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On the maximal p-ramified p-abelian extensions over Z_p^d-extensions

By

Masaru UEDA

Introduction.

Let p be an odd prime number and k be a finite algebraic number field. Let k_∞ be a $\boldsymbol{Z}_p^d$-extension of k, that is, a Galois extension of k with the Galois group $G(k_\infty/k)$ isomorphic to a product of d copies of the additive group of p-adic integers $\boldsymbol{Z}_p$. Let $M(k_\infty)$ denote the maximal p-ramified p-abelian extension of k_∞. Here, we say that a Galois extension is p-ramified if it is unramified outside the set of all prime divisors lying over p. Moreover, let $\tilde{X}(k_\infty)$ denote the Galois group of $M(k_\infty)$ over k_∞ and Λ_G the complete group ring of $G=G(k_\infty/k)$ over $\boldsymbol{Z}_p$. Then we can consider $\tilde{X}(k_\infty)$ as a finitely generated Λ_G-module. Let $\rho(k_\infty)$ denote the rank of $\tilde{X}(k_\infty)$ as a Λ_G-module and $r_2(k)$ the number of complex places of k.

Our main purpose in this paper is to study the following question.

(1) $$\rho(k_\infty) = r_2(k) \quad \text{for any} \quad \boldsymbol{Z}_p^d\text{-extension } k_\infty/k\ ?$$

On this question, Babaicev [1] and Greenberg [9] showed that this equality (1) is valid for almost all $\boldsymbol{Z}_p$-extensions of k. But for a given $\boldsymbol{Z}_p$-extension, only following criterions of the equality (1) are known:

(a) *k_∞/k is the cyclotomic $\boldsymbol{Z}_p$-extension.* (Iwasawa [11], Greenberg [8]).
(b) *Leopoldt's conjecture is valid for k.* (Greenberg [9]).
(c) *k_∞ contains a $\boldsymbol{Z}_p$-extension of k for which the equality* (1) *is valid* ([9]).

We shall first give another criterion under a certain assumption in section 1. The result is as follows.

Theorem I. *Let k be a finite algebraic number field and k_∞ be a $\boldsymbol{Z}_p$-extension of k. Assume that k contains a primitive p-th root of unity. If there exists no prime ideal of k over p which splits completely in k_∞ and if Iwasawa's μ-invariant $\mu(k_\infty/k)$ is zero, then we have*

$$\rho(k_\infty) = r_2(k)\,.$$

In this case we shall also give a necessary and sufficient condition for $\rho(k_\infty) = r_2(k)$ in terms of invariants of k_∞/k (see Corollary 1.10 in section 1).

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Next we shall consider the behavior of $\rho(k_\infty)$ when k varies in a family of finite algebraic number fields. Let H_∞ be a $\boldsymbol{Z}_p$-extension of k such that $k_\infty \cap H_\infty = k$. For all positive integers n, let H_n denote the fixed subfield of H_∞ for $G(H_\infty/k)^{p^n}$. Then we have a family of $\boldsymbol{Z}_p^d$-extensions $k_\infty H_n/H_n$ $(n=0, 1, 2\cdots)$. Let $M(k_\infty H_n)$ be the maximal p-ramified p-abelian extension of $k_\infty H_n$ and $\tilde{X}(k_\infty H_n)$ be its Galois group over $k_\infty H_n$. Then, every $\tilde{X}(k_\infty H_n)$ can be considered as a Λ_G-module. Let $\rho(k_\infty H_n)$ denote the rank of $\tilde{X}(k_\infty H_n)$ as a Λ_G-module. In this situation, we shall prove the following theorem in section 2.

Theorem II. *There exist non negative integers* ρ *and* $c=c(k_\infty, H_\infty)$, *which are independent of* n, *such that*

$$\rho(k_\infty H_n) = \rho\, p^n + c\,,$$

for all sufficiently large n.

We shall give an example for Theorem I in section 1 and two sufficient conditions for the constant c in Theorem II to vanish in section 2.

Notation and Terminology.

We shall use the notation in Introduction. Let μ_n denote the group consisting of all p^n-th roots of unity for a positive integer n and $\boldsymbol{F}_p$ the finite field with p elements. We use the following notations for any $\boldsymbol{Z}$-module A.

$$(A)_p = \{A \ni a \mid pa = 0\}\,,\ {}_p(A) = A/pA\,.$$

Moreover, if A is a discrete module, $r_p(A) = \dim_{\boldsymbol{F}_p}(A)_p$ will be called the p-rank of A and if A is a compact module, $r_p(A) = \dim_{\boldsymbol{F}_p}\, {}_p(A)$ will be called the p-rank of A.

Let $Q(R)$ denote the quotient field for any integral domain R. Then, for any R-module M, $\mathrm{rk}_R(M) = \dim_{Q(R)}(M \otimes_R Q(R))$ will be called the R-rank of M. Let $\mathrm{Ann}_R(M) = \{R \ni a \mid aM = 0\}$ and $\mathrm{Tor}_R(M) = \{M \ni m \mid am = 0\ (R \ni a \neq 0)\}$.

Let $C(k)$ denote the ideal class group of k and $C_0(k)$ the subgroup of $C(k)$ generated by all ideal classes containing prime ideals over p.

§1. The proof of Theorem I.

Throughout this section, we assume that $d=1$ and k contains μ_1. Let k_∞ be a $\boldsymbol{Z}_p$-extension of k and σ be a topological generator of $G = G(k_\infty/k)$. Let Λ_G denote the complete group ring of G over $\boldsymbol{Z}_p$. Then, we can identify Λ_G with the power series ring $\boldsymbol{Z}_p[[T]]$ in such a way as σ corresponds to $1+T$. From now on, we fix this identification.

Let k_n denote the unique cyclic extension of k of degree p^n in k_∞ for all $n \geqq 0$. Let $\tilde{X}^*(k_n)$ denote the Pontrjagin's dual group of $\tilde{X}(k_n)$. Then, $\tilde{X}(k_n)$ is a finitely generated $\boldsymbol{Z}_p$-module (see [11] Theorem 2), therefore ${}_p(\tilde{X}(k_n))$ is a finite p-abelian group. The dual group of ${}_p(\tilde{X}(k_n))$ is just $(\tilde{X}^*(k_n))_p$. Hence, we obtain

$$r_p(\tilde{X}(k_n)) = r_p(\tilde{X}^*(k_n))\,. \tag{2}$$

We shall give formulas for $r_p(\tilde{X}(k_n))$ and $r_p(\tilde{X}^*(k_n))$ in Proposition 1.6 and 1.7. Theorem I follows from these propositions and (2).

We know the following result.

Proposition 1.1. (Bertrandias-Payan [2])

$$r_p(\tilde{X}^*(k)) = g(k)+r_2(k)+r_p(P(k)), \tag{3}$$

where $g(k)$ is the number of prime ideals of k over p and $P(k)=C(k)/C_0(k)$.

Replacing k with k_n in (3), we have

$$r_p(\tilde{X}^*(k_n)) = g(k_n)+r_2(k_n)+r_p(P(k_n)). \tag{4}$$

We shall rewrite this formula (4) in such a way that we can see the asymptotic behavior of $r_p(\tilde{X}^*(k_n))$ as n grows.

Since any place outside p is unramified in any $\boldsymbol{Z}_p$-extension of k, we have

$$r_2(k_n) = r_2(k)\, p^n. \tag{5}$$

Next we shall consider the part of $g(k_n)$.

Proposition 1.2. *Let $\beta(k_\infty)$ be the number of the prime ideals of k over p which splits completely in k_∞. There is a non negative integer $\gamma(k_\infty)$, which is independent of n, such that*

$$g(k_n) = \beta(k_\infty)\, p^n+\gamma(k_\infty), \tag{6}$$

for all sufficiently large n.

Proof. Let A_i $(1\leqq i\leqq g(k))$ denote the prime ideals of k over p. Then, the number of prime ideals in k_n over p is the sum of the number $g_i(k_n)$ of prime ideals in k_n over A_i over all i. Let Z_i denote the decomposition group of A_i. Then, we have

$$g_i(k_n) = |G(k_n/k)|/|Z_i G(k_\infty/k_n): G(k_\infty/k_n)|.$$

Since Z_i is a closed subgroup of $G(k_\infty/k)\simeq\boldsymbol{Z}_p$, we have either $Z_i=\{0\}$ or $p^m\boldsymbol{Z}_p$ for some $m\geqq 0$. We have $Z_i=\{0\}$ if and only if A_i splits completely in k_∞. In this case, we have $g_i(k_n)=p^n$. On the other hand, if $Z_i\simeq p^m\boldsymbol{Z}_p$ for some $m\geqq 0$, then $g_i(k_n)=p^m$ if $n\geqq m$. So, the proof is complete.

Remark. In the above proof, if we use the inertia group in place of the decomposition group, we obtain the following result: *there exists a non negative integer e, such that each prime ideal of k_e over p is either unramified or totally ramified in k_∞.* Until the last of this section, we shall fix this number e.

Next, we shall consider the part of $P(k_n)$. By means of Artin map, $C_0(k)$ is mapped onto the composite group of decomposition groups of all prime ideals over p in k. Hence, we have the isomorphism $P(k)\simeq G(L'(k)/k)$, where for any algebraic extension K over $\boldsymbol{Q}$, we denote by $L'(K)$ the maximal unramified p-abelian extension of K in which every prime divisor of K over p splits completely. Now, we shall consider $X'=G(L'(k_\infty)/k_\infty)$. This Galois group is a finitely generated torsion Λ_G-module (see Iwasawa [11]). Using the structure theorem of a finitely generated Λ_G-module (see Bourbaki [3], chaptre 7, §4, n°4), we have the following decomposition of X'.

$$X' \sim \left(\bigoplus_{i=1}^{a'(k_\infty)} \Lambda_G/p^{c_i} \right) \oplus \left(\bigoplus_{j=1}^{b'(k_\infty)} \Lambda_G/(f_j^{d_j}) \right), \tag{7}$$

where, $a'(k_\infty)$ and $b'(k_\infty)$ are non negative integers, c_i and d_j are positive integers, f_j are distinguished polynomials and "$\sim$" denotes a pseudo-isomorphism from X' to the right-hand side of (7).

Then, we can prove the following proposition.

Proposition 1.3. *We have*

$$r_p(P(k_n)) = a'(k_\infty)\, p^n + O(1)\,, \tag{8}$$

where, $O(1)$ means the bounded number as n grows.

Proof. Let $Y'=G(L'(k_\infty)/k_\infty L'(k_e))$. It follows from Iwasawa [11] Theorem 8 that Y' is the Λ_G-submodule of X' and $G(L'(k_n)/k_n)\simeq G(k_\infty L'(k_n)/k_\infty)=X'/\nu_{e,n} Y'$ for all $n\geqq e$, where $\nu_{e,n}=1+\sigma^{p^e}+\sigma^{2p^e}+\cdots+\sigma^{(p^{n-e}-1)p^e}\in\Lambda_G$. Therefore, $r_p(P(k_n))=r_p(G(L'(k_n)/k_n))=r_p(X'/\nu_{e,n} Y')$. From the exact sequence:

$$0 \to Y'/\nu_{e,n} Y' \to X'/\nu_{e,n} Y' \to X'/Y' \to 0\,,$$

it follows that

$$r_p(Y'/\nu_{e,n} Y')\leqq r_p(X'/\nu_{e,n} Y')\leqq r_p(Y'/\nu_{e,n} Y')+r_p(X'/Y')\,.$$

Since $r_p(X'/Y')$ is obviously independent of n, we have the following formula.

$$r_p(P(k_n)) = r_p(Y'/\nu_{e,n} Y')+O(1)\,. \tag{9}$$

Now we shall calculate $r_p(Y'/\nu_{e,n} Y')$. For simplicity, let E denote the right-hand side of (7). We need the following two lemmas.

Lemma 1.4. *We have*

$$r_p(E/\nu_{e,n} E) = a'(k_\infty)\, p^n+O(1) \quad \text{for all} \quad n\geqq 0\,.$$

Proof. It is sufficient to calculate the p-rank of each direct summand. Since $\Lambda_G/(p^{c_i}, \nu_{e,n})\otimes_{Z_p} \boldsymbol{F}_p=\Lambda_G/(p, \nu_{e,n})$ and $\nu_{e,n}\equiv T^{p^n-p^e} \bmod p\Lambda_G$, we obtain

$$\Lambda_G/(p, \nu_{e,n}) = \Lambda_G/(p, T^{p^n-p^e}) \simeq \boldsymbol{F}_p^{p^n-p^e}\,.$$

From the definition of distinguished polynomial, it follows that $\Lambda_G/(f_j^{d_j}, \nu_{e,n})\otimes_{Z_p} \boldsymbol{F}_p \simeq\Lambda_G/(p, T^{d_j \deg(f_j)}, T^{p^n-p^e})$. If $p^n-p^e\geqq d_j \deg(f_j)$, the last group is isomorphic to

$F_p^{d_j \deg(f_j)}$. Combining these results, we obtain the assertion of Lemma 1.4.

Lemma 1.5. *Let X_1, X_2 be finitely generated torsion Λ_G-modules. Assume that X_1 is pseudo-isomorphic to X_2. Then, we have*

$$r_p(X_1/\nu_{e,n} X_1) - r_p(X_2/\nu_{e,n} X_2) = O(1) .$$

Proof. Since both X_1 and X_2 are torsion Λ_G-modules, $X_1 \sim X_2$ is equivalent to $X_2 \sim X_1$ (see Cuoco-Monsky [6]). Therefore it is sufficient to show that $r_p(X_1/\nu_{e,n} X_1) - r_p(X_2/\nu_{e,n} X_2)$ is bounded to the above. Let N_1 and N_2 denote the kernel and cokernel of the pseudo-isomorphism $X_1 \to X_2$. Then we resolve the exact sequence $0 \to N_1 \to X_1 \to X_2 \to N_2 \to 0$ into the following short exact sequences: $0 \to N_1 \to X_1 \to X_1' \to 0$ and $0 \to X_1' \to X_2 \to N_2 \to 0$. Taking the reduction of these sequences modulo $(p, \nu_{e,n})$ and estimating the p-rank of each term, we obtain $r_p(N_2) \geqq r_p(X_2/\nu_{e,n} X_2) - r_p(X_1/\nu_{e,n} X_1)$. Since $r_p(N_2)$ is independent of n and finite, the proof of Lemma 1.5 is complete.

We return to the proof of the proposition. Since $X'/Y' = G(L'(k_e)/k_e)$ is a finite group, we have $X' \sim Y'$. We have of course $X' \sim E$, so, $Y' \sim E$. From Lemma 1.4 and 1.5, it follows that $r_p(Y'/\nu_{e,n} Y') = r_p(E/\nu_{e,n} E) + O(1) = a'(k_\infty) p^n + O(1)$. Combining this result with formula (9), we obtain the proposition.

Combining (4), (5), (6), (8), we obtain the following expression of the asymptotic behavior of $r_p(\tilde{X}^*(k_n))$ as n grows.

Proposition 1.6. *We have*

$$r_p(\tilde{X}^*(k_n)) = (r_2(k) + a'(k_\infty) + \beta(k_\infty))\, p^n + O(1) , \tag{10}$$

for all $n \geqq 0$.

Now, we shall calculate $r_p(\tilde{X}(k_n))$. Since $\tilde{X}(k_\infty)$ is a finitely generated Λ_G-module, we have the following decomposition of $\tilde{X}(k_\infty)$,

$$\tilde{X}(k_\infty) \sim \Lambda_G^{\rho(k_\infty)} \oplus \left(\bigoplus_{i=1}^{s(k_\infty)} \Lambda_G/(p^{n_i}) \right) \oplus \left(\bigoplus_{j=1}^{t(k_\infty)} \Lambda_G/(g_j^{m_j}) \right) , \tag{11}$$

where $\rho(k_\infty)$, $s(k_\infty)$ and $t(k_\infty)$ are non negative integers, n_i and m_j are positive integers and g_j are distinguished polynomials. Then we can prove the following proposition.

Proposition 1.7. *We have*

$$r_p(\tilde{X}(k_n)) = (\rho(k_\infty) + s(k_\infty))\, p^n + O(1), \quad \text{for all} \quad n \geqq 0 . \tag{12}$$

Proof. For simplicity, we shall write $\tilde{X} = \tilde{X}(k_\infty)$ in this proof. We first consider the exact sequence of Galois groups: $0 \to G(M(k_n)/k_\infty) \to \tilde{X}(k_n) \to G(k_\infty/k_n) \to 0$. Since $G(k_\infty/k_n)$ is isomorphic to $\boldsymbol{Z}_p$, this exact sequence splits as a sequence of $\boldsymbol{Z}_p$-modules. We know also $G(M(k_n)/k_\infty) \simeq \tilde{X}/\omega_n \tilde{X}$ as a Λ_G-module, where $\omega_n =$

$\sigma^{p^n}-1\in\Lambda_G$ (see Iwasawa [11]). So, we obtain

$$r_p(\tilde{X}(k_n)) = r_p(\tilde{X}/\omega_n\tilde{X})+1\,. \tag{13}$$

Next, we shall consider $\tilde{X}/\omega_n\tilde{X}$. Let E denote the right-hand side of the decomposition (11) and N_1 and N_2 denote the kernel and cokernel of the pseudo-isomorphism $X\to E$. Then we resolve the decomposition (11) into the following short exact sequences: $0\to N_1\to\tilde{X}\to\tilde{X}'\to 0$ and $0\to\tilde{X}'\to E\to N_2\to 0$. By means of these exact sequences, we obtain the following estimation of the p-ranks,

$$r_p(\tilde{X}/\omega_n\tilde{X})+r_p(N_2/\omega_n N_2)\geqq r_p(E/\omega_n E)\,. \tag{14}$$

Since $\omega_n\equiv T^{p^n}\bmod(p\Lambda_G)$, in the same manner as the proof of Lemma 1.4, we can prove $r_p(E/\omega_n E)=(\rho(k_\infty)+s(k_\infty))\,p^n+O(1)$. Therefore we obtain

$$r_p(\tilde{X}/\omega_n\tilde{X})\geqq(\rho(k_\infty)+s(k_\infty))\,p^n+O(1)\,. \tag{15}$$

Here, we need the next lemma to show the converse inequality.

Lemma 1.8. *Let $R=\boldsymbol{Z}_p[[T_1,\cdots,T_d]]$ and M be a finitely generated torsion-free R-module. Assume that α is neither a zero nor a unit in R. Then there is a $\lambda\in R$ and a R-free submodule M' of M such that λ is prime to α and $\lambda M\subseteqq M'$.*

Proof. Let $P_1,\cdots,P_n$ denote all prime ideals containing α of height 1 and $S=R-\bigcup_{i=1}^{n}P_i$. Then the quotient ring R_S is a Dedekind domain. Since R is a U.F.D., for any prime ideal P of height 1, there is an element f in R such that $P=fR$. So, $P\cap S=\phi$ if and only if f divide α, hence R_S has only finite number of prime ideals. Therefore R_S is a principal ideal domain. Since $M\otimes_R R_S$ is a finitely generated torsion-free R_S-module, it is a R_S-free module. Now we can take a basis of $M\otimes_R R_S$ over R_S from $M\otimes 1$. Let $\{m_i\otimes 1\,|\,1\leqq i\leqq\rho\}$ be a basis. We define $M'=\bigoplus_{i=1}^{\rho}Rm_i$, so M' is a R-free module. It follows from the definition $M'\otimes_R R_S=M\otimes_R R_S$, hence $(M/M')\otimes_R R_S=0$. Since (M/M') is a finitely generated R-module, there is a $\lambda\in S$ such that $\lambda(M/M')=0$. So, the proof of the lemma is complete.

We return to the proof of the proposition. Let $T=\mathrm{Tor}_{\Lambda_G}\tilde{X}$ and $Z=\tilde{X}/T$. Then by means of the exact sequence: $0\to T\to\tilde{X}\to Z\to 0$, we obtain the following estimation of the p-ranks.

$$r_p(T/\omega_n T)+r_p(Z/\omega_n Z)\geqq r_p(\tilde{X}/\omega_n\tilde{X})\,. \tag{16}$$

We shall first consider the part of Z. We can apply Lemma 1.8 to Z and p. Therefore there is a $\lambda\in\Lambda_G-p\Lambda_G$ and a Λ_G-free submodule Z' of Z such that $\lambda Z\subseteqq Z'$. So, Z/Z' is a torsion Λ_G-module and $\mathrm{Ann}_{\Lambda_G}(Z/Z')\ni\lambda$, hence $\mathrm{Ann}_{\Lambda_G}(Z/Z')\subsetneqq p\Lambda_G$.

So, using the structure theorem, we have the decomposition:

$Z/Z'\sim\bigoplus_{i=1}^{a}\Lambda_G/(P_i^{b_i})$, where a is a non negative integer, b_i are positive integers and P_i are the prime ideals of height 1 of Λ_G which are not in $p\Lambda_G$. Therefore, using the method of the proof of Lemma 1.4 and 1.5, we obtain

$$(17)\qquad r_p((Z/Z')/\omega_n(Z/Z')) = O(1)\,.$$

On the other hand, we can easily show that $\rho(k_\infty)=\mathrm{rk}_{\Lambda_G}\tilde{X}=rk_{\Lambda_G}Z=rk_{\Lambda_G}Z'$. Hence, $Z'\simeq\Lambda_G^{\rho(k_\infty)}$. Therefore we obtain

$$(18)\qquad r_p(Z'/\omega_n Z') = \rho(k_\infty)\,p^n\,.$$

Next, by means of the exact sequence: $0\rightarrow Z'\rightarrow Z\rightarrow Z/Z'\rightarrow 0$, we obtain the estimation of p-ranks,

$$(19)\qquad r_p(Z'/\omega_n Z')+r_p((Z/Z')/\omega_n(Z/Z'))\geqq r_p(Z/\omega_n Z)$$

Combining (17), (18), (19), we obtain

$$(20)\qquad r_p(Z/\omega_n Z)\leqq\rho(k_\infty)\,p^n+O(1)\,.$$

Obviously, T is pseudo-isomorphic to the Λ_G-torsion part of E. Therefore, we obtain

$$(21)\qquad r_p(T/\omega_n T) = s(k_\infty)\,p^n+O(1)\,,$$

by means of the method of the proof of Lemma 1.4 and 1.5. Now, combining (13), (15), (16), (20), (21), the proof of the proposition is complete.

By virtue of the above calculations and the equality (2), we obtain the following theorem.

Theorem 1.9. *Under the above notations, we have*

$$(22)\qquad \rho(k_\infty)+s(k_\infty) = r_2(k)+\beta(k_\infty)+a'(k_\infty)\,.$$

Now, we shall derive the necessary and sufficient condition for $\rho(k_\infty)=r_2(k)$.

Corollary 1.10. *Under the above notations, we have* $s(k_\infty)\leqq\beta(k_\infty)+a'(k_\infty)$. *And the following conditions are equivalent:*

(i) $\rho(k_\infty)=r_2(k)$,
(ii) $s(k_\infty)=\beta(k_\infty)+a'(k_\infty)$,
(iii) $s(k_\infty)\geqq\beta(k_\infty)+a'(k_\infty)$.

Proof. We know that $\rho(k_\infty)\geqq r_2(k)$ (see Greenberg [9]). Combining this fact and Theorem 1.9, we obtain the above assertion.

Next, we shall give the proof of Theorem I.

The proof of Theorem I. We preserve the above notations and terminologies. Let $X=G(L(k_\infty)/k_\infty)$ and $Y=G(L(k_\infty)/k_\infty L(k_e))$, where, for any algebraic extension $K/\boldsymbol{Q}$, $L(K)$ denote the maximal unramified p-abelian extension of K. Since X is a finitely generated torsion Λ_G-module (see Iwasawa [11]), we have the following decomposition of X:

$$X \sim \left(\bigoplus_{i=1}^{a(k_\infty)} \Lambda_G/(p^{u_i})\right) \oplus \left(\bigoplus_{j=1}^{b(k_\infty)} \Lambda_G/(h_j^{v_j})\right),$$

where $a(k_\infty)$ and $b(k_\infty)$ are non negative integers, u_i and v_j are positive integers and h_j are distinguished polynomials. Then, Iwasawa's μ-invariant $\mu(k_\infty/k)$ is $\sum_{i=1}^{a(k_\infty)} u_i$. So, we obtain $\mu(k_\infty/k)=0$ if and only if $a(k_\infty)=0$. Therefore the assumptions of the Theorem I are equivalent to $\beta(k_\infty)=0$ and $a(k_\infty)=0$. On the other hand, we have also the isomorphism $C(k_n)\simeq G(L(k_n)/k_n)\simeq X/\nu_{e,n} Y$ for all $n\geqq e$ (see Iwasawa [11]). Using the method of the proof of Proposition 1.3, we obtain

$$r_p(C(k_n)) = a(k_\infty)\, p^n + O(1)\,. \tag{23}$$

Next, estimating the p-rank of each term of the exact sequence: $0\to C_0(k_n)\to C(k_n)\to P(k_n)\to 0$, we obtain

$$r_p(C_0(k_n))+r_p(P(k_n))\geqq r_p(C(k_n))\geqq r_p(P(k_n))\,. \tag{24}$$

$C_0(k_n)$ is generated by the ideal classes of the prime ideals of k_n over p. From this fact and Proposition 1.2, it follows that

$$r_p(C_0(k_n))\leqq \beta(k_\infty)\, p^n+O(1) \quad \text{for all} \quad n\geqq 0\,. \tag{25}$$

Combining (8), (23), (24), (25), we obtain $a'(k_\infty)+\beta(k_\infty)\geqq a(k_\infty)\geqq a'(k_\infty)$. From this inequality, it follows that $\beta(k_\infty)=0$ and $a(k_\infty)=0$ if and only if $\beta(k_\infty)=0$ and $a'(k_\infty)=0$. Obviously, $s(k_\infty)\geqq 0=\beta(k_\infty)+a'(k_\infty)$. Therefore using Corollary 1.10, we obtain Theorem I.

Remark. We can prove Corollary 1.10 and Theorem I by modifying method of Greenberg [8] Theorem 3. We can also express the necessary and sufficient condition by means of Galois cohomology. Let F denote the maximal p-ramified p-extension of k. Put $G_n=G(F/k_n)$. Since $H^3(G_n, \boldsymbol{Z})$ is divisible, we can express $H^3(G_n, \boldsymbol{Z})\simeq(\boldsymbol{Q}_p/\boldsymbol{Z}_p)^{a_n}$ for some $a_n\geqq 0$. Then, we have $\rho(k_\infty)=r_2(k)$ if and only if a_n is bounded with respect to n (see Brumer [4]).

On the assumptions of Theorem I, following results are known (see Greenberg [7], Babaicev [1], Monsky [14], Kuz'min [13]). Let $E(k)$ denote the set of all $\boldsymbol{Z}_p$-extensions of k. Let $E_1(k)=\{k_\infty\in E(k)\mid \mu(k_\infty/k)=0\}$ and $E_2(k)=\{k_\infty\in E(k)\mid \beta(k_\infty)=0\}$. Then, in a certain natural topology, $E_2(k)$ is a non empty open dense subset of $E(k)$. And if $E_1(k)\neq\phi$, then $E_1(k)$ is also an open dense subset of $E(k)$. Therefore if there is a $\boldsymbol{Z}_p$-extension k_∞ of k whose μ-invariant is zero, $E_1(k)\cap E_2(k)$ is an open dense subset of $E(k)$ and so there are many $\boldsymbol{Z}_p$-extensions which satisfy the assumption of Theorem I.

Finally, we shall give an example.

Example 1.11. We assume that the p-Sylow subgroup of $C(k)$ is trivial and that k has only one prime ideal over p. For example, if p is a regular prime, $\boldsymbol{Q}(\mu_1)$ satisfies these conditions. Let k_∞ be any $\boldsymbol{Z}_p$-extension of k and k' a finite Galois extension with degree a power of p such that all prime ideals of k, which is ramified in k', do not split completely in k_∞. Then, for $\boldsymbol{Z}_p$-extension $k'k_\infty$ of k', we can easily see $\beta(k'k_\infty)=0$ and have $\mu(k'k_\infty/k')=0$ by virtue of Iwasawa [10] and [12] Theorem 2. So, using Theorem I, we have $\rho(k'k_\infty)=r_2(k')$ for this $\boldsymbol{Z}_p$-extension $k'k_\infty$ of k'. It seems that the case of $k'k_\infty/k'$ cannot be treated by the criterion of Iwasawa and Greenberg.

§2. The proof of Theorem II.

In this section, we do not assume that k contains μ_1 and $d=1$. We shall use the notations in the statement of Theorem II in Introduction.

By means of the assumption $k_\infty \cap H_\infty=k$, we can identify naturally $G(k_\infty H_\infty/H_\infty)$ with $G(k_\infty H_n/H_n)$ for all $n\geqq 0$. We denote by G this group. Similarly, we identify $G(k_\infty H_\infty/k_\infty)$ with $G(H_\infty/k)$ and denote by B this group. Then we denote by Λ_G and Λ_B the complete group rings of G and B over $\boldsymbol{Z}_p$. Let Λ_A denote the complete group ring over $\boldsymbol{Z}_p$ of the Galois group $A=G(k_\infty H_\infty/k)$. Let $\{\sigma_1, \cdots, \sigma_d\}$ be a system of topological generators of G and τ a topological generator of B. Then, a system of topological generators of A is given by $\{\sigma_1, \cdots, \sigma_d, \tau\}$. Now we can identify Λ_A with $\boldsymbol{Z}_p[[S_1, \cdots, S_d, T]]$ in such a way as σ_i $(1\leqq i\leqq d)\to 1+S_i$ and $\tau\to T$. So, by means of this identification, we identify naturally Λ_G and Λ_B with $\boldsymbol{Z}_p[[S_1, \cdots, S_d]]$ and $\boldsymbol{Z}_p[[T]]$ respectively.

Our main purpose in this section is to study the asymptotic behavior of $\mathrm{rk}_{\Lambda_G} \tilde{X}(k_\infty H_n)$ as n grows. For simplicity, put $K=k_\infty H_\infty$, $A_n=G(K/H_n)$, $\tilde{X}=\tilde{X}(K)$, $M=M(K)$, $M_n=M(k_\infty H_n)$ and $\tilde{X}_n=\tilde{X}(k_\infty H_n)$. Now, we shall consider the exact sequence of Λ_G-modules:
$0\to G(M_n/K)\to \tilde{X}_n\to G(K/k_\infty H_n)\to 0$. Since K/k is an abelian extension, G acts trivially on $G(K/k_\infty H_n)$. So, $G(K/k_\infty H_n)$ is a torsion Λ_G-module, hence $\mathrm{rk}_{\Lambda_G} \tilde{X}_n=\mathrm{rk}_{\Lambda_G} G(M_n/K)$. We know also that $G(M_n/K)\simeq \tilde{X}/T_n\tilde{X}$ as Λ_A-modules, where $T_n=(1+T)^{p^n}-1\in\Lambda_A$. Hence, we have

$$\mathrm{rk}_{\Lambda_G} \tilde{X}_n = \mathrm{rk}_{\Lambda_G} \tilde{X}/T_n \tilde{X}\,. \tag{26}$$

In the case of $n=0$, we can calculate $\mathrm{rk}_{\Lambda_G} \tilde{X}/T_n \tilde{X}$ as follows.

Proposition 2.1. *Put* $\tilde{Y}=\mathrm{Tor}_{\Lambda_A}(\tilde{X})$ *for a finitely generated* Λ_A*-module* $\tilde{X}$, *then we have*

$$\mathrm{rk}_{\Lambda_G} \tilde{X}/T\tilde{X} = \mathrm{rk}_{\Lambda_A} \tilde{X}+\mathrm{rk}_{\Lambda_G} \tilde{Y}/T\tilde{Y}\,.$$

Proof. Let $\rho=\mathrm{rk}_{\Lambda_A} \tilde{X}$ and $\tilde{Z}=\tilde{X}/\tilde{Y}$. Since $\tilde{Z}$ is a torsion-free Λ_A-module, we obtain an exact sequence:

$$0\to \tilde{Y}/T\tilde{Y}\to \tilde{X}/T\tilde{X}\to \tilde{Z}/T\tilde{Z}\to 0\,. \tag{27}$$

So that we have $\mathrm{rk}_{\Lambda_G}\tilde{X}/T\tilde{X}=\mathrm{rk}_{\Lambda_G}\tilde{Y}/T\tilde{Y}+\mathrm{rk}_{\Lambda_G}\tilde{Z}/T\tilde{Z}$. It is sufficient to show $\rho=\mathrm{rk}_{\Lambda_G}\tilde{Z}/T\tilde{Z}$. We shall apply Lemma 1.8 to $\tilde{Z}$ and T. Then there exist a $\lambda\in\Lambda_A-T\Lambda_A$ and a free Λ_A-submodule $\tilde{Z}'$ of $\tilde{Z}$ such that $\tilde{Z}'$ contains $\lambda\tilde{Z}$. Let $W=\tilde{Z}'/\lambda\tilde{Z}$ and $W'=\tilde{Z}/\tilde{Z}'$. Since both W/TW and W'/TW' are annihilated by ($\lambda \bmod T\Lambda_A$) $\in\Lambda_A/T\Lambda_A\simeq\Lambda_G$, we obtain $\mathrm{rk}_{\Lambda_G}(W/TW)=\mathrm{rk}_{\Lambda_G}(W'/TW')=0$. By means of the following two exact sequences: $0\to\tilde{Z}\to\tilde{Z}'\to W\to 0$ and $0\to\tilde{Z}'\to\tilde{Z}\to W'\to 0$, we obtain the equality of Λ_G-ranks:
$\mathrm{rk}_{\Lambda_G}(\tilde{Z}'/T\tilde{Z}')=\mathrm{rk}_{\Lambda_G}(\tilde{Z}/T\tilde{Z})$. Since $\tilde{Z}'$ is a free Λ_A-module and $\lambda\tilde{Z}\subseteq\tilde{Z}'$, we have also $\rho=\mathrm{rk}_{\Lambda_A}\tilde{Z}=\mathrm{rk}_{\Lambda_A}\tilde{Z}'=\mathrm{rk}_{\Lambda_G}(\tilde{Z}'/T\tilde{Z}')$. So, the proof is complete.

Before we go to the general case, we shall consider the part of $\mathrm{rk}_{\Lambda_A}\tilde{X}$.

Proposition 2.2. *Λ_A is a free Λ_{A_n}-module of the rank p^n, and $\mathrm{rk}_{\Lambda_{A_n}}\tilde{X}=(\mathrm{rk}_{\Lambda_A}\tilde{X})$ p^n for all $n\geqq 0$.*

Proof. We can easily show that $\{\sigma_1, \cdots, \sigma_d, \tau^{p^n}\}$ is a system of topological generators of A_n and can identify Λ_{A_n} with $Z_p[[S_1, \cdots, S_d, T_n]]$ under the fixed identification $\Lambda_A=Z_p[[S_1, \cdots, S_d, T]]$, and T_n is the distinguished polynomial. From these facts and the use of Weierstrass's preparation Theorem, our assertions follow easily.

Since Λ_A is integral over Λ_{A_n}, we have easily $\tilde{Y}=\mathrm{Tor}_{\Lambda_{A_n}}(\tilde{X})$. So, if we rewrite the equality of Proposition 2.1 in the terms of Λ_{A_n}-module, we obtain

$$
\begin{aligned}
(28)\qquad \mathrm{rk}_{\Lambda_G}(\tilde{X}/T_n\tilde{X}) &= \mathrm{rk}_{\Lambda_{A_n}}(\tilde{X})+\mathrm{rk}_{\Lambda_G}(\tilde{Y}/T_n\tilde{Y})\\
&=(\mathrm{rk}_{\Lambda_A}\tilde{X})\,p^n+\mathrm{rk}_{\Lambda_G}(\tilde{Y}/T_n\tilde{Y}) \quad \text{for all} \quad n\geqq 0\,.
\end{aligned}
$$

Next, we shall calculate $\tilde{Y}/T_n\tilde{Y}$.

Proposition 2.3. *Let $c_n=\mathrm{rk}_{\Lambda_G}(\tilde{Y}/T_n\tilde{Y})$. The sequence of numbers $\{c_n|n=0, 1, 2, \cdots\}$ increases monotonely and bounded to the above. Hence, for all sufficiently large n, c_n is equal to a constant c.*

Proof. Since $\tilde{Y}$ is a finitely generated torsion Λ_A-module, we can take the pseudo-null modules N_1, N_2 such that the sequence of Λ_A-modules:
$0\to N_1\to\tilde{Y}\to\bigoplus_{i=1}^{a}\Lambda_A/(P_i^{n_i})\to N_2\to 0$ is exact, where a is a non negative integer, n_i are positive integers and P_i are the prime ideals of height 1 in Λ_A. We resolve this exact sequence into the two short exact sequences:
$0\to N_1\to\tilde{Y}\to\tilde{Y}'\to 0$ and $0\to\tilde{Y}'\to\bigoplus_{i=1}^{a}M_i\to N_2\to 0$, where $M_i=\Lambda_A/(P_i^{n_i})$ $(1\leqq i\leqq a)$.
By means of these exact sequences, we obtain the following estimation of Λ_G-ranks:

$$
(29)\qquad \mathrm{rk}_{\Lambda_G}(N_2/T_nN_2)\geqq\sum_{i=1}^{a}\mathrm{rk}_{\Lambda_G}(M_i/T_nM_i)-\mathrm{rk}_{\Lambda_G}(\tilde{Y}/T_n\tilde{Y})\,.
$$

By virtue of Bourbaki [3] chaptre 7, §4, n°8, Proposition 18, N_2 is also a pseudo-null Λ_{A_n}-module. From the definition of pseudo-null, it follows that $\mathrm{Ann}_{\Lambda_{A_n}} N_2 \subsetneqq T_n\Lambda_{A_n}$. Therefore $\mathrm{rk}_{\Lambda_G}(N_2/T_nN_2)=0$. So, we obtain

$$\mathrm{rk}_{\Lambda_G}(\tilde{Y}/T_n\tilde{Y}) \geqq \sum_{i=1}^{a} \mathrm{rk}_{\Lambda_G}(M_i/T_nM_i)\,.$$

On the other hand, since both $\tilde{Y}$ and $\bigoplus_{i=1}^{a} M_i$ are torsion Λ_A-modules, $\tilde{Y} \sim \bigoplus_{i=1}^{a} M_i$ implies $\bigoplus_{i=1}^{a} M_i \sim \tilde{Y}$.

So, the converse inequality can be shown in the same way.

Now, we obtain

$$\mathrm{rk}_{\Lambda_G}(\tilde{Y}/T_n\tilde{Y}) = \sum_{i=1}^{a} \mathrm{rk}_{\Lambda_G}(M_i/T_nM_i)\,. \tag{30}$$

Next we shall calculate $\mathrm{rk}_{\Lambda_G}(M/T_nM)$, where M is Λ_A/P^j for the prime ideal P of height 1 in Λ_A and a certain positive integer j. Since Λ_A is a U.F.D., P is generated over Λ_A by a prime element F. Let $f=F^j$, then $P^j=f\Lambda_A$. Moreover, we shall define a Λ_G-isomorphism: $\Lambda_A/T_n\Lambda_A \to (\Lambda_G)^{p^n}$ as follows. Using Weierstrass's preparation Theorem, for any element $h\in\Lambda_A$, there are $h_i\in\Lambda_G$ $(1\leqq i\leqq p^n-1)$ such that $h\equiv \sum_{i=0}^{p^n-1} h_iT^i \bmod T_n\Lambda_A$ and these h_i are determined uniquely. Then we define Λ_G-isomorphism by the correspondence $h \bmod T_n\Lambda_A \to (h_0, \cdots, h_{p^n-1}) \in (\Lambda_G)^{p^n}$.

For any element $g\in\Lambda_A$, we define the endomorphism $\theta(g)$ of $(\Lambda_G)^{p^n}$ in such a way as the following diagram is commutative:

$$\begin{array}{ccc} \Lambda_A/T_n\Lambda_A & \longrightarrow & (\Lambda_G)^{p^n} \\ \times g \downarrow & & \downarrow \theta(g) \\ \Lambda_A/T_n\Lambda_A & \longrightarrow & (\Lambda_G)^{p^n} \end{array}$$

where $\times g$ means the multiplication by g.

Then it is easy to show $M/T_nM \simeq \mathrm{coker}\ \theta(f)$. Let $Q(\Lambda_G)$ denote the quotient field of Λ_G. Then we easily obtain: $\mathrm{coker}\ \theta(f)$ is a torsion Λ_G-module if and only if $\mathrm{coker}\ \theta(f)\otimes_{\Lambda_G} Q(\Lambda_G)=0$ and this is so if and only if the $Q(\Lambda_G)$-linear extension of $\theta(f)$: $Q(\Lambda_G)^{p^n}\to Q(\Lambda_G)^{p^n}$ is surjective. The last statement is equivalent to $\det \theta(f)\neq 0$.

Next we shall calculate $\det\theta(f)$. We put $f=hT_n+g$, where $h\in\Lambda_A$ and $g=\sum_{i=0}^{p^n-1} b_iT^i$, $b_i\in\Lambda_G$ $(0\leqq i\leqq p^n-1)$. Obviously θ: $\Lambda_A\to \mathrm{End}_{\Lambda_G}((\Lambda_G)^{p^n})$ is a Λ_G-linear ring homomorphism.

So, $\theta(f)=\theta(g)=\sum_{i=0}^{p^n-1} b_i\theta(T)^i$. It follows easily from the direct calculation that the set of all eigen values of $\theta(T)$ is $\{\zeta-1\,|\,\zeta\in\mu_n\}$. So, we find that the set of all eigen values of $\theta(f)$ is $\{f(S_1, \cdots, S_d, \zeta-1)\,|\,\zeta\in\mu_n\}$. Therefore

$\det\theta(f) = \prod_{\zeta\in\mu_n} f(S_1, \cdots, S_d, \zeta-1)$. So, we obtain

(31) $$\mathrm{rk}_{\Lambda_G}(M/T_nM)\neq 0 \quad \text{if and only if}$$
$$F(S_1, \cdots, S_d, \zeta-1) = 0 \quad \text{for some} \quad \zeta\in\mu_n .$$

In this case, we can easily see also that such F is essentially the cyclotomic polynomial ψ_m, that is $F\Lambda_A=\psi_m\Lambda_A$, where ψ_m is T if $m=0$ and is T_m/T_{m-1} if $m\geqq 1$. This follows from that ψ_m is a distinguished polynomial and so we can divide F by ψ_m and (31).

Combining the above results, we obtain

(32) $$\mathrm{rk}_{\Lambda_G}(M/T_nM)\neq 0 \quad \text{if and only if} \quad P=\psi_m\Lambda_A \quad \text{for some} \quad n\geqq m\geqq 0 .$$

Next we shall calculate $\mathrm{rk}_{\Lambda_G}(\Lambda_A/(T_n\Lambda_A+\psi_m^j\Lambda_A))$. Let R_1 denote the ideal in $\Lambda_B=\boldsymbol{Z}_p[[T]]$ generated by T_n and ψ_m^j $(n\geqq m\geqq 0, j\geqq 1)$.
Let η: $\Lambda_B[[S_1, \cdots, S_d]]\to(\Lambda_B/R_1)[[S_1, \cdots, S_d]]$ denote the Λ_G-homomorphism obtained from the reduction modulo R_1 of coefficients of $\Lambda_A=\Lambda_B[[S_1, \cdots, S_d]]$.

Then we can easily show that $\ker\eta=T_n\Lambda_A+\psi_m^j\Lambda_A$. So, we obtain

(33) $$(\Lambda_B/R_1)[[S_1, \cdots, S_d]]\simeq\Lambda_A/(T_n\Lambda_A+\psi_m^j\Lambda_A) .$$

On the other hand, it is easily shown that

(34) $$\mathrm{rk}_{\Lambda_G}(\Lambda_B/R_1)[[S_1, \cdots, S_d]]=\mathrm{rk}_{\boldsymbol{Z}_p}(\Lambda_B/R_1) .$$

By virtue of Weierstrass's preparation Theorem, we obtain also $\Lambda_B/T_n\Lambda_B\simeq\boldsymbol{Z}_p[T]/T_n\boldsymbol{Z}_p[T]$. So, we have $\Lambda_B/R_1\simeq\boldsymbol{Z}_p[T]/R_2$, where $R_2=T_n\boldsymbol{Z}_p[T]+\psi_m^j\boldsymbol{Z}_p[T]$.

Next, we shall calculate $\mathrm{rk}_{\boldsymbol{Z}_p}(\boldsymbol{Z}_p[T]/R_2)$. We know that $(\boldsymbol{Z}_p[T]/R_2)\otimes_{\boldsymbol{Z}_p}\boldsymbol{Q}_p\simeq\boldsymbol{Q}_p[T]/(T_n\boldsymbol{Q}_p[T]+\psi_m^j\boldsymbol{Q}_p[T])$, $T_n=\prod_{i=0}^{n}\psi_i$ and that ψ_i is irreducible in $\boldsymbol{Q}_p[T]$. So, $\boldsymbol{Q}_p[T]T_n+\boldsymbol{Q}_p[T]\psi_m^j=\boldsymbol{Q}_p[T]\psi_m$. Therefore we have

(35) $$\mathrm{rk}_{\boldsymbol{Z}_p}(\boldsymbol{Z}_p[T]/R_2)=\dim_{\boldsymbol{Q}_p}(\boldsymbol{Q}_p[T]/\psi_m\boldsymbol{Q}_p[T]) =\begin{cases}1 & \text{if} \quad m=0 .\\ p^{m-1}(p-1) & \text{if} \quad m\geqq 1 .\end{cases}$$

Combining (33), (34), (35), we obtain

(36) $$\mathrm{rk}_{\Lambda_G}(\Lambda_A/(T_n\Lambda_A+\psi_m^j\Lambda_A))=\begin{cases}1 & \text{if} \quad m=0 .\\ p^{m-1}(p-1) & \text{if} \quad m\geqq 1 .\end{cases}$$

Let I_n denote the set of all prime ideals P_i which appear in the decomposition $\tilde{Y}\sim\bigoplus_{i=1}^{a}\Lambda_A/(P_i^{n_i})$ and are generated by ψ_m for some $n\geqq m\geqq 0$. By virtue of (30), (31) and (32), we have $c_n=\mathrm{rk}_{\Lambda_G}\tilde{Y}/T_n\tilde{Y}=\sum_{P_i\in I_n}\mathrm{rk}_{\Lambda_G}(\Lambda_A/(P_i^{n_i}+T_n\Lambda_A))$.

Obviously, I_n is a monotone increasing sequence of sets as n grows and I_n is stable for all sufficiently large n, because $I_n\subseteqq\{P_i\,|\,1\leqq i\leqq a\}$. Moreover, for any $P_i\in I_n$, $\mathrm{rk}_{\Lambda_G}(\Lambda_A/(P_i^{n_i}+T_n\Lambda_A))$ is independent of n by virtue of (36). Combining these results, the proof of the proposition is complete.

Proof of Theorem II. Theorem II follows from (26), (28) and Proposition 2.3.

Remark. We can prove the criterion of Greenberg mentioned in Introduction by means of Theorem II.

Corollary 2.4. *Under the notations of Theorem II in Introduction, the following two conditions are equivalent:*
(i) $\rho = r_2(k)$ *and* $c = 0$.
(ii) $\rho(k_\infty H_n) = r_2(H_n)$ *for all* $n \geqq 0$.

Proof. We first assume (i). By virtue of Proposition 2.3, we obtain $c_n=0$ for all $n\geqq 0$. Therefore the relations (26) and (28) imply (ii). Next, we assume (ii). Using Theorem II, we obtain $(\mathrm{rk}_{\Lambda_\Delta}\tilde{X})\, p^n + c = r_2(H_n) = r_2(k)\, p^n$ for all sufficiently large n. Dividing both sides by p^n and taking their limits as n tends to the infinity, we obtain (i).

Finally, we shall give two sufficient conditions for the constant $c=0$.

Proposition 2.5. *Assume that k contains μ_1 and k_∞/k is a $\boldsymbol{Z}_p$-extension such that $k_\infty \cap H_\infty = k$. If there exists no prime ideal of k over p which splits completely in k_∞ and if Iwasawa's μ-invariant $\mu(k_\infty/k)$ is zero, then the constant $c=c(k_\infty, H_\infty)$ in Theorem II is zero.*

Proof. By virtue of example 1.11, we have $\beta(k_\infty H_n) = \mu(k_\infty H_n/H_n) = 0$ for all $n\geqq 0$. Then our Theorem I assert that $\rho(k_\infty H_n) = r_2(H_n)$ for all $n\geqq 0$. So, our assertion follows from Corollary 2.4.

Proposition 2.6. *Assume that k contains μ_1 and $d=1$ and that H_∞ be the cyclotomic $\boldsymbol{Z}_p$-extension of k such that $k_\infty \cap H_\infty = k$. If Iwasawa's λ-invariant $\lambda(H_\infty k_m/k_m)$ is bounded with respect to m, then the constant $c=c(k_\infty, H_\infty)$ in Theorem II is zero, where k_n denotes the unique cyclic extension of k of degree p^n in k_∞.*

Proof. By virtue of the proof of Greenberg [8] Theorem 3, we have $\lambda(H_\infty k_m/k_m) \geqq \mathrm{rk}_{\boldsymbol{Z}_p} \tilde{X}(H_n k_m) - r_2(H_n k_m) - 1 = \delta(H_n k_m)$ for all $n\geqq 0$, $m\geqq 0$. Let $A = G(H_\infty k_\infty/k)$, $G = G(k_\infty/k)$, $\rho = \mathrm{rk}_{\Lambda_A} \tilde{X}(H_\infty k_\infty)$ and $\rho_n = \mathrm{rk}_{\Lambda_G} \tilde{X}(H_n k_\infty)$. Then we find from Greenberg [9], $\rho_n p^m + O(1) = \mathrm{rk}_{\boldsymbol{Z}_p} \tilde{X}(H_n k_m) = r_2(H_n)\, p^m + 1 + \delta(H_n k_m)$, where $O(1)$ means the bounded number with respect to m. So, we obtain

$$(37) \qquad (\rho_n\, p - \rho_{n+1})\, p^m = p\delta(H_n k_m) - \delta(H_{n+1}\, k_m) + O(1)\,.$$

Put $c=c(k_\infty, H_\infty)$, then Theorem II implies $\rho_n\, p - \rho_{n+1} = (p-1)\, c$. Combining (37) and Cuoco [5] Theorem 1.1, we obtain the following estimation:
$(p-1)\, c = \rho_n\, p - \rho_{n+1} \leqq p^{-m}(p\delta(H_n k_m) + \delta(H_{n+1} k_m)) + p^{-m}\, O(1) \leqq (p+1)\, p^{-m}(l_0\, p^m + l_1) + p^{-m}\, O(1)$, where l_0, l_1 are constants. So, taking their limits as m tends to the infinity, we obtain $(p-1)c \leqq (p+1)\, l_0$, so if $l_0=0$, then we obtain $c=0$. The proof is complete.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

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