J. Math. Kyoto Univ. (JMKYAZ)
27-1 (1987) 151-181

Representations of Weyl groups and their Hecke algebras on virtual character modules of a semisimple Lie group*)

By

Kyo NISHIYAMA**)

§ 0. Introduction.

Let G be a connected semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. In the preceeding paper ([16]), we defined a Weyl group action on virtual character modules with regular infinitesimal characters (recall that a virtual character is by definition a linear combination of irreducible characters on G). There, the representations of Weyl groups were completely decomposed by means of induced representations. However, in the case of singular infinitesimal character, representations of Weyl groups cannot be canonically realized on virtual character modules.

In this paper, we will define representations of Hecke algebras on virtual character modules with singular infinitesimal characters. These representations are natural ones and can be considered as the "limits" of the representations of Weyl groups.

Here we explain why we study the representations of Weyl groups or Hecke algebras on virtual character modules. The irreducible admissible representations of G were classified by R. Langlands ([11]) modulo tempered representations. Since irreducible tempered representations were classified by A. W. Knapp and G. J. Zuckerman ([10]), the classification of irreducible admissible representations of G is now complete. However, their parameters attached to each irreducible representation are very complicated, and do not make unitarizability or primitive ideal or its Gel'fand-Kirillov dimension etc. clear. We want to classify the irreducible representations of G into some different classes which make the invariants of representations as listed above much clearer. To achieve this, it is convenient to consider the Weyl group actions or Hecke algebra actions on virtual characters mentioned above.

Received December 2, 1985

*) This research was partially supported by Grant-in-Aid for Scientific Research (No. 60790018), Ministry of Education.

**) JSPS Fellowships for Japanese Junior Scientists.

Let us explain our definition of representations of Hecke algebras. The definition has three different interpretations which are interrelated each other.

Let H be a Cartan subgroup of G and $\lambda \in \mathfrak{h}_C^*$ an infinitesimal character not necessarily regular. We make some assumption on λ (see Assumption 2.1). This assumption is not essential, since it is satisfied for appropriate multiple of λ by a positive integer. Let $\lambda_0 \in \mathfrak{h}_C^*$ be a dominant regular infinitesimal character which satisfies: (1) $\mu = \lambda_0 - \lambda$ belongs to the root lattice of $(\mathfrak{g}_C, \mathfrak{h}_C)$. (2) μ satisfies Assumption 5.3. Such a λ_0 always exists. Then the representations of the Hecke algebras have three different constructions explained below.

Construction 1. Let τ be the representation of the integral Weyl group $W_H(\lambda_0)$ on $V_H(\lambda_0)$ defined in [16]. Here, $W_H(\lambda_0)$ is a certain subgroup of the complex Weyl group $W = W(\mathfrak{g}_C, \mathfrak{h}_C)$, and $V_H(\lambda_0)$ is a subspace of the virtual character module $V(\lambda_0)$ with infinitesimal character λ_0. We have

$$V(\lambda_0) = \sum_{[H] \in \mathrm{Car}(G)}^{\oplus} V_H(\lambda_0),$$

where Car(G) is the set of all the conjugacy classes of Cartan subgroups of G and $[H]$ denotes the class of H. Put $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$, the fixed subgroup of λ in W. Then W_λ is a subgroup of $W_H(\lambda) = W_H(\lambda_0)$ and we can define a Hecke algebra $\mathscr{H}(W_H(\lambda), W_\lambda)$ (see §3 for precise definition). Since $\mathscr{H}(W_H(\lambda), W_\lambda)$ is isomorphic to a subalgebra $e_\lambda \boldsymbol{C}[W_H(\lambda)]e_\lambda$ (where $e_\lambda = (\# W_\lambda)^{-1} \sum_{s \in W_\lambda} s$) of the group ring $\boldsymbol{C}[W_H(\lambda)]$, $\mathscr{H}(W_H(\lambda), W_\lambda)$ has natural action on $V_H(\lambda_0)$. We can prove

Theorem A (Theorem 4.2). *The vector space $V_H(\lambda)$ is isomorphic to the vector space $\tau(e_\lambda)V_H(\lambda_0)$ and we can define the representation of $\mathscr{H}(W_H(\lambda), W_\lambda)$ on the space $V_H(\lambda) \simeq \tau(e_\lambda)V_H(\lambda_0)$ naturally.*

This theorem is valid in purely algebraic situation (Proposition 3.5).

Construction 2. The above space $V_H(\lambda)$ is isomorphic to a certain subspace of analytic functions on H. We denote this space by $\mathfrak{C}(H; \lambda)$. For a canonical basis of $\mathfrak{C}(H; \lambda)$, we can define an action of $\mathscr{H}(W_H(\lambda), W_\lambda)$ analogous to the definition of the representation τ of $W_H(\lambda)$ (Theorem 4.2). This is the second construction of the representations.

Construction 3. Let $\varphi = \varphi_{\lambda_0}^{\lambda}$ and $\psi = \psi_{\lambda}^{\lambda_0}$ be Zuckerman's translation functors (see §5.1 for precise definition). These functors play an important role in representation theory ([10], [18]). We define an action σ of $e_\lambda w e_\lambda \in \mathscr{H}(W_H(\lambda), W_\lambda)$ on $V_H(\lambda)$ by

$$\sigma(e_\lambda w e_\lambda)v = (\# W_\lambda)^{-1} \psi \circ \tau(e_\lambda w e_\lambda) \circ \varphi(v),$$

where we consider τ as a representation of the group ring $\boldsymbol{C}[W_H(\lambda)] = \boldsymbol{C}[W_H(\lambda_0)]$. This action turns out to be a representation of $\mathscr{H}(W_H(\lambda), W_\lambda)$ (Theorem 5.6).

Since ψ is considered to be a "*limiting*" functor which sends a regular parameter to singular one, we can characterize σ as the "*limit*" of τ.

Theorem B. *The representations of $\mathscr{H}(W_H(\lambda), W_\lambda)$ constructed in the above three ways coincide with each other.*

We denote this representation by σ.

Theorem C. *If the infinitesimal character λ is integral, we have $W_H(\lambda)=W$ for each Cartan subgroup H. Therefore we can define a representation σ of a Hecke algebra $\mathscr{H}(W, W_\lambda)$ on the whole virtual character module $V(\lambda)$.*

Now we comment about applications of our theory. Using the equivalence of three definitions of σ, we can reproduce some results of D. Vogan about τ-invariants (see [19]), and get some new results. We think our representation σ will clarify Gel'fand-Kirillov dimensions of irreducible representations of G and some other invariants associated with primitive ideals of $U(\mathfrak{g}_C)$ (see [9]). These subjects are to be treated in future papers.

Our theory may supply a number of examples for representations of Hecke algebras. We carry out explicit calculations for the group $G=U(3, 1)$.

Now we explain the contents of this paper briefly. After some preparations in §1, we review the definition of the representation τ of integral Weyl groups $W_H(\lambda)$ shortly in §2 (see [16]). §3 is devoted to a general theory of Hecke algebras $\mathscr{H}(W, D)$, where W is a finite group acting on $\boldsymbol{R}^n$ faithfully and D is a subgroup of W. The algebraic part of the proof of Theorem A is contained in this section. In §4, we give the definition of the representation σ of $\mathscr{H}(W_H(\lambda), W_\lambda)$. Main theorem, Theorem 4.2, says Constructions 1 and 2 are equivalent. We study the commutativity of Zuckerman's functors and Hirai's method $\boldsymbol{T}$ of constructing invariant eigendistributions in the first half of §5 (Propositions 5.1 and 5.2). These results take an important part in the following theory. The main theorem in §5 is Theorem 5.6 which states Construction 3 is equivalent to Construction 2 (and hence to 1). Thus we establish Theorems B and C in this section. In §6, we apply our results to study τ-invariants and get several results. Some of them are already obtained by D. Vogan ([19]). In the final section §7, we give an example of the representations of Hecke algebras in case of $G=U(3, 1)$. Essentially, $G=U(n, 1)$ $(n\geqq 2)$ can be treated in the same way.

Hirai's method $\boldsymbol{T}$ is explained in Appendix A because it is an important tool for our theory. And, in Appendix B, we discuss Assumptions 2.1 and 5.3. One can conclude these assumptions are not essential.

The author is grateful to Professor T. Hirai for his constant encouragements and useful discussions.

§1. Notations and preliminaries.

1.1. Let G be a connected semisimple Lie group with finite centre. We always assume G is acceptable (see below). Let $\mathfrak{g}$ be the Lie algebra of G and $U(\mathfrak{g}_C)$ its enveloping algebra. In the following, we denote Lie groups by Roman capital letters and its Lie algebras by corresponding German small letters. The complexification of a Lie algebra will be denoted with the subscript C. Let H be a Cartan subgroup of G. Then the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$ has a root space decomposition with respect to $\mathfrak{h}_C$:

$$\mathfrak{g}_C=\mathfrak{h}_C\oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where Δ is the set of roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$ and $\mathfrak{g}_\alpha$ is the root space corresponding to α. We fix a positive system Δ^+ and put $\rho=\sum \alpha/2$ $(\alpha\in\Delta^+)$. Define an analytic function ξ_α $(\alpha\in\Delta)$ on H by $\mathrm{Ad}(h)X_\alpha=\xi_\alpha(h)X_\alpha$ $(h\in H)$, where X_α is a non-zero root vector for α. We call G acceptable if there exists a connected complex semisimple Lie group G_C with Lie algebra $\mathfrak{g}_C$ which has the following two properties. (1) The canonical injection from $\mathfrak{g}$ into $\mathfrak{g}_C$ can be lifted up to a homomorphism of G into G_C. (2) Let H_C be the analytic subgroup of G_C corresponding to $\mathfrak{h}_C$. Then $\xi_\rho(\exp x)=\exp \rho(x)$ $(x\in\mathfrak{h}_C)$ defines a character of H_C into C^*.

We denote the Weyl group of Δ by $W=W(\Delta)$ and call it the complex Weyl group. Let B be a subgroup of G and D be a subset of G (or of $\mathfrak{g}_C$). Then we define $W(B; D)=N_B(D)/Z_B(D)$, where $N_B(D)$ denotes the normalizer of D in B and $Z_B(D)$ the centralizer. We call $W(B; D)$ a Weyl group of D in B.

Let $\lambda\in\mathfrak{h}_C^*$ be a linear form on $\mathfrak{h}_C$. The complex Weyl group W acts on $\mathfrak{h}_C^*$ and consequently acts on $\mathfrak{h}_C$ in a contragredient manner. Let W_λ be a fixed subgroup of λ in W:

$$W_\lambda=\{w\in W \mid w\lambda=\lambda\}.$$

We call λ regular if $W_\lambda=\{e\}$ and otherwise call it singular.

We introduce an *integral Weyl group* $W_H(\lambda)$ for H and λ after [16]. Let $W_H^\sim(\lambda)$ be a subset of W defined by

$$W_H^\sim(\lambda)=\{w\in W \mid \xi_{w\lambda}(\exp x)=\exp w\lambda(x)\ (x\in\mathfrak{h}) \text{ defines a character of } H_0\},$$

where H_0 denotes the connected component of H containing the identity element e. Then $W_H(\lambda)$ is by definition the largest subgroup of W which leaves $W_H^\sim(\lambda)$ stable under the right multiplication (cf. [16, Prop. 1.5]). Let H_1 be a connected component of H. Then an element $w\in W(G; H_1)$ normalizes $\mathfrak{h}$. Therefore $w\in W(G; H_1)$ determines an element $\bar{w}$ of $W(G;\mathfrak{h})\subset W$. Similarly, for $w\in W(G; H)$, the element $\bar{w}\in W(G;\mathfrak{h})$ can be defined. We remark that $W_H^\sim(\lambda)$ is stable under the left multiplication by the elements of $W(G;\mathfrak{h})$. For $s\in W(G; H_1)$ (or $s\in W(G; H)$) and $t\in W_H^\sim(\lambda)$, we write $st\in W_H^\sim(\lambda)$ instead of $\bar{s}t$ for simplicity.

1.2. Invariant eigendistributions. We review the facts about invariant eigendistributions (IEDs) and characters on G briefly.

Let $(\pi, \mathfrak{H})$ be an irreducible representation of G on a Hilbert space $\mathfrak{H}$. We assume π be admissible, i.e., K-multiplicities are finite. Then π has a character Θ_π which is a distribution on G:

$$\Theta_\pi(f)=\mathrm{Trace}\int_G f(g)\pi(g)dg \quad (f\in C_0^\infty(G)),$$

where $C_0^\infty(G)$ is the space of C^∞-functions with compact supports. The irreducible character Θ_π has the following remarkable properties.

(1) It is invariant under the inner automorphisms of G.

(2) It is a simultaneous eigendistribution of two-sided invariant differential operators (*Laplace operators*) on G.

(3) Essentially, it coincides with a locally summable function f_π on G which is analytic on the open dence subset G' of regular elements of G.

Definition 1.1. We call a distribution Θ on G *invariant eigendistribution* (*IED*) if it satisfies the properties (1)–(2) above.

The property (3) follows from (1) and (2) (see [3, Th. 2]).

Take an IED Θ. Then Θ is an eigendistribution of Laplace operators:

$$z\Theta=\chi(z)\Theta \quad (z\in\mathfrak{Z}),$$

where $\mathfrak{Z}$ is the centre of $U(\mathfrak{g}_C)$ (identified with the space of Laplace operators). The algebra homomorphism χ of $\mathfrak{Z}$ into C is called the *inifinitesimal character* of Θ.

Let H be a Cartan subgroup of G. We give a local expression of Θ on H. By the Harish-Chandra map η we can identify $\mathfrak{Z}$ and $U(\mathfrak{h}_C)^W$, the space of W-invariant polynomials on $\mathfrak{h}_C$. Then χ defines an element of $\mathrm{Hom}_{alg}(U(\mathfrak{h}_C)^W, C)\simeq\mathfrak{h}_C^*/W$:

$$\begin{array}{ccc}\mathfrak{Z} & \xrightarrow{\sim} & U(\mathfrak{h}_C)^W \\ \chi=\chi_\lambda \searrow & & \swarrow \lambda\in\mathfrak{h}_C^* \\ & C & \end{array}$$

Corresponding element $\lambda\in\mathfrak{h}_C^*$ is also called an infinitesimal character of Θ and we denote this by $\chi=\chi_\lambda$. Remark that $\chi_\lambda=\chi_{w\lambda}$ for any $w\in W$.

Let $h\in H\cap G'$ be a regular element. Then we have for a sufficiently small $x\in\mathfrak{h}$,

$$D\Theta(h\exp x)=\sum_{w\in W} c(w,\,h;\,x)\exp w\lambda(x).$$

Here,

$$D(h)=\xi_\rho(h)\prod_{\alpha\in\Delta^+}(1-\xi_\alpha^{-1}(h))$$

is called the Weyl denominator. The coefficients $c(w, h; x)$ are polynomials in x. If all the coefficients can be taken as constants in x for any w, h and any Cartan subgroup H, we call Θ a *constant coefficient IED.*

1.3. Virtual characters and IEDs. A *virtual character* is by definition a linear combination of irreducible characters. The space of all the virtual characters with infinitesimal character λ is denoted by $V(\lambda)$. We proved the following in [14, 15].

Proposition 1.2. *The space $V(\lambda)$ of virtual characters coincides with the space of constant coefficient IEDs with infinitesimal character λ.*

By this proposition, virtual characters and constant coefficient IEDs are identified. Let us introduce the results on IEDs obtained by T. Hirai ([5, 6]). Let H be a Cartan subgroup of G and take an infinitasimal character $\lambda \in \mathfrak{h}_C^*$. Define a family of analytic functions on H as

$\mathfrak{B}(H;\lambda)=\{\zeta \mid \zeta$ is analytic on H, satisfying the following conditions (1) and (2)$\}$.

(1) ζ is an eigenfunction of $U(\mathfrak{h}_C)^W$ with eigenvalue λ.

(2) ζ is ε-symmetric under $W(G;H)$, i.e.,

$$\zeta(wh)=\varepsilon(h;w)\zeta(h) \quad (h \in H,\ w \in W(G;H)),$$

where $\varepsilon(h;w)$ is defined as follows:

$$\varepsilon(h;w)=(-1)^{N(w)} \prod_{\alpha \in R(w)} \operatorname{sgn}(\xi_{w^{-1}\alpha}(h)),$$

$$N(w)=\sharp\{\alpha \in \Delta^+ \mid \alpha \text{ is imaginary and } w^{-1}\alpha<0\},$$

$$R(w)=\{\alpha \in \Delta^+ \mid \alpha \text{ is real and } w^{-1}\alpha<0\}.$$

We say a root $\alpha \in \Delta$ is real (or imaginary) if it takes real (respectively, purely imaginary) values on $\mathfrak{h}$. The function $\varepsilon(h;w)$ is locally constant on H, with values in $\{\pm 1\}$.

Each element $\zeta \in \mathfrak{B}(H;\lambda)$ can be written as

$$\zeta(h \exp x)=\sum_{w \in W} a_w(h;x) \exp w\lambda(x) \quad (x \in \mathfrak{h},\ h \in H),$$

where $a_w(h;x)$ is a polynomial function in x depending on h and w. If $a_w(h;x)$ can be taken as constant in x for each h and w, we call ζ *of constant coefficients*. Put

$$\mathfrak{C}(H;\lambda)=\{\zeta \in \mathfrak{B}(H;\lambda) \mid \zeta \text{ is of constant coefficients}\}.$$

Theorem 1.3 (T. Hirai). (1) *There is a canonical linear isomorphism* $\boldsymbol{T}$ *of* $\mathfrak{B}(H;\lambda)$ *into the space of IEDs* $\mathfrak{A}(\lambda)$ *with infinitesimal character* λ. *Let* $\mathfrak{A}_H(\lambda)=\boldsymbol{T}(\mathfrak{B}(H;\lambda))$. *Then*

$$\mathfrak{A}(\lambda)=\sum_H{}^{\oplus} \mathfrak{A}_H(\lambda)$$

is a direct sum, where H runs through all the representatives of conjugacy classes of Cartan subgroups of G.

(2) *Let* $V_H(\lambda)=\boldsymbol{T}(\mathfrak{C}(H;\lambda))$. *Then*

$$V(\lambda)=\sum_H{}^{\oplus} V_H(\lambda)$$

gives a direct sum decomposition of the space of constant coefficient IEDs (*or the space of virtual characters*).

The definition of the linear map $\boldsymbol{T}$ is described in [6, §3]. We explain the construction of $\boldsymbol{T}$ in Appendix A for later use.

§ 2. The representations of integral Weyl groups $W_H(\lambda)$.

2.1. Let Car(G) be the set of all the conjugacy classes of Cartan subgroups of G. Take $[H] \in \mathrm{Car}(G)$, where $[H]$ denotes the conjugacy class of H.

At first, we describe generators of the space $\mathfrak{C}(H;\lambda)$. Let $\{H_i \mid 0 \leqq i \leqq l\}$ be a complete system of representatives of connected components of H under the inner automorphisms of G (we take H_0 as the connected component of e). For $t \in W_H^{\sim}(\lambda)$, $0 \leqq i \leqq l$ and $a_i \in H_i$, we define an analytic function $\zeta(a_i, t\lambda; h)$ on H as follows. Define $\zeta(a_i, t\lambda; h)$ first on H_i. Put for $h \in H_i$,

$$\zeta(a_i, t\lambda; h) = \sum_{s \in W(G; H_i)} \varepsilon(a_i; s)\xi_{t\lambda}(a_i^{-1}(sh)), \tag{2.1}$$

where $\xi_{t\lambda}$ is an analytic function on H_0 defined by $\xi_{t\lambda}(\exp x) = \exp t\lambda(x)$ $(x \in \mathfrak{h})$. On $W(G; H)$-orbit of H_i, we put $\zeta(a_i, t\lambda; h)$ as

$$\zeta(a_i, t\lambda; wh) = \varepsilon(h; w)\zeta(a_i, t\lambda; h) \quad (h \in H_i,\ w \in W(G; H)),$$

and for $h \in H$ outside of $W(G; H)$-orbit of H_i, put $\zeta(a_i, t\lambda; h) = 0$.

Easy calculations tell us that $\zeta(a_i, t\lambda; *) \in \mathfrak{C}(H;\lambda)$. Moreover, one knows that $\{\zeta(a_i, t\lambda; *) \mid 0 \leqq i \leqq l,\ t \in W_H^{\sim}(\lambda)\}$ spans $\mathfrak{C}(H;\lambda)$ for a fixed set $\{a_i \mid a_i \in H_i, 0 \leqq i \leqq l\}$.

In the following of this paper, we assume that $\{a_i\}$ can be taken nicely for λ. More precisely, we put the following assumption on λ.

Assumption 2.1. For each Cartan subgroup H of G, there exists $\{a_i\}$ such that
(0) $a_i \in H_i$ $(0 \leqq i \leqq l)$ and $a_0 = e$.
(1) $\xi_{t\lambda}(a_i^{-1}(sa_i)) = 1$ for any $t \in W_H^{\sim}(\lambda)$ and $s \in W(G; H_i)$.

Remark 2.2. For a special G, Assumption 2.1 is satisfied for any λ. For example, if $G = SL(n, \boldsymbol{R})$, $Sp(2n, \boldsymbol{R})$, $SO_0(p, q)$ $(p+q=2n)$ or a complex Lie group, then the assumption is satisfied. In general, if we replace λ by $m\lambda$ for some positive integer m, the assumption above is satisfied. More detailed discussion is given in Appendix B.

2.2. In the following of this section, we assume that λ is regular. Then it is known that $\mathfrak{B}(H;\lambda) = \mathfrak{C}(H;\lambda)$ and $V(\lambda) = \mathfrak{A}(\lambda)$. We recall the definition of the representations of integral Weyl group $W_H(\lambda)$ on $V_H(\lambda)$ (see [16, §3]).

Since $\mathfrak{B}(H;\lambda) = \mathfrak{C}(H;\lambda) = \langle \zeta(a_i, t\lambda; *) \mid 0 \leqq i \leqq l,\ t \in W_H^{\sim}(\lambda) \rangle$ (linear span over $\boldsymbol{C}$) and $V_H(\lambda) = \boldsymbol{T}(\mathfrak{B}(H;\lambda))$, we may identify $\mathfrak{B}(H;\lambda)$ and $V_H(\lambda)$ by $\boldsymbol{T}$. Then $w \in W_H(\lambda)$ acts on $\zeta(a_i, t\lambda; *)$ as

$$\mathscr{R}(w)\zeta(a_i, t\lambda; *) = \zeta(a_i, tw^{-1}\lambda; *).$$

An element $w \in W_H(\lambda)$ acts on $\boldsymbol{T}\zeta(a_i, t\lambda; *)$ as

$$\tau(w)(\boldsymbol{T}\zeta(a_i, t\lambda; *)) = \boldsymbol{T}(\mathscr{R}(w)\zeta(a_i, t\lambda; *)).$$

Assumption 2.1 assures that this definition of τ is well-defined. We can decompose the representation $(\tau, V_H(\lambda))$ of $W_H(\lambda)$ completely in terms of induced representations. Let us explain this. Let $\Gamma_i \subset W_H^{\sim}(\lambda)$ be a complete system of representatives of a coset space $W(G; H_i)\backslash W_H^{\sim}(\lambda)/W_H(\lambda)$ and put

$$W(i, \gamma) = W_H(\lambda) \cap \gamma^{-1} W(G; H_i)\gamma \quad (\gamma \in \Gamma_i),$$

$$\varepsilon(i, \gamma; w) = \varepsilon(a_i; \gamma w \gamma^{-1}) \quad (a_i \in H_i,\ w \in W(i, \gamma)).$$

Then $\varepsilon(i, \gamma; *)$ is a character of the group $W(i, \gamma)$.

Theorem 2.3 ([16, Th. 5.1]). *The representation τ of $W_H(\lambda)$ on $V_H(\lambda)$ given above is decomposed into a direct sum of induced representations*:

$$(\tau, V_H(\lambda)) = \sum_{i=0}^{l}{}^{\oplus} \sum_{\gamma \in \Gamma_i}{}^{\oplus} \operatorname{Ind}(\varepsilon(i, \gamma; *);\ W(i, \gamma) \uparrow W_H(\lambda)),$$

where $\operatorname{Ind}(\varepsilon; A \uparrow B) = \operatorname{Ind}_A^B \varepsilon$.

Now we remark the connection between our representations and the representations of Weyl groups which Zuckerman defined ([10, Appendix], see also [1]). In the case that λ is integral for G, i.e., $W_H(\lambda) = W$ for any H, our representation of W is defined on the whole space of virtual characters $V(\lambda) = \sum_H^{\oplus} V_H(\lambda)$. This representation is equivalent (under Assumption 2.1) to Zuckerman's one. But for general λ, his definition is only applied to a subgroup

$$W_0 = \{w \in W \mid w\lambda - \lambda \in \boldsymbol{Q}[\Delta]\}$$

of W, while our definition can be applied to a larger subgroup than W_0. Remark that Zuckerman's representation of W_0 and ours restricted to W_0 are almost equivalent (in fact, replacing λ by $m\lambda$ for some integer $m > 0$, we can prove they are equivalent).

§ 3. Generalities on Hecke algebras.

This section is devoted to explain general properties of Hecke algebras and their representations. We use notations independent of the other sections here.

3.1. Hecke algebras. Let W be a group (infinite or finite) and D its subgroup. We assume that

$$(3.1) \qquad [D; D \cap x^{-1}Dx] < \infty \qquad \text{for any} \quad x \in W.$$

Let $M = \{DxD \mid x \in W\}$ be the set of double cosets, and we denote by $\mathscr{H}^{Z}(W, D)$ a free abelian group generated by M. For $A, B, C \in M$, put $\mu_{A,B}^{C} = \sharp(D\backslash A^{-1}C \cap B) < \infty$ and define the product $A \circ B$ by

$$A \circ B = \sum_{C \in M} \mu_{A,B}^{C} C.$$

The algebra $\mathscr{H}^{Z}(W, D)$ with the above product $\circ$ is called the Hecke algebra of (W, D) over $\boldsymbol{Z}$([7, 8]). We simply call $\mathscr{H}(W, D) = \mathscr{H}^{Z}(W, D) \otimes_{Z} \boldsymbol{C}$ the *Hecke algebra of*

(W, D) in this paper.

Now we assume that W is a finite group. Remark that (3.1) is always satisfied. In this case we have more convenient interpretation of $\mathscr{H}(W, D)$. Let $C[W]$ be a group ring of W and put

$$e_D = \frac{1}{\sharp D} \sum_{d \in D} d \in C[W].$$

Then the subalgebra $e_D C[W] e_D$ of $C[W]$ is isomorphic to $\mathscr{H}(W, D)$ as an algebra. As a consequence, $\mathscr{H}(W, D)$ is a semisimple algebra. Since e_D is idempotent, $\mathscr{H}(W, D) \simeq e_D C[W] e_D$ has a unit element e_D. In the following, we always regard $\mathscr{H}(W, D)$ as the subalgebra $e_D C[W] e_D$ of $C[W]$.

Take a representation π of W on a finite dimensional vector space V. Then there corresponds a representation of the group ring $C[W]$ naturally. We denote it also by π. Since $\mathscr{H}(W, D)$ is a subalgebra, we can get a homomorphism

$$\pi|_{\mathscr{H}(W,D)} : \mathscr{H}(W, D) \longrightarrow \mathrm{End}(V).$$

But it does not send the unit element e_D to the unit element 1_V of $\mathrm{End}(V)$. To avoid this situation, we decompose V as

$$V = V_0 \oplus V_1 \quad \text{(direct sum of } D\text{-modules)},$$

where $V_1 = V^D = \{v \in V \mid \pi(d)v = v \text{ for any } d \in D\}$ and V_0 is the complement of V_1. Since $\pi(e_D)V = V_1$, we have

$$\pi|_{\mathscr{H}(W,D)} : \mathscr{H}(W, D) \longrightarrow \mathrm{End}(V_1) \subset \mathrm{End}(V),$$

and $\pi(e_D) = 1_{V_1}$. Therefore we get a representation of $\mathscr{H}(W, D)$ on V_1 from a representation (π, V) of W. We call this representation of $\mathscr{H}(W, D)$ the *reduction* of (π, V) to $\mathscr{H}(W, D)$ and denote it by $\mathrm{Red}_D^W \pi$. The representation space of $\mathrm{Red}_D^W \pi$ is $V_1 \cong V/V_0$ as described above.

Lemma 3.1. *If π is irreducible, then* $\mathrm{Red}_D^W \pi$ *is irreducible.*

Proof. It is easy to see that every vector of V_1 except 0 is cyclic, and consequently $\mathrm{Red}_D^W \pi$ is irreducible. Q. E. D.

3.2. The representations of the Hecke algebra $\mathscr{H}(W, W_\lambda)$.
Let us consider the following case. Take a finite group W' acting on R^n faithfully.

(*) For a subset $W^\sim$ of W', let A and W be subgroups such that $A \subset \{a \in W' \mid aW^\sim = W^\sim\}$ and $W = \{b \in W' \mid W^\sim b = W^\sim\}$. Then there exists $\lambda \in R^n$ such that $W_\lambda = \{w \in W' \mid w\lambda = \lambda\}$ is a subgroup of W.

Now we treat the Hecke algebra $\mathscr{H}(W, W_\lambda)$ and their representations. Take a character χ of A. Define an element of the group ring of R^n by

$$\zeta(t, \lambda_0) = \sum_{a \in A} \chi(a) \exp at\lambda_0 \quad (t \in W^\sim),$$

and put $\mathfrak{B}(\lambda_0) = \langle \zeta(t, \lambda_0) \mid t \in W^\sim \rangle$ (linear span over C), where $\lambda_0 \in R^n$ is a regular element, i.e., $W_{\lambda_0} = \{w \in W' \mid w\lambda_0 = \lambda_0\} = \{e\}$.

Lemma 3.2. *Linear transformations* $\tau(w)$ $(w \in W)$ *on* $\mathfrak{B}(\lambda_0)$ *defined by*

$$\tau(w) : \zeta(t, \lambda_0) \longrightarrow \zeta(tw^{-1}, \lambda_0)$$

give a representation $(\tau, \mathfrak{B}(\lambda_0))$ *of* W.

As described in 3.1, we get a representation Red $^W_{W_\lambda}\tau$ of $\mathscr{H}(W, W_\lambda)$ from $(\tau, \mathfrak{B}(\lambda_0))$. In the following, we will give another interpretation of Red $^W_{W_\lambda}\tau$ in the above situation. This is achieved by *translating* regular parameter λ_0 to singular one. Returning to $\lambda \in \boldsymbol{R}^n$ in (∗), we define $\zeta(t, \lambda)$ $(t \in W^\sim)$ and $\mathfrak{B}(\lambda)$ as $\zeta(t, \lambda_0)$ and $\mathfrak{B}(\lambda_0)$, using λ instead of λ_0. Define a linear map P of $\mathfrak{B}(\lambda_0)$ to $\mathfrak{B}(\lambda)$ by

$$P\zeta(t, \lambda_0) = \zeta(t, \lambda).$$

Remark that P is onto but not injective in general.

We construct a representation σ of $\mathscr{H}(W, W_\lambda)$ on the space $\mathfrak{B}(\lambda)$ as follows. Recall that $\mathscr{H}(W, W_\lambda) = e_\lambda \boldsymbol{C}[W] e_\lambda$, where $e_\lambda = (\# W_\lambda)^{-1} \sum_{s \in W_\lambda} s$. For $e_\lambda w e_\lambda \in \mathscr{H}(W, W_\lambda)$, we put

$$\sigma(e_\lambda w e_\lambda)\zeta(t, \lambda) = P(\tau(e_\lambda w e_\lambda)\zeta(t, \lambda_0)). \tag{3.2}$$

Lemma 3.3. *The linear operators* $\sigma(e_\lambda w e_\lambda)$ $(w \in W)$ *define a representation of the Hecke algebra* $\mathscr{H}(W, W_\lambda)$.

Proof. At first we prove $\sigma(e_\lambda w e_\lambda)$ is well-defined. That is to say, we prove that if

$$\sum_{t \in W^\sim} c_t \zeta(t, \lambda) = 0,$$

then it holds

$$P(\tau(e_\lambda w e_\lambda) \sum_{t \in W^\sim} c_t \zeta(t, \lambda_0)) = 0 \tag{3.3}$$

for any $w \in W$. We use the following lemma.

Lemma 3.4. *Let* $\mathfrak{B}(\lambda_0)_1$ *be the space of all the* W_λ*-fixed vectors and* $\mathfrak{B}(\lambda_0)_0$ *the complement in* $\mathfrak{B}(\lambda_0)$ *as* W_λ*-module. Then we have* Ker $P = \mathfrak{B}(\lambda_0)_0$.

We will prove this lemma after the proof of Lemma 3.3.

Now apply Lemma 3.4 to the element $\sum c_t \zeta(t, \lambda_0)$. Since it belongs to Ker P by assumption, it generates a W_λ-module that contains no non-zero fixed vector. So we have

$$\tau(e_\lambda)(\sum_{t \in W^\sim} c_t \zeta(t, \lambda_0)) = 0$$

and we have proved (3.3).

To verify that σ defines a representation is now an easy task. Take $w_1, w_2 \in W$. Then we have

$$\sigma(e_\lambda w_1 e_\lambda)\sigma(e_\lambda w_2 e_\lambda)\zeta(t, \lambda) = \sigma(e_\lambda w_1 e_\lambda)P(\tau(e_\lambda w_\lambda e_\lambda)\zeta(t, \lambda_0))$$

$$=P(\tau(e_\lambda w_1 e_\lambda)\tau(e_\lambda w_2 e_\lambda)\zeta(t, \lambda_0))$$

$$=P(\tau(e_\lambda w_1 e_\lambda w_2 e_\lambda)\zeta(t, \lambda_0))$$

$$=\sigma(e_\lambda w_1 e_\lambda w_2 e_\lambda)\zeta(t, \lambda). \qquad \text{Q.E.D.}$$

Proof of Lemma 3.3. At first we show that Ker P contains $\mathfrak{B}(\lambda_0)_0$. For any $s \in W_\lambda$, we have

$$P(\tau(s^{-1})\zeta(t, \lambda_0)) = P(\zeta(ts, \lambda_0))$$
$$= \zeta(ts, \lambda) = \zeta(t, \lambda) = P(\zeta(t, \lambda_0)).$$

Therefore, for any $v \in \mathfrak{B}(\lambda_0)_0$, we have $\tau(e_\lambda)v = 0$ and

$$0 = P(\tau(e_\lambda)v) = (\sharp W_\lambda)^{-1} \sum_{s \in W_\lambda} P(\tau(s)v) = (\sharp W_\lambda)^{-1} \sum_{s \in W_\lambda} P(v) = P(v).$$

Thus we have $P(v) = 0$.

Now we prove the reversed inclusion. Assume that $P(v) = 0$. Decompose $v = v_0 \oplus v_1$ along the direct sum $\mathfrak{B}(\lambda_0) = \mathfrak{B}(\lambda_0)_0 \oplus \mathfrak{B}(\lambda_0)_1$. Since $P(v) = P(v_0) + P(v_1) = P(v_1)$ from the above, we can assume that $v = v_1 \in \mathfrak{B}(\lambda_0)_1$. Let $\{t_i \mid i \in I\}$ be a complete system of representatives of $A \backslash W^\sim$. Clearly, $\{\zeta(t_i, \lambda_0) \mid i \in I\}$ is a basis of $\mathfrak{B}(\lambda_0)$. So we can write

$$v = \sum_{i \in I} c_i \zeta(t_i, \lambda_0) \quad (c_i \in \boldsymbol{C}).$$

Using this expression for v, we rewrite the equality $\tau(s)v = v$ for any $s \in W_\lambda$. We have

$$\tau(s^{-1})v = \sum_{i \in I} c_i \zeta(t_i s, \lambda_0) = \sum_{i \in I} c_i \sum_{a \in A} \chi(a) \exp a t_i s \lambda_0.$$

If we write $t_i s = a(i, s) t_{i(s)} \in A\{t_i\} = W^\sim$, then the above formula becomes

$$\sum_i c_i \sum_a \chi(a) \exp a a(i, s) t_{i(s)} \lambda_0$$
$$= \sum_i c_i \chi(a(i, s)^{-1}) \sum_a \chi(a) \exp a t_{i(s)} \lambda_0$$
$$= \sum_i c_i \chi(a(i, s)^{-1}) \zeta(t_{i(s)}, \lambda_0).$$

This is equal to $v = \sum c_i \zeta(t_i, \lambda_0)$. Therefore we have $c_i = \chi(a(i, s)) c_{i(s)}$ for any $s \in W_\lambda$.

Now, since

$$0 = P(v) = \sum_{i \in I} c_i \zeta(t_i, \lambda) = \sum_{i \in I} c_i \sum_{a \in A} \chi(a) \exp a t_i \lambda,$$

the coefficients of $\exp a t_i \lambda$ must be zero. Remark that $a_1 t_i \lambda = a_2 t_j \lambda$ $(a_1, a_2 \in A)$ is equivalent to that there exists an $s \in W_\lambda$ such that $a_1 t_i = a_2 t_j s$. Therefore the coefficients of $\exp a t_i \lambda$ is equal to

$$\sum_{s \in W_\lambda} c_{i(s)} \chi(a a(i, s)) = \sum_{s \in W_\lambda} c_i \chi(a) = (\sharp W_\lambda) c_i \chi(a),$$

where we used $c_i = c_{i(s)} \chi(a(i, s))$. Now we proved that $P(v) = 0$ and $v \in \mathfrak{B}(\lambda_0)_1$ give

$c_i=0$, and therefore $v=0$. Q.E.D.

Proposition 3.5. *The representation* $(\sigma, \mathfrak{B}(\lambda))$ *of* $\mathscr{H}(W, W_\lambda)$ *is equivalent to* $\operatorname{Red}_{W_\lambda}^{W}(\tau, \mathfrak{B}(\lambda_0))$.

Proof. By Lemma 3.4, we have $\operatorname{Ker} P=\mathfrak{B}(\lambda_0)_0$. Therefore P defines a linear map of the representation space of $\operatorname{Red}_{W_\lambda}^{W}\tau$ to $\mathfrak{B}(\lambda)$. It is easy to see that P intertwines $\operatorname{Red}_{W_\lambda}^{W}\tau$ and σ. Q.E.D.

§ 4. Representations of Hecke algebras on virtual character modules.

4.1. After the general theory in §3, we now return to the notations and subjects in §§1 and 2. Let H be a Cartan subgroup of G and $\{H_i \mid 0\leqq i\leqq l\}$ a system of representatives of conjugacy classes of connected components of H under the inner automorphisms of G. Let $\lambda\in\mathfrak{h}_C^*$ be an infinitesimal character not necessarily regular, and W_λ its fixed subgroup in W. We choose λ to be dominant with respect to Δ^+ in the sense that $\operatorname{Re}\langle\lambda, \alpha\rangle\geqq 0$ for $\alpha\in\Delta^+$. As is mentioned in §2, the virtual character module $V(\lambda)$ with infinitesimal character λ is decomposed as a vector space over $\boldsymbol{C}$:

$$V(\lambda)=\sum_{[H]\in\operatorname{Car}(G)}^{\oplus} V_H(\lambda).$$

Each $V_H(\lambda)$ is isomorphic to the vector space $\mathfrak{C}(H; \lambda)$ of ε-symmetric λ-eigenfunctions on H which are of constant coefficients. Put $\mathfrak{C}_i(H; \lambda)=\langle\zeta(a_i, t\lambda; *) \mid t\in W_H^{\sim}(\lambda)\rangle$ and $V_H^i(\lambda)=\boldsymbol{T}(\mathfrak{C}_i(H; \lambda))$. Then clearly it holds that

$$\mathfrak{C}(H; \lambda)=\sum_{0\leqq i\leqq l}^{\oplus}\mathfrak{C}_i(H; \lambda), \quad V_H(\lambda)=\sum_{0\leqq i\leqq l}^{\oplus} V_H^i(\lambda).$$

Take a $\mu\in\mathfrak{h}_C^*$ such that (i) μ belongs to the root lattice $\boldsymbol{Q}[\Delta]$ and (ii) $\lambda_0=\lambda+\mu$ is dominant regular. Then we have the following lemma.

Lemma 4.1. (1) *The subset* $W_H^{\sim}(\lambda)$ *coincides with* $W_H^{\sim}(\lambda_0)$.
(2) *The integral Weyl group* $W_H(\lambda)$ *coincides with* $W_H(\lambda_0)$.
(3) *The subgroup* W_λ *is contained in* $W_H(\lambda)$.

The proof is easy. So we omit it.

4.2. Now we apply the results of §3 to this case. Take a character $\varepsilon(a_i; *)$ of $W(G; H_i)$ and form an analytic function $\zeta(a_i, t\lambda_0; *)$ $(a_i\in H_i,\ t\in W_H^{\sim}(\lambda))$ on H_i as

$$\zeta(a_i, t\lambda_0; a_i\exp x)=\sum_{s\in W(G; H_i)}\varepsilon(a_i; s)\exp st\lambda_0(x).$$

Then $\mathfrak{C}_i(H; \lambda_0)=\langle\zeta(a_i, t\lambda_0; *) \mid t\in W_H^{\sim}(\lambda)\rangle$ is a $W_H(\lambda)$-module as described in §2 (under the Assumption 2.1). Define a linear operator $P: \mathfrak{C}_i(H; \lambda_0)\to\mathfrak{C}_i(H; \lambda)$ by $P(\zeta(a_i, t\lambda_0; *))=\zeta(a_i, t\lambda; *)$. Then we come to the situation of §3.2, if we replace W', W, A, $W^{\sim}$, W_λ and χ in §3.2 by W, $W_H(\lambda)$, $W(G; H_i)$, $W_H^{\sim}(\lambda)$, W_λ and $\varepsilon(a_i; *)$ in this section respectively. We get the following.

Theorem 4.2. (1) *For* $e_\lambda w e_\lambda \in \mathscr{H}(W_H(\lambda), W_\lambda)$, *put*

$$\sigma(e_\lambda w e_\lambda) T\zeta(a_i, t\lambda; h) = (\# W_\lambda)^{-1} \sum_{s \in W_\lambda} T\zeta(a_i, tsw^{-1}\lambda; h) \quad (h \in H).$$

Then σ *is a representation of* $\mathscr{H}(W_H(\lambda), W_\lambda)$ *which carries the unit element of* $\mathscr{H}(W_H(\lambda), W_\lambda)$ *to the unit element of* End $V_H^i(\lambda)$. *Denote again by* σ *this representation of* $\mathscr{H}(W_H(\lambda), W_\lambda)$ *on the virtual character module* $V_H(\lambda) = \sum^{\oplus} V_H^i(\lambda)$ $(0 \leqq i \leqq l)$.

(2) *The representation* $(\sigma, V_H(\lambda))$ *of the Hecke algebra* $\mathscr{H}(W_H(\lambda), W_\lambda)$ *is equivalent to the reduction (with respect to the subgroup* W_λ*) of the representation* $(\tau, V_H(\lambda_0))$ *of* $W_H(\lambda_0) = W_H(\lambda)$, *the integral Weyl group*:

$$(\sigma, V_H(\lambda)) \simeq \mathrm{Red}\,{}^{W_H(\lambda_0)}_{W_\lambda}(\tau, V_H(\lambda_0)).$$

Using Theorem 2.3, we can decompose $(\sigma, V_H(\lambda))$ into a direct sum of "*induced*" representations. Namely, if we write

$$\mathrm{RI}\,(\varepsilon; A \uparrow B \downarrow C) = \mathrm{Red}\,{}^B_C\, \mathrm{Ind}\,{}^B_A\, \varepsilon,$$

we have the following.

Corollary 4.3. *The representation* $(\sigma, V_H(\lambda))$ *of* $\mathscr{H}(W_H(\lambda), W_\lambda)$ *defined in the above is decomposed as follows*:

$$(\sigma, V_H(\lambda)) = \sum_{i=0}^{l}{}^{\oplus} \sum_{\gamma \in \Gamma_i}{}^{\oplus} \mathrm{RI}\,(\varepsilon(i, \gamma; *); W(i, \gamma) \uparrow W_H(\lambda) \downarrow W_\lambda),$$

where Γ_i, $W(i, \gamma)$ *and* $\varepsilon(i, \gamma; *)$ *is given as in* §2.2.

Let

$$(4.1) \qquad (\tau, V_H(\lambda_0)) = \sum_{\eta \in W_H(\lambda)^\wedge} m_\eta \eta$$

be the decomposition into irreducible components, where m_η is the multiplicity of η. Remark that we can get (4.1) from Theorem 2.3 easily for explicit cases. We put

$$\mathscr{F}(\lambda) = \{\eta \in W_H(\lambda)^\wedge \mid \eta \text{ has non-trivial fixed vector for } W_\lambda\}$$
$$= \{\eta \in W_H(\lambda)^\wedge \mid [\eta;\ \mathrm{Ind}\,(1; W_\lambda \uparrow W_H(\lambda))] \neq 0\}.$$

Then we have

Corollary 4.4. *The representation* $(\sigma, V_H(\lambda))$ *of* $\mathscr{H}(W_H(\lambda), W_\lambda)$ *has the decomposition into irreducible components*:

$$(\sigma, V_H(\lambda)) \cong \sum_{\eta \in \mathscr{F}(\lambda)} m_\eta\, \mathrm{Red}{}^{W_H(\lambda_0)}_{W_\lambda}\, \eta.$$

Proof. This is clear from Lemma 3.1 and the fact that Red $\eta \neq (0)$ is equivalent to $\eta \in \mathscr{F}(\lambda)$. Q. E. D.

In the case where λ is integral, i.e., $W_H(\lambda) = W$ for each Cartan subgroup H of G, we have the representation

$$(\sigma, V(\lambda)) = \sum_{[H]\in \mathrm{Car}(G)}^{\oplus} (\sigma, V_H(\lambda))$$

of $\mathscr{H}(W, W_\lambda)$. Then Corollary 4.3 is reduced to the following (see [16, Th. 5.2]).

Corollary 4.5. *If λ is integral, the representation $(\sigma, V(\lambda))$ of $\mathscr{H}(W, W_\lambda)$ is decomposed as follows:*

$$(\sigma, V(\lambda)) = \sum_{[H]\in \mathrm{Car}(G)}^{\oplus} \sum_{i=0}^{l}{}^{\oplus} \mathrm{RI}\,(\varepsilon(a_i; *); W(G; H_i) \uparrow W \downarrow W_\lambda).$$

Theorem 4.2 says that "if we know $W_H(\lambda_0)$-module structures completely for arbitrary regular infinitesimal character λ_0, then we know the $\mathscr{H}(W_H(\lambda), W_\lambda)$-module structure for singular infinitesimal character λ", by translating the regular parameter λ_0 to the singular one λ. This theorem is useful to study the properties of the virtual characters (or irreducible representations of G) at singular parameters. For example, we have the following result about the dimension of $V(\lambda)$.

Corollary 4.6. *Let λ and $\lambda_0 = \lambda + \mu$ be as before. For a Cartan subgroup H, put*

$$n(H; \lambda_0, \lambda) = \dim \{v \in V_H(\lambda_0) \mid \tau(s)v = v \textit{ for any } s \in W_\lambda\}.$$

Then we have

$$\dim V(\lambda) = \sum_{[H]\in \mathrm{Car}(G)} n(H; \lambda_0, \lambda).$$

Remark. Recall that $\dim V(\lambda)$ is equal to the number of (equivalence classes of) irreducible admissible representations which have infinitesimal character λ.

§5. Relation to Zuckerman's translation functions: another interpretation of the representation σ.

5.1. Zuckerman's functors. We use the notations of §4 (and, of course, we suppose Assumption 2.1). Let $\varphi = \varphi_{\lambda_0}^{\lambda}$ and $\psi = \psi_{\lambda}^{\lambda_0}$ be Zuckerman's translation functors (see [20]). Here we explain the properties of φ and ψ briefly for later uses. Originally, Zuckerman defined them using the tensor products with finite dimensional representations of G. Functors φ and ψ are defined as

$$\varphi = \mathrm{Proj}\,(\lambda_0) \circ (F_\mu \otimes (\cdot)) \circ \mathrm{Proj}\,(\lambda),$$

$$\psi = \mathrm{Proj}\,(\lambda) \circ (F_\mu^* \otimes (\cdot)) \cdot \mathrm{Proj}\,(\lambda_0),$$

where F_μ is the irreducible finite dimensional representation of G with highest weight μ, and F_μ^* is its contragredient. Notations $\mathrm{Proj}\,(\lambda)$ and $\mathrm{Proj}\,(\lambda_0)$ mean "*projections*" to the components with infinitesimal character λ and λ_0 respectively. So φ and ψ are by definition the functors of categories of $(\mathfrak{g}_C, K)$-modules. Since both of them are exact functors, they induce linear maps between the virtual character modules $V(\lambda)$ and $V(\lambda_0)$. Here we denote these linear maps by the same letters φ and ψ:

$$\varphi\colon V(\lambda) \longrightarrow V(\lambda_0), \quad \psi\colon V(\lambda_0) \longrightarrow V(\lambda).$$

Take $\Theta_0 \in V(\lambda_0)$ and $[H] \in \mathrm{Car}(G)$. Then Θ_0 has a local expression arround a regular element $h \in H' = H \cap G'$ as explained in 1.2:

$$D\Theta_0(h \exp x) = \sum_{w \in W} c_w(h) \exp w\lambda_0(x) \quad (x \in \mathfrak{h}),$$

where $c_w(h)$ is a locally constant function on H'. By (3.8) in [20], we have

$$D(\psi\Theta_0)(h \exp x) = \sum_{w \in W} \xi_{-w\mu}(h) c_w(h) \exp w\lambda(x) \quad (x \in \mathfrak{h}).$$

Similarly, if we express $\Theta \in V(\lambda)$ as

$$D\Theta(h \exp x) = \sum_{w \in W} a_w(h) \exp w\lambda(x) \quad (x \in \mathfrak{h}),$$

then by (3.7) in [20], we have

$$D(\varphi\Theta)(h \exp x) = \sum_{t \in W_\lambda} \sum_{w \in W} \xi_{wt\mu}(h) a_w(h) \exp wt\lambda_0(x),$$

for $h \in H'$ and $x \in \mathfrak{h}$.

5.2. Relation to Hirai's method T. Let $\mathscr{P}$ be a linear map from $\mathfrak{B}(H; \lambda_0) = \mathfrak{C}(H; \lambda_0)$ to $\mathfrak{C}(H; \lambda)$ defined as follows. For $0 \leqq i \leqq l$ and $t \in W_{\tilde{H}}(\lambda_0)$, put

$$\mathscr{P}\zeta(a_i, t\lambda_0;\, h) = \xi_{-t\mu}(a_i)\zeta(a_i, t\lambda;\, h) \quad (h \in H),$$

where $\mu = \lambda_0 - \lambda$ is an element of $Q[\Delta]$.

Proposition 5.1. *For any $\zeta \in \mathfrak{B}(H; \lambda_0) = \mathfrak{C}(H; \lambda_0)$, we have $\psi(\boldsymbol{T}\zeta) = \boldsymbol{T}(\mathscr{P}(\zeta))$, where the notation $\boldsymbol{T}$ means Hirai's method $\boldsymbol{T}$* (see [6] and Appendix A).

Proof. It is sufficient to show the proposition for $\zeta = \zeta(a_i, t\lambda_0;\, *)$. Let D be the Weyl denominator as in §1. Then for $a_i \exp x \in H_i$ ($x \in \mathfrak{h}$), we have

$$\varepsilon_{\boldsymbol{R}} D\psi(\boldsymbol{T}\zeta)(a_i \exp x) = \sum_{s \in W(G; H_i)} \xi_{-s^{-1}t\mu}(a_i)\varepsilon(a_i;\, s) \exp(t\lambda, sx)$$

$$= \xi_{-t\mu}(a_i) \sum_{s \in W(G; H_i)} \varepsilon(a_i;\, s) \exp(t\lambda, sx),$$

by the results of [20] and the definition of $\boldsymbol{T}$. Here we used

$$\xi_{-s^{-1}t\mu}(a_i) = \xi_{-t\mu}(a_i) \qquad \text{for any} \quad s \in W(G;\, H_i).$$

This follows from Assumption 2.1. On the other hand, we have

$$\varepsilon_{\boldsymbol{R}} D\,\boldsymbol{T}(\mathscr{P}(\zeta))(a_i \exp x) = \mathscr{P}(\zeta)(a_i \exp x)$$

$$= \xi_{-t\mu}(a_i) \sum_{s \in W(G; H_i)} \varepsilon(a_i;\, s) \exp(t\lambda, sx).$$

Thus we proved

$$\psi(T\zeta)|_H = T(\mathscr{P}(\zeta))|_H .$$

Since $\psi(T\zeta)$ and $T(\mathscr{P}(\zeta))$ are extremal IEDs of height H, we can prove $\psi(T\zeta)|_J = T(\mathscr{P}(\zeta))|_J$ for another Cartan subgroup J inductively on the order on $\mathrm{Car}(G)$ as given below. The proof depends fully on the construction of T. We explain about T in Appendix A.

At first, we prepare notations. Let J_1 be a connected component of J and F a connected component of $J_1'(R)=\{h\in J_1 \mid \xi_\alpha(h)\neq 1$ for $\alpha\in\Delta_R\}$. Denote by $\Sigma=\Sigma(J_1)$ the root system consisting of all the real roots $\alpha\in\Delta(\mathfrak{g}_C, \mathfrak{j}_C)$ for which $\xi_\alpha(h)>0$ on J_1. Let $S=S(J_1)$ be the subgroup of $W(G; J_1)$ generated by s_α $(\alpha\in\Sigma)$, where s_α denotes the reflection with respect to α. Put $P(F)=\{\alpha\in\Sigma \mid \xi_\alpha(F)>1\}$. Then $P(F)$ is a positive system in Σ and we denote by $\Pi=\Pi(F)=\{\alpha_1,\dots,\alpha_r\}$ the simple system in $P(F)$. Let B^m $(1\leqq m\leqq r)$ be a Cartan subgroup obtained from J by the Cayley transform $\nu_{\alpha_m}=\nu_m$ with respect to the real simple root $\alpha_m\in\Pi$. Then $[B^m]>[J]$ holds. By the induction hypothesis, we have

(5.1) $$\psi(T\zeta)|_{B^m}=T(\mathscr{P}(\zeta))|_{B^m} \quad (1\leqq m\leqq r).$$

Put $f^m=D\psi(T\zeta)|_{B^m}=DT(\mathscr{P}(\zeta))|_{B^m}$. We devide the proof for $\psi(T\zeta)|_J=T(\mathscr{P}(\zeta))|_J$ into two steps as in the proof of Theorem 4.3 in [16].

Step $\boldsymbol{R}$. Put

$$\Sigma_m=\{h\in J \mid \xi_{\alpha_m}(h)=1\},$$
$$\Sigma_m'=\{h\in\Sigma_m \mid \xi_\alpha(h)\neq 1 \text{ for any root } \alpha\neq\pm\alpha_m\}.$$

Then for $a\in\Sigma_m'\cap J_1$ and $x\in\mathfrak{j}$, we define

$$\left(\boldsymbol{R}_{\alpha_m}f^m\right)(a\exp x)=f^m(a\exp\nu_m(x)).$$

On the other hand, if we write $g^m=D(\zeta T)|_{B^m}$ as

$$g^m(a\exp x)=\sum_{w\in W} c_w\exp w\lambda_0(x) \quad (c_w\in\boldsymbol{C}),$$

then, by (5.1) and the results of [20], we have

(5.2) $$f^m(a\exp x)=\sum_{w\in W} c_w\xi_{-w\mu}(a)\exp w\lambda(x).$$

For a function g on J of the form:

$$g(a\exp x)=\sum_{w\in W} c_w\exp w\lambda_0(x) \quad (a\in J',\ x\in\mathfrak{j}),$$

we define an operation ψ_J by

$$\psi_J(g)(a\exp x)=\sum_{w\in W} c_w\xi_{-w\mu}(a)\exp w\lambda(x).$$

Then, by (5.2) clearly it holds that

(5.3) $$\psi_J(\boldsymbol{R}_{\alpha_m}g^m)=\boldsymbol{R}_{\alpha_m}f^m \quad (1\leqq m\leqq r).$$

Step S. For a function g on J_1 and $s \in S$, we define sg as $sg(h)=g(s^{-1}h)$ ($h \in J_1$). For each $s_m = s_{\alpha_m}$ ($1 \leqq m \leqq r$), we put

$$\mathscr{A}(f^m;\, s_m)=(1-s_m)(\boldsymbol{R}_{\alpha_m} f^m).$$

Each element $s \in S$ can be written in the form $s=s_{i_1}s_{i_2}\cdots s_{i_k}$. Then we put

$$\mathscr{A}(f^1,\dots,f^r;\, s)=\mathscr{A}(f^{i_1};\, s_{i_1})+s_{i_1}\mathscr{A}(f^{i_2};\, s_{i_2})+\dots+s_{i_1}s_{i_2}\dots s_{i_{k-1}}\mathscr{A}(f^{i_k};\, s_{i_k}).$$

It can be proved $\mathscr{A}(f^1,\dots,f^r;\, s)$ is independent of a choice of expressions for $s \in S$. Finally, we put

$$\mathscr{B}(f^1,\dots,f^r)=(\# S)^{-1}\sum_{s\in S}\mathscr{A}(f^1,\dots,f^r;\, s).$$

Similarly, $\mathscr{B}(g^1,\dots, g^r)$ can be defined. Then, we have $D\boldsymbol{T}(\mathscr{P}(\zeta))|_F=\mathscr{B}(f^1,\dots,f^r)$ and $DT(\zeta)|_F=\mathscr{B}(g^1,\dots, g^r)$. Since $D\psi(\boldsymbol{T}\zeta)|_F=\psi_J(\mathscr{B}(g^1,\dots, g^r))$ holds, it is enough to show that $\psi_J(\mathscr{B}(g^1,\dots, g^r))=\mathscr{B}(f^1,\dots,f^r)$. But this is reduced to the fact that

$$\psi_J(s(\boldsymbol{R}_{\alpha_m} g^m))=s(\boldsymbol{R}_{\alpha_m} f^m).$$

Let us prove this. Taking (5.3) into consideration, it is enough to show

$$\psi_J(sg)=s(\psi_J(g)) \tag{5.4}$$

for an analytic function g on J_1 of the following type: for $a \in F \cap J'$ and $x \in \mathfrak{j}$, g has an expression

$$g(a \exp x)=\sum_{w\in W} c_w \exp w\lambda_0(x) \quad (c_w \in \boldsymbol{C}).$$

Put $b=sa$. Since

$$sg(b \exp x)=g(a \exp s^{-1}x)=\sum_{w\in W} c_w \exp sw\lambda_0(x),$$

We have for $x \in \mathfrak{j}$,

$$\psi_J(sg)(b \exp x)\overset{(*)}{=}\sum_{w\in W} c_w \xi_{-sw\mu}(b)\exp sw\lambda(x)$$
$$=\sum_{w\in W} c_w \xi_{-w\mu}(a)\exp sw\lambda(x).$$

In the equality (*), ψ_J is applied to the expansion of sg at $b \in F \cap J'$. On the other hand, we have for the right hand side for (5.4),

$$s(\psi_J(g))(b \exp x)=\psi_J(g)(a \exp s^{-1}x)$$
$$\overset{(*)}{=}\sum_{w\in W} c_w \xi_{-w\mu}(a)\exp w\lambda(s^{-1}x)$$
$$=\sum_{w\in W} c_w \xi_{-w\mu}(a)\exp sw\lambda(x).$$

Here, in the equality (*), ψ_J is applied to the expansion of g at the regular point $a \in F \cap J'$. Thus we proved $\boldsymbol{T}(\mathscr{P}(\zeta))|_F=\psi(\boldsymbol{T}\zeta)|_F$.

Now, since F is arbitrary, we proved $T(\mathscr{P}(\zeta))|_J=\psi(T\zeta)|_J$ and the induction step is completed. Q. E. D.

Let $\mathscr{Q}$ be a linear map of $\mathfrak{C}(H;\lambda)$ into $\mathfrak{C}(H;\lambda_0)$ (from singular λ to regular λ_0) defined by

$$\mathscr{Q}\zeta(a_i, t\lambda; h)=\sum_{w\in W_\lambda}\xi_{tw\mu}(a_i)\zeta(a_i, tw\lambda_0; h)$$

$$=\sum_{w\in W_\lambda}\xi_{tw\mu}(a_i)\mathscr{R}(w^{-1})\zeta(a_i, t\lambda_0; h).$$

Then we can prove the following, similarly as in the proof of the preceeding proposition.

Proposition 5.2. *For any $\zeta\in\mathfrak{C}(H;\lambda)$, we have $\varphi(T\zeta)=T(\mathscr{Q}(\zeta))$.*

We omit the proof to avoid the repetition of the same sentences.

5.3. Representations of Hecke algebras. To consider relations between Zuckerman's translation functors and our representation σ, there appears always the trifling constants $\{\xi_{t\mu}(a_i)\}$. In the following, we want to consider the case where these constants are all reduced to 1. We assume:

Assumption 5.3. For any $t\in W_H^{\sim}(\lambda)$ and $0\leqq i\leqq l$, $\xi_{t\mu}(a_i)=1$ holds.

This assumption is not essential. In fact, we can take μ and $\{a_i\}$ so that Assumption 5.3 holds (see Lemma B.4 in Appendix B).

Corollary 5.4. *Under Assumptions* 2.1 *and* 5.3. *we have*

$$\operatorname{Ker}\psi_\lambda^{\lambda_0}=\operatorname{Ker}\tau(e_\lambda)\quad\text{on}\quad V(\lambda_0).$$

Proof. By Proposition 5.1, we have $\operatorname{Ker}\psi=T(\operatorname{Ker}\mathscr{P})$. Since P in §3 is equal to $\mathscr{P}$ by Assumption 5.3, we have

$$\operatorname{Ker}\mathscr{P}=\sum_{[H]\in\operatorname{Car}(G)}^{\oplus}\mathfrak{C}(H;\lambda_0)_0$$

from Lemma 3.4. The subspace $\mathfrak{C}(H;\lambda_0)_0$ is given by

$$\mathfrak{C}(H;\lambda_0)_0=\{\zeta\in\mathfrak{C}(H;\lambda_0)\,|\,\mathscr{R}(e_\lambda)\zeta=0\},$$

where $\mathscr{R}$ is defined as in 2.2. Clearly, it holds that $T(\mathfrak{C}(H;\lambda_0)_0)=\operatorname{Ker}\tau(e_\lambda)$ (in $V_H(\lambda_0)$) and, summing up through $[H]\in\operatorname{Car}(G)$, we have the corollary. Q. E. D.

One can prove the following lemma similarly as in the proof of Theorem C.2 in [10].

Lemma 5.5. *For $\Theta\in V(\lambda_0)$, we have*

$$\varphi\psi(\Theta)=\sum_{s\in W_\lambda}\tau(s)\Theta=(\# W_\lambda)\tau(e_\lambda)\Theta.$$

Using Lemma 5.5, we introduce another interpretation of the representation σ of the Hecke algebra $\mathscr{H}(W_H(\lambda), W_\lambda)$ in §4.

Theorem 5.6. *For $e_\lambda w e_\lambda \in H(W_H(\lambda), W_\lambda)$ and $\Theta \in V_H(\lambda)$, put*

$$\sigma'(e_\lambda w e_\lambda)\Theta = (\# W_\lambda)^{-1}(\psi \circ \tau(e_\lambda w e_\lambda) \circ \varphi)(\Theta). \tag{5.5}$$

Then $(\sigma', V_H(\lambda))$ defines a representation of the Hecke algebra $\mathscr{H}(W_H(\lambda), W_\lambda)$, and moreover σ' is equal to σ.

Proof. Since $\boldsymbol{T} \circ P = \boldsymbol{T} \circ \mathscr{P}$ is surjective, there exists $\zeta_0 \in \mathfrak{C}(H; \lambda_0)$ such that $\boldsymbol{T}(\mathscr{P}(\zeta_0)) = \Theta$. Then we have

$$\begin{aligned}
&(\# W_\lambda)^{-1}\psi \circ \tau(e_\lambda w e_\lambda) \circ \varphi(\boldsymbol{T}(\mathscr{P}\zeta_0)) \\
&= (\# W_\lambda)^{-1}\psi \circ \tau(e_\lambda w e_\lambda) \circ \varphi\psi(\boldsymbol{T}\zeta_0) \quad \text{(by Proposition 5.1)} \\
&= (\# W_\lambda)^{-1}\psi \circ \tau(e_\lambda w e_\lambda) \circ (\# W_\lambda)\tau(e_\lambda)(\boldsymbol{T}\zeta_0) \quad \text{(by Lemma 5.5)} \\
&= \psi \circ \tau(e_\lambda w e_\lambda)(\boldsymbol{T}\zeta_0).
\end{aligned}$$

The last formula and Proposition 5.1 tell us that this is equal to $\sigma(e_\lambda w e_\lambda)(\boldsymbol{T}(P\zeta_0)) = \sigma(e_\lambda w e_\lambda)\Theta$. Q. E. D.

§ 6. τ-invariants for admissible representations.

In this section, we show some applications of representations of Weyl groups or Hecke algebras to study admissible representations of G. Our representations τ and σ are closely related to so-called τ-invariants of an irreducible admissible representation of G.

Let $(\pi, \mathfrak{H})$ be an irreducible admissible representation of G on a Hilbert space $\mathfrak{H}$. We denote by $(\pi, \mathfrak{H}_K)$ the corresponding irreducible $(\mathfrak{g}_C, K)$-module on the K-finite vectors of $\mathfrak{H}$. Then we can define a grobal character $\Theta(\pi)$ of $(\pi, \mathfrak{H}_K)$ as in §1. Here we suppose that $\Theta(\pi)$ has a dominant regular infinitesimal character $\lambda_0 \in \mathfrak{h}_C^*$.

Definition 6.1. Let Π be the simple system in Δ^+. Then *τ-invariants* $\mathscr{S}(\pi)$ of $(\pi, \mathfrak{H})$ is a subset of Π defined as

$$\mathscr{S}(\pi) = \left\{\alpha \in \Pi \;\middle|\; \frac{\langle \alpha, \lambda_0 \rangle}{\langle \alpha, \alpha \rangle} \in \boldsymbol{Z} \text{ and } \tau(s_\alpha)\Theta(\pi) = -\Theta(\pi)\right\},$$

where $\langle\ ,\ \rangle$ is an inner product on $\mathfrak{h}_C^*$ invariant under the action of W. Remark that if $\langle \alpha, \lambda_0 \rangle / \langle \alpha, \alpha \rangle$ is an integer, then $s_\alpha \in W_H(\lambda_0)$ holds for any H.

Remark. Our definition of τ-invariants may slightly differ from that of Vogan's (see [19]). The difference between our representations of Weyl groups and Vogan's ([1, 19]) is the cause of the difference of τ-invariants. However, most of the results obtained by D. Vogan are valid in our situation (for example, see Propositions 6.2 and 6.4).

Put $\mu = (\langle \alpha, \lambda_0 \rangle / \langle \alpha, \alpha \rangle)\alpha$. Let p be a positive integer such that $\xi_{p\mu}(a_i) = 1$ for any i on each Cartan subgroup H ($[H] \in \mathrm{Car}(G)$). The existence of such a p is assured for a special choice of $\{a_i\}$ (see Appendix B).

Proposition 6.2 (D. Vogan). *Take an* $\alpha \in \Pi$ *such that* $\langle \alpha, \lambda_0 \rangle / \langle \alpha, \alpha \rangle \in p\boldsymbol{Z}$. *Put* $\lambda = \lambda_0 - \mu$, $\mu = (\langle \alpha, \lambda_0 \rangle / \langle \alpha, \alpha \rangle)\alpha$. *Then* $W_\lambda = \{e, s_\alpha\}$ *and the following two conditions are equivalent.*

(1) $\psi_\lambda^{\lambda_0}(\Theta(\pi)) = 0$.

(2) $\alpha \in \mathscr{S}(\pi)$.

Proof. This proposition is essentially known (see [19, Prop. 3.2]). But here we give a proof because it shows usefulness of our theory. The proof is very short, if we use the results of preceeding sections.

We know from Corollary 5.6, $\operatorname{Ker} \psi = \{v \in V(\lambda_0) \mid \tau(e_\lambda)v = 0\}$. The equation $\tau(e_\lambda)v = 0$ means $\tau(s_\alpha)v = -v$ because $e_\lambda = (e + s_\alpha)/2$. Q. E. D.

Example 6.3. (1) If π_f is a finite dimensional representation, then $\mathscr{S}(\pi_f) = \Pi$.

(2) If π_d is a discrete series representation with Harish-Chandra parameter $\lambda_0 \in \mathfrak{h}_C^*$, where $\mathfrak{h}$ is a compact Cartan subalgebra of $\mathfrak{g}$. Remark that G has discrete series representations if and only if G has a compact Cartan subgroup. Choose a positive system Δ^+ so that λ_0 is dominant regular with respect to Δ^+. Then we have

$$\mathscr{S}(\pi_d) = \{\alpha \in \Pi \mid \alpha \text{ is a compact simple root}\}.$$

This is a deep result of W. Schmid ([17, Th. 9.4]).

Take a regular dominant infinitesimal character $\lambda_0 \in \mathfrak{h}_C^*$. If necessary, replacing λ_0 by a multiple of λ_0 by some positive integer, we can assume:

(1) For suitable choice of $\{a_i\}$, λ_0 satisfies Assumption 2.1.

(2) If $\langle \alpha, \lambda_0 \rangle / \langle \alpha, \alpha \rangle \in \boldsymbol{Z}$ for an $\alpha \in \Pi$, then $\mu = (\langle \alpha, \lambda_0 \rangle / \langle \alpha, \alpha \rangle)\alpha$ satisfies Assumption 5.3.

This is clear from the argument given in Appendix B.

Let $\{\Theta_j \mid j \in J\}$ be the set of all the irreducible characters of G with infinitesimal character λ_0. Take an $\alpha \in \Pi$ such that $\langle \alpha, \lambda_0 \rangle / \langle \alpha, \alpha \rangle \in \boldsymbol{Z}$. We put $s = s_\alpha \in W$, the reflection with respect to α and $\lambda(\alpha) = \lambda_0 - (\langle \alpha, \lambda_0 \rangle / \langle \alpha, \alpha \rangle)\alpha$. Then $W_{\lambda(\alpha)} = \{e, s_\alpha\} \subset W_H(\lambda_0)$ and $\lambda(\alpha)$ satisfies Assumption 2.1 for the same $\{a_i\}$. Let $J(s)$, $s = s_\alpha$, be the subset of J defined as

$$J(s) = \{j \in J \mid \tau(s)\Theta_j = -\Theta_j\} = \{j \in J \mid s \in \mathscr{S}(\Theta_j)\}.$$

Proposition 6.4 (D. Vogan). (1) *For* $k \in J \backslash J(s)$, *we have*

$$\tau(s)\Theta_k = \Theta_k + \sum_{j \in J(s)} z_j \Theta_j,$$

where z_j $(j \in J(s))$ *is a non-negative integer. Consequently,* $\tau(s)\Theta_k$ *is a true character.*

(2) *If we put* $V_{\boldsymbol{Z}}(\lambda_0) = \sum_{j \in J} \boldsymbol{Z}\Theta_j$, *then* $\tau(s)$ *preserves* $V_{\boldsymbol{Z}}(\lambda_0)$.

Proof. The proof is carried out similarly as in the proof of Lemma 3.11 in [19]. So we omit it.

Now we return to the situation in §5, i.e., start from a dominant λ not necessarily regular and put $\lambda_0=\lambda+\mu$ dominant regular. Of course, we assume Assumptions 2.1 and 5.3. Put $\Pi(\lambda)=\{\alpha\in\Pi\mid\langle\lambda,\alpha\rangle=0\}$. Then W_λ is generated by $\{s_\alpha\mid a\in\Pi(\lambda)\}$.

Theorem 6.5. (1) *Put*

$$V_{\boldsymbol{Z}}(\lambda_0;\,\Pi(\lambda))=\langle\Theta_j\mid j\in J(s_\alpha)\textit{ for some }\alpha\in\Pi(\lambda)\rangle/\boldsymbol{Z}$$

generated as a $\boldsymbol{Z}$*-module. Then* $V_{\boldsymbol{Z}}(\lambda_0;\,\Pi(\lambda))$ *is stable under the action of* W_λ *and* $V(\lambda_0;\,\Pi(\lambda))=V_{\boldsymbol{Z}}(\lambda_0;\,\Pi(\lambda))\otimes_{\boldsymbol{Z}}\boldsymbol{C}$ *is the kernel of* $\tau(e_\lambda)$: $V(\lambda_0)\to V(\lambda_0)$.

(2) *For an irreducible character* Θ, *it holds that* $\psi_\lambda^{\lambda_0}(\Theta)=0$ *if and only if* $\tau(s_\alpha)\Theta=-\Theta$ *for some* $\alpha\in\Pi(\lambda)$.

Proof. (1) At first, we show that $V_{\boldsymbol{Z}}(\lambda_0;\,\Pi(\lambda))$ is stable under the action of W_λ. It is enough to show that, for any $j\in J(\lambda)=\bigcup_{\alpha\in\Pi(\lambda)}J(s_\alpha)$ and any $\alpha\in\Pi(\lambda)$, it holds that

$$\tau(s_\alpha)\Theta_j\in V_{\boldsymbol{Z}}(\lambda_0;\,\Pi(\lambda)).$$

This is trivial, if $j\in J(s_\alpha)$. Suppose $j\notin J(s_\alpha)$. Then we have from Proposition 6.4,

$$\tau(s_\alpha)\Theta_j=\Theta_j+\sum_{k\in J(s_\alpha)}z_k\Theta_k\quad(z_k\in\boldsymbol{Z}).$$

The second term of the right hand side of the above equation is contained in $V_{\boldsymbol{Z}}(\lambda_0;\,\Pi(\lambda))$ by definition. Since Θ_j is originally taken from $V_{\boldsymbol{Z}}(\lambda_0;\,\Pi(\lambda))$, we proved $\tau(s_\alpha)\Theta_j\in V_{\boldsymbol{Z}}(\lambda_0;\,\Pi(\lambda))$, hence $V_{\boldsymbol{Z}}(\lambda_0;\,\Pi(\lambda))$ is W_λ-invariant.

Now we prove that $V_{\boldsymbol{Z}}(\lambda_0;\,\Pi(\lambda))$ contains no non-zero fixed vector for W_λ. Put

$$V_1(\alpha)=(1+\tau(s_\alpha))V(\lambda_0),$$

$$V_0(\alpha)=(1-\tau(s_\alpha))V(\lambda_0).$$

Then $V(\lambda_0)=V_0(\alpha)\oplus V_1(\alpha)$ is a direct sum decomposition. From Proposition 6.4, $V_0(\alpha)$ has a basis $\{\Theta_j\mid j\in J(s_\alpha)\}$. If $\Theta\in V(\lambda_0)$ is a fixed vector for W_λ, Θ is contained in $V_1(\alpha)$ for every $\alpha\in\Pi(\lambda)$, that is to say

$$\Theta\in\bigcap_{\alpha\in\Pi(\lambda)}V_1(\alpha).$$

Therefore, if we denote by (,) a W_λ-invariant inner product on $V(\lambda_0)$, we have $(\Theta,\,V_0(\alpha))=0$ for any $\alpha\in\Pi(\lambda)$. Consequently, $(\Theta,\,\Theta_j)=0$ holds for any $j\in J(\lambda)$ and we have $(\Theta,\,V(\lambda_0;\,\Pi(\lambda))=0$.

From the above, we see that $V(\lambda_0;\,\Pi(\lambda))\subset\operatorname{Ker}\tau(e_\lambda)$. Remark that $\dim V(\lambda_0;\,\Pi(\lambda))=\sharp J(\lambda)$. From Proposition 6.4, we have for $j\in J\backslash J(\lambda)$ and $\alpha\in\Pi(\lambda)$,

$$\tau(s_\alpha)\Theta_j\equiv\Theta_j\quad\operatorname{mod}V(\lambda_0;\,\Pi(\lambda)).$$

Since $\{\Theta_j\mid j\in J\backslash J(\lambda)\}$ is linearly independent modulo $V(\lambda_0;\,\Pi(\lambda))$, the dimension of the space of W_λ-fixed vectors is $\sharp(J\backslash J(\lambda))$. Now, since the complement of $\operatorname{Ker}\tau(e_\lambda)$ is precisely the space of W_λ-fixed vectors, we have $\dim V(\lambda_0;\,\Pi(\lambda))=\sharp J(\lambda)=\sharp J-\sharp(J\backslash J(\lambda))=\dim\operatorname{Ker}\tau(e_\lambda)$. Thus we proved $V(\lambda_0;\,\Pi(\lambda))=\operatorname{Ker}\tau(e_\lambda)$.

(2) is clear from (1) and Corollary 5.4. Q. E. D.

From this theorem, we know that the subspace $\operatorname{Ker}\tau(e_\lambda)$ of $V(\lambda_0)$ (or equivalently, the direct sum of all the non-trivial representations of W_λ in $V(\lambda_0)$) has a basis consisting of irreducible characters. This is a remarkable fact and maybe is useful for picking up irreducible characters from the space of IEDs.

§7. The case of $U(3, 1)$.

In this section, we give some examples of representations of Hecke algebra on the virtual character modules of $G=U(3, 1)$ (cf. [16, §6]). The results of this section is valid (with appropriate modifications) for $U(n, 1)$ ($n \geqq 2$), however, we restrict ourselves to the case $n=3$ for simplicity of notations.

7.1. Irreducible representations of $U(3, 1)$. Let $G=U(3, 1)$ be the group of "unitary" matrix with respect to the Hermitian form $x_1\bar{x}_1+x_2\bar{x}_2+x_3\bar{x}_3-x_4\bar{x}_4$. That is to say, we put

$$G=\{g \in GL(4, \boldsymbol{C}) \mid gJ^t\bar{g}=J\},$$

$$J=\begin{pmatrix} 1_3 & 0 \\ 0 & -1 \end{pmatrix},$$

where 1_3 denotes the identity matrix of size 3. All the irreducible admissible representations of G are classified by T. Hirai ([4]). We follow after his notations. Irreducible representations of G are described as follows.

a) *Irreducible principal series representations*: $\mathfrak{D}(\alpha; c_1, c_2)$, where $\alpha=(l_1, l_2)$ $(l_1>l_2)$ is a row of integers and (c_1, c_2) a pair of complex numbers such that $c_1+c_2=$an integer, and neither c_1 nor c_2 are equal to an integer, or else, both c_1 and c_2 are equal to some of integers l_1, l_2. The infinitesimal character of $\mathfrak{D}(\alpha; c_1, c_2)$ is (l_1, l_2, c_1, c_2).

b) *Irreducible subquatients of reducible principal series representations*: $D_\alpha^{i,j}$, where $\alpha=(l_0, l_1, l_2, l_3)$ is a row of integers such that $l_0>l_1>l_2>l_3$ and (i, j) is a pair of integers such that $0 \leqq i<j \leqq 4$. The infinitesimal character of $D_\alpha^{i,j}$ is α. The representations $D_\alpha^{i,i+1}$ $(0 \leqq i \leqq 3)$ are discrete series representations and $D_\alpha^{0,4}$ is a finite dimensional representation.

c) *The limit representations of the representations of type b)*. We denote these representations by the same letters as in b), while the parameters are degenerate.

	1	2	3	4
0	D^{01}	D^{02}	D^{03}	D^{04}
1		D^{12}	D^{13}	D^{14}
2			D^{23}	D^{24}
3				D^{34}

Figure A.

The representations with regular integral infinitesimal character belong to the class b) and, in the following, we consider this class of irreducible representations. Of course, irreducible representations of type c) and some of type a) naturally appear when we consider the representations of Hecke algebras.

7.2. Representations of the Weyl group. Let W be the complex Weyl group, then $W \simeq \mathfrak{S}_4$ (symmetric group of degree 4). Take a regular integral infinitesimal character $\alpha_0 = (l_0, l_1, l_2, l_3)$, $l_0 > l_1 > l_2 > l_3$. Then its integral Weyl group is precisely W, and we realize the action of $W \simeq \mathfrak{S}_4$ on $C^4 = \mathfrak{h}_C^*$ by the permutation of coordinates. Simple reflections, which make α_0 dominant, are transpositions:

$$\{s_1 = (0, 1), s_2 = (1, 2), s_3 = (2, 3)\}.$$

Since we only consider the virtual characters, we denote by the same letters D^{ij} the corresponding irreducible characters. We have, from 7.1,

$$V(\alpha_0) = \sum_{0 \leqq i < j \leqq 4}^{\oplus} CD_{ij},$$

and the action of $\tau(s_k)$ on $V(\alpha_0)$ is given by

$$\tau(s_k)D^{ij} = \begin{cases} -D^{ij} & \text{if } k \neq i, j \\ D^{i-1,j} + D^{ij} + D^{i+1,j} & \text{if } k = i \\ D^{i,j-1} + D^{ij} + D^{i,j+1} & \text{if } k = j, \end{cases}$$

where D^{ii} is considered to be 0. This action of $\tau(s_k)$ defines a representation of W. The decomposition of τ into the irreducible components is given in [16, §6]: $V(\alpha_0) = [1^4] \oplus 2[2 \cdot 1^2] \oplus [3 \cdot 1]$ (for notations see [13]).

Remark. The above formula of $\tau(s_k)$ is valid for $U(n, 1)$ $(n \geqq 2)$ without modifications for regular integral infinitesimal character α_0. In this case, simple reflections are $\{s_i = (i-1, i) \mid 1 \leqq i \leqq m\}$ and the irreducible representations of G with infinitesimal character α_0 are $\{D^{ij} \mid 0 \leqq i < j \leqq n+1\}$ (see [4]). The decomposition of τ is given by $(\tau, V(\alpha_0)) \simeq [1^{n+1}] \oplus 2[2 \cdot 1^{n-1}] \oplus [3 \cdot 1^{n-2}]$ ([16, §6]).

By the formula of $\tau(s_k)$, we know the τ-invariants of D^{ij}.

	1	2	3	4
0	$s_2\ s_3$	$s_1\ \ s_3$	$s_1\ s_2$	$s_1\ s_2\ s_3$
1		s_3	s_2	$s_2\ s_3$
2			s_1	$s_1\ \ s_3$
3				$s_1\ s_2$

Figure B.

We explain how to read Figure B. For example, $\mathscr{S}(D^{02}) = \{s_1, s_3\}$ is the τ-invariants of D^{02} (we identify the simple system with simple reflections by usual manner). From Figure B and Theorem 6.5, we know the irreducible characters with infinitesimal character (l_0', l_1', l_2, l_3), $l_0'=l_1'>l>_2>l_3$, are $\{\psi(D^{ij})\mid i=1 \text{ or } j=1\}$. The other singular infinitesimal characters can be treated similarly.

From Proposition 6.5, we know

(i) The space $\sum^{\oplus} \boldsymbol{C}D^{ij}$ $(i\neq 1, j\neq 1)$ is invariant under the action of $W_1=\{1, s_1\}$. This space is a multiple of the sign representations of W_1.

(ii) The space $\sum^{\oplus} \boldsymbol{C}D^{ij}$ $((i, j)\neq(1, 2))$ is invariant under the action of $W_{12}=\langle s_1, s_2\rangle\simeq\mathfrak{S}_3$. This space is decomposed as $3[1^3]\oplus 3[2\cdot 1]$ (for notations, see [13]). The decomposition is calculated from [16, Lemma 6.2].

(iii) The space $\sum^{\oplus} \boldsymbol{C}D^{ij}$ $((i, j)\neq(1, 3))$ is invariant under the action of $W_{13}=\langle s_1, s_3\rangle\cong\mathfrak{S}_2\times\mathfrak{S}_2$. This space is decomposed as $3(\mathrm{sgn}\otimes\mathrm{sgn})\oplus 3(\mathrm{sgn}\otimes 1)\oplus 3(1\otimes \mathrm{sgn})$.

7.3. Representations of the Hecke algebras. Essentially we have three different types of Hecke algebras for $G=U(3, 1)$.

(i) At first, we consider the case where the singular infinitesimal character is of the form $\alpha_1=(l_0, l_1, l_2, l_3)$, $l_0=l_1>l_2>l_3$. In this case, the irreducible characters with infinitesimal character α_1 is given by $\{\psi(D^{ij})\mid i=1 \text{ or } j=1\}$ as commented in 7.2. We denote also by the same letters *degenerate* characters. Then we have $V(\alpha_1)=\langle D^{01}, D^{12}, D^{13}, D^{14}\rangle/\boldsymbol{C}$ where D^{01} and D^{12} are limits of discrete series representations. The fixed subgroup W_1 of α_1 is given by $W_1=\{1, s_1\}$ and we put $e_1=(1+s_1)/2$. Then a Hecke algebra $\mathscr{H}(W, W_1)=e_1\boldsymbol{C}[W]e_1$ is of dimension 7 and for the generators of $\mathscr{H}(W, W_1)$ we can take $\{h_2=e_1s_2e_1,\ h_3=e_1s_3e_1\}$. The relations of generators are given as follows:

$$h_2^2=\frac{1}{2}(1+h_2),\quad h_3^2=1,$$

$$(h_2h_3)^2=h_2h_3h_2+\frac{1}{2}(h_3h_2-h_3h_2h_3),$$

$$(h_3h_2)^2=h_2h_3h_2+\frac{1}{2}(h_2h_3-h_3h_2h_3).$$

The actions of generators on $V(\alpha_1)$ are given as below:

$$\sigma(e_1s_2e_1)=\begin{pmatrix} -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & -1/2 \end{pmatrix},$$

$$\sigma(e_1s_3e_1)=\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

where the matrix is expressed with respect to the basis $\{D^{01}, D^{12}, D^{13}, D^{14}\}$ in this order. This representation is reducible and has three irreducible components. Invariant subspaces are precisely,

$$\langle D^{01}\rangle/C, \quad \langle D^{14}\rangle/C, \quad \langle D^{12}+(D^{01}-D^{14})/2, D^{13}-(D^{01}-3D^{14})/4\rangle/C.$$

Corresponding to this basis, operators are diagonarized as

$$\sigma(e_1s_2e_1)=\begin{pmatrix} -1/2 & & & \\ & 1 & 0 & \\ & 1 & -1/2 & \\ & & & -1/2 \end{pmatrix},$$

$$\sigma(e_1s_3e_1)=\begin{pmatrix} -1 & & & \\ & -1 & 1 & \\ & 0 & 1 & \\ & & & -1 \end{pmatrix}.$$

(ii) Next, consider the case where the singular infinitesimal character is of the form $\alpha_{12}=(l_0, l_1, l_2, l_3)$, $l_0=l_1=l_2>l_3$. In this case, the only irreducible character with infinitesimal character α_{12} is $\psi(D^{12})$. This is a *degenerate* principal series representation (type a) of 7.1) and, in the same time, is a limit of discrete series representations. We write this irreducible character by the same letter D^{12}. Since the fixed subgroup W_{12} of α_{12} is generated by s_1 and s_2, the Hecke algebra $\mathscr{H}(W, W_{12})$ has dimension 2. A generator of $\mathscr{H}(W, W_{12})$ is $h_3=e_{12}s_3e_{12}$, where $e_{12}=(1/6)\cdot\sum_{s\in W_{12}}s$. The relation of the generator is given by

$$3h_3^2-2h_3-1=0.$$

Non-trivial element $e_{12}s_3e_{12}\in\mathscr{H}(W, W_{12})$ acts on D^{12} as

$$\sigma(e_{12}s_3e_{12})D^{12}=-\frac{1}{3}D^{12}.$$

(iii) This case treats the singular infinitesimal character of the form $\alpha_{13}=(l_0, l_1, l_2, l_3)$, $l_0=l_1>l_2=l_3$. The only irreducible character with infinitesimal character α_{13} is $\psi(D^{13})$. This is a degenerate principal series representation of type a) in 7.1. We write this character by the same letter D^{13}. The fixed subgroup of α_{13} is $W_{13}=\langle s_1, s_3\rangle\simeq\mathfrak{S}_2\times\mathfrak{S}_2$ and the Hecke algebra $\mathscr{H}(W, W_{13})$ has dimension 3. Put $e_{13}=(1/4)\sum_{s\in W_{13}}s$. Then the action of the generator $e_{13}s_2e_{13}$ of $\mathscr{H}(W, W_{13})$ is given by

$$\sigma(e_{13}s_2e_{13})D^{13}=0.$$

In this case, the relation of the generator $h_2=e_{13}s_2e_{13}$ is given by

$$2h_2^3-h_2^2-h_2=0.$$

Therefore one dimensional representations of $\mathscr{H}(W, W_{13})$ consist of three equivalent classes. The other two classes are given by

$$\sigma'(e_{13}s_2e_{13}) = -\frac{1}{2} \quad \text{or} \quad 1$$

respectively, and do not appear in the virtual character modules.

The above three types (i), (ii) and (iii) correspond to (i), (ii) and (iii) in 7.2.

Remark. For $G=U(n, 1)$ $(n \geqq 2)$, one can claculate out the representations of Hecke algebras using the formula of τ in 7.2. The details will be discussed elsewhere.

Appendix A.

This appendix is devoted to describe Hirai's method $\boldsymbol{T}$ for the usage in §5. For detailed arguments see [6, §3].

A.1. Let $\mathfrak{A}(\lambda)$ $(\lambda \in \mathfrak{h}_C^*)$ be the space of all the IEDs with infinitesimal character λ. Since $\Theta \in \mathfrak{A}(\lambda)$ is essentially a locally summable function on G which is analytic on G', it is determined by the values on the set of regular elements G'. Moreover, Θ is determined by the values on the finite system of Cartan subgroups $\{H \mid [H] \in \mathrm{Car}(G)\}$ because Θ is invariant under the inner automorphisms of G.

To understand Hirai's method $\boldsymbol{T}$, it is essential to consider some kind of order on $\mathrm{Car}(G)$. Let us explain this order on $\mathrm{Car}(G)$ (see [5, §3]). Take $[A] \in \mathrm{Car}(G)$, where $[A]$ means the conjugacy class of a Cartan subgroup A. For $\alpha \in \Delta_{\boldsymbol{R}} = \Delta_{\boldsymbol{R}}(\mathfrak{g}_C, \mathfrak{a}_C)$, let H be the element of $\mathfrak{a}_C$ for which $\alpha(X) = B(H_\alpha, X)$, where $B(\ ,\ )$ denotes the Killing form on $\mathfrak{g}_C$. Take root vectors X_α, $X_{-\alpha}$ from $\mathfrak{g}$ in such a way that $[X_\alpha, X_{-\alpha}] = H_\alpha$ and we put

$$H'_\alpha = \frac{2}{|\alpha|^2} H_\alpha, \quad X'_{\pm\alpha} = \frac{\sqrt{2}}{|\alpha|} X_{\pm\alpha}.$$

Let $\nu = \nu_\alpha$ be the automorphism of $\mathfrak{g}_C$ defined by

$$\nu = \nu_\alpha = \exp\left\{-\sqrt{-1}\,\frac{\pi}{4}\,\mathrm{ad}\,(X'_\alpha + X'_{-\alpha})\right\},$$

so-called *Cayley transform* with respect to α. Then $\mathfrak{b} = \nu(\mathfrak{a}_C) \cap \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ not conjugate to $\mathfrak{a}$ under any automorphism of $\mathfrak{g}$, and $\beta = \nu(\alpha)$ is a singular imaginary root of $\mathfrak{b}$. We have

$$\mathfrak{a} = \Sigma_\alpha + \boldsymbol{R}H'_\alpha, \quad \mathfrak{b} = \Sigma_\alpha + \sqrt{-1}\,\boldsymbol{R}H'_\beta,$$

where Σ_α is the hyperplane of $\mathfrak{a}$ defined by $\alpha = 0$ and

$$H'_\beta = \nu(H'_\alpha) = \sqrt{-1}\,(X'_\alpha - X'_{-\alpha}).$$

This relation between $\mathfrak{a}$ and $\mathfrak{b}$ is denoted by $(\mathfrak{a}, \alpha) \rightarrow (\mathfrak{b}, \beta)$ or simply by $\mathfrak{a} \rightarrow \mathfrak{b}$. We introduce the order $<$ on $\mathrm{Car}(G)$ by defining $[A] < [B]$ when $\mathfrak{a} \rightarrow \mathfrak{b}$ for an appropriate choice of a representative B of the class $[B]$, and extend it transitively.

For $\Theta \in \mathfrak{A}(\lambda)$, we put

$$\mathrm{Supp}(\Theta)=\{[H]\in \mathrm{Car}(G) \mid \Theta|_H \not\equiv 0\},$$

$$\mathrm{Hght}(\Theta)=\{[H]\in \mathrm{Supp}(\Theta) \mid [H] \text{ is maximal in } \mathrm{Supp}(\Theta)\}.$$

and call $[H]\in \mathrm{Hght}(\Theta)$ a *height* of Θ. If Θ has the unique height $[H]$, then Θ is called an *extremal IED* of height $[H]$ (or simply, H).

For a Cartan subgroup H, put

$$D^H(h)=\xi_\rho(h)\prod_{\alpha\in\Delta^+}(1-\xi_\alpha(h)^{-1}) \quad (h\in H),$$

$$D^H_{\boldsymbol{R}}(h)=\prod_{\alpha\in\Delta^+_{\boldsymbol{R}}}(1-\xi_\alpha(h)^{-1}) \quad (h\in H).$$

For a given IED Θ on G we put

$$C_H(\Theta)(h)=D^H(h)\Theta(h) \quad (h\in H'),$$

$$C'_H(\Theta)(h)=\varepsilon^H_{\boldsymbol{R}}(h)D^H(h)\Theta(h) \quad (h\in H'),$$

where $\varepsilon^H_{\boldsymbol{R}}(h)=\mathrm{sgn}(D^H_{\boldsymbol{R}}(h))$ $(h\in H')$.

Define a family of analytic functions $\mathfrak{B}(H;\lambda)$ as in §1.3. Then we have

Theorem A.1 (Hirai [5, Th. 1]). *Let Θ be an IED on G with eigenvalue λ. If Θ has a height $[H]\in \mathrm{Car}(G)$, then $C'_H(\Theta)$ can be extended to an analytic function on the whole group H. Moreover, it belongs to $\mathfrak{B}(H;\lambda)$.*

A.2. Hirai's method T is the method to construct an extremal IED with height H from an element $\zeta\in\mathfrak{B}(H;\lambda)$. This is done by induction on the order on $\mathrm{Car}(G)$, and has two different steps $\boldsymbol{R}$ and $\boldsymbol{S}$. Roughly speaking, the step $\boldsymbol{R}$ corresponds to boundary conditions to be satisfied by IEDs, and the step $\boldsymbol{S}$ corresponds to Weyl group symmetry which assures the invaraince of IEDs. As is mentioned above, an IED Θ is determined by the system of functions $C_J(\Theta)$ ($[J]\in \mathrm{Car}(G)$). So, in order to give an IED $\boldsymbol{T}\zeta$ for $\zeta\in\mathfrak{B}(H;\lambda)$, it is sufficient to give functions $C_J(\boldsymbol{T}\zeta)$ for every $[J]\in \mathrm{Car}(G)$. T. Hirai gave necessary and sufficient conditions for the system of functions $C_J(\Theta)$ ($[J]\in \mathrm{Car}(G)$) obtained from an IED Θ in his works ([5, 6]). Using his results one can verify that constructed functions $C_J(\boldsymbol{T}\zeta)$ ($[J]\in \mathrm{Car}(G)$) really determine an IED $\boldsymbol{T}\zeta$.

Let us explain the construction in detail. Take an element $\zeta\in\mathfrak{B}(H;\lambda)$. We put

$$C_H(\boldsymbol{T}\zeta)=\varepsilon^H_{\boldsymbol{R}}\cdot\zeta \qquad \text{for } H \text{ itself},$$

$$C_J(\boldsymbol{T}\zeta)\equiv 0 \qquad \text{for } [J]\not\geqq[H].$$

Let A be a Cartan subgroup of G and assume that we have already constructed $C_B(\boldsymbol{T}\zeta)$ for $[B]>[A]$. Let A_1 be a connected component of A and F a connected component of $A'_1(\boldsymbol{R})=A_1\cap A'(\boldsymbol{R})$, where $A'(\boldsymbol{R})=\{h\in A \mid \xi_\alpha(h)\neq 1 \text{ for any } \alpha\in\Delta_{\boldsymbol{R}}\}$. Denote by $\Sigma=\Sigma(A_1)$ the set of all the real roots $\alpha\in\Delta_{\boldsymbol{R}}$ for which $\xi_\alpha(h)>0$ on A_1. Then Σ is a root system. Let $S=S(A_1)$ be the subgroup of $W(G;A_1)$ generated by $\omega_\alpha|_{A_1}$ $(\alpha\in\Sigma)$, where ω_α is the conjugation by an element $g_\alpha = \exp\frac{1}{2}\pi(X'_\alpha - X'_{-\alpha}) \in G$. We put $\omega_\alpha|_{A_1} = s_\alpha$. Let $P(F)$ be the set of $\alpha \in \Sigma$ for which $\xi_\alpha(F) > 1$. Then $P(F)$ is the set of all the positive roots of Σ with respect to a certain order of roots. Let $\Pi = \Pi(F) = \{\alpha_1, \ldots, \alpha_r\}$ be the simple system in $P(F)$.

*Step **R**.* Denote by $\mathfrak{b}^m$ a Cartan subalgebra obtained from $\mathfrak{a}$ by the Cayley transform $\nu_{\alpha_m} = \nu_m$ with respect to the real root α_m ($1 \leqq m \leqq r$). By assumption, the functions $C_{B^m}(\boldsymbol{T}\zeta)$ have been already determined. We write C_m instead of $C_{B^m}(\boldsymbol{T}\zeta)$ for brevity.

We put

$$\Sigma_m = \{h \in A \mid \xi_{\alpha_m}(h) = 1\},$$

$$\Sigma'_m = \{h \in \Sigma_m \mid \xi_\alpha(h) \neq 1 \text{ for any root } \alpha \neq \pm\alpha_m\}.$$

Then for $a \in \Sigma'_m \cap A_1$ and $x \in \mathfrak{a}$, we put

$$(\boldsymbol{R}_{\alpha_m} C_m)(a \exp x) = C_m(a \exp \nu_m(x)).$$

Here $\nu_m(x)$ may not be contained in $\mathfrak{b}^m$, but C_m is locally a linear combination of the form $\exp \mu(x)$ ($\mu \in (\mathfrak{b}^m_{\mathbf{C}})^*$) (or its multiple by a certain polynomial function), so $C_m(a \exp \nu_m(x))$ has natural meaning.

*Step **S**.* For a function f on A_1 and $s \in S$, we define sf as $(sf)(h) = f(s^{-1}h)$ ($h \in A_1$). For each $s_m = s_{\alpha_m}$ ($1 \leqq m \leqq r$), we put

$$\mathscr{A}_{s_m} = (1 - s_m)(\boldsymbol{R}_{\alpha_m} C_m).$$

Each element $s \in S$ can be written in the form $s = s_{i_1} s_{i_2} \cdots s_{i_k}$ (see, for example, [2]). Then we put

$$\mathscr{A}_s = \mathscr{A}_{s_{i_1}} + \mathscr{A}_{s_{i_2}} + \cdots + s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \mathscr{A}_{s_{i_k}}.$$

It can be proved that $\mathscr{A}_s$ is independent of a choice of expressions for $s \in S$. Finally we put

$$\mathscr{B} = \mathscr{B}(C_1, C_2, \ldots, C_m) = (\# S)^{-1} \sum_{s \in S} \mathscr{A}_s.$$

Denote by E_{A_1} the union of wA_1 over $w \in W(G; A)$. Define $C_A(\boldsymbol{T}\zeta)$ on $E_{A_1} \cap A'(\boldsymbol{R})$ by

$$C_A(\boldsymbol{T}\zeta)(wh) = \det(w) \mathscr{B}(h) \quad (w \in W(G; A),\ h \in F).$$

Let $A_1, A_2, \ldots$ be a complete system of representatives of connected components of A under the conjugation of $W(G; A)$. Then A is the disjoint union of $E_{A_1}, E_{A_2}, \ldots$. Repeating the same construction for every A_i, we get $C_A(\boldsymbol{T}\zeta)$ on the whole A. Thus we can define $C_J(\boldsymbol{T}\zeta)$ ($[J] \in \mathrm{Car}(G)$) inductively. We see that they altogether define an IED $\boldsymbol{T}\zeta$ by Hirai's arguments.

Our proof of Proposition 5.1 is carried out along the above construction of $\boldsymbol{T}$. Steps $\boldsymbol{R}$ and $\boldsymbol{S}$ there correspond to the same parts of this appendix.

Appendix B.

Here we remark about the Assumptions 2.1 and 5.3.

At first we prepare some notations. Let H be a Cartan subgroup of G. We can choose a θ-stable Cartan subgroup from $[H]$, where θ is a Cartan involution with respect to a maximal compact subgroup K. So, we may assume H is θ-stable. Then the Lie algebra $\mathfrak{h}$ of H is also θ-stable and we define $\mathfrak{h}^+ = ((+1)$-eigenspace of $\theta)$ and $\mathfrak{h}^- = ((-1)$-eigenspace of $\theta)$. Put

$$H^+ = H \cap K, \quad H^- = \exp \mathfrak{h}^-.$$

Then $H = H^+ H^-$ (direct product) and H^- is connected. Denote the adjoint representation of G by $\mathrm{Ad}\colon G \to \mathrm{Int}(\mathfrak{g})$. The kernel of Ad is the centre of G. Put

$$\Gamma = \Gamma_H = \mathrm{Ad}^{-1}(\mathrm{Ad}(K) \cap \exp(\sqrt{-1}\,\mathfrak{h}^-)).$$

Then we have the following lemma (see, for example, [12]).

Lemma B.1. (1) *Γ is a finite group and commutes with $Z_G(H^-)_0$, identity component of the centralizer of H^- in G.*

(2) *It holds that $H = \Gamma H_0 = H_0 \Gamma$ and $\Gamma \subset H^+$.*

(3) *Γ is stable under the action of $W(G; H)$.*

Put $M = \cap \{\mathrm{Ker}\,|\chi| \mid \chi\colon Z_G(H^-) \to \boldsymbol{R}^*$, a continuous homomorphism$\}$. Then M is a reductive subgroup of G containing a compact Cartan subgroup H^+.

Lemma B.2. *Let $\Gamma_0 = \Gamma \cap H_0$. Then we have*

(1) *The finite group Γ_0 is contained in the centre of M_0, the connected component of M containing e.*

(2) *For $\alpha \in \Delta_{\sqrt{-1}\boldsymbol{R}} \cup \Delta_{\boldsymbol{R}}$ and $a \in \Gamma_0$, $\xi_\alpha(a) = 1$ holds.*

Proof. (1) is clear from Lemma B.1 (1). Let us prove (2). For $\alpha \in \Delta_{\sqrt{-1}\boldsymbol{R}}$, take a non-zero root vector X_α. By the definition of Γ, $\mathrm{Ad}(a)$ $(a \in \Gamma)$ has the form $\exp(\sqrt{-1}\,\mathrm{ad}\,x)$ $(x \in \mathfrak{h}^-)$. We have $\mathrm{Ad}(a)X_\alpha = \exp(\sqrt{-1}\,\mathrm{ad}\,x)X_\alpha = \exp(\sqrt{-1}\,\alpha(x)) \cdot X_\alpha = X_\alpha$, since $\alpha(x) = 0$. So $\xi_\alpha(a) = 1$ holds. The proof of $\xi_\alpha(a) = 1$ $(\alpha \in \Delta_{\boldsymbol{R}})$ is carried out similary, since $a \in H_0^+$. Q.E.D.

Lemma B.3. *For $\lambda \in \mathfrak{h}_{\boldsymbol{C}}^*$ and $t \in W_{\tilde{H}}(\lambda)$, there exists a positive integer m such that*

$$\xi_{tm\lambda}(a) = 1 \quad (a \in \Gamma_0).$$

The integer m can be taken as $m \leqq \sharp\Gamma_0$.

Proof. Since Γ_0 is a finite group, there exists an m such that $a^m = e$ for any $a \in \Gamma_0$. Then $\xi_{t\lambda}(a^m) = \xi_{tm\lambda}(a) = 1$ holds. Q.E.D.

Let $\{H_i \mid i \in I\}$ be a complete system of representatives of the conjugacy classes of connected components of H, under the action of $W(G; H)$. As for Assumptions

2.1 and 5.3, we have the following lemma.

Lemma B.4. *There exists a subset* $\{a_i \mid i \in I\}$ *for each Cartan subgroup* H ($[H] \in \mathrm{Car}(G)$) *satisfying the following conditions.*

(1) *It holds that* $a_i \in H_i$ *for* $i \in I$.

(2) *For an arbitrary infinitesimal character* $\chi = \chi_\lambda$ ($\lambda \in \mathfrak{h}_C^*$), *there exists a positive integer* m *such that*

$$\xi_{tm\lambda}(a_i^{-1} s a_i) = 1 \quad (t \in W_H^{\sim}(m\lambda),\ i \in I,\ s \in W(G; H_i)).$$

(3) *There exists a positive integer* p *depending only on* G, *such that*

$$\xi_{p\mu}(a_i) = 1 \qquad \textit{for any} \quad \mu \in \boldsymbol{Q}[\Delta].$$

Proof. It follows from Lemma B.3 and its proof that there exist a positive integer p depending only on G, and $\{a_i \mid i \in I\}$ a subset of H such that

(a) It holds that $a_i \in H_i$ for $i \in I$.

(b) For any $\lambda \in \mathfrak{h}_C^*$, we have

$$\xi_{tp\lambda}(a_i^{-1} s a_i) = 1 \quad (t \in W_H^{\sim}(\lambda),\ i \in I,\ s \in W(G; H_i)).$$

(c) For any $\mu \in \boldsymbol{Q}[\Delta]$, it holds that $\xi_{p\mu}(a_i) = 1$.

In fact, we can take $p = \prod_{[H] \in \mathrm{Car}(G)} \sharp \Gamma_H$ and $\{a_i \mid i \in I\}$ can be taken from Γ_H.

By the definition of $W_H^{\sim}(\lambda)$, it is clear that $W_H^{\sim}(p\lambda) \subset W_H^{\sim}(\lambda)$. Therefore, for some positive integer r, $W_H^{\sim}(p^r\lambda) = W_H^{\sim}(p^{r-1}\lambda)$ holds. Put $\lambda' = p^r\lambda$. By the above argument, we have

$$\xi_{t\lambda'}(a_i^{-1} s a_i) = 1 \quad (t \in W_H^{\sim}(p^{r-1}\lambda),\ i \in I,\ s \in W(G; H_i)).$$

Since $W_H^{\sim}(p^{r-1}\lambda) = W_H^{\sim}(\lambda')$, we have

$$\xi_{t\lambda'}(a_i^{-1} s a_i) = 1 \quad (t \in W_H^{\sim}(\lambda'),\ i \in I,\ s \in W(G; H_i)).$$

Q. E. D.

Remark B.5. An integer m in Lemma B.4(2) can be taken as $m \leqq p^r$ ($r = \sharp W$ and p as in the proof). This is clear from the above.

For an arbitrary $\lambda \in \mathfrak{h}_C^*$, if necessary, take $m\lambda$ instead of λ. Then Assumption 2.1 is satisfied.

Also, we can take $\mu \in \mathfrak{h}_C^*$ which satisfies Assumption 5.3 as follows. It is clear that there exists a $\mu' \in \boldsymbol{Q}[\Delta]$ such that $\lambda_0' = \lambda + \mu'$ is dominant regular. Then it holds that

$$\xi_{t\mu}(a_i) = 1 \quad (t \in W_H^{\sim}(\lambda),\ i \in I),$$

where $\mu = p\mu' \in \boldsymbol{Q}[\Delta]$. Clearly $\lambda_0 = \lambda + \mu$ is dominant regular and Assumption 5.3 is satisfied for this μ.

Department of Mathematics
Kyoto University

Current Address
Department of Mathematical Sciences
Tokyo Denki University

References

[1] D. Barbasch and D. Vogan, Weyl group representations and nilpotent orbits, in Representation theory of reductive groups, Birkhäuser, 1983.

[2] N. Bourbaki, Groupes et algèbres de Lie, Chapter 4, 5 et 6, Hermann, Paris, 1968.

[3] Harish-Chandra, Invariant eigendistributions on a semi-simple Lie group, Trans. Amer. Math. Soc., **119** (1965), 457–508.

[4] T. Hirai, Classification and the characters of irreducible representations of $SU(p, 1)$, Proc. Japan Acad., **42** (1966), 907–912.

[5] T. Hirai, Invariant eigendistributions of Lapalace operators on real simple Lie groups II, General theory for semisimple Lie groups, Japan. J. Math., New Series, **2** (1976), 27–89.

[6] T. Hirai, ———, III, Methods for construction for semisimple Lie groups, Japan. J. Math., New Series, **2** (1976), 269–341.

[7] N. Iwahori, Representation theory of symmetric groups and general linear groups I, Seminary Note of Univ. of Tokyo, 11 (1965), in Japanese.

[8] N. Iwahori, On the structure of a Hecke ring of a Chevalley group over a finite field, J. Faculty of Sci., Univ. of Tokyo, **10** (1964), 215–236.

[9] A. Joseph, Goldie rank in the enveloping algebra of a semisimple Lie algebra I, II, III, J. Algebra, 65 (1980), 269–283, 284–306; J. Algebra, **73** (1981), 295–326.

[10] A. W. Knapp and G. J. Zuckerman, Classification of irreducible tempered representations of semisimple groups, Ann. Math., **116** (1982), 389–501.

[11] R. Langlands, On the classification of irreducible representations of real algebraic groups, mimeographed notes, Institute for Advanced Study, 1973.

[12] R. L. Lipsman, On the characters and equivalence of continuous series representations, J. Math. Soc. Japan, **23** (1971), 452–480.

[13] D. E. Littlewood, The theory of group characters and matrix representations of groups, Oxford, 1950.

[14] K. Nishiyama, Virtual characters and constant coefficiet invariant eigendistributions on a semisimple Lie group, Proc. Japan Acad., **61** (1985), 168–171.

[15] K. Nishiyama, ———, Japan. J. Math., New Series, **12** (1986), 75–94.

[16] K. Nishiyama, Virtual character modules of semisimple Lie groups and representations of Weyl groups, J. Math. Soc. Japan, **37** (1985), 719–740.

[17] W. Schmid, On the characters of discrete series; The Hermitian symmetric case, Inv. Math., **30** (1975), 47–144.

[18] B. Speh and D. Vogan, Reducibility of generalized principal series, Acta Math., **145** (1980), 227–299.

[19] D. Vogan, Irreducible characters of semisimple Lie groups I, Duke Math. J., **46** (1976), 61–108.

[20] G. J. Zuckerman, Tensor products of finite and infinite dimensional representations of semisimple Lie groups, Ann. Math., **106** (1977), 295–308.