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On boundary behaviours of holomorphic functions

By

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1. Introduction.

Let $D=\{z: |z|<1\}$ and let $w=S(z)$ denote an arbitrary one-to-one conformal mapping from D onto itself. A function $f(z)$, meromorphic in D, is said to be normal in D [12, p. 53], if the family of functions $\{f(S(z))\}$ is normal in D in the sense of Montel, where convergence is defined in terms of the spherical metric. Following Bagemihl and Seidel [3, p. 10], we call $e^{i\theta}$ a Fatou point of f if f has an angular limit v (possibly ∞) at $e^{i\theta}$. This limit v will be called a Fatou value of f.

There is a normal meromorphic function in D which possesses no Fatou points [12, p. 58]. On the other hand, Bagemihl and Seidel proved that any normal holomorphic function f in D must have Fatou points everywhere dense on the circle $C=\{z: |z|=1\}$ [3, Corollary 1]. In fact, in [3, Theorem 3], they proved that if the set of Fatou points of f on an arc Γ of C is of measure zero, then the arc Γ contains a Fatou point at which the corresponding Fatou value is ∞. Moreover, they constructed a normal holomorphic function f for which the measure of the set of all Fatou points of f is less than any prescribed small number and the function f has no infinite Fatou values [3, Theorem 4]. They then asked that if f is normal holomorphic in D and if there is an arc Γ on C such that the measure of the set of Fatou points of f on every subarc of Γ is less than the length of that subarc, does the arc Γ contain a Fatou point of f whose Fatou value is ∞? The answer turns out to be negative due to S. Dragosh [6].

A problem related to the above one was asked by MacLane [13]. Following MacLane [13, p. 8], we denote by $\mathscr{A}$ the class of all non-constant holomorphic functions f in D such that f has asymptotic values on a set S which is dense in C, namely, for each $p \in S$, there is a Jordan arc J lying in D and tending to p such that $f(z)$ tends to a value along J. In contrast to the notion of Fatou points, we shall call a point $p \in C$ an asymptotic point of f if f has an asymptotic value at p. Let A be the set of all asymptotic points of f on C and let A_∞ be the subset of A containing all points of A with the asymptotic value ∞. In [13, p. 77], MacLane asked that if $f \in \mathscr{A}$ and if Γ is an arc on C with $\Gamma \cap A_\infty = \emptyset$, is it true that the arc Γ contains a subarc γ such that almost every point of γ is an asymptotic point of f. Again, the answer is negative due to Dragosh (He has not mentioned this assertion in [6]).

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The above two problems give us the motivation to study the following problem in the positive sense.

Problem. Let Γ be an arc on C such that $\Gamma \cap A_\infty = \emptyset$. Under what additional conditions for $f(z)$, would almost every point of Γ is a Fatou point of f (to avoid the trivial case we require f to be unbounded).

In order to answer the above problem, we shall first recall the definition of class $\mathscr{L}$ of MacLane. For a function f defined in D and a number $\lambda > 0$, we denote by $L(\lambda) = \{z: z \in D \text{ and } |f(z)| = \lambda\}$ the level set of f, and we call $C(\lambda)$ a level curve if it is a component of $L(\lambda)$. We say that a level set $L(\lambda)$ (or curve $C(\lambda)$) ends at points on the circle C, if the diameter of every component in the ring $\{z: r < |z| < 1\}$ tends to zero as $r \to 1$. We then denote by $\mathscr{L}(\mathscr{L}^*)$ the class of all non-constant holomorphic functions in D such that every level set (curve) $L(\lambda)$ $(C(\lambda))$ ends at points on C.

We now let $D^l(\lambda) = \{z: z \in D \text{ and } |f(z)| < \lambda\}$ and $D^u(\lambda) = \{z: z \in D \text{ and } |f(z)| > \lambda\}$ be the lower and upper level domain respectively. We decompose both of them into disjoint components

$$D^i(\lambda) = \bigcup_n D_n^i(\lambda), \quad \text{where} \quad i = l \quad \text{and} \quad u.$$

Clearly, each boundary $\partial D_n^i(\lambda)$ of $D_n^i(\lambda)$ consists of two parts: one from the level set $L(\lambda)$ and another from the circle C. Denote by $L_n^i(\lambda)$ the part from $L(\lambda)$, then clearly we have

$$L(\lambda) = \bigcup_n L_n^i(\lambda), \quad (i = l \text{ and } u)$$

where each component of $L_n^i(\lambda)$ can be either a crosscut or a Jordan curve. We remark that the boundary $\partial D_n^i(\lambda)$ is rectifiable if and only if the level set $L_n^i(\lambda)$ is.

For each level domain, we define the associated boundary sets as follows

$$\gamma^i(\lambda) = C \cap \bar{D}^i(\lambda) \quad \text{and} \quad \gamma_n^i(\lambda) = C \cap \bar{D}_n^i(\lambda),$$

where the set $\bar{G}$ denotes the closure of the domain G. Clearly, each of the above sets is closed on C and hence is measurable. For convenience, we shall use $|S|$ to denote the length of a curve S as well as the measure of a boundary set S and use T^* to denote the smallest simply connected domain containing the set T. As in [11], we say that the number $\lambda > 0$ is an admissible value for f if the following three conditions are satisfied:

(1) If $\gamma_n^i(\lambda) \neq \emptyset$, then $|\partial(D_n^i(\lambda))^*| < \infty$, $i \in \{l, u\}$.

(2) If $\{n_j\}$ is a sequence of positive integers such that $n_j \to \infty$ and $\gamma_{n_j}^l(\lambda) = \emptyset$ for each j, then diam $D_{n_j}^l(\lambda) \to 0$.

(3) $\sum_m |\gamma_m^l(\lambda)| + \sum_n |\gamma_n^u(\lambda)| = 2\pi$.

We say that f is in the class $\mathscr{L}_1$ if there exists a sequence $\{\lambda_k\}$ of admissible values for f such that $\lambda_k \to \infty$. In [11], the author and Lappan proved the following two results.

Theorem 1. *The class $\mathscr{L}_1$ is a proper subclass of $\mathscr{L}$.*

The notion of this subclass $\mathscr{L}_1$ allows us to answer the problem we posed as follows.

Theorem 2. *If $f \in \mathscr{L}_1$, and if Γ is an arc on C with $\Gamma \cap A_\infty = \emptyset$, then almost every point of Γ is a Fatou point of f.*

As a consequence of Theorem 2, we see that under the hypothesis of Bagemihl and Seidel's problem if the function f there belongs to the class $\mathscr{L}_1$ then the arc Γ there contains a Fatou point of f whose Fatou value is ∞. With regard to the property of Theorem 2, we posed the following question in [11, p. 297].

Question. Let $\mathscr{L}_2 = \{f \in \mathscr{L}_1$ and if Γ is an arc on C with $\Gamma \cap A_\infty = \emptyset$ then almost every point of Γ is a Fatou point of $f\}$, can $\mathscr{L}_2$ be characterized in terms of $L(\lambda)$ or $D^i(\lambda)$, $i \in \{l, u\}$?

In this paper, we shall present a negative answer as follows.

Theorem 3. *The class $\mathscr{L}_2$ cannot be characterized by the rectifiability of $L(\lambda)$.*

As for Theorem 1, we shall extend it by omitting Condition (1) and relaxing Condition (3) by

(4) The set $\gamma(\lambda) = \{\bigcup_m \gamma_m^l(\lambda)\} \cup \{\bigcup_n \gamma_n^u(\lambda)\}$ is dense on C.

Theorem 4. *Let $\mathscr{L}_1^*$ be the class of all analytic functions in D satisfying* (2) *and* (4) *for a sequence of positive numbers $\lambda_k \to \infty$, then $\mathscr{L}_1 \subset \mathscr{L}_1^* \subset \mathscr{L}$.*

The first inclusion $\mathscr{L}_1 \subset \mathscr{L}_1^*$ is obvious and only the second will be proved. After Theorem 4, we study some boundary behaviours of MacLane's class. We then extend some classical theorems of Fatou, Lindelöf, and Riesz from the disk to an arbitrary simply connected domain whose boundary is rectifiable. Finally, we study a connection between annular functions and MacLane's class.

2. Proof of Theorem 3.

It suffices to construct a function $f \in \mathscr{L}_2$ whose level set $L(\lambda)$ is not rectifiable. In fact, we shall prove that the elliptic modular function $M(z)$ does have the desired property. We first observe that the set A_∞ is dense on C so that $\Gamma \cap A_\infty \neq \emptyset$ for any arc Γ in C and hence by definition we must have that $M \in \mathscr{L}_2$.

Finally, the non-rectifiability of $L(\lambda)$ for the function $M(z)$ was proved in the last remark [11, p. 298]. This completes the proof.

3. Proof of Theorem 4.

Let $f(z)$ be a function in the class $\mathscr{L}_1^*$. We shall prove that $f \in \mathscr{L}$. For this, we need the following theorem of MacLane [13, Theorem 1].

Theorem M. *$\mathscr{A}=\mathscr{B}=\mathscr{L}$, where $\mathscr{B}$ is the class defined in* [13] *which contains all non-constant holomorphic functions bounded on arcs ending at points of a dense subset of C.*

We now let Γ be an arbitrary arc on C and let $\{\lambda_k\}$ be a sequence of positive numbers satisfying (2) and (4). Then by virtue of (4), the arc Γ contains a point p such that either $p\in\gamma_m^l(\lambda_1)$ or $p\in\gamma_n^u(\lambda_1)$ for some m and n. In view of the definition, the first case implies that $p\in\bar{D}_m^l(\lambda_1)$, so that there is an arc $\gamma\subset\bar{D}_m^l(\lambda_1)$ ending at the point p for which the function f is bounded by λ_1 on the arc γ. Since the arc Γ on C is arbitrary, if the first case occurs on Γ, then $f\in\mathscr{B}$, so that $f\in\mathscr{L}$ due to Theorem M.

It remains to consider the second case. In this case, we have $p\in\bar{D}_{n_1}^u(\lambda_1)$ for some n_1. Let $D_r(p)=\{z\colon z\in D \text{ and } |z-p|<r\}$. If there is an $0<r<1$ such that $D_r(p)$ is disjoint from the level set $L(\lambda_1)$, then we have $D_r(p)\subset D_{n_1}^u(\lambda_1)$, so that the function $g=1/f$ is bounded by λ_1 in $D_r(p)$. By a simple extension of Fatou's theorem [5, Theorem 2.1], we see that almost every point on the boundary $\Gamma_r=D_r(p)\cap C$ is a Fatou point of g as well as f and we are done. We may therefore assume that $D_r(p)\cap L(\lambda_1)\neq\emptyset$ for each $r>0$. In this case, if the set $L(\lambda_1)$ contains an arc ending at points on Γ_r for some $r>0$, we are done again due to Theorem M. Thus, we may assume that no arcs on $L(\lambda_1)$ can end at points on Γ_r. This in turn implies that the arc Γ_r lies on the boundary $\partial D_{n_1}^u(\lambda_1)$ of $D_{n_1}^u(\lambda_1)$. Clearly, the boundary $\partial D_{n_1}^u(\lambda_1)$ consists of some level arcs $A_j(\lambda_1)$ (crosscuts), level curves $C_j(\lambda_1)$, and some arcs on the circle C. By what we have just assumed, we know that only level curves can meet $D_r(p)$. Furthermore, from (2) we see that

$$\operatorname{diam} D_j^l(\lambda_1)\longrightarrow 0, \quad \text{as} \quad j\longrightarrow\infty, \tag{5}$$

where $D_j^l(\lambda_1)$ is the interior of $C_j(\lambda_1)$.

Let $\lambda_2>\lambda_1$, then by the same argument as before, we may assume that no arcs on the second level set $L(\lambda_2)$ can end at points on the arc Γ_r, so that the arc Γ_r lies on the boundary $\partial D_{n_2}^u(\lambda_2)$ of some component $D_{n_2}^u(\lambda_2)$. Since $\lambda_2>\lambda_1$, we must have $D_{n_2}^u(\lambda_2)\subset D_{n_1}^u(\lambda_1)$. Furthermore, we have the same property as in (5) with λ_2 in place of λ_1.

Inductively, there can be chosen a sequence of positive integers n_k such that

$$\begin{aligned} &\Gamma_r\subset D_{n_k}^u(\lambda_k), \quad k=1,2,\dots,\\ &D_{n_1}^u(\lambda_1)\supset D_{n_2}^u(\lambda_2)\supset\cdots, \end{aligned} \tag{6}$$

and (5) holds with λ_k in place of λ_1.

We now let q be an arbitrary point on Γ_r and let R_q be the radius ending at q. Denote by $C_j^q(\lambda_k)$ the set of all level curves meeting R_q and contained in the ring $\{z\colon 1-2^{-k}\leq|z|<1\}$, where $j=j(k)$ and j, $k=1,2,\dots$. We begin with $k=1$. We replace each portion of R_q lying in the interior of $C_j^q(\lambda_1)$, $j=1,2,\dots$, by a half of $C_j^q(\lambda_1)$ which joins two points on R_q in the obvious way. Let the resulting arc be denoted by R_q^1 and let $\gamma_1=R_q^1\cap\{z\colon |z|\leq 1/2\}$. By the same argument, we have the resulting arcs R_q^k and we let $\gamma_k=R_q^k\cap\{z\colon 1-2^{-k+1}<|z|\leq 1-2^{-k}\}$. Clearly, the endpoints of γ_k and γ_{k+1} on the circle $|z|=1-2^{-k}$ can be joined by an arc γ_k^* such that $|f(z)|\geq\lambda_k$ for all $z\in\gamma_k^*$. Let

$$\gamma=\gamma_1\cup\gamma_1^*\cup\gamma_2\cup\gamma_2^*\cup\cdots.$$

Then by (5) and (6), we see that the arc γ ends at the point q and the function $f(z)$ tends to infinity as $z\to q$ and $z\in\gamma$. This shows that $q\in A_\infty$, so that $f\in\mathscr{A}=\mathscr{L}$. Hence $\mathscr{L}_1^*$ is a subclass of $\mathscr{L}$ and the proof is complete.

Note that the cases considered in (5) and (6) can actually occur and in fact, the following function, constructed by Bagemihl, Erdös, and Seidel [1], does have these properties

$$f(z)=\prod_{j=1}^{\infty}\{1-[z/(1-1/n_j)]^{n_j}\}, \tag{7}$$

where n_j are positive integers and $n_{j+1}/n_j\to\infty$.

4. Boundary sets of functions in $\mathscr{L}$.

In this section, we shall prove the following boundary behaviours of functions in the class $\mathscr{L}$ in terms of the boundary sets.

Theorem 5. *Let $f\in\mathscr{L}$ and let $\gamma(\lambda)$ be defined in* (4). *Then for each $\lambda>0$, the set $\gamma(\lambda)$ is dense on C and the total measure $m_f(\lambda)$ on the left side of* (3) *is not greater than* 2π.

Proof. The proof of the first assertion will be the same as that of Theorem 4 and therefore we sketch it. Let Γ be an arbitrary arc on C. If either $\Gamma\cap\gamma_m^l(\lambda)\neq\emptyset$, or f is bounded below by λ in a neighborhood of Γ, we are done, otherwise, Γ lies on the boundary $\partial D_n^u(\lambda)$ of some component $D_n^u(\lambda)$. This yields that $\Gamma\subset\gamma(\lambda)$ and therefore the set $\gamma(\lambda)$ is dense on C.

It remains to show that the measure $m_f(\lambda)\leq 2\pi$. To do this, we shall first show that the intersection

$$I=\gamma_m^i(\lambda)\cap\gamma_n^j(\lambda),\quad\text{where}\quad i,j=l\quad\text{or}\quad u,$$

contains at most two points for any m, $n=1,2,\ldots$. Clearly, we may assume that I contains two points p_1 and p_2, otherwise, there is nothing more to prove. Let $D_m^i(\lambda)$ and $D_n^j(\lambda)$ be the associated domains of $\gamma_m^i(\lambda)$ and $\gamma_n^j(\lambda)$ respectively. Then these two domains must be disjoint. Since $f\in\mathscr{L}$, it follows that the boundaries of these two domains must end at points on C. Hence there can be chosen two disjoint Jordan arcs $J_m\subset D_m^i(\lambda)$ and $J_n\subset D_n^j(\lambda)$ such that both of them end at p_1 and p_2. It follows that there is another Jordan arc $J\subset D$ disjoint from both $D_m^i(\lambda)$ and $D_n^j(\lambda)$, and ending at p_1 and p_2. Clearly, this arc J separates one of the above two domains from the other, and therefore the intersection I contains no more points than the set $\{p_1, p_2\}$. This in turn implies that the family $\{\gamma_m^i(\lambda)\}$, $i=l$, u, and $m=1$, $2,\ldots$, consists of subsets of C which are mutually disjoint except two points in common. Hence the total measure $m_f(\lambda)$ of them is not greater than 2π. This completes the proof.

Note that the hypothesis $f \in \mathscr{L}$ is necessary. Without this condition, by using a theorem of Bagemihl-Seidel or Rudin (see [5, Theorem 8.11]), there can be constructed a holomorphic function f which tends to zero along a sequence of disjoint spirals in D. It then follows that for all sufficiently large λ the upper level domain $D^u(\lambda)$ contains a sequence of spiral-like components $D_n^u(\lambda)$, so that

$$\gamma_n^u(\lambda) = C \cap \bar{D}_n^u(\lambda) = C \quad \text{or} \quad |\gamma_n^u(\lambda)| = 2\pi, \quad n = 1, 2, \dots.$$

Hence the measure $m_f(\lambda)$ on the left side of (3) becomes infinite.

Also note that the set $\gamma(\lambda)$ is a subset of C so that the measure $|\gamma(\lambda)| \leq 2\pi$. This measure, in general, is different from the total measure $m_f(\lambda)$ on the left side of (3) which can possible be infinite as was shown in the above example.

Now, using the same argument we obtain the following result.

Corollary 1. *Let $f \in \mathscr{L}$, then for each $\lambda > 0$ the measures of the boundary sets satisfy*

$$|\gamma^i(\lambda)| \geq \sum_m |\gamma_m^i(\lambda)|, \quad i = l, u.$$

Proof. As before, we know that the family $\{\gamma_m^i(\lambda)\}$ is a partition of $\gamma^i(\lambda)$ except a countable set. This gives the inequality.

Note that the above inequality cannot be replaced by equality. In fact, if $f(z)$ is the function defined in (7), then we have

$$|\gamma_m^l(\lambda)| = 0, \quad m = 1, 2, \dots,$$

but $|\gamma^l(\lambda)| = 2\pi$, because

$$\gamma^l(\lambda) = C \cap \bar{D}^l(\lambda) = C, \quad \text{where} \quad D^l(\lambda) = \bigcup_m D_m^l(\lambda).$$

Similarly, if $M(z)$ is the modular function defined in Theorem 3, then we have

$$|\gamma^u(\lambda)| = 2\pi \quad \text{and} \quad |\gamma_n^u(\lambda)| = 0, \quad n = 1, 2, \dots.$$

For application, we shall give a different proof to the following result [11, Lemma 2] which was proved via a deep theorem of Bagemihl (see [5, p. 83]).

Corollary 2. *Let $f \in \mathscr{L}$ and let $T(\lambda)$ be the set of all points on C such that each point in $T(\lambda)$ belongs to more than one $\gamma_m^i(\lambda)$, where $\lambda > 0$, $i = l, u$, and $m = 1, 2, \dots$. Then the set $T(\lambda)$ is at most countable.*

Proof. For simplicity, we write $\gamma_j = \gamma_j^i(\lambda)$. We let γ^λ be the family of all γ_j and let the Cartesian product of γ^λ be defined by

$$\gamma^\lambda \times \gamma^\lambda = \{(\gamma_m, \gamma_n) : \gamma_m, \gamma_n \in \gamma^\lambda\}.$$

For each $p \in T(\lambda)$, we can associate with two members γ_m^p and γ_n^p from the family

γ^λ such that both of them contain the point p. We then define the following product

$$\gamma_T^\lambda \times \gamma_T^\lambda = \{(\gamma_m^p, \gamma_n^p):\ p \in T(\lambda)\}.$$

Since the family $\gamma^\lambda \times \gamma^\lambda$ is countable, if we can show that each member in the family $\gamma_T^\lambda \times \gamma_T^\lambda$ repeats at most twice, then the cardinality of $T(\lambda)$ is less than twice of that of the family $\gamma^\lambda \times \gamma^\lambda$ so that $T(\lambda)$ is countable. Suppose on the contrary that there is a member in $\gamma_T^\lambda \times \gamma_T^\lambda$ repeating three times, say

$$(\gamma_{m1}^{p_1}, \gamma_{n1}^{p_1}) = (\gamma_{m2}^{p_2}, \gamma_{n2}^{p_2}) = (\gamma_{m3}^{p_3}, \gamma_{n3}^{p_3}), \quad p_i \neq p_j, \quad i \neq j.$$

Since each pair is unordered, we may, without loss of generality, assume that

$$\gamma_{mj}^{p_j} = \gamma_m \quad \text{and} \quad \gamma_{nj}^{p_j} = \gamma_n, \quad j = 1, 2, 3.$$

This in turn implies that

$$\{p_1, p_2, p_3\} \subset \gamma_m \cap \gamma_n,$$

contradicting to what we have proved in Theorem 5. We thus conclude that the set $T(\lambda)$ is countable.

5. Remark on Condition (1).

We ask whether or not the simply connected domain in Condition (1) can be omitted? In other words, whether Condition (1) can be replaced by

$$\text{(1)}' \qquad \text{If } \gamma_n^i(\lambda) \neq \emptyset, \text{ then } |\partial D_n^i(\lambda)| < \infty, \ i \in \{l, u\}.$$

The answer turns out to be no. In fact, we shall construct a function $h \in \mathscr{L}_1$ for which Condition (1)$'$ is no longer true for $i = u$. Our construction is analogous to [1, Theorem 1], but much more complicated than that one.

Theorem 6. *The following function $h \in \mathscr{L}_1$, but Condition (1)$'$ is false for $i = u$*

$$h(z) = \prod_{j=1}^{\infty} \{1 - [z/(1 - 1/n_j)]^{n_j}\}^{p_j}, \tag{8}$$

where n_j and p_j are positive integers satisfying

$$e^{1/4} < (1 + 1/4(n_1 - 1))^{n_1} \leq 1.3, \tag{9}$$

$$n_{j+1}/n_j \geq a, \quad j = 1, 2, \ldots, \tag{10}$$

$$p_j = j/2^b, \quad \textit{if} \quad j = 2^m \quad \textit{for some} \quad m > b, \tag{11}$$

$$= 1, \quad \textit{otherwise},$$

and the positive numbers a and b are large integers satisfying

$$e^{1-3/a} - 1 > 5^{1/2^b}. \tag{12}$$

Proof. To prove the assertion, we shall need two-side estimates instead of one-side in [1]. For convenience, we separate the product in (8) into the following three subproducts

$$h(z)=\prod_{q=1}^{3} P_q(z), \tag{13}$$

where $P_q(z)$ correspond to the ranges over $1\le j\le k-1$, $j=k$, and $j\ge k+1$ respectively, and the number $k=2^m$ for some m.

We begin with the first product. For this, we let z_0 be an arbitrary point lying in the ring

$$1-5/4n_k\le |z_0|<1.$$

Then we have

$$\prod_{j=1}^{k-1}(x_j-1)\le |P_1(z_0)|\le \prod_{j=1}^{k-1}(x_j^*+1), \tag{14}$$

where $x_j=[(1-5/4n_k)/(1-1/n_j)]^{n_j}$ and $x_j^*=1/(1-1/n_j)^{n_j}$. Using (10) and some basic inequalities, we get

$$x_j\ge e(1-5/4n_k)^{n_j}\ge e^{1-3n_j/n_k}\ge e^{1-3/a},$$

and

$$x_j^*<1/(1-1/n_1)^{n_1}<4, \quad j=1, 2,\dots, k-1.$$

Substituting these two inequalities into (14), we obtain

$$(e^{1-3/a}-1)^{k-1}\le |P_1(z_0)|\le 4^{k-1}. \tag{15}$$

Turning to the second product, we let z_{jt}, $t=1, 2,\dots, n_j$, be the set of all zeros on the circle

$$C_j=\{z\colon |z|=1-1/n_j\},$$

and let

$$D_{jt}=\{z\colon |z-z_{jt}|<1/4n_j\}, \quad j=1, 2,\dots,$$

$$d_{kt}=\{z\colon |z-z_{kt}|<1/(mn_k)\}, \quad k=2^m.$$

We shall prove two key properties, namely, the function h tends to infinity uniformly on the domain D^* complement to all D_{jt} while h is bounded by one inside each d_{kt}. Since the set of all zeros on each circle C_j is equally distributed over C_j, it follows that the local properties of h at each zero on C_j will be the same as at the one on the positive axis. For simplicity, we denote this zero and the associated disks by z_j, D_j, and d_j respectively. We shall devote on the index $j=k$ and we rewrite

$$D_k=\{z\colon |z-z_k|<1/4n_k\}, \quad d_k=\{z\colon |z-z_k|<1/(mn_k)\}, \tag{16}$$

where $k=2^m$ and $z_k=1-1/n_k$. According to the minimum principle we know that to prove $h(z)\to\infty$ uniformly on D^* it is sufficient to prove this property to be true on the boundaries ∂D_{jt} of D_{jt}. Since the multiple zeros of h occur only at the zeros

z_{jt} corresponding to the index $j=k$, we thus need only show the above property to be true on ∂D_k. Let ζ be an arbitrary point on ∂D_k, then by (8), (13), and (16), we have

$$P_2(\zeta)=\left\{1-\left(1+\frac{e^{i\theta}}{4(n_k-1)}\right)^{n_k}\right\}^{p_k}, \quad \zeta=z_k+e^{i\theta}/4n_k. \tag{17}$$

To estimate the lower bound of $P_2(\zeta)$, we first observe from (9) that

$$e^{1/4}<\left(1+\frac{1}{4(n-1)}\right)^n=1+\frac{n}{4(n-1)}+S_n\leq 1.3, \quad \text{for each} \quad n\geq n_1.$$

Then by (17), we obtain

$$|P_2(\zeta)|\geq\left\{\frac{n_k}{4(n_k-1)}-S_{n_k}\right\}^{p_k}\geq(1.5-1.3)^{p_k}=5^{-p_k}. \tag{18}$$

We now consider an arbitrary point $\eta\in C_k\cap d_k$. In view of (16), we have

$$|P_2(\eta)|\leq|1-e^{i/m}|^{p_k}\leq m^{-p_k}, \quad k=2^m. \tag{19}$$

As for the last product, we have for each $z\in\bar{D}_k$,

$$\prod_{j=k+1}^{\infty}(1-y_j)^{p_j}\leq|P_3(z)|\leq\prod_{j=k+1}^{\infty}(1+y_j)^{p_j}, \tag{20}$$

where $y_j=[(1-3/4n_k)/(1-1/n_j)]^{n_j}$. Since the inequality (10) gives

$$n_j/n_k\geq a^{j-k}, \quad j=k+1,\ k+2,\ldots,$$

it follows that

$$\begin{aligned} y_j&\leq 4(1-3/4n_k)^{n_j}\leq 4\exp(-3n_j/4n_k)\\ &\leq 4\exp(-3a^{j-k}/4)=4\exp(-3a^t/4), \quad t=1,\ 2,\ldots. \end{aligned}$$

This in turn implies that

$$\sum_{j=k+1}^{2k-1} y_j\leq\sum_{t=1}^{\infty}4\exp(-3a^t/4)=c_1<\infty, \tag{21}$$

and

$$\sum_{j=2k}^{\infty} jy_j\leq\sum_{t=1}^{\infty}4(2k+t-1)\exp(-3a^{k+t-1}/4)=c_2<\infty. \tag{22}$$

In view of (11), we see that

$$\begin{aligned} p_j&=1, && \text{for} \quad j=k+1,\ k+2,\ldots,\ 2k-1,\\ &\leq j, && \text{for all} \quad j=1,\ 2,\ldots. \end{aligned}$$

Substituting into (20) and then applying (21) and (22), we obtain for each $z\in\bar{D}_k$,

$$|P_3(z)|\leq\prod_{j=k+1}^{2k-1}(1+y_j)\prod_{j=2k}^{\infty}(1+y_j)^j \tag{23}$$

$$\leq \exp\left(\sum_{j=k+1}^{2k-1} y_j\right) \exp\left(\sum_{j=2k}^{\infty} j y_j\right) \leq e^{c_1+c_2} = c,$$

and

$$|P_3(z)| \geq e^{-(c_1+c_2)} = c^{-1}.$$

Finally, by combining (8), (11), (12), (13), (15), (18), (19), and (23), we obtain for each $\zeta \in \partial D_k$ and when $k \to \infty$,

$$|h(\zeta)| \geq (e^{1-3/a}-1)^{(k-1)} 5^{-k/2^b} c^{-1} \tag{24}$$

$$= [(e^{1-3/a}-1)/5^{1/2^b}]^k / [c(e^{1-3/a}-1)] \longrightarrow \infty,$$

and for each $\eta \in C_k \cap d_k$ and all sufficiently large m

$$|h(\eta)| \leq 4^{k-1} m^{-k/2^b} c \leq 1, \quad \text{where} \quad k = 2^m. \tag{25}$$

Since the domain D^* contains no zeros of h it follows from (24) and the minimum principle that the function $h(z)$ tends to infinity uniformly on D^*. It follows from (16) that (2) holds and further, for each $\lambda > 0$ no arcs on the level set $L(\lambda)$ can go out to the boundary C, so that the upper level domain $D^u(\lambda)$ is connected and is the only one component in itself. Hence the set $\gamma^u(\lambda) = C$ so that (3) holds which is due to $\gamma^l(\lambda) = \emptyset$. Since $(D^u(\lambda))^* = D$, (1) holds and hence $h \in \mathscr{L}_1$.

Finally, we shall prove that the inequality (1)′ is false. To do this, it is sufficient to show that the total length $l(\lambda)$ of $L(\lambda)$ is infinite for all $\lambda \geq 1$. For this, we let the lower level domain be $D^l(\lambda)$ and let $D_{jt}(\lambda)$ be the component of $D^l(\lambda)$ containing the zero z_{jt}, $t = 1, 2, \ldots, n^j$. Denote by $l_{jt}(\lambda)$ the length of the level curve $\partial D_{jt}(\lambda)$. Then this length is longer than the diameter of the curve. It follows from (16) and (25) that there is a positive integer m_0 such that for each $\lambda \geq 1$ and each $m \geq m_0$, the length

$$l_{kt}(\lambda) \geq 1/(mn_k), \quad k = 2^m \quad \text{and} \quad t = 1, 2, \ldots, n_k.$$

Since there are n_k zeros on the circle C_k, by summing up all of the lengths we get

$$\sum_{t=1}^{n_k} l_{kt}(\lambda) \geq 1/m, \quad m = m_0, m_0+1, \ldots.$$

This in turn implies that the total length

$$l(\lambda) \geq \sum_{m=m_0}^{\infty} \sum_{t=1}^{n_k} l_{kt}(\lambda) \geq \sum_{m=m_0}^{\infty} \frac{1}{m} = \infty.$$

Thus the inequality (1)′ is false. Hence Condition (1) cannot be replaced by (1)′ and the proof is complete.

6. Conformal invariance.

In this section, we shall come back to prove our extension of Fatou and Lindelöf's theorem which was used in the proof of Theorem 4. For this, we shall first extend Riesz theorem (see [5, Theorem 3.3]). Let $w = f(z)$ map D conformally onto a domain G bounded by a rectifiable Jordan curve ∂G. Then by Riesz theorem, we know that under the homeomorphism $w=f(e^{i\theta})$ of the frontiers induced by the mapping $w=f(z)$, a set of measure zero on C is mapped onto a set of measure zero on ∂G, and vice versa. For our purpose, we shall now extend the above Riesz theorem from a Jordan domain bounded by a rectifiable curve to a simply connected domain bounded by a set of curves whose total length is finite. To do this, we shall need to define the measure of a set on the boundary of the domain considered. For this, we first observe from Carathéodory's correspondence theorem (see [5, Theorem 9.4]) which says that under a conformal mapping $w=f(z)$ from D onto a simply connected domain G, there exists a one-to-one correspondence between points on C and prime ends on ∂G. Moreover, for each $e^{i\theta}\in C$, the cluster set $C(f, e^{i\theta})$ of $f(z)$ at $e^{i\theta}$ and the impression $I(P(e^{i\theta}))$ of the prime end $P(e^{i\theta})$ are the same, that is $C(f, e^{i\theta})=I(P(e^{i\theta}))$, where we refer the definitions of cluster set, prime end and impression to [5, p. 3, 168, and 170].

Theorem 7. *Let $w=f(z)$ map D conformally onto a simply connected domain G bounded by a set of rectifiable curves. Then for each $e^{i\theta}$, we have*

$$C(f, e^{i\theta})=I(P(e^{i\theta}))=f(e^{i\theta}),\quad \text{a single point.} \tag{26}$$

Moreover, let J be the set of (junction points) all points $w\in\partial G$ such that for each $w\in J$, there are at least two distinct prime ends $P(e^{i\theta_1})$ and $P(e^{i\theta_2})$ whose impressions equal w, that is,

$$I(P(e^{i\theta_1}))=I(P(e^{i\theta_2}))=w. \tag{27}$$

Then the set J is at most countable.

Proof. Suppose on the contrary that there is a point $e^{i\theta}\in C$ such that the impression $I(P(e^{i\theta}))$ is not a single point. Then it is a continuum so that in any neighborhood of $I(P(e^{i\theta}))$ there is a sequence of Jordan arcs $J_n\subset G$ such that the length

$$|J_n|\geq a>0,\quad \text{for some } a \text{ and}\quad n=1, 2,\dots.$$

This implies that the length $|\partial G|=\infty$, contradicting to the hypothesis. Hence (26) must be true.

It remains to prove that the set J is at most countable. To do this, we shall need the notion of plane triod T (see [5, p. 106]), which is the union of two arcs Γ_1 and Γ_2 whose intersection is a single point P which is simultaneously an endpoint of Γ_1 and an interior point of Γ_2. The point P is called the junction point of T. A theorem of Moore [14] asserts that any set of mutually disjoint triods in the plane is countable.

Now, from (27), it is easy to see that for any $w\in J$, there can be chosen three disjoint arcs γ_1, γ_2, and γ_3 from ∂G all of them meet at w. One of them, say γ_1 lies between γ_2 and γ_3. Let $\Gamma_1=\gamma_1$ and $\Gamma_2=\gamma_2\cup\gamma_3$, then $\{\Gamma_1, \Gamma_2\}$ forms a triod which corresponds to only one $w\in J$. The assertion now follows from Moore's theorem.

From the above Theorem 7, we can now define the measure of an arbitrary set B

on the boundary ∂G of a domain G bounded by a set of curves whose total length is finite. Note that the second part of Theorem 7 simply says that the set J of all junction points on ∂G is countable and therefore is of measure zero. The set B can be represented by

$$B=(B\cap J)\cup(B\cap\mathscr{C}J), \tag{28}$$

where $\mathscr{C}J$ is the complement of J. Clearly $\mathscr{C}J$ consists of countably many disjoint Jordan arcs Γ_n whose total length is finite. We say that B is a measurable subset of ∂G if and only if each intersection $B\cap\Gamma_n$ is measurable in Γ_n, $n=1, 2,\dots$, where the measurability of $B\cap\Gamma_n$ on Γ_n is well defined because each Γ_n is rectifiable. If B is measurable on ∂G then its measure

$$|B|=\sum_{n=1}^{\infty}|B\cap\Gamma_n|\leq\sum_{n=1}^{\infty}|\Gamma_n|<\infty. \tag{29}$$

With the help of the above definitions, we are now able to state and prove the following extension of Riesz theorem.

Theorem 8. *Let $w=f(z)$ maps D conformally onto a simply connected domain G defined in Theorem 7. Then under the one-to-one correspondence between points on C and prime ends on ∂G, a set of measure zero on C is mapped onto a set of measure zero on ∂G, and vice versa. Furthermore, the total length $|\partial G|$ satisfies*

$$|\partial G|\leq\int_0^{2\pi}|f'(e^{i\theta})|d\theta\leq 2|\partial G|. \tag{30}$$

Proof. Let E_z be a set of measure zero on C. Then by Theorem 7, the image E_w of E_z can be represented by

$$E_w=\{f(e^{i\theta}):\ e^{i\theta}\in E_z\}.$$

Let J and Γ_n be defined in (28) and (29). Then the measure of E_w can be written as

$$|E_w|=\sum_{n=1}^{\infty}|E_w\cap\Gamma_n|, \tag{31}$$

where each Γ_n is a Jordan arc containing no junction points. By joining the two endpoints of Γ_n by a Jordan arc k_n lying in G, we obtain a rectifiable Jordan domain $G_n\subset G$. Now, let $z=f^{-1}(w)$ be the inverse of f. Since Γ_n contains no junction points, it follows that the image $f^{-1}(\Gamma_n)$ is an arc on C. It is easy to see that k_n can be chosen so that the image $f^{-1}(k_n)$ is a rectifiable Jordan arc in D. Therefore the image $f^{-1}(G_n)$ is a rectifiable Jordan domain in D. It then follows from Riesz theorem that a set of measure zero on ∂G_n is mapped onto a set of measure zero on $\partial f^{-1}(G_n)$, and vice versa. Since

$$f^{-1}(E_w\cap\Gamma_n)\subset E_z \quad\text{and}\quad |E_z|=0,$$

we thus have the measure

$$|E_w\cap\Gamma_n|=0,\quad n=1, 2,\dots. \tag{32}$$

Substituting into (31), we obtain $|E_w|=0$.

Conversely, if $|E_w|=0$, then by (31) we obtain (32). By applying Riesz theorem again, we have

$$|f^{-1}(E_w \cap \Gamma_n)|=0, \quad n=1, 2, \ldots,$$

so that

$$|E_z|=\sum_{n=1}^{\infty} |f^{-1}(E_w \cap \Gamma_n)|=0.$$

This proves the first assertion.

Finally, we shall prove (30). To do this, we divide the boundary ∂G into inner and outer boundary, that is

$$\partial G=\partial G_i \cup \partial G_0 \quad \text{and} \quad |\partial G|=|\partial G_i|+|\partial G_0|. \tag{33}$$

It is easy to see that for any open arc $A\subset\partial G_i$, there correspond two disjoint open arcs A_1 and A_2 on C such that the images $f(A_1)=f(A_2)=A$. From this, we find that the integration in (30) contains

$$\int_{A_1} |f'(e^{i\theta})|d\theta+\int_{A_2} |f'(e^{i\theta})|d\theta=2|A|.$$

This together with (33) implies (30), namely

$$|\partial G|\leq\int_0^{2\pi} |f'(e^{i\theta})|d\theta=2|\partial G_i|+|\partial G_0|\leq 2|\partial G|.$$

With the help of Theorem 8, we are now able to extend the Fatou and Lindelöf theorem (see [5, Theorems 2.1 and 2.2]). For this, we let G be a domain and let $f(w)$ be a function holomorphic in G. We say that a point $p\in\partial G$ is a Fatou point of f if for each $\varepsilon>0$ there is a Stolz angle $\Delta(p)\subset G$ with one vertex at p whose subtending angle is greater than $\pi-\varepsilon$ such that the function $f(w)$ tends to a value when w tends to p and $w\in\Delta(p)$.

Theorem 9. *Let G be a simply connected domain as in Theorem 7 and let $f(w)$ be a function bounded and holomorphic in G. Then almost every point on ∂G is a Fatou point of f.*

Proof. Let $w=\psi(z)$ be a conformal mapping from D onto G and let $F(z)=f(\psi(z))$. Then F is a function bounded and holomorphic in D. It follows from Fatou and Lindelöf's theorem that there is a set $S\subset C$ such that the measure $|S|=2\pi$ and each point on S is a Fatou point of F.

According to a theorem of Beurling (see [5, Theorem 3.5]), there is a set $T\subset C$ such that the measure $|T|=2\pi$ and the derivative $\psi'(z)\neq 0$ for each $z\in T$. Using the same argument as ours [7, p. 454], it is easy to see that for each $z_0\in T$ the mapping ψ is conformal at z_0 from the interior of C, so that for each $\varepsilon>0$ there is a Stolz angle $\Delta(w_0)\subset G$ whose subtending angle is greater than $\pi-\varepsilon$, where $w_0=\psi(z_0)$.

Now, let $U=S\cap T$, then clearly the measure $|U|=2\pi$. Let $z_0\in U$, then z_0 is a Fatou point of F and the mapping ψ preserves angles at z_0. Hence the function f tends to a value when w tends to w_0 and $w \in \Delta(w_0)$. This shows that w_0 is a Fatou point of f on ∂G. Since the complement $C-U$ is of measure zero on C, it follows from Theorem 8 that the image $\psi(C-U)$ is of measure zero on ∂G, so that almost every point on ∂G is a Fatou point of f. This completes the proof.

7. Annular functions.

In this last section, we shall study a connection between annular functions and MacLane's class, and then prove that Condition (1) is necessary in Theorem 1. As usual, a function $f(z)$ holomorphic in D is said to be annular if there is sequence $\{J_n\}$ of disjoint Jordan curves about 0, converging out to C, such that

$$\lim_{n\to\infty} \min_{z \in J_n} |f(z)| = \infty. \tag{34}$$

For instance, the function defined in Theorem 6 is annular.

In [16, Theorem 1], Sons and Campbell answered our question [8, p. 188] by showing that any gap series

$$f(z)=\sum_{k=0}^{\infty} c_k z_k^{n_k}, \quad z_{k+1}/n_k \geq q > 1$$

is normal if and only if $\limsup |c_k| < \infty$. They then asked whether $\limsup |c_k| = \infty$ implies f is annular. We recently answered this question in the affirmative sense (see [9] or [10]). Meanwhile, T. Murai [15, Theorem 1] proved that any gap series belongs to MacLane's class. Thus any gap series with $\limsup |c_k| = \infty$ is both annular and in MacLane's class. This leads to ask the question as to whether any annular function belongs to MacLane's class. The answer turns out to be no due to Bagemihl and Erdös [2, Theorem 3]. In this connection, it is worth to study the condition under which an annular function cannot be in MacLane's class. This gives the motivation to prove the following result which will be needed in the last theorem.

Theorem 10. *If $f(z)$ is annular and omits a value in a relative neighborhood of a point on C, then $f \notin \mathscr{L}$.*

Note that annular functions of this kind do exist due to Barth and Schneider [4].

Proof. To prove the assertion, we suppose on the contrary that $f \in \mathscr{L}$. Let $f(z)$ omit the value v and let $g(z)=f(z)-v$. Then by Theorem M we see that $g \in \mathscr{A}$. Clearly, g is also annular, and has no zeros in a relative neighborhood of a point on C. This contradicts a theorem of Bagemihl and Erdös [2, Theorem 1].

The function described in Theorem 10 allows us to prove the necessity of Condition (2) in Theorem 1.

Theorem 11. *If $f(z)$ is annular and $f \notin \mathscr{L}$, then f satisfies Condition* (3) *but not* (2).

Proof. Let $\lambda>0$ be given, then by (34) there is an integer $I(\lambda)$ such that

$$\min_{z\in J_n}|f(z)|>\lambda, \quad \text{for each} \quad n\geq I(\lambda).$$

It follows that no components of the lower level domain $D^l(\lambda)$ can tend to C meanwhile the upper level domain $D^u(\lambda)$ is itself a component. By definition, this implies that

$$\sum|\gamma_m^l(\lambda)|=0 \quad \text{and} \quad \sum|\gamma_n^u(\lambda)|=|\gamma^u(\lambda)|=2\pi,$$

so that Condition (3) holds. Since the function $f\notin \mathscr{L}$, Condition (2) is no longer true due to Theorem 1. This completes the proof.

Note that the above theorem says that any annular function not in the MacLane class will automatically satisfy (3) but not (2).

In closing this paper, let us pose the following problem related to the converse of Theorem 10. What is the necessary and sufficient condition that an annular function belong to MacLane's class.

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References

[1] F. Bagemihl, P. Erdös, and W. Seidel, Sur quelques propriétés Frontières des fonctions holomorphes définies par certains produits dans le cercle-unité, Ann. Scient. École Norm. Sup. (3), **70** (1953), 135–149.

[2] F. Bagemihl and P. Erdös, A problem concerning the zeros of a certain kind of holomorphic function in the unit disk, J. Reine Angew. Math., **214** (1964), 340–344.

[3] F. Bagemihl and W. Seidel, Koebe arcs and Fatou points of normal functions, Comment. Math. Helv., **36** (1961), 9–18.

[4] K. Barth and W. Schneider, On a problem of Bagemihl and Erdös concerning the distribution of zeros of an annular function, J. Reine Angew. Math., **234** (1969), 179–183.

[5] E. F. Collingwood and A. J. Lohwater, The Theory of Cluster Sets, Cambridge Univ. Press, 1966.

[6] S. Dragosh, The distribution of Fatou points of bounded and normal analytic functions, Ann. Acad. Scien. Fenn. Ser. AI, **496** (1971), 1–11.

[7] J. S. Hwang, On the recurrence property of Gaussian Taylor series, Ann. of Probability, **4** (1976), 453–455.

[8] J. S. Hwang, Plessner points, Julia points, and ρ^*-points, J. Math. Kyoto Univ., **18** (1978), 173–188.

[9] J. S. Hwang, On Hadamard gap series and errata to Plessner, Julia, and ρ^*-points, J. Math. Kyoto Univ., **26** (1986), 213–221.

[10] J. S. Hwang and D. M. Campbell, Annular functions and gap series, London Math. Soc., **14** (1982), 415–418.

[11] J. S. Hwang and Peter Lappan, Analytic functions with large sets of Fatou points, Proc. Amer. Math. Soc., **90** (1984), 293–298.

[12] O. Lehto and K. I. Virtanen, Boundary behaviour and normal meromorphic functions, Acta Math., **97** (1957), 47–65.

[13] G. R. MacLane, Asymptotic values of holomorphic functions, Rice Univ. Studies **49** (1963), 83 pp.

[14] R. L. Moore, Concerning triods in the plane and the junction points of plane continua, Proc. Nat. Acad. Sci. U.S.A., **14** (1928), 85–88.

[15] T. Murai, The boundary behaviour of Hadamard lacunary series, Nagoya Math. J., **89** (1983), 65–76.

[16] L. R. Sons and D. M. Campbell, Hadamard gap series and normal functions, Bull. London Math. Soc., **12** (1980), 115–118.