J. Math. Kyoto Univ. (JMKYAZ)
27-1 (1987) 131-140

The regular discrete models of the Boltzmann equation

Dedicated to Professor Sigeru Mizohata on his sixtieth birthday

By

Yasushi SHIZUTA and Shuichi KAWASHIMA

§ 1. Introduction.

In this paper we study the discrete models of the Boltzmann equation. It is shown that there exist regular models with k moduli of velocities for an arbitrary integer $k \geq 2$.

Let $M = \{v_1, \dots, v_m\}$ be the set of velocities, i.e., the constant vectors in $\boldsymbol{R}^3$. We assume that the linear span of M coincides with $\boldsymbol{R}^3$. The model M is essentially three dimensional in this sense. First of all, we introduce the notion of the collision. Let us denote by Σ the set of all unordered pairs of distinct velocities. We may set

$$\Sigma = \{(v_i, v_j);\ 1 \leq i < j \leq m\}.$$

Let $\alpha, \beta \in \Sigma$. Then, $\alpha = (v_i, v_j)$, $\beta = (v_k, v_l)$, for some i, j, k, l. The ordered couple formed by α and β is called a collision, if

(i) $\alpha \neq \beta$, i.e., the trivial collision is excluded,

(ii) the momentum of α equals the momentum of β, i.e., $v_i + v_j = v_k + v_l$,

(iii) the energy of α equals the energy of β, i.e., $|v_i|^2 + |v_j|^2 = |v_k|^2 + |v_l|^2$.

It is usual to denote collision formed by α and β by $\alpha \rightarrow \beta$. We call α and β the initial and the final states of the collision $\alpha \rightarrow \beta$, respectively. It is assumed in the following that there exists at least one collision for the given model M. Now let $\mathscr{C}$ be the set of all collisions for the model M. We obtain a partition of $\mathscr{C}$ by the equivalence relation given below.

We introduce the group of transformations acting in M. We set

$$G = \{T;\ T \in \boldsymbol{O}(3),\ TM = M\}.$$

Here, $\boldsymbol{O}(3)$ denotes the orthogonal transformation group. G induces naturally a group of isometric transformations on M, which we denote again by G. It is easily seen that G is determined uniquely as the maximal set of isometric transformations on M. Since M is a finite set, we may regard G as a permutation group. We define that $\alpha \rightarrow \beta$ and $\alpha' \rightarrow \beta'$ are equivalent if these collisions are obtained from each other by performing a transformation which belongs to G or by interchanging the initial

Communicated by Prof. T. Ikebe. Received November 2, 1985.

and the final states or by applying these two kinds of operations successively. The constants A_{ij}^{kl} appearing in the definition of the collision term may be identified with a "step function" subordinate to the partion of $\mathscr{C}$, which is induced from the equivalence relation defined above (See [3] for details). Thus, if $\mathscr{C}$ consists of q equivalence classes, we have q arbitrary constants in defining the collision term. The general form of the discrete Boltzmann equation is given by

$$\frac{\partial F_i}{\partial t}+v_i\cdot \nabla_x F_i=\frac{1}{2}\sum_{j,k,l}(A_{kl}^{ij}F_kF_l-A_{ij}^{kl}F_iF_j),\qquad i=1,2,\dots,m. \tag{1.1}$$

Setting $F={}^t(F_1,\dots,F_m)$ and denoting the right-hand side of (1.1) by $Q_i(F, F)$, we rewrite the equation as

$$\frac{\partial F}{\partial t}+\sum_{j=1}^{3}v^j\frac{\partial F}{\partial x_j}=Q(F,F), \tag{1.2}$$

where $V^j=\mathrm{diag}(v_1^j,\dots,v_m^j)$, $j=1, 2, 3$. $Q(F, F)={}^t(Q_1(F, F),\dots, Q_m(F, F))$ is called the collision term. Here we understand that A_{ij}^{kl} is set to be zero if the formal expression $(v_i, v_j)\to(v_k, v_l)$ does not correspond to a collision.

We say that (1.1) is regular if the following properties hold:

1°) The equation (1.1) is irreducible in the sense that the system cannot be decomposed into two decoupled subsystesm.

2°) The collision term $Q(F, F)$ is invariant under the associated transformation group G. Namely, $TQ(F, F)=Q(TF, TF)$ for any $T\in G$, $F\in \boldsymbol{R}^m$. Here F is regarded as a function on M.

3°) The stability condition for Maxwellians is satisfied (The precise statement of this condition is given in §3. We refer the reader to [3] for detailed discussions).

Note that 2°) is always satisfied when A_{ij}^{kl} is chosen according to the procedure described above. The other conditions 1°) and 3°) can be verified without the knowledge of A_{ij}^{kl}. Therefore we may also say that the discrete model M is regular by abuse of the terminology.

Our main result is summarized in the following theorem.

Theorem 1.1. *Let $k\geq 2$ be an integer. Then there exists a regular discrete model with k moduli of velocities.*

We shall obtain a certain refinement of this theorem in §3.

§2. Construction of the models.

In this section we define a series of discrete models and compute the dimension of the space of summational invariants of these models. The regularity will be shown in the next section. The models considered here are invariant under the transformation group $G=S_4\times I$. Here I denotes the group of order two generated by the central inversion (For the notations of the symmetry groups, we refer the reader to [2]). First we give two discrete models as follows.

$u_1^{(1)}=(1, 0, 0)$, $u_2^{(1)}=(0, 1, 0)$, $u_3^{(1)}=(0, 0, 1)$, $u_4^{(1)}=(-1, 0, 0)$,

$u_5^{(1)}=(0, -1, 0)$, $u_6^{(1)}=(0, 0, -1)$;

$u_1^{(2)}=(1, 0, 1)$, $u_2^{(2)}=(0, 1, 1)$, $u_3^{(2)}=(-1, 0, 1)$, $u_4^{(2)}=(0, -1, 1)$,

$u_5^{(2)}=(-1, 0, -1)$, $u_6^{(2)}=(0, -1, -1)$, $u_7^{(2)}=(1, 0, -1)$, $u_8^{(2)}=(0, 1, -1)$,

$u_9^{(2)}=(1, 1, 0)$, $u_{10}^{(2)}=(-1, 1, 0)$, $u_{11}^{(2)}=(-1, -1, 0)$, $u_{12}^{(2)}=(1, -1, 0)$.

We set $M^{(k)}=\{u_i^{(k)};\ 1\leq i\leq m(k)\}$ for $k=1, 2$, where $m(1)=6$, $m(2)=12$. We define $N_2=M^{(1)}\cup M^{(2)}$. Then N_2 is a model with two moduli of velocities. The complete list of the collisions of this model is given in Appendix 1 (We have 192 collisions for N_2). Let $\mathscr{N}_2$ be the space of summational invariants. We recall that any summational invariant ${}^t(\phi_1, \phi_2,\dots, \phi_{18})$ satisfies

$$\phi_i+\phi_j-\phi_k-\phi_l=0$$

for all i, j, k, l such that $(v_i, v_j)\rightarrow(v_k, v_l)$ is a collision of N_2 and vice versa. $\mathscr{N}_2$, the set of all summational invariants of N_2 forms a subspace of $\mathscr{V}_2=\boldsymbol{R}^{18}$. It is shown by a straightforward computation that $\dim\mathscr{N}_2=5$. Next we consider the discrete model $M^{(3)}=\{u_i^{(3)};\ 1\leq i\leq m(3)\}$. Here, $m(3)=8$ and

$u_1^{(3)}=(1, 1, 1)$, $u_2^{(3)}=(-1, 1, 1)$, $u_3^{(3)}=(-1, -1, 1)$, $u_4^{(3)}=(1, -1, 1)$,

$u_5^{(3)}=(-1, -1, -1)$, $u_6^{(3)}=(1, -1, -1)$, $u_7^{(3)}=(1, 1, -1)$, $u_8^{(3)}=(-1, 1, -1)$.

The extremities of these vectors form the vertices of a cube. The extremities of 6 vectors $u_1^{(1)},\dots, u_6^{(1)}$ coincide with the centers of the faces of the cube, while the extremities of 12 vectors $u_1^{(2)},\dots, u_{12}^{(2)}$ are the middle points of the edges of the cube. We note that $M^{(1)}$ and $M^{(3)}$ are Broadwell's 6- and 8-velocity models, respectively. We define $N_3=M^{(1)}\cup M^{(2)}\cup M^{(3)}$ and denote by $\mathscr{N}_3$ the space of summational invariants of N_3. Then, $\mathscr{N}_3\subset\mathscr{V}_3=\boldsymbol{R}^{26}$. We shall show that $\dim\mathscr{N}_3=\dim\mathscr{N}_2=5$. First we note that

$$u_1^{(1)}+u_1^{(3)}=u_1^{(2)}+u_9^{(2)}$$

and that

$$|u_1^{(1)}|^2+|u_1^{(3)}|^2=|u_1^{(2)}|^2+|u_9^{(2)}|^2.$$

Hence

$$(u_1^{(1)}, u_1^{(3)})\longrightarrow(u_1^{(2)}, u_9^{(2)})$$

is a collision of N_3 (Note that only $u_1^{(3)}$ belongs to $M^{(3)}$ here). By applying transformations of G, we obtain similar collisions. Each of these collisions contains only one velocity of $M^{(3)}$. Any velocity of $M^{(3)}$ is a component of one of these collisions. Now we write down the defining equations for $\mathscr{N}_3$ and look at the coefficient matrix C_3. We may suppose that the first 192 rows of C_3 are corresponding to the collisions of N_2 (It is enough to consider just one half of the 192 collisions because the restitution of a collision does not lead to a new equation. But here we consider the whole system of equations without reducing the size). We assume also that the first 18 entries of the vectors of $\mathscr{V}_3=\boldsymbol{R}^{26}$ correspond to $\mathscr{V}_2=\boldsymbol{R}^{18}$. Let D_3 be the submatrix of C_3 obtained by striking out the first 192 rows and the first 18 columns. Then the above observation shows that D_3 contains an 8×8 submatrix which is identical with the unit matrix after a rearrangement of rows. It follows that

$$\text{rank of } C_3 \geq \text{rank of } C_2+8,$$

where C_2 is the submatrix of C_3 corresponding to the collisions of N_2. But, $\mathscr{N}_3$ has at least dimension 5. Hence $\dim \mathscr{N}_3=\dim \mathscr{N}_2=5$. We define $M^{(4)},\dots, M^{(7)}$ as follows. Let

$$u_i^{(4)}=2u_i^{(1)} \quad (1\leq i\leq 6);$$

$$u_1^{(5)}=(2,\ 1,\ 1),\quad u_2^{(5)}=(2,\ -1,\ 1),\quad u_3^{(5)}=(2,\ -1,\ -1),$$

$$u_4^{(5)}=(2,\ 1,\ -1),\quad u_5^{(5)}=(-2,\ -1,\ -1),\quad u_6^{(5)}=(-2,\ 1,\ -1),$$

$$u_7^{(5)}=(-2,\ 1,\ 1),\quad u_8^{(5)}=(-2,\ -1,\ 1),\quad u_9^{(5)}=(1,\ 2,\ 1),$$

$$u_{10}^{(5)}=(1,\ 2,\ -1),\quad u_{11}^{(5)}=(-1,\ 2,\ -1),\quad u_{12}^{(5)}=(-1,\ 2,\ 1),$$

$$u_{13}^{(5)}=(-1,\ -2,\ -1),\quad u_{14}^{(5)}=(-1,\ -2,\ 1),\quad u_{15}^{(5)}=(1,\ -2,\ 1),$$

$$u_{16}^{(5)}=(1,\ -2,\ -1),\quad u_{17}^{(5)}=(1,\ 1,\ 2),\quad u_{18}^{(5)}=(-1,\ 1,\ 2),$$

$$u_{19}^{(5)}=(-1,\ -1,\ 2),\quad u_{20}^{(5)}=(1,\ -1,\ 2),\quad u_{21}^{(5)}=(-1,\ -1,\ -2),$$

$$u_{22}^{(5)}=(1,\ -1,\ -2),\quad u_{23}^{(5)}=(1,\ 1,\ -2),\quad u_{24}^{(5)}=(-1,\ 1,\ -2);$$

$$u_j^{(6)}=2u_j^{(2)} \quad (1\leq j\leq 12);$$

$$u_k^{(7)}=2u_k^{(3)} \quad (1\leq k\leq 8).$$

For $k=4,\dots, 7$, we set $M^{(k)}=\{u_j^{(k)};\ 1\leq j\leq m(k)\}$, where $m(4)=6$, $m(5)=24$, $m(6)=12$, $m(7)=8$. We define $N_k=M^{(1)}\cup\cdots\cup M^{(k)}$ for $k=4,\dots, 7$ and denote by $\mathscr{N}_k$ the space of summational invariants of N_k. Thus, $\mathscr{N}_k\subset\mathscr{V}_k=\boldsymbol{R}^{m(1)+\cdots+m(k)}$. It is shown that $\dim \mathscr{N}_k=5$ for $k=4,\dots, 7$. We observe that

$$u_2^{(1)}+u_1^{(4)}=u_7^{(2)}+u_1^{(3)}$$

and that

$$|u_2^{(1)}|^2+|u_1^{(4)}|^2=|u_7^{(2)}|^2+|u_1^{(3)}|^2.$$

Therefore,

$$(u_2^{(1)},\ u_1^{(4)})\longrightarrow(u_7^{(2)},\ u_1^{(3)})$$

is a collision of N_4. By repeating a similar argument as before, we conclude that $\dim \mathscr{N}_4=\dim \mathscr{N}_3=5$. Next we look at the following collision of N_5,

$$(u_1^{(1)},\ u_1^{(5)})\longrightarrow(u_1^{(3)},\ u_1^{(4)}),$$

and the collisions derived from this by performing a transformation of G. To proceed from $k=5$ to $k=6$, we consider

$$(u_9^{(2)}, u_1^{(6)}) \longrightarrow (u_1^{(4)}, u_{17}^{(5)})$$

and the other collisions of N_6 like this. Finally we notice that

$$(u_9^{(2)}, u_1^{(7)}) \longrightarrow (u_{17}^{(5)}, u_9^{(6)})$$

is a collision of N_7 and by employing the same argument as above at each stage we conclude that $\dim \mathcal{N}_7 = \dim \mathcal{N}_6 = \dim \mathcal{N}_5 = \dim \mathcal{N}_4 = 5$. For $k \geq 8$, we define $M^{(k)} = \{u_i^{(k)};\ 1 \leq i \leq m(k)\}$ by $u_i^{(k)} = 2u_i^{(k-4)}$ $(1 \leq i \leq m(k))$ where $m(k) = m(k-4)$. Let $N_k = M^{(1)} \cup \cdots \cup M^{(k)}$ and let $\mathcal{N}_k$ be the space of summational invariants of N_k. Note that $\mathcal{N}_k$ is a subspace of $\mathcal{V}_k = \boldsymbol{R}^{m(1)+\cdots+m(k)}$. It is shown by induction that $\dim \mathcal{N}_k = 5$ for any $k \geq 8$. Since N_k, $k=2, 3, \ldots$, is a discrete model with k moduli of velocities, the proof of Theorem 1.1 is complete if the regularity is established for these models.

§ 3. Proof of the regularity.

Since the irreducibility is checked easily for all models constructed in the preceeding section, it is enough to show that the stability condition holds. The verification of this condition for the model N_2 is given in Appendix 2. To treat the model N_k for arbitrary $k > 2$, we proceed as follows. Let i be a positive integer. Then there exist integers k, l such that $i = m(1) + \cdots + m(k) + l$, where $1 \leq l \leq m(k+1)$ (We set $d(0) = 0$). Let $v_i = u_l^{(k+1)}$ for $k \geq 0$ and $n(k) = m(1) + \cdots + m(k)$ for $k \geq 1$. Then, $N_k = \{v_i;\ 1 \leq i \leq n(k)\}$ for $k \geq 2$. We write $N = N_k$ for brevity's sake. Let us assume that $k > 2$ in the following. Thus N_2 is a submodel of N. The space of summational invariants of N is denoted by $\mathcal{N}$. Hence, $\mathcal{N} \subset \mathcal{V} = \boldsymbol{R}^{n(k)}$. Let $v_i = (v_{i1}, v_{i2}, v_{i3})$ for $i = 1, \ldots, n(k)$. We define $V^j, j = 1, 2, 3$, to be the $n(k) \times n(k)$ diagonal matrix with diagonal elements v_{ij}, $i = 1, \ldots, n(k)$. Then the stability condition for N reads as follows: Let $\phi \in \mathcal{N}$. Let $\omega \in S^2$ and let $V(\omega) = \sum_{j=1}^{3} \omega_j V^j$. Then, $\mu\phi + V(\omega)\phi = 0$ for some $\mu \in \boldsymbol{R}$, implies that $\phi = 0$.

We take a basis of $\mathcal{N}$. To be precise, we define $\phi^1, \ldots, \phi^5$ as follows. Let $\phi^1 \in \mathcal{V}$ be the vector whose entries are all equal to 1. We define $\phi^2, \phi^3, \phi^4 \in \mathcal{V}$ to be the vectors obtained by projecting the velocities v_i, $i = 1, \ldots, n(k)$, onto the three axes of $\boldsymbol{R}^3$, respectively. Hence the i-th entry of ϕ^{j+1} equals v_{ij}, $i = 1, \ldots, n(k)$, for $j = 1, 2, 3$. Finally $\phi^5 \in \mathcal{V}$ is the vector whose i-th entry is given by $|v_i|^2$, $i = 1, \ldots, n(k)$. The set of vectors $\phi^1, \ldots, \phi^5$ defined above forms a basis of $\mathcal{N}$. Now we assume that

$$\mu\phi + V(\omega)\phi = 0$$

for some $\mu \in \boldsymbol{R}$, $\omega \in S^2$ and that $\phi \in \mathcal{N}$. We write $\phi = \sum_{i=1}^{5} \alpha_i \phi^i$ and substitute this into the above equation. Then we have

$$\sum_{i=1}^{5} \mu\alpha_i\phi^i + \sum_{i=1}^{5} \sum_{j=1}^{3} \omega_j \alpha_i V^j \phi^i = 0. \tag{3.1}$$

Now let P be the orthogonal projection of $\mathscr{V}=\boldsymbol{R}^{n(k)}$ onto $\mathscr{V}_2=\boldsymbol{R}^{n(2)}$. We denote also by P the $n(k)\times n(k)$ diagonal matrix corresponding to this projection in the standard basis. Then, multiplying both sides of (3.1) by P, we obtain

$$\sum_{i=1}^{5}\mu\alpha_i P\phi^i+\sum_{i=1}^{5}\sum_{j=1}^{3}\omega_j\alpha_i PV^jP\phi^i=0. \tag{3.2}$$

Here we used $PV^j=P^2V^j=PV^jP$. The set of vectors $P\phi^1,\dots, P\phi^5$ forms a basis of $\mathscr{N}_2$. Therefore, (3.2) can be written as

$$\mu\psi+V_2(\omega)\psi=0,$$

where we set $\psi=\sum_{i=1}^{5}\alpha_i P\phi^i$ and $V_2(\omega)=\sum_{j=1}^{3}\omega_j PV^j$. Since the stability condition is already verified for the model N_2, we deduce that $\psi=0$. Hence $\alpha_i=0$ for $1\le i\le 5$. Therefore $\phi=\sum_{i=1}^{5}\alpha_i\phi^i=0$. This means that the stability condition holds for the model N. The proof of the regularity is completed.

Finally we consider a modification of the models N_k, $k=2, 3,\cdots$. Let $\tilde{M}^{(3)}=\{u_1^{(3)}, u_3^{(3)}, u_6^{(3)}, u_8^{(3)}\}$. Then $\tilde{M}^{(3)}$ is a submodel of $M^{(3)}$. The transformation group for the model $\tilde{M}^{(3)}$ is not $S_4\times I$ but S_4A_4, because the extremities of the 4 vectors of $\tilde{M}^{(3)}$ form the vertices of a tetrahedron. We set that $\tilde{N}_2=M^{(2)}\cup\tilde{M}^{(3)}$. Then it is shown that $\tilde{N}_2$ is a regular model. We define $\tilde{N}_3=M^{(1)}\cup M^{(2)}\cup\tilde{M}^{(3)}$ and $\tilde{N}_k=\tilde{N}_3\cup M^{(4)}\cup\cdots\cup M^{(k)}$ for $k>3$. By the same argument as before, we conclude that $\tilde{N}_k$ is a regular model for any $k>2$. The transformation group for $\tilde{N}_k$, $k=3, 4,\dots$, is also S_4A_4. Thus we obtain a refinement of Theorem 1.1, which is as follows.

Theorem 3.1. *Let $k\ge 2$ be an arbitrary integer. Let $\mathscr{R}(G)$ be the set of regular discrete models invariant under the prescribed transformation group G. Then, if G is either S_4A_4 or $S_4\times I$, $\mathscr{R}(G)$ contains a model with k moduli of velocities.*

The counterpart of this theorem for other transformation groups will be given elsewhere. See [5], [6] for the case of $G=A_5\times I$.

Appendix 1.

We set

$$\Sigma_1=\{(v_i, v_j);\ 1\le i<j\le 6\},$$

$$\Sigma_2=\{(v_i, v_j);\ 7\le i<j\le 18\},$$

$$\Sigma_{1,2}=\{(v_i, v_j);\ 1\le i\le 6,\ 7\le j\le 18\}.$$

Note that $M^{(1)}=\{v_1,\dots, v_6\}$, $M^{(2)}=\{v_7,\dots, v_{18}\}$. We denote by $\mathscr{C}_1$ and $\mathscr{C}_2$ the sets of all collisions which have the initial and the final states in Σ_1 and Σ_2, respectively. Similarly $\mathscr{C}_{1,2}$ denotes the totality of collisions whose initial and hence finall states are the elements of $\Sigma_{1,2}$. Any collision is contained in one of the three categories $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_{1,2}$. The following is the complete list of collisions for the model N_2.

1) 6 collisions of $\mathscr{C}_1$.

$$(v_1, v_4) \rightleftarrows (v_2, v_5), \quad (v_1, v_4) \rightleftarrows (v_3, v_6), \quad (v_2, v_5) \rightleftarrows (v_3, v_6).$$

2) 120 collisions of $\mathscr{C}_{1,2}$

$$(v_1, v_8) \rightleftarrows (v_2, v_7), \quad (v_1, v_8) \rightleftarrows (v_3, v_{15}), \quad (v_2, v_7) \rightleftarrows (v_3, v_{15}),$$

$$(v_1, v_{14}) \rightleftarrows (v_2, v_{13}), \quad (v_1, v_{14}) \rightleftarrows (v_6, v_{15}), \quad (v_2, v_{13}) \rightleftarrows (v_6, v_{15}),$$

$$(v_2, v_{18}) \rightleftarrows (v_3, v_{13}), \quad (v_2, v_{18}) \rightleftarrows (v_5. v_{15}), \quad (v_2, v_{18}) \rightleftarrows (v_6, v_7),$$

$$(v_3, v_{13}) \rightleftarrows (v_5, v_{15}), (v_3, v_{13}) \rightleftarrows (v_6, v_7), \quad (v_5, v_{15}) \rightleftarrows (v_6, v_7),$$

$$(v_1, v_{10}) \rightleftarrows (v_5, v_7), \quad (v_1, v_{10}) \rightleftarrows (v_3, v_{18}), \quad (v_5, v_7) \rightleftarrows (v_3, v_{18}),$$

$$(v_1, v_{12}) \rightleftarrows (v_5, v_{13}), \quad (v_1, v_{12}) \rightleftarrows (v_6, v_{18}), \quad (v_5, v_{13}) \rightleftarrows (v_6, v_{18}),$$

$$(v_1, v_{16}) \rightleftarrows (v_3, v_{14}), \quad (v_1, v_{16}) \rightleftarrows (v_4, v_{15}), \quad (v_1, v_{16}) \rightleftarrows (v_6, v_8),$$

$$(v_3, v_{14}) \rightleftarrows (v_4, v_{15}), \quad (v_3, v_{14}) \rightleftarrows (v_6, v_8), \quad (v_4, v_{15}) \rightleftarrows (v_6, v_8),$$

$$(v_1, v_9) \rightleftarrows (v_2, v_{10}), \quad (v_1, v_9) \rightleftarrows (v_4, v_7), \quad (v_1, v_9) \rightleftarrows (v_5, v_8),$$

$$(v_2, v_{10}) \rightleftarrows (v_4, v_7), \quad (v_2, v_{10}) \rightleftarrows (v_5, v_8), \quad (v_4, v_7) \rightleftarrows (v_5, v_8),$$

$$(v_1, v_{11}) \rightleftarrows (v_2, v_{12}), \quad (v_1, v_{11}) \rightleftarrows (v_4, v_{13}), \quad (v_1, v_{11}) \rightleftarrows (v_5, v_{14})$$

$$(v_2, v_{12}) \rightleftarrows (v_4, v_{13}), \quad (v_2, v_{12}) \rightleftarrows (v_5, v_{14}), \quad (v_4, v_{13}) \rightleftarrows (v_5, v_{14}),$$

$$(v_1, v_{17}) \rightleftarrows (v_3, v_{12}), \quad (v_1, v_{17}) \rightleftarrows (v_4, v_{18}), \quad (v_1, v_{17}) \rightleftarrows (v_6, v_{10}),$$

$$(v_3, v_{12}) \rightleftarrows (v_4, v_{18}), \quad (v_3, v_{12}) \rightleftarrows (v_6, v_{10}), \quad (v_4, v_{18}) \rightleftarrows (v_6, v_{10}),$$

$$(v_2, v_9) \rightleftarrows (v_3, v_{16}), \quad (v_2, v_9) \rightleftarrows (v_4, v_8), \quad (v_3, v_{16}) \rightleftarrows (v_4, v_8),$$

$$(v_2, v_{11}) \rightleftarrows (v_4, v_{14}), \quad (v_2, v_{11}) \rightleftarrows (v_6, v_{16}), \quad (v_4, v_{14}) \rightleftarrows (v_6, v_{16}),$$

$$(v_2, v_{17}) \rightleftarrows (v_3, v_{11}), \quad (v_2, v_{17}) \rightleftarrows (v_5, v_{16}), \quad (v_2, v_{17}) \rightleftarrows (v_6, v_9),$$

$$(v_3, v_{11}) \rightleftarrows (v_5, v_{16}), \quad (v_3, v_{11}) \rightleftarrows (v_6, v_9), \quad (v_5, v_{16}) \rightleftarrows (v_6, v_9),$$

$$(v_3, v_{17}) \rightleftarrows (v_4, v_{10}), \quad (v_3, v_{17}) \rightleftarrows (v_5, v_9), \quad (v_4, v_{10}) \rightleftarrows (v_5, v_9),$$

$$(v_4, v_{12}) \rightleftarrows (v_5, v_{11}), \quad (v_4, v_{12}) \rightleftarrows (v_6, v_{17}), \quad (v_5, v_{11}) \rightleftarrows (v_6, v_{17}).$$

3) 66 collisions of $\mathscr{C}_2$.

$$(v_7, v_{13}) \rightleftarrows (v_{15}, v_{18}), \quad (v_7, v_{14}) \rightleftarrows (v_8, v_{13}), \quad (v_8, v_{18}) \rightleftarrows (v_{10}, v_{15}),$$

$$(v_{12}, v_{15}) \rightleftarrows (v_{14}, v_{18}), \quad (v_7, v_{12}) \rightleftarrows (v_{10}, v_{13}), \quad (v_8, v_{14}) \rightleftarrows (v_{15}, v_{16}),$$

$$(v_7, v_{16}) \rightleftarrows (v_9, v_{15}), \quad (v_{11}, v_{15}) \rightleftarrows (v_{13}, v_{16}), \quad (v_7, v_9) \rightleftarrows (v_8, v_{10}),$$

$$(v_7, v_{11}) \rightleftarrows (v_8, v_{12}), \quad (v_7, v_{11}) \rightleftarrows (v_9, v_{13}), \quad (v_7, v_{11}) \rightleftarrows (v_{10}, v_{14}),$$

$(v_7, v_{11}) \rightleftarrows (v_{15}, v_{17}),\quad (v_7, v_{11}) \rightleftarrows (v_{16}, v_{18}),\quad (v_8, v_{12}) \rightleftarrows (v_9, v_{13}),$

$(v_8, v_{12}) \rightleftarrows (v_{10}, v_{14}),\quad (v_8, v_{12}) \rightleftarrows (v_{15}, v_{17}),\quad (v_8, v_{12}) \rightleftarrows (v_{16}, v_{18}),$

$(v_9, v_{13}) \rightleftarrows (v_{10}, v_{14}),\quad (v_9, v_{13}) \rightleftarrows (v_{15}, v_{17}),\quad (v_9, v_{13}) \rightleftarrows (v_{16}, v_{18}),$

$(v_{10}, v_{14}) \rightleftarrows (v_{15}, v_{17}),\quad (v_{10}, v_{14}) \rightleftarrows (v_{16}, v_{18}),\quad (v_{15}, v_{17}) \rightleftarrows (v_{16}, v_{18}),$

$(v_{11}, v_{13}) \rightleftarrows (v_{12}, v_{14}),\quad (v_7, v_{17}) \rightleftarrows (v_9, v_{18}),\quad (v_{11}, v_{18}) \rightleftarrows (v_{13}, v_{17}),$

$(v_{10}, v_{12}) \rightleftarrows (v_{17}, v_{18}),\quad (v_6, v_9) \rightleftarrows (v_7, v_{12}),\quad (v_8, v_{17}) \rightleftarrows (v_{10}, v_{16}),$

$(v_{12}, v_{16}) \rightleftarrows (v_{14}, v_{17}),\quad (v_9, v_{12}) \rightleftarrows (v_{10}, v_{11}),\quad (v_9, v_{11}) \rightleftarrows (v_{16}, v_{17}).$

Appendix 2.

We use the argument of Cercignani [1]. We have, by definition,

$$\phi^1 = {}^t(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$
$$\phi^2 = {}^t(1, 0, 0, -1, 0, 0, 1, 0, -1, 0, -1, 0, 1, 0, 1, -1, -1, 1),$$
$$\phi^3 = {}^t(0, 1, 0, 0, -1, 0, 0, 1, 0, -1, 0, -1, 0, 1, 1, 1, -1, -1),$$
$$\phi^4 = {}^t(0, 0, 1, 0, 0, -1, 1, 1, 1, 1, -1, -1, -1, -1, 0, 0, 0, 0),$$
$$\phi^5 = {}^t(1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2),$$

for the model N_2. We set

$$\phi^0 = (1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$

and use $\phi^0, \phi^1, \dots, \phi^4$ as a basis of $\mathscr{N}_2$. Note that $\phi^0 = 2\phi^1 - \phi^5$. Let $\phi \in \mathscr{N}_2$ and let $\mu\phi + V(\omega)\phi = 0$ for some $\mu \in \boldsymbol{R}$, $\omega \in S^2$. Then, substituting $\phi = \sum_{i=0}^{4} \alpha_i \phi^i$ into this equation, we obtain

$$\sum_{i=0}^{4} \mu\alpha_i\phi^i + \sum_{i=0}^{4}\sum_{j=1}^{3} \omega_j\alpha_i V^j\phi^i = 0. \tag{A.1}$$

We denote by $\mathscr{S}$ the set of 20 vectors ϕ^i, $V^j\phi^i$, $0 \le i \le 4$, $1 \le j \le 3$. Then, a set of 13 elements of $\mathscr{S}$ is linearly independent and every set of 14 vectors from $\mathscr{S}$ is linearly dependent. To show this, we note the following equalities

$$\begin{cases} V^1\phi^3 = V^2\phi^2, \quad V^1\phi^4 = V^3\phi^2, \quad V^2\phi^4 = V^3\phi^3, \\ \phi^{j+1} = V^j\phi^1, \quad j = 1, 2, 3, \\ \phi^0 = 2\phi^1 - (V^1\phi^2 + V^2\phi^3 + V^3\phi^4), \end{cases} \tag{A.2}$$

and set

$$\begin{aligned} &\psi^1 = V^1\phi^0, \quad \psi^2 = V^2\phi^0, \quad \psi^3 = V^3\phi^0, \quad \psi^4 = V^1\phi^1, \quad \psi^5 = V^2\phi^1, \\ &\psi^6 = V^3\phi^1, \quad \psi^7 = V^1\phi^2, \quad \psi^8 = V^2\phi^2, \quad \psi^9 = V^3\phi^2, \quad \psi^{10} = V^2\phi^3, \\ &\psi^{11} = V^3\phi^3, \quad \psi^{12} = V^3\phi^4, \quad \psi^{13} = \phi^1. \end{aligned} \tag{A.3}$$

We define B to be the 18×13 matrix $(\psi^1,\ldots,\psi^{13})$, where $\psi^1,\ldots,\psi^{13}$ are the columns of B. We compute the rank of B and conclude that B is of full rank. Hence $\psi^1,\ldots,\psi^{13}$ are linearly independent. The expressions for other vectors of $\mathscr{S}$ by means of $\psi^1,\ldots,\psi^{13}$ are already given above. This completes the proof of the assertion. We substitute (A.2) into (A.1) and use (A.3). Then, we obtain

$$\begin{aligned}&\omega_1\alpha_0\psi^1+\omega_2\alpha_0\psi^2+\omega_3\alpha_0\psi^3+(\omega_1\alpha_1+\mu\alpha_2)\psi^4\\&\quad+(\omega_2\alpha_1+\mu\alpha_3)\psi^5+(\omega_3\alpha_1+\mu\alpha_4)\psi^6+(-\mu\alpha_0+\omega_1\alpha_2)\psi^7\\&\quad+(\omega_2\alpha_2+\omega_1\alpha_3)\psi^8+(\omega_3\alpha_2+\omega_1\alpha_4)\psi^9+(-\mu\alpha_0+\omega_2\alpha_3)\psi^{10}\\&\quad+(\omega_3\alpha_3+\omega_2\alpha_4)\psi^{11}+(-\mu\alpha_0+\omega_3\alpha_4)\psi^{12}+(2\mu\alpha_0+\mu\alpha_1)\psi^{13}=0.\end{aligned}$$

Equating the coefficients of ψ^i, $i=1,\ldots,13$, with zero, we get

$$T(\mu,\omega_1,\omega_2,\omega_3)\tilde{\phi}=0,$$

where $\tilde{\phi}={}^t(\alpha_0.\ \alpha_1,\ldots,\alpha_4)$ and $T=T(\mu,\omega_1,\omega_2,\omega_3)$ is the following 13×5 matrix,

$$\begin{bmatrix}\omega_1 & 0 & 0 & 0 & 0\\ \omega_2 & 0 & 0 & 0 & 0\\ \omega_3 & 0 & 0 & 0 & 0\\ 0 & \omega_1 & \mu & 0 & 0\\ 0 & \omega_2 & 0 & \mu & 0\\ 0 & \omega_3 & 0 & 0 & \mu\\ -\mu & 0 & \omega_1 & 0 & 0\\ 0 & 0 & \omega_2 & \omega_1 & 0\\ 0 & 0 & \omega_3 & 0 & \omega_1\\ -\mu & 0 & 0 & \omega_2 & 0\\ 0 & 0 & 0 & \omega_3 & \omega_2\\ -\mu & 0 & 0 & 0 & \omega_3\end{bmatrix}$$

Let D_1 be the 5×5 minor of T formed by the 5 rows including ω_1 as an entry. Then, it is shown that $D_1=\omega_1^5$. We define similarly D_2 and D_3 and observe that $D_2=\omega_2^5$, $D_3=\omega_3^5$. Since $\omega_1^2+\omega_2^2+\omega_3^2=1$, one of the three minors D_1, D_2, D_3 does not vanish. This implies that the rank of $T(\mu,\omega_1,\omega_2,\omega_3)$ is 5 for any $\mu\in\boldsymbol{R}$, $\omega\in S^2$. It follows that $\alpha_i=0$ for $i=0,1,\ldots,4$. Hence $\phi=\sum_{i=0}^{4}\alpha_i\phi^i=0$. This is the desired result. The proof of the stability condition for N_2 is completed.

DEPARTMENT OF MATHEMATICS
NARA WOMEN'S UNIVERSITY

References

[1] C. Cercignani, private communication.

[2] H. S. M. Coxeter, Introduction to Geometry, 2nd edition, New York, John Wiley & Sons, 1965.

[3] Y. Shizuta and S. Kawshima, Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation, Hokkaido Math. J., **14** (1985), 249–275.

[4] Y. Shizuta and S. Kawashima, The regularity of discrete models of the Boltzmann equation, Proc. Japan Acad., Ser. A, **61** (1985), 252–254.

[5] Y. Shizuta, M. Maeji, A. Watanabe and S. Kawashima, Regularity of the 90-velocity model of the Boltzmann equation, Proc. Japan Acad., Ser, A, **62** (1986), 171–173.

[6] Y. Shizuta, M. Maeji, A. Watanabe and S. Kawashima, The 102-velocity model and the related discrete models of the Boltzmann equation, Proc. Japan Acad., Ser. A, **62** (1986), 367–370.

Added in proof: The result cited as [1] was published in C. R. Acad. Sci. Paris Sér, I, **301** (1985), 89–92.