A limit theorem for sums of i.i.d. random variables with slowly varying tail probability

By

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§1. Introduction.

Let X_1, X_2, $\cdots$ be nonnegative independent random variables with common distribution function $F(x)$ belonging to the domain of attraction of stable law with index α ($0<\alpha<1$); i.e.,

$$(1.1) \qquad 1-F(x) \sim 1/(x^\alpha L(x)) \quad \text{as} \quad x\to\infty ,$$

for some slowly varying $L(x)$ ($\geqq 0$). Here $f(x)\sim g(x)$ means that $\lim f(x)/g(x)=1$. Let $S_n=X_1+X_2+\cdots+X_n$, $n=1, 2, \cdots$. It is well known as Skorohod's invariance principle that, for suitably chosen $v(t)$, $S_{[nt]}/v(n)$ converges in law to the one-sided stable process in the function space $D([0, \infty)\to\boldsymbol{R})$ endowed with J_1-topology (see [7] for the definition and [9] for the result).

In this paper we will consider the extreme case of (1.1) as $\alpha\downarrow 0$; we treat the case where

$$(1.2) \qquad 1-F(x) \sim 1/L(x) \quad \text{as} \quad x\to\infty .$$

As is well known, under condition (1.2), every linear normalization $a_n S_n+b_n$ leads to a degenerate limiting distribution. However, if we allow non-linear normalizations, we have

Darling's Theorem ([2]). *If* (1.2) *is satisfied, then*

$$\lim_{n\to\infty} P[\frac{1}{n} L(S_n)\leqq x] = e^{-1/x}, \quad x>0 .$$

(In [2] some technical conditions are assumed but they may be removed.) The convergence of the semi-group of the process $t\wedge\!\!\!\to\dfrac{1}{n} L(S_{[nt]})$ was obtained by S. Watanabe [10], and the purpose of this paper is to show the weak convergence of this process in J_1-topology. Our idea of the proof is quite different from that of Darling or Watanabe and so may be read independently.

§2. Main theorem.

Let $\xi=(\xi(t))_{t\geq0}$ be the symmetric Cauchy process such that $E[e^{i\theta\xi(t)}]=e^{-\pi t|\theta|}$, $\theta\in\boldsymbol{R}$. In other words, ξ is the Lévy process with Lévy measure $x^{-2}dx$. Define

$$(2.1) \qquad m(t) = \max_{0<s\leq t} \varDelta\xi(s) \quad \text{where} \quad \varDelta\xi(s) = \xi(s)-\xi(s-) .$$

Thus $m=(m(t))_{t\geq0}$ $(m(0)=0)$ is the maximum process of the Poisson point process with intensity measure $x^{-2}dx\,dt$, and therefore it is easy to compute the finite-dimensional marginal distributions; for $0\leq t_1<\cdots<t_n$ and $0\leq x_1<\cdots<x_n$,

$$(2.2) \qquad P[m(t_1)\leq x_1, \cdots, m(t_n)\leq x_n] = F(x_1)^{t_1} F(x_2)^{t_2-t_1} \cdots F(x_n)^{t_n-t_{n-1}} ,$$

where $F(x)=e^{-1/x}$, $x>0$. Processes with the property (2.2) are often referred to as *extremal processes* (see [3]). Our main result is

Theorem 2.1. *Let $L(x)\geq0$, $x\geq0$, be a nondecreasing function varying slowly at infinity. Then, under the assumption* (1.2),

$$\frac{1}{n} L(S_{[nt]}) \overset{\mathscr{D}}{\to} m(t) \quad as \quad n\to\infty \quad in \quad D([0, \infty)\to\boldsymbol{R})$$

where $\overset{\mathscr{D}}{\to}$ denotes the weak convergence in J_1-topology.

Proof. Since the convergence of finite-dimensional marginal distributions are obtained by [10], it remains to prove the tightness of the processes. However, it does not seem easy to check the well-known conditions for tightness such as Chentsov's moment condition (see page 128 of [1]), so we will adopt a direct method: On the probability space where ξ (and hence m) is defined we will construct processes ζ_n, $n=1, 2, \cdots$ which are distributed like $(1/n)L(S_{[nt]})$, $n=1, 2, \cdots$ and which converge to m almost surely in J_1-topology.

Now let $\eta(t)=\sum_{s\leq t} (\varDelta^+\xi(s))^2$ where $\varDelta^+\xi(s)=\max\{\varDelta\xi(s), 0\}$. In other words, $\eta(t)=\int_0^{t+}\int_{x>0} x^2 N(du\,dx)$, $N(du\,dx)$ being the Poisson random measure defined by the jumps of the Cauchy process ξ. Clearly $\eta=(\eta(t))_{t\geq0}$ is a one-sided stable process with index $1/2$ and the Lévy measure is given by $\mu[x, \infty)=x^{-1/2}$, $x>0$. Observe that $m(t)$ may be expressed by using η instead of ξ;

$$(2.3) \qquad m(t) = \{\max_{0\leq s\leq t} \varDelta\eta(s)\}^{1/2}, \quad \varDelta\eta(s) = \eta(s)-\eta(s-) .$$

Indeed this is immediate from (2.1) and the definition of η. Let $F_0(x)=P[\eta(1)\leq x]$, $x\in\boldsymbol{R}$. It is well known that

$$(2.4) \qquad 1-F_0(x) \sim x^{-1/2} \quad as \quad x\to\infty .$$

This may be confirmed as follows. Since $\eta(n)/n^2$ clearly converges in law to $\eta(1)$,

appealing to Theorem 4 in page 124 of [5] we have that for every $x>0$,

$$n\{1-F_0(n^2 x)\} \to x^{-1/2} \quad \text{as} \quad n\to\infty ,$$

which proves (2.4). We also define $\phi(x)=F^{-1}(F_0(x))$ $(x\in\mathbf{R})$. Throughout the inverse of a nondecreasing function is chosen so that it is right-continuous;

$$F^{-1}(x) = \inf\{u: F(u)>x\}, \quad 0<x<1 .$$

Let, for $n=1, 2, \cdots$,

(2.5) $$\eta_{ni} = \eta\left(\frac{i}{n}\right)-\eta\left(\frac{i-1}{n}\right), \quad i = 1, 2, \cdots ,$$

(2.6) $$\zeta_n(t) = \frac{1}{n} L(\textstyle\sum_{i\leq nt} \phi(n^2 \eta_{ni})), \quad t\geq 0 ,$$

(2.7) $$m_n(t) = \begin{cases} (\max_{i\leq nt} \eta_{ni})^{1/2}, & t\geq 1/n \\ 0, & 0\leq t<1/n . \end{cases}$$

We are now ready to explain the story of the proof. We will prove the following three.

(1) $\qquad \zeta_n = (\zeta_n(t))_{t\geq 0}$ is identical in law to

$$\left(\frac{1}{n} L(\textstyle\sum_{i\leq nt} X_i)\right)_{t\geq 0}, \quad n = 1, 2, \cdots .$$

(2) $\qquad m_n \to m \quad (in \ J_1) \text{ a.s.}$

(3) $\qquad \sup_{0\leq t\leq T} |\zeta_n(t)-m_n(t)| \to 0, \quad \text{a.s., for every } T>0.$

Indeed, (2) and (3) imply that $\zeta_n\to m$ (in J_1) a.s., which together with (1) clearly proves our Theorem.

The proof of (1) and (2) are easy. To see (1) it suffices to show that $\phi(\eta(1))$ (and hence $\phi(n^2 \eta_{ni})$) is distributed like X_1. However, this is obvious from the definition $\phi(x)=F^{-1}(F_0(x))$. In fact ϕ was defined so that (1) holds. To see (2) define $\eta_n(t)=\eta([nt]/n)$, $n=1, 2, \cdots$. Since $\eta_n(t)\to\eta(t)$ (in J_1), we have from the continuity theorem that

$$\max_{i\leq nt} \eta_{ni} = \max_{0\leq s\leq t} \Delta\eta_n(t) \to \max_{0\leq s\leq s} \Delta\eta(t) \quad (in \ J_1) .$$

Therefore, considering the square root of each side, we obtain from (2.3) and (2.7) that $m_n(t)\to m(t)$ (in J_1) a.s., which is the desired result.

For the proof of (3) we need the next two key lemmas.

Lemma 2.2. *For every $\lambda>0$ and $T>0$,*

$$\lim_{n\to\infty} \sup_{0\leq x\leq T} |\frac{1}{n} L(\lambda\phi(n^2 x^2))-x| = 0 .$$

Lemma 2.3. *For every* $\varepsilon > 0$ *and* $T > 0$ ($\varepsilon < T$) *and for almost sure* ω, *there exist* K_1, $K_2 > 0$ *and* n_0 *depending on* $\{\varepsilon, T, \omega\}$ *such that, for every* $t \in [\varepsilon, T]$ *and* $n \geq n_0$,

$$(2.8) \qquad K_1 \, \phi(n^2 \, m_n(t)^2) \leq \sum_{i \leq nt} \phi(n^2 \, \eta_{ni}) \leq K_2 \, \phi(n^2 \, m_n(t)^2) \, .$$

We postpone the proof until next section and return to the proof of (3). Since $L(x)$ is nondecreasing, applying $L(\cdot)$ to each side of (2.8) and dividing by n, we obtain

$$\frac{1}{n} \, L(K_1 \, \phi(n^2 \, m_n(t)^2)) \leq \zeta_n(t) \leq \frac{1}{n} \, L(K_2 \, \phi(n^2 \, m_n(t)^2)) \, .$$

Therefore, we have

$$(2.9) \qquad \sup_{\varepsilon \leq t \leq T} |\zeta_n(t) - m_n(t)|$$

$$\leq \sup_{0 \leq t \leq T} \max_{i=1,2} |\frac{1}{n} \, L(K_i \, \phi(n^2 \, m_n(t)^2) - m_n(t)|$$

$$\leq \max_{i=1,2} \sup_{0 \leq x \leq m_n(T)} |\frac{1}{n} \, L(K_i \, \phi(n^2 \, x^2)) - x|$$

which converges to 0 a.s. by Lemma 2.2. and (2). Since $\zeta_n(t)$ and $m_n(t)$ are monotone in t, we also see

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \sup_{0 \leq t \leq \varepsilon} |\zeta_n(t) - m_n(t)|$$

$$\leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} (\zeta_n(\varepsilon) + m_n(\varepsilon))$$

$$= \lim_{\varepsilon \to 0} 2m(\varepsilon) = 0 \quad \text{a.s.}$$

This completes the proof of (3) and hence of Theorem 2.1.

Remark 2.4. We have a very simple proof for the special case where $L(x) = \log(x+1)$. Let $M_0 = 0$ and $M_n = \max(X_1, \cdots, X_n)$, $n \geq 1$. Then we have

$$(2.10) \qquad \frac{1}{n} \, L(M_{[nt]}) \leq \frac{1}{n} \, L(S_{[nt]}) \leq \frac{1}{n} \, L(nM_{[nt]}) \, .$$

Since $\log(nx) = \log n + \log x$, it is easy to see that the first and the third process (and hence the second) should have the same limiting processes. Thus our problem may be reduced to the study of M_n, which was treated by [4] and [6]. However, for general slowly varying $L(x)$, we cannot apply this argument. For example, let $L(x) = \exp\sqrt{\log(x+1)} \cdot$ $L(x)$ is a slowly varying function and we can show that the first and the second processes in (2.10) converge to $m(t)$ but the third to $\sqrt{e} \, m(t)$.

§3.   Proofs of Lemmas.

Proof of Lemma 2.2.   Since $\phi(x)=F^{-1}(F_0(x))$, we have $1-F(\phi(x))=1-F_0(x)$ provided that $F(x)$ is continuous.   In general it holds that

$$(3.1) \qquad 1-F(\phi(x)) \leqq 1-F_0(x) \leqq 1-F(\phi(x)/2) \;.$$

By (1.2) both $1-F(\phi(x))$ and $1-F(\phi(x)/2)$ are asymptotically equal to $1/L(\phi(x))$ as $x \to \infty$, while $1-F_0(x) \sim x^{-1/2}$ (see (2.4)).   Therefore, (3.1) implies that $L(\phi(x)) \sim x^{1/2}$, or equivalently, $L(\phi(x^2)) \sim x$ as $x \to \infty$.   This also proves that

$$(3.2) \qquad \lim_{x \to \infty} L(\lambda\phi(x^2))/x = 1, \;\; \lambda > 0 \;,$$

since $L(x)$ is a slowly varying function.   Thus we have for every $\lambda > 0$ and $x \geqq 0$,

$$(3.3) \qquad \lim_{n \to \infty} \frac{1}{n} L(\lambda\phi(n^2 x^2)) = x \;.$$

Since $\dfrac{1}{n} L(\lambda\phi(n^2 x^2))$ is monotone in x and the limiting function is continuous, the convergence in (3.3) is automatically uniform for x on every finite interval, which completes the proof.

Proof of Lemma 2.3.   It is easy to find K_1; $K_1 = 1$ is the desired number.   To find K_2, define $\tilde{L}(x)=F_0^{-1}(F(x))$ $(=\phi^{-1}(x))$.   By (2.4) we see that $F_0^{-1}(x) \sim 1/(1-x)^2$ as $x \to 1-0$.   Therefore, $\tilde{L}(x) \sim 1/(1-F(x))^2 \sim L(x)^2$ as $x \to \infty$, from which it follows that $\tilde{L}(x)$ is also a slowly varying function, and so has the canonical representation $\tilde{L}(x)=c(x) \exp \displaystyle\int_1^x \frac{\varepsilon(t)}{t}\,dt$, where $c(x) \to c > 0$ and $\varepsilon(x) \to 0$ as $x \to \infty$ (see [8] page 2).   Therefore $\phi(x)=\tilde{L}^{-1}(x)$ may be expressed as

$$(3.4) \qquad \phi(x) = \exp \int_1^{q(x)x} \frac{\bar{\varepsilon}(t)}{t}\,dt$$

where $\bar{\varepsilon}(x) \to \infty$ and $q(x) \to 1/c$ as $x \to \infty$.   Using (3.4) we easily see that, for every $a > 0$,

$$(3.5) \qquad \lim_{n \to \infty} n^2 \phi(n^2 a)/\phi(2n^2 a) = 0 \;.$$

Now let $a = a(\omega) = m(\varepsilon)^2/3$.   Note that $a > 0$, *a.s.*   Since $m_n(\varepsilon) \to m(\varepsilon)$ a.s., we may and do assume that there exists $n_1 > 0$ such that

$$(3.6) \qquad m_n(t)^2 \geqq m_n(\varepsilon)^2 \geqq 2a, \;\; t \geqq \varepsilon, \;\; n \geqq n_1 \;.$$

We divide $\sum_{i \leqq nt} \phi(n^2 \eta_{ni})$ into three parts;

$$S_{1,n}(t) = \sum_{i \leqq nt} \phi(n^2 \eta_{ni}) \, I(\eta_{ni} < n^{-2}) \;,$$

$$S_{2,n}(t) = \sum_{i \leqq nt} \phi(n^2 \eta_{ni}) \, I(n^{-2} \leqq \eta_{ni} \leqq a) \;,$$

$$S_{3,n}(t) = \sum_{i \leqq nt} \phi(n^2 \eta_{ni}) \, I(\eta_{ni} > a) \;.$$

Here, $I(\cdot)$ denotes the indicator function. Since ϕ is monotone, we clearly have

$$(3.7) \qquad\qquad S_{1,n}(t) \leq nT\phi(1), \quad t \leq T.$$

On the other hand, using (3.4), we see that $\lim_{n\to\infty} n/\phi(2n^2 a)=0$. Therefore, by (3.7) and (3.6) there exists $n_2 > 0$ such that

$$(3.8) \qquad S_{1,n}(t) \leq \phi(2n^2 a) \leq \phi(n^2 m_n(t)^2), \quad \varepsilon \leq t \leq T, \ n > n_2.$$

We next consider $S_{2,n}(t)$. If $n^{-2} \leq x \leq a$, then we have

$$(3.9) \qquad\qquad 0 \leq \phi(n^2 x) \leq \phi(n^2 a)\, n^2 x.$$

Combining this with (3.5), we see that there exists $n_3 > 0$ such that

$$(3.10) \qquad\qquad \phi(n^2 x) \leq \phi(2n^2 a)\, x, \quad n \geq n_3.$$

Therefore, we obtain that for $\varepsilon \leq t \leq T$ and $n \geq n_3$,

$$(3.11) \qquad 0 \leq S_{2,n}(t) \leq \phi(2n^2 a) \sum_{i \leq nt} \eta_{ni}$$

$$= \phi(2n^2 a)\, \eta([nt]/n) \leq \phi(n^2 m_n(t)^2)\, \eta(T).$$

In the last inequality we used (3.6). Finally let us consider $S_{3,n}(t)$. Notice that there are only finitely many $t \in [0, T]$ such that $\Delta\eta(t) > a$. Thus if we denote by $K_3 = K_3(\omega)$ the number of such t's, then $\sum_{i \leq nT} I(\eta_{ni} > a) = K_3$ for all sufficiently large n (a.s.). Thus we have

$$(3.12) \qquad 0 \leq S_{3,n}(t) \leq K_3 \max_{i \leq nt} \phi(n^2 \eta_{ni})$$

$$= K_3\, \phi(n^2 m_n(t)^2), \quad \varepsilon \leq t \leq T, \ n \geq n_4.$$

See (2.7) for the last equality. Now combining (3.8), (3.11) and (3.12) we obtain

$$(3.13) \qquad \sum_{i \leq nt} \phi(n^2 \eta_{ni}) \leq K_2\, \phi(n^2 m_n(t)^2), \quad \varepsilon \leq t \leq T, \ n \geq n_0,$$

where $K_2 = 1 + \eta(T) + K_3$ and $n_0 = \max(n_2, n_3, n_4)$.
Thus Lemma 2.3 is proved.

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