\Omega)$. For another weight function λ' (with same ρ), we assume*
i) $\lambda' \leqq \lambda$ *in* Ω,
ii) $|\partial_\xi^\alpha D_x^\beta a(x, \xi)| \leqq C\, \lambda'^{k-\alpha\cdot\rho}\, \lambda^{m-k}$ *in* Ω, *if* $k - \alpha\cdot\rho > 0$.
Then we have

$$a(x, \xi)\, \lambda^{-(m-k)} \in S^k_{\lambda',\rho}(\Omega).$$

Proof. Let $k - \alpha\cdot\rho > 0$, then we have

$$\begin{aligned} &|\partial_\xi^\alpha D_x^\beta(a(x, \xi)\, \lambda^{-m+k})| \\ &\quad \leqq C \sum_{\alpha' \leqq \alpha} |\partial_\xi^{\alpha-\alpha'} D_x^\beta a|\, |\partial_\xi^{\alpha'} \lambda^{-m+k}| \\ &\quad \leqq C' \sum \lambda'^{k-(\alpha-\alpha')\cdot\rho}\, \lambda^{m-k}\, \lambda^{-m+k-\alpha'\cdot\rho} \\ &\quad \leqq C' \sum \lambda'^{k-\alpha\cdot\rho}. \end{aligned}$$

Let $k - \alpha\cdot\rho \leqq 0$, then we have

$$|\partial_\xi^\alpha D_x^\beta(a(x, \xi)\, \lambda^{-m+k})| \leqq C\, \lambda^{k-\alpha\cdot\rho} \leqq C\, \lambda'^{k-\alpha\cdot\rho}. \quad \blacksquare$$

Now we say that $a(x, \xi) \in S^{m,k}_{\lambda,\lambda',\rho}(\Omega)$ for two weight functions λ, λ' $(\lambda' \leqq \lambda)$, if
i) $a(x, \xi) \in S^m_{\lambda,\rho}(\Omega)$,
ii) $|\partial_\xi^\alpha D_x^\beta a(x, \xi)| \leqq C\, \lambda^{m-k}\, \lambda'^{k-\alpha\cdot\rho}$ in Ω, if $0 < \alpha\cdot\rho < k$.

Lemma 1.2. *Let $a(x, \xi) \in S^{m,k}_{\lambda,\lambda',\rho}(\Omega)$ and $b(x, \xi) \in S^{m',k}_{\lambda,\lambda',\rho}(\Omega)$, then we have $a(x, \xi)\, b(x, \xi) \in S^{m+m',k}_{\lambda,\lambda',\rho}(\Omega)$.*

Proof. It is obvious that $ab \in S^{m+m'}_\lambda(\Omega)$. From the Leibniz law, we have

$$\begin{aligned} \partial_\xi^\alpha D_x^\beta(ab) &= \sum_{0 \leqq \beta' \leqq \beta} C_{\beta,\beta'}\, a^{(\alpha)}_{(\beta-\beta')}\, b_{(\beta')} \\ &+ \sum_{0 \leqq \beta' \leqq \beta} C_{\beta,\beta'}\, a_{(\beta-\beta')}\, b^{(\alpha)}_{(\beta')} \\ &+ \sum_{\substack{0 < \alpha' < \alpha \\ 0 \leqq \beta' \leqq \beta}} C_{\alpha,\alpha',\beta,\beta'}\, a^{(\alpha-\alpha')}_{(\beta-\beta')}\, b^{(\alpha')}_{(\beta')}. \end{aligned}$$

Let $0 < \alpha\cdot\rho < k$, then we have

$$|a^{(\alpha)}_{(\beta-\beta')}\, b_{(\beta')}| \leqq C(\lambda'^{k-\alpha\cdot p}\, \lambda^{m-k})\, \lambda^{m'},$$
$$|a_{(\beta-\beta')}\, b^{(\alpha)}_{(\beta')}| \leqq C\, \lambda^{m}(\lambda'^{k-\alpha\cdot p}\, \lambda^{m'-k}),$$
$$\sum_{0<\alpha'<\alpha} |a^{(\alpha-\alpha')}_{(\beta-\beta')}\, b^{(\alpha')}_{(\beta)}| \leqq C(\lambda'^{k-(\alpha-\alpha')\cdot p}\, \lambda^{m-k})(\lambda'^{k-\alpha'\cdot p}\, \lambda^{m'-k}),$$

therefore we have

$$|\partial_\xi^\alpha D_x^\beta(ab)| \leqq C\, \lambda'^{k-\alpha\cdot p}\, \lambda^{m+m'-k}. \quad ■$$

1.2. In the following, we consider the polynomial

$$A(t, x; \tau, \xi) = \sum_{k+\sum_{j=1}^{n}\nu_j/p_j=\mu} a_{k\nu}(t, x)\, \tau^k\, \xi^\nu \quad (a_{\mu 0}=1)$$
$$= \sum_{k=0}^{\mu} a_{\mu-k}(t, x; \xi)\, \tau^k,$$

where $\{p_j\}$ are positive integers such that $p_1=\cdots=p_s=1$, $p_j>1$ $(j=s+1, \cdots, n)$, and $a_{k\nu}\in\mathcal{B}^\infty(R^{n+1})$. We denote $p=(p_1, \cdots, p_n)$.

Let us denote $\xi=(\xi', \xi'')$, where

$$\xi'=(\xi_1, \cdots, \xi_s) \quad\text{and}\quad \xi''=(\xi_{s+1}, \cdots, \xi_n).$$

Moreover, denoting

$$|\xi|_p = (\sum_{j=1}^{n} |\xi_j|^{2p_j})^{1/2}, \quad |\xi''|_p = (\sum_{j=s+1}^{n} |\xi_j|^{2p_j})^{1/2},$$

we define

$$\Lambda(\tau, \xi) = (|\tau|^2+|\xi|_p^2)^{1/2} = (\sigma^2+\gamma^2+|\xi|_p^2)^{1/2} \quad (\tau=\sigma-i\gamma),$$
$$\Lambda'(\tau, \xi) = (|\sigma|^2+|\xi'|^2)^{1/2},$$
$$\Lambda''(\tau, \xi) = (\gamma^2+|\xi''|_p^2)^{1/2}.$$

Moreover, we define

$$\Lambda_0(\xi) = (1+|\xi|_p^2)^{1/2}, \quad \Lambda_0'(\xi) = (1+|\xi'|^2)^{1/2},$$
$$\Lambda_0''(\xi) = (1+|\xi|_p^2)^{1/2}.$$

Immediately, we have

$$A(t, x; \tau, \xi)\in S^{\mu}_{\Lambda,(1,q)},$$

where $q=(1/p_1, \cdots, 1/p_n)$.

Lemma 1.3. *Assume that μ_1 of the roots of $A=0$ w.r.t. τ, $\{\tau_j\}_{j=1,\cdots,\mu_1}$, are inside of Γ_1 and μ_2 of those $(\mu_1+\mu_2=\mu)$, $\{\tau_j\}_{j=\mu_1+1,\cdots,\mu}$, are inside of Γ_2, where Γ_1 and Γ_2 are simple closed curves on τ-plane contained inside of a circle with radius*

$R\Lambda_0(R>0)$ and $\mathrm{dis}(\Gamma_1, \Gamma_2)=\delta\Lambda_0(\delta>0)$ *if* $\xi\in\Omega$. *Then we have*

$$A_1=\prod_{j=1}^{\mu_i}(\tau-\tau_j)=\sum_{j=0}^{\mu_i}a_{1\,\mu_1-j}(t, x; \xi)\,\tau^j$$

and

$$A_2=\prod_{j=\mu_1+1}^{\mu}(\tau-\tau_j)=\sum_{j=0}^{\mu_2}a_{2\,\mu_2-j}(t, x; \xi)\,\tau^j\,,$$

where $a_{i\,k}(t, x; \xi)\in S^{k,1}_{\Lambda_0,\Lambda_0'',q}(\Omega)$.

Proof. Setting

$$c_k(t, x; \xi)=\frac{1}{2\pi i}\oint_{\Gamma_1}\frac{A_\tau(t, x; \tau, \xi)\,\tau^k}{A(t, x; \tau, \xi)}\,d\tau\,,$$

we shall see $c_k\in S^{k,1}_{\Lambda_0,\Lambda_0'',q}(\Omega)$. Since we have

$$\delta_1\,\Lambda_0^\mu\leqq|A(t, x; \tau, \xi)|\leqq\delta_2\,\Lambda_0^\mu\qquad(\delta_i>0)\,,$$

$$|\partial_\tau^h\,\partial_\xi^\alpha\,D_x^\beta\,A(t, x; \tau, \xi)|\leqq C\Lambda_0^{\mu-h-\alpha\cdot q}$$

for $\tau\in\Gamma_1$, we have

$$|\partial_\xi^\alpha\,D_{t,x}^\beta\left\{\frac{A_\tau(t, x; \tau, \xi)}{A(t, x; \tau, \xi)}\right\}|\leqq C\Lambda_0^{-1-\alpha\cdot q}\,.$$

Therefore we have

$$|\partial_\xi^\alpha\,D_{t,x}^\beta\,c_k(t, x; \xi)|\leqq C\,\Lambda_0^{k-\alpha\cdot q}\,.$$

Especially when $0<\alpha\cdot q\leqq 1$, since

$$|\partial_\xi^\alpha\,D_{t,x}^\beta\left\{\frac{A_\tau(t, x; \tau, \xi)}{A(t, x; \tau, \xi)}\right\}|\,\Lambda_0^2\leqq C\,\Lambda_0''^{1-\alpha\cdot q}\,,$$

we have

$$|\partial_\xi^\alpha\,D_{t,x}^\beta\,c_k(t, x; \xi)|\,\Lambda_0^{-k+1}\leqq C\,\Lambda_0''^{1-\alpha\cdot q}\,.\quad\blacksquare$$

Lemma 1.4. *Let* $a(\xi)\in S^{k,1}_{\Lambda_0,\Lambda_0'',q}(\Omega)$, *where* $\Omega\subset\{\Lambda_0'\geqq c\,\Lambda_0\}$ $(c>0)$, *then we have*

$$\{a(\xi)-a(\xi', 0)\}\;\Lambda_0^{-k+1}\in S^1_{\Lambda_0'',q}(\Omega)\,.$$

Proof. It is obvious that $a(\xi', 0)\in S^k_{\Lambda_0,q}(\Omega)$, because $\Lambda_0'(\xi)\geqq c\,\Lambda_0(\xi)$ in Ω. Then, we have

$$\tilde{a}(\xi)=a(\xi)-a(\xi', 0)=\sum_{j=s+1}^{n}\partial_{\xi_j}\,a(\xi', \tilde{\xi}'')\,\xi_j\,,$$

where $|\tilde{\xi}''|\leqq|\xi''|$. Since

$$|\partial_{\xi_j} a(\xi)| \leqq C\, \Lambda_0''^{1-q_j}\, \Lambda_0^{k-1},$$

we have

$$|\sum_{j=s+1}^{n} \partial_{\xi_j} a(\xi', \tilde{\xi}'')\, \xi_j| \leqq C\, \Lambda_0^{k-1} \sum_{j=s+1}^{n} \Lambda_0''^{1-q_j} |\xi_j|$$
$$\leqq C'\, \Lambda_0^{k-1}\, \Lambda_0''$$

Moreover, we have

$$|\tilde{a}_{(\beta)}^{(\alpha)}(\xi)| \leqq |a_{(\beta)}^{(\alpha)}(\xi)| + |a_{(\beta)}^{(\alpha)}(\xi', 0)|$$
$$\leqq C\, \Lambda_0''^{1-\alpha\cdot q}\, \Lambda_0^{k-1} + C\, \Lambda_0^{k-1} \leqq C'\, \Lambda_0''^{1-\alpha\cdot q}\, \Lambda_0^{k-1}$$

for $0<\alpha\cdot q<1$. ■

Corollary. *Let us assume*

$$A(\tau, \xi) = \prod_{j=1}^{\mu} (\tau - \tau_j(\xi)),$$

where $\{\tau_j(\xi', 0)\}_{j=1,\cdots,\mu}$ *are distinct for* $\xi' \in S^{s-1}$. *Then we have*

$$\tau_j(\xi', 0) \in S^{1,1}_{\Lambda_0, \Lambda_0'', q}(\Omega)$$

from Lemma 1.3, *and we have*

$$\tilde{\tau}_j(\xi) = \tau_j(\xi) - \tau_j(\xi', 0) \in S^1_{\Lambda_0'', q}(\Omega)$$

from Lemma 1.4, *where* $\Omega = \{|\xi'| \geqq c|\xi|_p\}$ $(c>0)$.

1.3. Let us specialize the direction of x_1-axis in x-space, where we assume $A(0, 1, 0, \cdots, 0) \neq 0$. We may assume $A(0, 1, 0, \cdots, 0)=1$, rewriting $A(\tau, \xi)/A(0, 1, 0, \cdots, 0)$ as A. Denoting

$$\xi = (\zeta, \eta_2, \cdots, \eta_n) = (\zeta, \eta),$$

we have

$$A(\tau, \zeta, \eta) = \sum_{j=0}^{\mu} b_{\mu-j}(\tau, \eta)\, \zeta^j.$$

Moreover we denote

$$\Lambda_1(\tau, \eta) = (|\tau|^2 + |\eta|_p^2)^{1/2},$$
$$\Lambda_1'(\tau, \eta) = (|\sigma|^2 + |\eta'|^2)^{1/2},$$

and

$$\Lambda_1''(\tau, \eta) = (\gamma^2 + |\eta''|_p^2)^{1/2},$$

where $\eta' = (\eta_2, \cdots, \eta_s)$ and $\eta'' = (\eta_{s+1}, \cdots, \eta_n)$.

Lemma 1.5. *Let m_1 of the roots of $A(\tau, \zeta, \eta)=0$ with respect to ζ, $\{\zeta(\tau, \eta)\}_{k=1,\cdots,m_1}$, be inside of Γ_1 and let the rest m_2 $(=\mu-m_1)$, $\{\zeta_1(\tau, \eta)\}_{j=m_1+1,\cdots,\mu}$, be inside of Γ_2 for $(\tau, \eta)\in\Omega$, where* $\mathrm{dis}(\Gamma_1, \Gamma_2)=\delta\Lambda_1$ $(\delta>0)$. *Then we have*

$$A_1 = \prod_{j=1}^{m_1} (\zeta-\zeta_j(\tau, \eta)) = \zeta^{m_1}+b_{1\,1}(\tau, \eta)\, \zeta^{m_1+1}+\cdots+b_{1\,m_1}(\tau, \eta)$$

and

$$A_2 = \prod_{j=m+1}^{\mu} (\zeta-\zeta_j(\tau, \eta)) = \zeta^{m_2}+b_{2\,1}(\tau, \eta)\, \zeta^{m_2}+\cdots+b_{2\,m_2}(\tau, \eta)\,,$$

where

$$b_{ij}(\tau, \eta)\in S^{j,1}_{\Lambda_1,\Lambda_1'',q}\,(\Omega)\,.$$

Proof. It is proved in the same way as in Lemma 1.3. ■

Lemma 1.6. *Let $b(\tau, \eta)\in S^{k,1}_{\Lambda_1,\Lambda_1'',q}\,(\Omega)$, where $\Omega\subset\{\Lambda_1'\geqq c\,\Lambda_1\}$ $(c>0)$. Then we have*

$$\{b(\tau, \eta)-b(\sigma, \eta', 0)\}\ \Lambda_1^{-k+1}\in S^{1}_{\Lambda_1'',q}\,(\Omega)\,.$$

Proof. It is proved in the same way as in Lemma 1.4. ■

§2. Cauchy problem for pluriparabolic operators.

2.1. Pluriparabolic. Let us consider a polynomial with respect to (τ, ξ):

$$A(\tau, \xi) = \sum_{k+\Sigma\nu_j/p_j\leqq\mu} a_{k\nu}(t, x)\, \tau^k\, \xi^\nu\,,$$

where $a_{k\nu}(t, x)\in\mathcal{B}^\infty(R^{n+1})$ and $a_{k\nu}(\tau, \xi)=$constant outside a ball in R^{n+1}. Let us call

$$A_0(\tau, \xi) = \sum_{k+\Sigma\nu_j/p_j=\mu} a_{k\nu}(t, x)\, \tau^k\, \xi^\nu$$

as the *principal part* of $A(\tau, \xi)$, considered in §1. Let us use the same notation $\xi=(\xi', \xi'')$ as in §1.

Assumption (A). *A* is *pluriparabolic*, that is,

i) $A_0(1, 0)\neq0$,

ii) the roots $\{\tau_j(\xi)\}_{j=1,\cdots,\mu}$ of $A_0(\tau, \xi)=0$ for $\xi\in S^{n-1}$ satisfy

$$\mathrm{Im}\ \tau_j(\xi)\geqq c\,|\xi''|_p\quad (c>0)\,,$$

iii) the roots of $A_0(\tau, \xi', 0)=0$ are real and distinct for $\xi'\in S^{s-1}$.

First, we consider the 0-Cauchy problem:

$$\text{(C.P.)}\quad \begin{cases} A(t, x; D_t, D_x)\, u(t, x) = f(t, x) & \text{in}\quad R^1\times R^n\,, \\ u(t, x) = 0 & \text{in}\quad (-\infty, 0)\times R^n\,. \end{cases}$$

Theorem 2.1. *There exist $\gamma_0>0$, $C>0$ such that for $u\in H^\infty(R^{n-1})$ and $\gamma>\gamma_0$, we have*

$$\gamma^{1/2} \sum_{k+\Sigma\mu_j/p_j\leqq\mu-1} ||(D_t-i\gamma)^k D_x^\nu u|| + \sum_{\substack{k+\Sigma\nu_j/p_j\leqq\mu-1/2 \\ k+\Sigma'\nu_j\leqq\mu-1}} ||(D_t-i\gamma)^k D_x^\nu u|| \leqq C||\Lambda''^{-1/2}A(D_t-i\gamma, D_x)u|| ,$$

where $\sum'$ means the summation about j for which $p_j=1$.

By the usual way, energy inequalities of higher or lower orders are obtained from the basic energy inequality stated in Theorem 2.1. Therefore, concerning to the dual problem, we have the following existence theorem:

Theorem 2.2. *For $f\in H^\infty(R^{n+1})$ with* $\mathrm{supp}[f]\subset\{t\geqq 0\}$, *there exists a unique solution $u\in H^\infty(R^{n+1})$ with* $\mathrm{supp}[u]\subset\{t\geqq 0\}$ *satisfying*

$$A(t, x; D_t-i\gamma, D_x)\, u(t, x) = f(t, x) \quad in \quad R^1\times R^n ,$$

where $\gamma>\gamma_0$.

2.2. Energy inequality. We denote, for $\tau=\sigma-i\gamma$ $(\gamma\geqq 1)$,

$$D' = \{(\sigma, \xi)\in R^1\times R^n;\ \Lambda''(\tau, \xi)\leqq\varepsilon_1\,\Lambda(\tau, \xi)\} ,$$

$$D'' = \{(\sigma, \xi)\in R^1\times R^n;\ \Lambda''(\tau, \xi)\geqq\varepsilon_1\,\Lambda(\tau, \xi)\} ,$$

$$D'_j = D'\cap\{|\sigma-\tau_j(\xi', 0)|\leqq\varepsilon_2\,\Lambda(\tau, \xi)\} ,$$

$$D'_0 = D'\setminus\bigcup_{j=1}^{\mu} D'_j .$$

Taking ε_1, ε_2 small enough, we have

$$|A_0(\tau, \xi)| = |\Pi(\tau-\tau_j(\xi))|\geqq c\,\Lambda(\tau, \xi)^\mu \quad \text{in} \quad D''\cup D'_0 .$$

Denoting

$$A_0(\tau, \xi) = (\tau-\tau_j(\xi))\, A_j(\tau, \xi) ,$$

we have

$$|A_j(\tau, \xi)|\geqq c\,\Lambda(\tau, \xi)^{\mu-1} \quad \text{in} \quad D'_j .$$

Since $\{\tau_j(\xi', 0)\}$ are real and distinct, we have from Corollary of Lemma 1.4

$$A_0(\tau, \xi) = \prod_{j=1}^{\mu}\{\tau-(\tau_j(\xi', 0)+\tilde{\tau}_j(\xi))\} \quad \text{in} \quad D' ,$$

where

$$\tau_j(\xi', 0)\in S^1_{\Lambda_0, q}(D'), \quad \tilde{\tau}_j(\xi)\in S^1_{\Lambda''_0, q}(D') ,$$

and

$$\mathrm{Im}\ \tilde{\tau}_j(\xi) \geqq c\,|\xi''|_p\,.$$

Now, let $\chi(\tau, \xi)$ be a C^∞-function with support in D'_j is defined by

$$\chi(\tau, \xi) = \phi\left(\frac{\sigma-\tau_j(\xi', 0)}{\Lambda(\tau, \xi)}\right) \phi\left(\frac{\Lambda''(\tau, \xi)}{\Lambda(\tau, \xi)}\right),$$

where $\phi(s)$ is a C^∞-function with support in ε-neighbourhood of the origin, $0 \leqq \phi(s) \leqq 1$, and $\phi(s)=1$ near the origin. Then we have

Lemma 2.3.

$$\chi(\tau, \xi) \in S^{0,1}_{\Lambda,\Lambda'',(1,q)}.$$

Set

$$\begin{aligned} T(\tau, \xi) &= \chi(\tau, \xi)\ \{(\sigma-\mathrm{Re}\ \tau_j(\xi))-i(\gamma+\mathrm{Im}\ \tau_j(\xi))\} \\ &\quad -i(1-\chi(\tau, \xi))\ \Lambda''(\tau, \xi) \\ &= \chi(\tau, \xi)\ (\sigma-\mathrm{Re}\ \tau_j(\xi)) \\ &\quad -i\,\{\chi(\tau, \xi)\ (\gamma+\mathrm{Im}\ \tau_j(\xi))+(1-\chi(\tau, \xi))\ \Lambda''(\tau, \xi)\} \\ &= T'(\tau, \xi)-i\,T''(\tau, \xi)\,, \end{aligned}$$

then we have from Lemma 2.3

Lemma 2.4.

i) $T'(\tau, \xi) \in S^{1,1}_{\Lambda,\Lambda'',(1,q)}$,

ii) $T''(\tau, \xi) \in S^{1}_{\Lambda'',(1,q)}$ *and* $T''(\tau, \xi) \geqq c\,\Lambda''$ $(c>0)$.

Lemma 2.5. *There exist* $\gamma_0>0$ *and* $C>0$ *such that*

$$\|\Lambda''^{1/2}(D_t-i\gamma, D_x)\,u\| \leqq C\|\Lambda''^{-1/2}\,T(D_t-i\gamma, D_x)\,u\|$$

for $u \in H^\infty(R^1 \times R^n)$ *and* $\gamma>\gamma_0$.

Proof. We denote $T=T\ (D_t-i\gamma,\ D_x)$ and e.c.. We consider the integral form in $R^1 \times R^n$:

$$\begin{aligned} (Tu, u)-(u, Tu) &= ((T'-iT'')\,u, u)-(u, (T'-iT'')\,u) \\ &= \{(T'\,u, u)-(u, T'\,u)\} -i\,\{(T''\,u, u)+(u, T''\,u)\}\ . \end{aligned}$$

Since

$$T'^{*}-T' \sim \sum_{|\alpha|>0} (\alpha\,!)^{-1}\ \overline{T'^{(\alpha)}_{(\alpha)}}\,,$$

we have from Lemma 2.4

$$\begin{cases} |((T'^{*}-T')u, u)| \leqq C\|\Lambda''^{1/2-q/2}u\|^2 \leqq C\gamma^{-q}\|\Lambda''^{1/2}u\|^2, \\ (T''u, u)+(u, T''u) \geqq c\|\Lambda''^{1/2}u\|^2 - C\gamma^{-q}\|\Lambda''^{1/2}u\|^2, \end{cases}$$

where $q=\min_i q_i$. Therefore we have $\|\Lambda''^{1/2}u\| \leqq C\|\Lambda''^{-1/2}Tu\|$, if γ is large enough. ■

Proof of Theorem 2.1. Let us define $\tilde{A}_j$ from A_j in the same way that we defined $T=T_j$ from $\tau-\tau_j(\xi)$. Let $\tilde{\chi}_j$ have the same properties as χ_j and moreover $\mathrm{supp}[\tilde{\chi}_i] \subset \{\chi_j=1\}$. Let us denote $\cdot$ as the product of operators and denote $\circ$ as the product in symbols. Then we have

$$\tilde{\chi}_j \circ A_0 = T_j \circ \tilde{A}_j \circ \tilde{\chi}_j,$$

$$\|\Lambda''^{-1/2}(\tilde{\chi}_j \cdot A_0 - \tilde{\chi}_j \circ A_0)u\| \leqq C\|\Lambda''^{1/2-q}\Lambda^{\mu-1}u\|,$$

$$\|\Lambda''^{-1/2}(T_j \circ \tilde{A}_j \circ \tilde{\chi}_j - T_j \cdot \tilde{A}_j \cdot \tilde{\chi}_j)u\| \leqq C\|\Lambda''^{1/2-q}\Lambda^{\mu-1}u\|,$$

and moreover from Lemma 2.5

$$\|\Lambda''^{-1/2}T_j \cdot \tilde{A}_j \cdot \tilde{\chi}_j u\| \geqq c\|\Lambda''^{1/2} \cdot \tilde{A}_j \cdot \tilde{\chi}_j u\|.$$

On the other hand, we have

$$\|(\Lambda''^{1/2} \cdot \tilde{A}_j - \tilde{A}_j \cdot \Lambda''^{1/2})\tilde{\chi}_j u\| \leqq C\|\Lambda''^{1/2-q}\Lambda^{\mu-1}u\|$$

and

$$\|\tilde{A}_j \cdot \Lambda''^{1/2} \cdot \tilde{\chi}_j u\| \geqq c\|\Lambda^{\mu-1}\Lambda''^{1/2} \cdot \tilde{\chi}_j u\|.$$

Let $\tilde{\chi}_0$ be the localization symbol on D_0', then we have the estimations of $\|\Lambda''^{-1/2}(\chi_0 \cdot A_0 - \chi_0 \circ A_0)u\|$ etc. in the same way as in D_j'. And moreover we have

$$\|\Lambda''^{-1/2}A_0 \cdot \tilde{\chi}_0 u\| \geqq c\|\Lambda''^{-1/2}\Lambda^{\mu}\tilde{\chi}_0 u\|.$$

Hence, summing up the estimations of $\{\tilde{\chi}_j u\}_{j=0,1,\cdots,\mu}$, we have

$$\|\Lambda''^{1/2}\Lambda^{\mu-1}u\| \leqq C\|\Lambda''^{-1/2}Au\|. \quad ■$$

§3. Initial-boundary value problems for pluriparabolic operators.

3.1. Problems and results.

Let A be a pluriparabolic operator defined in §2. We consider the initial-boundary value problem for A in a half space $x_1>0$, where we assume the boundary $\{x_1=0\}$ is non-characteristic of A in the weighted sense, i.e.

Assumption (C): $A_0(0, 1, 0, \cdots, 0) \neq 0$.

From the Assumption (A), the roots of $A_0=0$ with respect to ξ_1 are non-real when $\mathrm{Im}\,\tau<0$ and $(\xi_2, \cdots, \xi_n) \in R^{n-1}$. Hence, the number μ_+ of the roots with

Im $\xi_1>0$ and the number μ_- ($\mu_+ + \mu_- = \mu$) of the roots with Im $\xi_1<0$ are independent of variables. We denote

$$A_0(\tau, \xi) = A_+(\tau, \xi)\, A_-(\tau, \xi)\,,$$

where the roots of $A_\pm=0$ are in $\{\mathrm{Im}\, \xi_1 \gtrless 0\}$.

Now, the boundary conditions are given by

$$B_j(D_t, D_x)\, u|_{x_1=0} = g_j \quad (j=1, \cdots, \mu_+)\,,$$

where

$$B_j(\tau, \xi) = \sum_{h+\Sigma\nu_j/p_j \leqq r_j} b_{jh\,\nu}(t, x)\, \tau^h\, \xi^\nu$$
$$(r_j \leqq \mu-1,\ r_j \neq r_k \text{ if } j\neq k)\,,$$

where $b_{j\,h\nu}(t, x) \in \mathcal{B}^\infty(R^{n+1})$ and $b_{j\,h\nu}$=constant outside a ball in R^{n+1}. We say that

$$B_{j0}(\tau, \xi) = \sum_{h+\nu\cdot q=r_j} b_{jh\,\nu}(t, x)\, \tau^h\, \xi^\nu$$

is the principal part of B_j. Moreover we say that

$$B_j^\sharp(\tau, \xi) = \Lambda_1(\tau, \xi_2, \cdots, \xi_n)^{\mu-r_j-1}\, B_j(\tau, \xi)$$

is the standardization of B. Here we introduce the Lopatinski determinant

$$R(\tau, \xi_2, \cdots, \xi_n) = \det\left[\frac{1}{2\pi i}\oint \frac{B_{j0}(\tau, \xi)\, \xi_1^{k-1}}{A_+(\tau, \xi)}\, d\,\xi_1\right]_{j,k=1,\cdots,\mu_+},$$

and we assume

Assumption (B) (Uniform Lopatinski Condition): $R(\tau, \xi_2, \cdots, \xi_n) \neq 0$ for $\{\mathrm{Im}\, \tau \leqq 0,\ \ (\xi_2, \cdots, \xi_n) \in R^{n-1},\ \ (\tau, \xi_2, \cdots, \xi_n) \neq 0\}$.

Theorem 3.1. *Under the Assumptions (A), (B), (C), there exists positive numbers γ_0 and C such that we have*

$$\langle\!\langle u\rangle\!\rangle + |||u||| \leqq C\{||\Lambda''^{-1/2}\, A(D_t - i\gamma, D_x)\, u||^2_{L^2(R\times R^n_+)}$$
$$+ \sum_{j=1}^{+\mu} <B_j^\sharp(D_t - i\gamma, D_x)\, u>^2_{L^2(R^1\times R^{n-1}),\, x=0}\}$$

for $u \in H^\infty(R^1 \times R^n_+)$ and $\gamma > \gamma_0$, where

$$\langle\!\langle u\rangle\!\rangle^2 = \sum_{h+\nu\cdot q \leqq \mu-1} <(D_t - i\gamma)^h\, D_x^\nu\, u>^2_{L^2(R\times R^{n-1}),\, x=0}\,,$$

$$|||u|||^2 = \sum_{h+\nu\cdot q \leqq \mu-1} ||\Lambda_1'^{\prime 1/2}(D_t - i\gamma)^h\, D_x^\nu\, u||^2_{L^2(R^1\times R^n_+)}.$$

We can get the energy inequalities of higher orders or lower orders from the above Theorem 3.1 in the same way as in the hyperbolic case ([7]). Moreover, we can get the same type of energy inequalities for the adjoint problem. Hence we have

Theorem 3.2. *Under the Assumptions (A), (B), (C), there exists a unique solution*

$$u\in H^{\infty}(R^1\times R^n_+),\quad \mathrm{supp}\,[u]\subset\{t\geqq 0\}\ ,$$

for the problem:

$$\begin{cases} A(D_t-i\gamma,\ D_x)\,u=f & \textit{for}\ (t,\ x)\in R^1\times R^n_+\ ,\\ B_j(D_t-i\gamma,\ D_x)\,u=g_j\quad (j=1,\ \cdots,\ \mu_+) & \textit{for}\ (t,\ x)\in R^1\times\partial R^n_+ \end{cases}$$

for any given datas:

$$\begin{cases} f\in H^{\infty}(R^1\times R^n_+), & \mathrm{supp}\,[f]\subset\{t\geqq 0\}\ ,\\ g_j\in H^{\infty}(R^n), & \mathrm{supp}\,[g_j]\subset\{t\geqq 0\}\ , \end{cases}$$

where $\gamma>\gamma_0$.

3.2. (*H-P*)-property. We say that

$$P(\tau,\ \zeta,\ \eta)=\zeta^h+c_1(\tau,\ \eta)\,\zeta^{h-1}+c_2(\tau,\ \eta)\,\zeta^{h-2}+\cdots+c_h(\tau,\ \eta)$$

is a polynomial of ζ of order h with $S_{\Lambda_1,q}\,(U)$-coefficients, if

i) $c_j(\tau,\ \eta)\in S^{j,1}_{\Lambda_1,\Lambda_1'',q}\,(U)$,

ii) $\mathrm{Im}\,c_j(\tau,\ \eta)\ \Lambda_1^{-j+1}\in S^1_{\Lambda_1'',q}\,(U)$.

Here we denote

$$P'(\tau,\ \zeta,\ \eta)=\zeta^h+\mathrm{Re}\,c_1(\tau,\ \eta)\,\zeta^{h-1}+\cdots+\mathrm{Re}\,c_h(\tau,\ \eta)\ ,$$

$$P''(\tau,\ \zeta,\ \eta)=\mathrm{Im}\,c_1(\tau,\ \eta)\,\zeta^{h-1}+\cdots+\mathrm{Im}\,c_h(\tau,\ \eta)\ ,$$

then we have

$$P(\tau,\ \zeta,\ \eta)=P'(\tau,\ \zeta,\ \eta)+iP''(\tau,\ \zeta,\ \eta)\ .$$

We say that P has *(H-P)-property* in U, if the roots $\zeta(\tau,\ \eta)$ of $P=0$ satisfy the following properties (*P*) or (*H*). We say that $\zeta(\tau,\ \eta)$ satisfies the property (*P*) if $|\mathrm{Im}\,\zeta|\geqq c\,\Lambda_1$ in U $(c>0)$. We say that $\zeta(\tau,\ \eta)$ satisfies the property (*H*) if

$$|P''(\tau,\ \zeta(\sigma,\ \eta',\ 0),\ \eta)|\geqq c\ \Lambda_1''\ \Lambda_1^{h-1}\quad \text{in}\ U\ .$$

Moreover, we say that P has (*P*)-*property* in U if all the roots of $P=0$ has the property (*P*) in U, and that P has (*H*)-*property* in U if all the roots of $P=0$ has the property (*H*) in U.

Denoting a ε-neighbourhood of $(t_0,\ x_0,\ \tau_0,\ \eta_0)$ by

$$\tilde{U}_\varepsilon = \{(t, x, \tau, \eta) \in R^1 \times R^n \times C^1 \times R^{n-1};\ |t-t_0|^2+|x-x_0|^2<\varepsilon^2,$$
$$|\tau-\tau_0|^2+|\eta-\eta_0|_p^2<\varepsilon^2\},$$

we define the corresponding conic ε-neighbourhood by

$$U_\varepsilon = \{(t, x, \lambda\tau, \lambda^{q_2}\eta_2, \cdots, \lambda^{q_n}\eta_n);\ (t, x, \tau, \eta) \in \tilde{U}_\varepsilon, \lambda>0\}.$$

Lemma 3.3. *Let $P(\tau, \zeta, \eta)$, $P_1(\tau, \zeta, \eta)$, $P_2(\tau, \zeta, \eta)$ be ploynomials of orders h, h_1, h_2 with respect to ζ with $S_{\Lambda_1,q}(U)$-coefficients, and*

$$P(\tau, \zeta, \eta) = P_1(\tau, \zeta, \eta)\, P_2(\tau, \zeta, \eta),$$

where the distance between the roots of $P_1=0$ and the roots of $P_2=0$ is $\delta\Lambda_1$ ($\delta>0$). Then, P has (H-P)-property in a conic neighbourhood of $(t_0, x_0, \tau_0, \eta_0', 0)$ ($\in U$) iff P_1 and P_2 have (H-P)-property in a conic neighbourhood of $(t_0, x_0, \tau_0, \eta_0', 0)$ ($\in U$).

Proof. Assuming that P has (H-P)-property near $(t_0, x_0, \tau_0, \eta_0', 0)$, we shall see that P_1 has (H-P)-property near $(t_0, x_0, \tau_0, \eta_0', 0)$. Let (σ, ζ_1, η') be real and $P_1(\sigma, \zeta_1, \eta', 0)=0$, then we have

$$P(\sigma, \zeta_1, \eta', 0) = 0$$

and

$$|P''(\tau, \zeta_1, \eta)| \geqq c\, \Lambda_1'' \, \Lambda_1^{h-1}.$$

On the other hand, we have

$$\begin{aligned} &|P''(\tau, \zeta_1, \eta)| \\ &= |P_1''(\tau, \zeta_1, \eta)\, P_2'(\tau, \zeta_1, \eta)+P_1'(\tau, \zeta_1, \eta)\, P_2''(\tau, \zeta_1, \eta)| \\ &\leqq |P_1''(\tau, \zeta_1, \eta)|\, |P_2(\sigma, \zeta_1, \eta', 0)| + C\, \Lambda_1''^2\, \Lambda_1^{h-2}. \end{aligned}$$

Hence we have

$$|P_1''(\tau, \zeta_1, \eta)| \geqq c\, \Lambda_1''\, \Lambda_1^{h_1-1}$$

in a small conic neighbourhood of $(\tau_0, \eta_0', 0)$. ■

Lemma 3.4. *$A_0(\tau, \zeta, \eta)$ has (H-P)-property in a conic neighbourhood of $(t_0, x_0, \sigma_0, \eta_0', 0)$, where $(\sigma_0, \eta_0') \in S^{s-1}$.*

Proof. Let $(\sigma_0, \zeta_0, \eta_0')$ ($\neq 0$) be real and $A_0(\sigma_0, \zeta_0, \eta_0', 0)=0$, then we have $(\zeta_0, \eta_0') \neq 0$. Denoting

$$A_0(\tau, \zeta, \eta) = \prod_k (\tau - \tau_k(\zeta, \eta)),$$

since $\{\tau_k(\zeta_0, \eta_0', 0)\}$ are distinct, there exists a number k_0 such that

$$\sigma_0 = \tau_{k_0}(\zeta_0, \eta_0', 0).$$

Therefore, we have

$$\operatorname{Im} A_0(\tau, \zeta, \eta) = \operatorname{Im} (\tau-\tau_{k_0}(\zeta, \eta)) \operatorname{Re} \prod_{k\neq k_0} (\tau-\tau_k(\zeta, \eta))$$
$$+\operatorname{Re} (\tau-\tau_{k_0}(\zeta, \eta)) \operatorname{Im} \prod_{k\neq k_0} (\tau-\tau_k(\zeta, \eta)) .$$

Hence the rest of the proof will be carried in the same way as in Lemma 3.3. ■

Let us consider the behavior of the roots of $A_0(\tau, \zeta, \eta)$ with respect to ζ in

$$\{\operatorname{Im} \tau \leqq 0, \eta \in R^{n-1}, (\tau, \eta) \neq 0\} ,$$

which is devided into three parts:

① $\operatorname{Im} \tau < 0, \eta \in R^{n-1}$,
② $(\tau, \eta) \in R^n, \eta'' \neq 0$,
③ $(\tau, \eta') \neq 0$: real, $\eta''=0$.

In case ① and ②, the roots ζ of $A_0(\tau, \zeta, \eta)=0$ are non-real. In fact, let ζ be a real root, then we have $\operatorname{Im} \tau < 0$, $(\zeta, \eta) \in R^n$ and $A_0(\tau, \zeta, \eta)=0$ in case ①, which is a contradiction to the Ass. (A). In case ②, let ζ be a real root of $A_0(\tau, \zeta, \eta)=0$, then we have $(\tau, \zeta, \eta) \in R^{n-1}$, $\eta'' \neq 0$ and $A_0(\tau, \zeta, \eta)=0$, which is a contradiction to the Ass. (A). Hence we shall consider only the case ③ in the following.

Let us fix $(\sigma_0, \eta_0') \in S^{s-1}$, and let $\{\zeta_j\}_{j=1,\cdots,d}$ be the real roots of $A_0(\sigma_0, \zeta, \eta_0', 0)=0$ with respect to ζ, whose multiplicities are $\{h_j\}$. From Lemma 1.5, Lemma 1.6, Lemma 3.3 and Lemma 3.4, we have

Lemma 3.5. *We have the local factorization*

$$A_0(\tau, \zeta, \eta) = P(\tau, \zeta, \eta)\, P_1(\tau, \zeta, \eta) \cdots P_d(\tau, \zeta, \eta)$$

in a conic neighbourhood U of $(\sigma_0, \eta_0', 0)$, where P is a polynomial of ζ of order $h(=\mu-\sum h_j)$ with (P)-property and P_j is a polynomial of ζ of order h_j with (H)-property, satisfying

$$P_j(\tau_0, \zeta, \eta_0', 0) = (\zeta-\zeta_j)^{h_j} .$$

§4. Energy estimates.

First, we can easily get energy estimates for P with (P)-property, that is,

Proposition 4.1. *Let $P(\tau, \zeta, \eta)$ be a polynomial of ζ of order h with (P)-property, then there exist $\gamma_0 > 0$, $C > 0$ such that*

$$\sum_{k=0}^{h-1} \langle \Lambda_1^{h-k-1} D_{x_1}^k P_+(D_t - i\gamma, D_x) u \rangle$$
$$+\sum_{k=0}^{h} \|\Lambda_1^{h-k-1/2} D_{x_1}^k P_+(D_t - i\gamma, D_x) u\|$$
$$\leqq C \|\Lambda_1^{-1/2} P(D_t - i\gamma, D_x) u\|$$

and

$$\sum_{k=0}^{h} ||\Lambda_1^{h-k-1/2} D_{x_1}^k P_-(D_t-i\gamma, D_x)u||$$
$$\leqq C\{||\Lambda_1^{-1/2} P(D_t-i\gamma, D_x)u||$$
$$+\sum_{k=0}^{h-1} <\Lambda_1^{h-k-1} D_{x_1}^k P_-(D_t-i\gamma, D_x)u>\}$$

for $u\in H^\infty(R^1\times R_+^n)$ *and* $\gamma>\gamma_0$, *where* $\Lambda_1=\Lambda_1(D_t-i\gamma, D_{x_2}, \cdots, D_{x_n})$ *and* $P=P_+P_-$.

Next, we consider energy estimates for P with (H)-property. To get energy estimates, we shall see that the method used in hyperbolic case ([7]) is applicable also in our case. As is shown easily, we have

Lemma 4.2. *Let* $P(\tau, \zeta, \eta)$ *be a polynomial of* ζ *of order h with* (H)-*property, satisfying*

$$P(\sigma_0, \zeta, \eta_0', 0)=(\zeta-\zeta_0)^h,$$

where $(\sigma_0, \zeta_0, \eta_0')$ *is real and* $|\sigma_0|^2+|\eta_0'|^2=1$. *Then we have*

$$P(\tau, \zeta, \eta)=(\zeta-\zeta_0\Lambda_1)^h+c_1(\tau, \eta)(\zeta-\zeta_0\Lambda_1)^{h-1}+\cdots+c_h(\tau, \eta),$$

where

i) $c_j(\tau, \eta)\in S^{j,1}_{\Lambda_1,\Lambda_1'',q}$,
ii) $\operatorname{Im} c_j(\tau, \eta)\Lambda_1^{1-j}\in S^1_{\Lambda_1'',q}$,
iii) $|\operatorname{Im} c_h(\tau, \eta)|\geqq c\Lambda_1''\Lambda_1^{h-1}$ $(c>0)$.

Moreover, denoting $c_h''=\operatorname{Im} c_h$, we have

Lemma 4.3. *There exist* $c>0$ *and* $C>0$ *such that we have*

i) $\operatorname{Re}(c_h''u, \Lambda_1^{-1+h}u)\geqq c||\Lambda_1''^{1/2}\Lambda_1^{-1+h}u||^2-CR$

$$\text{if } \partial_\tau c_h(\sigma_0, \eta_0', 0)>0,$$

ii) $-\operatorname{Re}(c_h''u, \Lambda_1^{-1+h}u)\geqq c||\Lambda_1''^{1/2}\Lambda_1^{-1+h}u||^2-CR$

$$\text{if } \partial_\tau c_h(\sigma_0, \eta_0', 0)<0,$$

where

$$R=\sum_{k=0}^{h}||\Lambda_1''^{1/2-\varepsilon}\Lambda_1^{-1+h-k}L^k u||^2\ (\varepsilon=q/2),\ L=D_{x_1}-\zeta_0\Lambda_1,$$

and

$$c_h''=c_h''(D_t-i\gamma, D_{x_2}, \cdots, D_{x_n}),\ \Lambda_1=\Lambda_1(D_t-i\gamma, D_{x_2}, \cdots, D_{x_n}), \cdots.$$

Proposition 4.4. *Let* $P(\tau, \zeta, \eta)$ *be a polynomial of* ζ *of order h with* (H)-*property, satisfying*

$$P(\tau, \zeta, \eta) = (\zeta-\zeta_0\,\Lambda_1)^h+c_j(\tau, \eta)\,(\zeta-\zeta_0\,\Lambda_1)^{h-1}+\cdots+c_h(\tau, \eta)\,,$$

where

$$|c_j(\tau, \eta)| \leqq \delta\,\Lambda_1^j \quad (j=1, \cdots, h)\,.$$

Then there exists κ_0 *as follows. For any* $0<\kappa<\kappa_0$, *there exist* $C_\kappa>0$, $\delta_\kappa>0$, $\gamma_\kappa>0$ *such that we have*

$$\begin{aligned}&\kappa^{-1}\sum_{j=h}^{h-1} <\Lambda_1^{h-j-1}\,L^j\,u>^2\\ &+\sum_{j=0}^{h} \|\Lambda_1^{\prime\prime 1/2}\,\Lambda_1^{h-j-1}\,L^j\,u\|^2\\ &\leqq C_\kappa\|\Lambda_1^{\prime\prime -1/2}\,Pu\|^2+\kappa\sum_{j=0}^{h_+-1} <\Lambda_1^{h-j-1}\,L^j\,u>^2\end{aligned}$$

for $u\in H^\infty(R^1\times R^n_+)$, $0<\delta<\delta_\kappa$ *and* $\gamma>\gamma_\kappa$, *where*

① $h_+=h_-=h/2$ (*if h is even and* $\partial_\tau\,c_h(\sigma_0, \eta_0', 0)>0$),

② $h_+=h_-=h/2$ (*if h is even and* $\partial_\tau\,c_h(\sigma_0, \eta_0', 0)<0$),

③ $h_+=h_-+1=\dfrac{h+1}{2}$ (*if h is odd and* $\partial_\tau\,c_h(\sigma_0, \eta_0', 0)>0$),

④ $h_++1=h_-=\dfrac{h+1}{2}$ (*if h is odd and* $\partial_\tau\,c_h(\sigma_0, \eta_0', 0)<0$),

and

$$P=P(D_t-i\gamma, D_x), \quad \Lambda_1=\Lambda_1(D_t-i\gamma, D_{x_2}, \cdots, D_{x_n}), \cdots\,.$$

First, we introduce energy forms, which will be used to prove Prop. 4.4:

$$\begin{aligned}I_j &= 2\,\mathrm{Im}(Pu, \Lambda_1^{h-j-1}\,L^j\,u)\\ &= 2\,\mathrm{Im}(\{L^h+c_1'\,L^{h-1}+\cdots+c_h'\}\,u, \Lambda_1^{h-j-1}\,L^j\,u)\\ &\quad -2\,\mathrm{Re}(\{c_1''\,L^{h-1}+c_2''\,L^{h-2}+\cdots+c_h''\}\,u, \Lambda_1^{h-j-1}\,L^j\,u)\\ &= \{\langle L^{h-1}\,u, \Lambda_1^{h-j-1}\,L^j\,u\rangle+\langle L^{h-2}\,u, \Lambda_1^{h-j-1}\,L^{j+1}\,u\rangle\\ &\qquad +\cdots+\langle L^j\,u, \Lambda_1^{h-j-1}\,L^{h-1}\,u\rangle\}\\ &\quad +\{[\langle c_1'\,L^{h-2}\,u, \Lambda_1^{h-j-1}\,L^j\,u\rangle+\langle c_1'\,L^{h-3}\,u, \Lambda_1^{h-j-1}\,L^{j+1}\,u\rangle\\ &\qquad +\cdots+\langle c_1'\,L^j\,u, \Lambda_1^{h-j-1}\,L^{h-2}\,u\rangle]+\cdots-\cdots\\ &\quad -[\langle c_h'\,u, \Lambda_1^{h-j-1}\,L^{j-1}\,u\rangle+\cdots+\langle c_h'\,L^{j-1}\,u, \Lambda_1^{h-j-1}\,u\rangle]\}\\ &\quad +V_j\\ &= W_j+W_j'+V_j \quad (j=0, 1, \cdots, h)\,,\end{aligned}$$

then we have

Lemma 4.5. i) *We have*

$$|V_j| \leqq C(R+V) \quad (j=1, \cdots, h)\,,$$

$$|V_0+2\,\mathrm{Re}(c_h'' u,\ \Lambda_1^{h-1} u)|\leqq C(R+V)\,,$$

where R is the same one stated in Lemma 4.3 *and*

$$V=\sum_{k=0}^{h}||\Lambda_1''^{1/2}\,\Lambda_1^{h-k-1}\,L^k u||\sum_{k=1}^{h}||\Lambda_1''^{1/2}\,\Lambda_1^{h-k-1}\,L^k u||\,.$$

ii) *We have*

$$|W_j'|\leqq c(\delta)\,(W_+ + W_-)\quad (j=1,\cdots,h)\,,$$

where $c(\delta)\rightarrow 0$ *as* $\delta\rightarrow 0$,

$$W_+=\sum_{k=0}^{h_+-1}\langle\Lambda_1^{h-k-1}\,L^k u\rangle^2\,,$$

and

$$W_-=\sum_{k=h_+}^{h-1}\langle\Lambda_1^{h-k-1}\,L^k u\rangle^2\,.$$

iii) *In case* ①, ②, *or* ③, *we have*

$$|W_0|\leqq C\,W_+^{1/2}\,W_-^{1/2}\,.$$

Lemma 4.6. *There exists* δ_0 *as follows. For any* $0<\delta<\delta_0$, *there exist* $c(\delta)>0$ *and* $C_\delta>0$ *such that* $c(\delta)\rightarrow 0$ *(as* $\delta\rightarrow 0$*) and*

$$\sum_{j=1}^{h}||\Lambda_1''^{1/2}\,\Lambda_1^{h-j-1}\,L^j u||^2\leqq c(\delta)||\Lambda_1''^{1/2}\,\Lambda_1^{h-1}\,u||^2+C_\delta(||\Lambda_1''^{-1/2}\,Pu||^2+R)\,.$$

Proof. First we remark the interpolation inequality:

$$\sum_{j=1}^{h}||\Lambda_1''^{1/2}\,\Lambda_1^{h-j-1}\,L^j u||\leqq C\sum_{j=1}^{h}||\Lambda_1''^{1/2}\,\Lambda_1^{-1}\,L^h u||^{1-\varepsilon_j}||\Lambda_1''^{1/2}\,\Lambda_1^{h-1}\,u||^{\varepsilon_j}$$

($0\leqq\exists\varepsilon_j<1$), that is,

$$\sum_{j=1}^{h}||\Lambda_1''^{1/2}\,\Lambda_1^{h-j-1}\,L^j u||\leqq C_\varepsilon||\Lambda_1''^{1/2}\,\Lambda_1^{-1}\,L^h u||+\varepsilon||\Lambda_1''^{1/2}\,\Lambda_1^{h-1}\,u||$$

for any $\varepsilon>0$. On the order hand, since

$$P=L^h+c_1\,L^{h-1}+\cdots+c_h\,,$$

we have

$$\begin{aligned}||\Lambda_1''^{1/2}\,\Lambda_1^{-1}\,L^h u||&\leqq C(||\Lambda_1''^{1/2}\,\Lambda_1^{-1}\,P u||+\sum_{k=1}^{h}||\Lambda_1''^{1/2}\,\Lambda_1^{-1}\,c_k\,L^{h-k}\,u||)\\&\leqq C||\Lambda_1''^{-1/2}\,P u||+c(\delta)\sum_{k=1}^{h}||\Lambda_1''^{1/2}\,\Lambda_1^{k-1}\,L^{h-k}\,u||+C\,R^{1/2}\,.\end{aligned}$$

Hence, applying the interpolation inequality to the second term of the right hand side of the above inequality, we have

$$||\Lambda_1''^{1/2}\Lambda_1^{-1}L^h u||\leqq C(||\Lambda_1''^{-1/2}Pu||+R^{1/2})$$
$$+c(\delta)(||\Lambda_1''^{1/2}\Lambda_1^{-1}L^h u||+||\Lambda_1''^{1/2}\Lambda_1^{h-1}u||)\,.$$

Therefore, taking δ_0 small enough, we have

$$||\Lambda_1''^{1/2}\Lambda_1^{-1}L^h u||\leqq C(||\Lambda_1''^{-1/2}Pu||+R^{1/2})+c(\delta)||\Lambda_1''^{1/2}\Lambda_1^{h-1}u||\,.$$

Moreover, applying the above inequality to the interpolation inequality, we have

$$\sum_{j=1}^{h}||\Lambda_1''^{1/2}\Lambda_1^{i-j-1}L^j u||$$
$$\leqq C_\delta(||\Lambda_1''^{-1/2}Pu||+R^{1/2})+c(\delta)||\Lambda_1''^{1/2}\Lambda_1^{h-1}u||\,. \quad \blacksquare$$

Corollary. *We have*

$$R\leqq C\,\gamma^{-2\varepsilon}(||\Lambda_1''^{1/2}\Lambda_1^{h-1}u||^2+||\Lambda_1''^{-1/2}Pu||^2)$$

and

$$V\leqq(c(\delta)+C_\delta\,\gamma^{-2\varepsilon})||\Lambda_1''^{1/2}\Lambda_1^{h-1}u||^2+C_\delta||\Lambda_1''^{-1/2}Pu||^2\,.$$

Lemma 4.7. (see [7]) *There exists κ_0 as follows. For any $0<\kappa<\kappa_0$, there exist positive constants $\{\lambda_j=\lambda_j(\kappa)\}$ such that*

i) $\lambda_{h-1}W_{h-1}+\lambda_{h-3}W_{h-3}+\cdots+\lambda_1 W_1\geqq\kappa^{-1}W_- -\kappa W_+$ *in case* ①, ②

ii) $\lambda_{h-1}W_{h-1}+\lambda_{h-3}W_{h-3}+\cdots+\lambda_2 W_2\geqq\kappa^{-1}W_- -\kappa W_+$ *in case* ③,

iii) $\lambda_{h-1}W_{h-1}+\lambda_{h-3}W_{h-3}+\cdots+\lambda_0 W_0\geqq\kappa^{-1}W_- -\kappa W_+$ *in case* ④.

Proof of Prop. 4.4. From Lemma 4.3, 4.5, 4.6, 4.7, we have

$$\lambda_{h-1}I_{h-1}+\lambda_{h-3}I_{h-3}+\cdots+\lambda_1 I_1-\lambda_0 I_0$$
$$\geqq\kappa^{-1}W_- -\kappa W_+ -C_\kappa(R+V+c(\delta)W)+||\Lambda_1''^{1/2}\Lambda_1^{h-1}u||^2$$

in case ①,

$$\lambda_{h-1}I_{h-1}+\lambda_{h-3}+\cdots+\lambda_1 I_1+\lambda_0 I_0$$
$$\geqq\kappa^{-1}W_- -\kappa W_+ -C_\kappa(R+V+c(\delta)W)+||\Lambda_1''^{1/2}\Lambda_1^{h-1}u||^2$$

in case ②,

$$\lambda_{h-1}I_{h-1}+\lambda_{h-3}I_{h-3}+\cdots+\lambda_1 I_1-\lambda_0 I_0$$
$$\geqq\kappa^{-1}W_- -\kappa W_+ -C_\kappa(R+V+c(\delta)W)+||\Lambda_1''^{1/2}\Lambda_1^{h-1}u||^2$$

in case ③,

$$\lambda_{h-1}I_{h-1}+\lambda_{h-3}I_{h-3}+\cdots+\lambda_1 I_1+\lambda_0 I_0$$
$$\geqq\kappa^{-1}W_- -\kappa W_+ -C_\kappa(R+V+c(\delta)W)+||\Lambda_1''^{1/2}\Lambda_1^{h-1}u||^2$$

in case ④, for $0<\kappa<\kappa_1$ ($W=W_+ + W_-$).

On the other hand, since

$$\sum_{i=0}^{h-1} |I_i| \leqq C||\Lambda_1''^{-1/2} P u|| (||\Lambda_1''^{-1/2} Pu|| + ||\Lambda_1''^{1/2} \Lambda_1^{h-1} u|| + R^{1/2})$$
$$\leqq C_\varepsilon(||\Lambda_1''^{-1/2} P u||^2 + R) + \varepsilon ||\Lambda_1''^{1/2} \Lambda_1^{h-1} u||^2 ,$$

we have

$$\kappa^{-1} W_- - \kappa W_+ + ||\Lambda_1''^{1/2} \Lambda_1^{h-1} u||^2 \leqq C_\kappa(||\Lambda_1''^{-1/2} P u||^2 + R + V + c(\delta) W) .$$

For fixed κ, taking δ_κ small enough, we have

$$\kappa^{-1} W_- - \kappa W_+ + ||\Lambda_1''^{1/2} \Lambda_1^{h-1} u||^2 \leqq C_\kappa ||\Lambda_1''^{-1/2} P u||^2$$

for $0<\delta<\delta_\kappa$ and $\gamma>\gamma_\kappa$. ■

§5. Boundary energy estimates under the uniform Lopatinski condition.

Now we remember the local factorization of A_0 in a conic neighbourhood U of $(\sigma_0, \eta_0', 0)$, discussed in §3:

$$A_0(\tau, \zeta, \eta) = P(\tau, \zeta, \eta) P_1(\tau, \zeta, \eta) \cdots P_d(\tau, \zeta, \eta) ,$$

where P is a polynomial of ζ of order h with (P)-property and P_j is of order h with (H)-property. Let $\tilde{A}_0$ be the extension of A_0 by the representation of the right hand side. Using the usual notation of $P = P_+ \cdot P_-$, we define

$$V_j^\pm = \Lambda_1^{h^\pm - j} \zeta^{j-1} \frac{\tilde{A}_0}{P_\pm} \qquad (j = 1, \cdots, h^\pm) ,$$

$$V_{ij}^+ = \Lambda_1^{h_i^+ - j} L_i^{j-1} \frac{\tilde{A}_0}{P_i} \qquad (j = 1, \cdots, h_i^+, \quad i = 1, \cdots, d) ,$$

$$V_{ij}^- = \Lambda_1^{h_i^- - j} L_i^{h_i^+ + j - 1} \frac{\tilde{A}_0}{P_i} \qquad (j = 1, \cdots, h_i^-, \quad i = 1, \cdots, d)$$

and

$$V_\pm = {}^t(V_1^\pm, \cdots, V_{h^\pm}^\pm, V_{11}^\pm, \cdots, V_{1h_1^\pm}^\pm, \cdots, V_{d1}^\pm, \cdots, V_{dh_d^\pm}^\pm) .$$

Lemma 5.1. *We have*

$$ {}^t(\zeta^{\mu-1}, \Lambda_1 \zeta^{\mu-2}, \cdots, \Lambda_1^{\mu-1})$$
$$= C_+(\tau, \eta) V_+(\tau, \zeta, \eta) + C_-(\tau, \eta) V_-(\tau, \zeta, \eta) ,$$

where

$$C_\pm(\tau, \eta) \in S^{0,1}_{\Lambda_1, \Lambda_1'', q} .$$

Now, we denote

$$B_0 = {}^t\Big(B_{10}^\sharp, \cdots\cdots, B_{\mu_+0}^\sharp \Big) ,$$

then we have

$$B_0(\tau, \zeta, \eta) = B_+(\tau, \eta)\ V_+(\tau, \zeta, \eta) + B_-(\tau, \eta)\ V_-(\tau, \zeta, \eta)\,,$$

where $B_\pm(\tau, \eta) \in S^{0,1}_{\Lambda_1, \Lambda_1'', q}$ and $\det B_+(\tau_0, \eta_0', 0) \neq 0$ from the Uniform Lopatinski condition. Therefore, there exists a conic neighbourhood U of $(\tau_0, \eta_0', 0)$ such that $|\det B_+(\tau, \eta)| \geqq c\ (c>0)$. Let $\tilde{B}_\pm$ be extensions of $B_\pm$ outside of U, preserving the above properties, and let

$$\tilde{B}_0(\tau, \zeta, \eta) = \tilde{B}_+(\tau, \eta)\ V_+(\tau, \zeta, \eta) + \tilde{B}_-(\tau, \eta)\ V_-(\tau, \zeta, \eta)\,,$$

then we have

$$\begin{aligned} &V_+(\tau, \zeta, \eta) \\ &\quad = \tilde{B}_+^{-1}(\tau, \eta)\ \tilde{B}_0(\tau, \zeta, \eta) - \tilde{B}_+^{-1}(\tau, \eta)\ \tilde{B}_-(\tau, \eta)\ V_-(\tau, \zeta, \eta) \\ &\quad = \tilde{C}_+(\tau, \eta)\ \tilde{B}_0(\tau, \zeta, \eta) + \tilde{C}_-(\tau, \eta)\ V_-(\tau, \zeta, \eta)\,, \end{aligned}$$

where $\tilde{C}_\pm \in S^{0,1}_{\Lambda_1, \Lambda_1'', q}$.

Taking κ small enough in Proposition 4.4, we have

Proposition 5.2. *There exist* $C>0$, $\delta_0>0$, $\gamma_0>0$ *such that*

$$\begin{aligned} &\sum_{k+\nu\cdot q \leqq \mu-1} \langle (D_t - i\gamma)^k D_x^\nu u \rangle \\ &\quad + \sum_{k+\nu\cdot q \leqq \mu-1} \|\Lambda_1'^{\prime 1/2} (D_t - i\gamma)^k D_x^\nu u\| \\ &\quad \leqq C(\|\Lambda_1'^{\prime -1/2} \tilde{A}_0 u\| + \langle \tilde{B}_0 u \rangle) \end{aligned}$$

for $u \in H^\infty(R^1 \times R^n_+)$, $0<\delta<\delta_0$ *and* $\gamma>\gamma_0$.

Proof of Theorem 3.1. In the same way as in §2, the proof is carried by applying the above Proposition 5.2 to the finite number of local factorizations of A_0. ■

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