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On the Hausdorff dimension of spherical limit sets

By

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1. Introduction.

In this paper we shall determine the Hausdorff dimension of a certain class of limit sets in $X=\boldsymbol{R}^p$ with the usual Euclidean distance. The spherical limit sets, whose definition will be given in Section 2, contain the spherical Cantor sets investigated by M. Tsuji [9] and A. F. Beardon [1] as a generalization of Cantor's ternary set in connection with function theory. For an application to a singular set of some properly discontinuous group, see A. F. Beardon [2]. Moreover we shall study the metric dimension of such sets.

2. Definitions aud Theorems.

Throughout this paper we shall use the following notations: the diameter of a set E is denoted by $|E|$; the distance between two sets U and V is denoted by $\operatorname{dist}(U, V)$; the interior of a set E is denoted by $\mathring{E}$. We need some definitions.

Definition 2.1. For any $\alpha>0$, $\varepsilon>0$, and $E\subset X$, we shall denote by $\Lambda_\alpha^\varepsilon(E)$ the lower bound of the sums $\sum_{n\geq 1}|S_n|^\alpha$ where $\{S_n\}_{n\geq 1}$ is an arbitrary covering of E consisting of closed spheres of diameters less than ε. When $\varepsilon\to 0+$, $\Lambda_\alpha^\varepsilon(E)$ tends to a unique limit $\Lambda_\alpha(E)$ (finite or infinite), which we call the *α-dimensional outer measure*. Then there exists a uniquely determined number such that $\sup\{\alpha\,;\,\Lambda_\alpha(E)=\infty\}=\inf\{\alpha\,;\,\Lambda_\alpha(E)=0\}$, which we shall call the *Hausdorff dimension* of E and denote by $\dim_H(E)$.

Let $\nu_n\geqq 2$ be an integer. Then $\Sigma_n=\{w=(w_1\cdots w_n)\,;\,1\leqq w_j\leqq\nu_j \text{ for } 1\leqq j\leqq n\}$ is called the set of finite words with length n.

Definition 2.2. A set K is said to be a *spherical limit set* provided that it can be expressed in the form

$$K=\bigcap_{n=1}^{\infty}\bigcup_{w\in\Sigma_n}S(w), \tag{2.1}$$

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where $\{S(w)\}$ are p-dimensional closed spheres satisfying

(a) $S(w_1 \cdots w_n) \supset S(w_1 \cdots w_n w_{n+1})$ for any $(w_1 \cdots w_n w_{n+1}) \in \Sigma_{n+1}$;

(b) $\mathring{S}(w) \cap \mathring{S}(w') = \emptyset$ for any $w \neq w' \in \Sigma_n$;

(c) $M_n \equiv \max_{w \in \Sigma_n} |S(w)| \longrightarrow 0$ as $n \to \infty$.

Remark 2.3. If the fundamental closed spheres $\{S(w)\}$ satisfy the following condition, instead of (b):

(b′) $S(w) \cap S(w') = \emptyset$ for any $w \neq w' \in \Sigma_n$,

then the set K is said to be a *spherical Cantor set*. Such sets were studied by Tsuji and Beardon, although they required some separation conditions in addition.

Remark 2.4. Consider the compact metric space $\Sigma_\infty = \{w = (w_1 w_2 \cdots); 1 \leqq w_j \leqq \nu_j$ for $j \geqq 1\}$ with the metric $d_\Sigma(u, v) = \sum_{n \geq 1} 2^{-n} \tau(u_n, v_n)$ for $u = (u_n)$, $v = (v_n)$ where $\tau(i, j) = 0$ for $i \neq j$ and $\tau(i, j) = 1$ for $i = j$. Then it is easily seen that the mapping $\phi: \Sigma_\infty \to K$ defined by

$$\phi(w_1 w_2 \cdots) = \bigcap_{n \geq 1} S(w_1 \cdots w_n)$$

is continuous. Moreover, if the set K is a spherical Cantor set, the mapping ϕ becomes a homeomorphism.

We are now ready to state our first theorem. Put $m_n \equiv \min_{w \in \Sigma_n} |S(w)|$, $n \geqq 1$ for brevity. Then we have

Theorem 2.5. *Let K be any spherical limit set expressed in the form* (2.1). *Then*

$$\liminf_{n \to \infty} \frac{\log \nu_1 \cdots \nu_{n-2}}{-\log m_n} \leqq \dim_H(K) \leqq \liminf_{n \to \infty} \frac{\log \nu_1 \cdots \nu_n}{-\log M_n}. \tag{2.2}$$

In particular, if $\log \nu_n = o(\log \nu_1 \cdots \nu_n)$ *and* $\log M_n \sim \log m_n$ *as* $n \to \infty$, *then*

$$\dim_H(K) = \alpha = \liminf_{n \to \infty} \frac{\log \nu_1 \cdots \nu_n}{-\log M_n}. \tag{2.3}$$

Moreover, if in addition

$$0 < \liminf_{n \to \infty} (m_n^\alpha \nu_1 \cdots \nu_{n-2}) \leqq \liminf_{n \to \infty} (M_n^\alpha \nu_1 \cdots \nu_n) < \infty, \tag{2.4}$$

then we have $0 < \Lambda_\alpha(K) < \infty$.

Remark 2.6. In the case where K is a spherical Cantor set, $X = \boldsymbol{R}$, and $M_n = m_n$, that is, $|S(w)| = |S(w')|$ for any $w \neq w' \in \Sigma_n$, the set K is known as "un ensemble parfait de translation" (J. P. Kahane-R. Salem [6, p. 19]). If in addition ν_n is independent of n, then the formula (2.3) was first proved by A. F. Beardon [3].

We now define another dimension as follows:

Definition 2.7. For a totally bounded set E, the *metric dimension* of E, which we denote by $\dim_M(E)$, is defined by

$$\dim_M(E)=\limsup_{\varepsilon\to 0+}\frac{\log N_E(\varepsilon)}{-\log\varepsilon},$$

where $N_E(\varepsilon)$ is the minimal number of sets in a finite ε-covering of E.

Note that we have clearly $\dim_H(E)\leqq\dim_M(E)$ for any E. Then

Theorem 2.8. *For any spherical limit set K, we have*

$$\limsup_{n\to\infty}\frac{\log\nu_1\cdots\nu_{n-1}}{-\log m_n}\leqq\dim_M(K)\leqq\limsup_{n\to\infty}\frac{\log\nu_1\cdots\nu_{n+1}}{-\log M_n}. \tag{2.5}$$

In particular, if $\log\nu_n=o(\log\nu_1\cdots\nu_n)$ *and* $\log M_n\sim\log m_n$ *as* $n\to\infty$, *then we have*

$$\dim_M(K)=\limsup_{n\to\infty}\frac{\log\nu_1\cdots\nu_n}{-\log M_n}. \tag{2.6}$$

As a corollary, we have immediately

Corollary 2.9 *Suppose that* $\log\nu_n=o(\log\nu_1\cdots\nu_n)$ *and* $\log M_n\sim\log m_n$ *as* $n\to\infty$. *Then* $\dim_H(K)=\dim_M(K)$ *if and only if the sequence*

$$\frac{\log\nu_1\cdots\nu_n}{-\log M_n}$$

converges to a finite limit.

3. Preliminaries.

In this section we will give an elementary 'geometrical lemma'.

Lemma 3.1. *Let S_1, S_2, S_3 be three closed spheres in $X=\boldsymbol{R}^p$ such that $\mathring{S}_i\cap\mathring{S}_j=\emptyset$ for any $i\neq j$. Suppose that each S_j contains a small closed sphere σ_j for $1\leqq j\leqq 3$. Then we have*

$$\sum_{i<j}\operatorname{dist}(\sigma_i,\sigma_j)\geqq 3\left(1-\frac{\sqrt{3}}{2}\right)\min_j(|S_j|-|\sigma_j|). \tag{3.1}$$

Proof. Let O_1, O_2, O_3 be the centers of S_1, S_2, S_3 respectively. If the points O_1, O_2, O_3 lie on a straight line in this order, it follows that $\operatorname{dist}(\sigma_1,\sigma_3)\geqq\operatorname{dist}(S_1,S_3)\geqq|S_2|$. Hence, in particular, (3.1) holds true in the case $X=\boldsymbol{R}$. Thus we can assume that $X=\boldsymbol{R}^p$, $p\geqq 2$ and that the triangle with vertices O_1, O_2, and O_3 is contained in the 2-dimensional plane $\Pi=\{x=(x_1,\cdots,x_p);\,x_3=\cdots=x_p=0\}$.

Let $P:X=\boldsymbol{R}^p\to\Pi$ be the projection definned by $P(x_1,\cdots,x_p)=(x_1,x_2,0,\cdots,0)$ Let o_1, o_2, o_3 be the centers of σ_1, σ_2, σ_3 respectively. Then it follows that

$$\begin{aligned}\sum_{i<j}\operatorname{dist}(\sigma_i,\sigma_j)&=\sum_{i<j}\|o_i-o_j\|-\sum_j|\sigma_j|\\&\geqq\sum_{i<j}\|P(o_i)-P(o_j)\|-\sum_j|\sigma_j|.\end{aligned} \tag{3.2}$$

On the other hand, let Q_{ij} be the projection of the plane Π to the straight line through the points O_i and O_j for $i<j$. Then,

$$\sum_{i<j}\|P(o_i)-P(o_j)\| \geqq \sum_{i<j}\|Q_{ij}P(o_i)-Q_{ij}P(o_j)\|$$
$$\geqq \sum_{i<j}\|O_i-O_j\|-\sum_j\|O_j-o_j\|(\cos\alpha_j+\cos\beta_j), \tag{3.3}$$

where $\alpha_j+\beta_j>0$ for $1\leqq j\leqq 3$ and $\sum_j(\alpha_j+\beta_j)=\pi$ (see Figure 1).

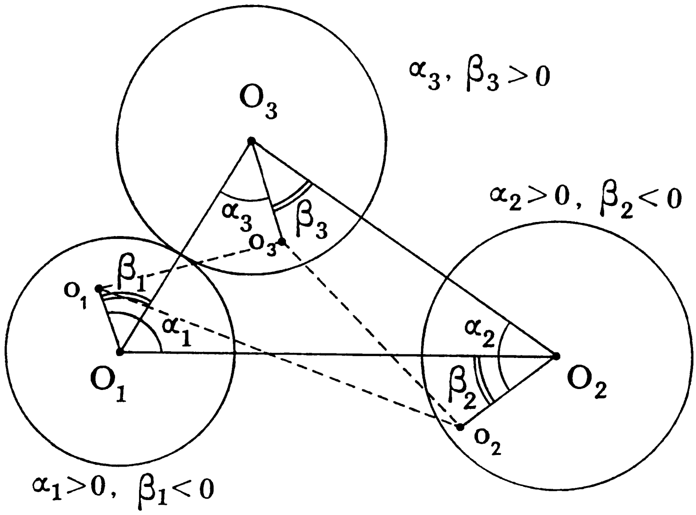

Fig. 1.

Put $\theta_j=1/2(\alpha_j+\beta_j)$, $1\leqq j\leqq 3$ for brevity. Since $\cos\alpha_j+\cos\beta_j\leqq 2\cos\theta_j$, it follows from (3.3) that

$$\sum_{i<j}\|P(o_i)-P(o_j)\|\geqq\sum_j|S_j|-\sum_j(|S_j|-|\sigma_j|)\cos\theta_j$$
$$\geqq\sum_j|\sigma_j|+\sum_j(1-\cos\theta_j)(|S_j|-|\sigma_j|).$$

Since $\sum_j\theta_j=\pi/2$, we have from (3.2),

$$\sum_{i<j}\operatorname{dist}(\sigma_i,\sigma_j)\geqq\sum_j(1-\cos\theta_j)\cdot\min_j(|S_j|-|\sigma_j|)$$
$$\geqq 3\left(1-\frac{\sqrt{3}}{2}\right)\min_j(|S_j|-|\sigma_j|).$$

This completes the proof. □

As a corollary, we have immediately

Corollary 3.2. *Let* S_1, S_2, S_3 *be three closed spheres in* $X=\boldsymbol{R}^p$ *satisfying* $\mathring{S}_i\cap\mathring{S}_j=\emptyset$ *for any* $i\neq j$. *Let* σ *be another closed sphere such that* $\sigma\cap S_j\neq\emptyset$ *for* $1\leqq j\leqq 3$. *Then we have*

$$|\sigma|\geqq\left(1-\frac{\sqrt{3}}{2}\right)\min_j|S_j|.$$

4. Proof of Theorem 2.5.

We will first show the upper estimate of (2.2). Put $\beta=\liminf_{n\to\infty}(-\log\nu_1\cdots\nu_n/\log M_n)$ for brevity. For any $\delta>0$, there exists a subsequence $\{n_j\}$ such that $M_{n_j}^{\beta+\delta}\nu_1\cdots\nu_{n_j}<1$. Since $\{S(w)\}$, $w\in\Sigma_{n_j}$ becomes an ε-covering of the set K with $\varepsilon=M_{n_j}$, we have

$$\Lambda_{\beta+\delta}^{\varepsilon}(K)\leqq\sum_{w\in\Sigma_{n_j}}|S(w)|^{\beta+\delta}\leqq M_{n_j}^{\beta+\delta}\nu_1\cdots\nu_{n_j}<1. \tag{4.1}$$

Hence $\Lambda_{\beta+\delta}(K)\leqq 1$. Since δ is arbitrary, we get $\dim_H(K)\leqq\beta$ as required.

We next show the lower estimate of (2.2). To this end, we will introduce the set-function $\Phi:\mathcal{F}\to\boldsymbol{R}$ defined by

$$\Phi(\sigma)=\lim_{n\to\infty}\frac{1}{\nu_1\cdots\nu_n}\#\{w\in\Sigma_n;\ \sigma\cap S(w)\neq\emptyset\}, \tag{4.2}$$

for any $\sigma\in\mathcal{F}$, where $\mathcal{F}$ is the collection of all finite unions of closed spheres in $X=\boldsymbol{R}^p$. First of all, the limit (4.2) certainly exists since $T_n=\#\{w\in\Sigma_n;\sigma\cap S(w)\neq\emptyset\}$ clearly satisfies $T_{n+1}\leqq\nu_{n+1}T_n$. Then it easily follows that Φ is monotone and subadditive and that $\Phi(\sigma)=1$ for every $\sigma\in\mathcal{F}$ satisfying $\sigma\supset K$. Put $\gamma=\liminf_{n\to\infty}(-\log\nu_1\cdots\nu_{n-2}/\log m_n)$ for brevity. Then for any $\delta>0$, $m_n^{\gamma-\delta}\nu_1\cdots\nu_{n-2}>1$ for all sufficiently large integer n. Therefore there exists a constant $c(\delta)>0$ such that $m_n^{\gamma-\delta}\nu_1\cdots\nu_{n-2}\geqq c(\delta)$ for any n.

Consider now an arbitrary closed sphere $S\in\mathcal{F}$ satisfying $\Phi(S)>0$. Then there exists a unique integer N such that

$$\frac{1}{\nu_1\cdots\nu_{N-1}}\geqq\Phi(S)>\frac{1}{\nu_1\cdots\nu_N}. \tag{4.3}$$

We will show $|S|\geqq c_0m_{N+1}$ where $c_0=1-\sqrt{3}/2$. Suppose, on the contrary, that $|S|<c_0m_{N+1}$. Then it easily follows that $T_{N+1}=\#\{w\in\Sigma_{N+1};\ S\cap S(w)\neq\emptyset\}\leqq 2$ from Corollary 3.2. Hence

$$\Phi(S)\leqq\frac{T_{N+1}}{\nu_1\cdots\nu_{N+1}}\leqq\frac{2}{\nu_1\cdots\nu_{N+1}}\leqq\frac{1}{\nu_1\cdots\nu_N},$$

contrary to (4.3). Therefore we get $|S|\geqq c_0m_{N+1}$. Thus,

$$\Phi(S)\leqq\frac{1}{\nu_1\cdots\nu_{N-1}}\leqq\frac{c_0^{\delta-\gamma}}{m_{N+1}^{\gamma-\delta}\nu_1\cdots\nu_{N-1}}|S|^{\gamma-\delta}\leqq c_1|S|^{\gamma-\delta}, \tag{4.4}$$

where $c_1=c_0^{\delta-\gamma}/c(\delta)$ is a constant independent of the choice of S. Note that the estimate (4.4) holds true for every closed sphere $S\in\mathcal{F}$.

Consider now an arbitrary finite ε-covering $\{S_n\}$ of K. Then we have

$$\sum|S_n|^{\gamma-\delta}\geqq c_1^{-1}\sum\Phi(S_n)\geqq c_1^{-1}\Phi(\sum S_n)\geqq c_1^{-1}. \tag{4.5}$$

Since K is compact, it follows from (4.5) that $\Lambda_{\gamma-\delta}^{\varepsilon}(K)\geqq c_1^{-1}$; therefore $\Lambda_{\gamma-\delta}(K)\geqq c_1^{-1}$. Since δ is arbitrary, we get $\dim_H(K)\geqq\gamma$ as required.

We now suppose that $\log\nu_n=o(\log\nu_1\cdots\nu_n)$ and $\log M_n\sim\log m_n$ as $n\to\infty$. Since it follows that

$$c_1 \equiv \sum_{j \in \Sigma_1} |S(j)|^p \geqq \sum_{w \in \Sigma_n} |S(w)|^p \geqq \nu_1 \cdots \nu_n m_n^p ,$$

we have

$$0 < \frac{\log \nu_{n-1}\nu_n}{-\log m_n} \leqq \frac{\log \nu_{n-1}}{-\log m_{n-1}} + \frac{\log \nu_n}{-\log m_n}$$

$$\leqq \frac{p \log \nu_{n-1}}{\log \nu_1 \cdots \nu_{n-1} - \log c_1} + \frac{p \log \nu_n}{\log \nu_1 \cdots \nu_n - \log c_1} \longrightarrow 0 \quad \text{as} \quad n \to \infty .$$

Hence the formula (2.3) follows immediately. Finally, if we suppose (2.4) in addition, one can take δ to be zero in the estimates (4.1), (4.4), and (4.5); therefore we obtain $0 < \Lambda_\alpha(K) < \infty$. This completes the proof of Theorem 2.5. □

5. Proof of Theorem 2.8.

We first show the upper estimate of (2.5). For any $\varepsilon > 0$, there exists a unique integer $n = n(\varepsilon)$ such that $M_{n+1} \leqq \varepsilon < M_n$. Since $\{S(w)\}$, $w \in \Sigma_{n+1}$ becomes an ε-covering of K, we have

$$\frac{\log N_K(\varepsilon)}{-\log \varepsilon} \leqq \frac{\log \nu_1 \cdots \nu_{n+1}}{-\log \varepsilon} < \frac{\log \nu_1 \cdots \nu_{n+1}}{-\log M_n},$$

as required.

We next show the lower estimate of (2.5). For any $\varepsilon > 0$, there exists an ε-covering $\{U_j\}$ of K in which the number of sets is $N_K(\varepsilon)$. Then for each U_j, we can choose a closed sphere S_j such that $U_j \subset S_j$ and $|S_j| = 2|U_j|$. Now define a unique integer $n = n(\varepsilon)$ by $c_2 m_n \leqq \varepsilon < c_2 m_{n-1}$ where $c_2 = 1/2(1 - \sqrt{3}/2)$. Since each S_j intersects at most two members of $\{S(w)\}$, $w \in \Sigma_{n-1}$ by Corollary 3.2, we have $N_K(\varepsilon) \geqq (1/2)\nu_1 \cdots \nu_{n-1}$; therefore

$$\frac{\log N_K(\varepsilon)}{-\log \varepsilon} \geqq \frac{\log \nu_1 \cdots \nu_{n-1} - \log 2}{-\log m_n - \log c_2} \sim \frac{\log \nu_1 \cdots \nu_{n-1}}{-\log m_n} \quad \text{as} \quad n \to \infty ,$$

as required.

The second part of the theorem is easily verified since

$$\frac{\log \nu_{n+1}}{-\log M_n} \sim \frac{\log \nu_{n+1}}{-\log m_n} = O\left(\frac{\log \nu_{n+1}}{\log \nu_1 \cdots \nu_n} \right) = O\left(\left(\frac{\log \nu_1 \cdots \nu_{n+1}}{\log \nu_{n+1}} - 1 \right)^{-1} \right) \quad \text{as} \quad n \to \infty .$$

This completes the proof of Theorem 2.8. □

6. Uniformity.

In this section we will discuss the local structure of spherical limit sets. We need some definitions.

Definition 6.1. For any $x \in X$, the *local Hausdorff dimension* of a set E at x, which we denote by $\mathrm{d}_H(x, E)$, is defined by

$$\mathrm{d}_H(x, E) = \lim_{\varepsilon \to 0+} \dim_H (E \cap S(x, \varepsilon)) , \tag{6.1}$$

where $S(x, \varepsilon)$ is the closed sphere of radius ε centered at x. Similarly we will define the *local metric dimension* $\mathrm{d}_M(x, E)$ of E at x replacing $\dim_H$ by $\dim_M$ in (6.1).

Note that $\mathrm{d}_H(x, E)$ and $\mathrm{d}_M(x, E)$ are both upper semi-continuous functions of x. For further properties on $\mathrm{d}_H(x, E)$, see [1].

Definition 6.2. A set E is said to be *H-uniform* (resp. *M-uniform*) provided that $\mathrm{d}_H(x, E)$ (resp. $\mathrm{d}_M(x, E)$) is constant on the set E (as a function of x).

Let K be a spherical limit set. For any $\varepsilon>0$ and $x\in K$, there exists a word $w\in\Sigma_n$ such that $K\cap S(w)\subset K\cap S(x, \varepsilon)\subset K$. Therefore the set K is H-uniform if and only if $\dim_H(K\cap S(w))=\dim_H(K)$ for all finite words w. Obviously the same result holds true for the M-uniformity. Then we have

Theorem 6.3. *Suppose that* $\log \nu_n=o(\log \nu_1 \cdots \nu_n)$ *and* $\log M_n \sim \log m_n$ *as* $n\to\infty$. *Then the set K is both H-uniform and M-uniform.*

Proof. Let $u=(u_1 \cdots u_k)\in\Sigma_k$ be an arbitrary finite word. Put

$$m_n^*=\min_{u\circ w\in\Sigma_n} |S(u\circ w)| \quad \text{and} \quad M_n^*=\max_{u\circ w\in\Sigma_n} |S(u\circ w)|$$

for $n>k$, where $u\circ w$ is the composite word defined by $u\circ w=(u_1 \cdots u_k w_1 \cdots w_{n-k})$. Since $m_n\leqq m_n^*\leqq M_n^*\leqq M_n$, we have $\log M_n^*\sim\log m_n^*$ as $n\to\infty$. Then it follows from Theorem 2.5 that

$$\dim_H(K\cap S(u))=\liminf_{n\to\infty}\frac{\log \nu_{k+1}\cdots\nu_n}{-\log M_n^*}$$

$$\geqq\liminf_{n\to\infty}\frac{\log \nu_1\cdots\nu_n}{-\log m_n}=\dim_H(K).$$

Similarly we have the same inequality as above for $\dim_M$. Since the converse inequality is obvious, this completes the proof. □

Remark 6.4. A spherical limit set K is said to be *self-similar* provided that ν_m is independent of m, say ν, and that $|S(w\circ j)|/|S(w)|$ is independent of $w\in\Sigma_n$, say t_j, for $1\leqq j\leqq\nu$. Such self-similar sets were studied by P. A. P. Moran [8] who showed that $\alpha=\dim_H(K)$ is given by a unique positive root of the equation $\sum_{j=1}^{\nu} t_j^{\alpha}=1$. Then it is clear that such sets are H-uniform. See also J. Marion [7] for further results of self-similar sets.

Example 6.5. Let $2\leqq m_1<m_2<\cdots<m_k$ be k integers and $(\nu_1\nu_2\cdots)\in\{m_1, \cdots, m_k\}^N$. Suppose that the limit

$$p_j=\lim_{n\to\infty}\frac{1}{n}\#\{1\leqq i\leqq n\ ;\ \nu_i=m_j\}$$

exists for each $1\leqq j\leqq k$. Suppose further that $|S(w\circ i)|/|S(w)|$ is independent of $w\circ i\in\Sigma_n$, say τ_j, where $\nu_n=m_j$. Since $\{\nu_n\}$ and $\{M_n/m_n\}$ are both bounded sequences, it follows from Theorem 2.5 and 2.8 that

$$\dim_H(K)=\dim_M(K)=\frac{\sum_{j=1}^{k} p_j \log m_j}{-\sum_{j=1}^{k} p_j \log \tau_j}$$

and the set K is both H-uniform and M-uniform by Theorem 6.3.

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References

[1] A. F. Beardon, On the Hausdorff dimension of general Cantor sets, Proc. Camb. Phil. Soc., **61** (1965), 679-694.

[2] A. F. Beardon, The Hausdorff dimension of singular sets of properly discontinuous groups, Amer. J. Math., **88** (1966), 722-736.

[3] A. F. Beardon, The generalized capacity of Cantor sets, Quart. J. Math. Oxford, **19** (1968), 301-304.

[4] A. S. Besicovitch and S. J. Taylor, On the complementary intervals of a linear closed set of zero Lebesgue measuro, J. London Math. Soc., **29** (1954), 449-459.

[5] R. T. Bumby, Hausdoff dimensions on Cantor sets, J. für Math., **331** (1981), 192-206.

[6] J. P. Kahane and R. Salem, Ensembles parfaits et séries trigonométriques, Hermann, 1963.

[7] J. Marion, Le calcul de la mesure de Hausdorff des sous-ensembles parfaits isotypiques de $\boldsymbol{R}^m$, C. R. Acad. Sc. Paris, **289** (1979), A65-A68.

[8] P. A. P. Moran, Additive functions of intervals and Hausdorff measure, Proc. Camb. Phil. Soc., **42** (1946), 15-23.

[9] M. Tsuji, On the capacity of general Cantor sets, J. Math. Soc. Japan, **5** (1953), 235-252.