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On two theorems concerning reductions in local rings

By

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1. The purpose of this paper is twofold. First we wish to generalize a recent theorem of Rees [7, Theorem 1.3] on the existence of so-called complete reductions of a finite set of ideals in a Noetherian local ring. Secondly we wish to give a proof of the basic theorem of Eakin and Sathaye [1, Theorem 1] on the existence of reductions which avoids the adding on of an infinite set of indeterminates and the resulting action of a permutation group. (All rings considered here are commutative with an identity element.)

Reductions of ideals were introduced by Northcott and Rees [5] and have proved extremely useful in local algebra. In fact reductions have a strong geometrical content; the connections between reductions of an ideal and the homogeneous affine coordinate ring of the fibre over the closed point, in the blow-up of the ring along the given ideal, have been spelled out explicitly in [3, 6], for example. Indeed in this paper it is this ring which plays a crucial rôle in our proofs. Finally we remark that in the proof of his theorem Rees used a multi-graded Rees ring; in our generalization we use a general position result in a singly-graded ring, together with a result from Nagata [4] on superficial elements in such rings.

2. We collect together some properties of reductions which will be used without comment in the rest of the paper.

Let (R, m, k) be a quasi-local ring with infinite residue field k (this is usually a trivial restriction on k; see [8, p. 10], for example). Let I and J be finitely generated ideals in R with $J \subseteq I$. Then J is called a *reduction* [5] of I if $JI^n = I^{n+1}$ for some positive integer n.

Consider the associated homogeneous affine ring G over k, where

$$G = k \oplus I/mI \oplus I^2/mI^2 \oplus \cdots .$$

Then J is a reduction of I if and only if the images in I/mI of the given finite set of generators of J themselves generate a homogeneous affine subalgebra of G over which G is integral, and conversely every reduction of I arises in this way; further, the affine dimension of G is called the *analytic spread* $l(I)$ of I, and if R is Noetherian we have Böger's inequality:

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$$l(I) \leq \dim R. \tag{1}$$

(See [6], for example, for a discussion of these facts.) Thus $l(I)=0$ if and only if I is nilpotent.

The ring G is the coordinate ring of the fibre referred to in §1, and reductions can be thought of either algebraically or geometrically. As an interesting example of this, one can contrast the proofs of [7, Lemma 1.1] and [3, Proposition (5.5)].

Finally we recall from [5, Theorem 2] that the analytic spread $l(I)$ of I is the smallest number of elements required to generate a reduction of I. (This also follows immediately from the properties of G above.)

3. We now give our generalization of Rees' theorem [7, Theorem 1.3].

Theorem 1. *Let (R, m, k) be a Noetherian local ring such that k is infinite, and let $I_1, \ldots, I_s$ be ideals of R. Let $r=l(I)$, the analytic spread of I, where $I=I_1 \cdots I_s$. Then there exist elements x_{ij} $(j=1,\ldots, r)$ of I_i for $i=1,\ldots, s$ such that, if $y_j=x_{1j}x_{2j}\cdots x_{sj}$ $(j=1,\ldots, r)$, then $J=(y_1,\ldots, y_r)$ is a reduction of I.*

Proof. The proof will be by induction on r, the result being trivial if $r=0$, since the zero ideal is then a reduction of I.

As in [7], it suffices to consider the case where I contains a non-zero-divisor, by the usual device of factoring out

$$\begin{aligned} H_I^0(R) &= \cup \{0: I^n \mid n \geq 1\} \\ &= 0: I^q, \quad \text{for some} \quad q \geq 1. \end{aligned}$$

The details are as follows: set $K=0: I^q$ and pass to $\bar{R}=R/K$, using '$^-$' to denote images in $\bar{R}$. If $\bar{I}=0$, then I is nilpotent and $r=0$ in this case; this is the trivial situation dealt with above. If $\bar{I}\neq 0$, then $0:_{\bar{R}}\bar{I}=0$ so $\bar{I}$ contains a non-zero-divisor. It is clear from the discussion at the end of §2, since reductions are preserved by homomorphisms, that if $\bar{r}=l(\bar{I})$ then $\bar{r}\leq r$. Hence even if $\bar{r}=r$ and we suppose the result known when the ideal contains a non-zero-divisor, or if $\bar{r}<r$ and we use induction, we can find elements x_{ij} $(j=1,\ldots, \bar{r})$ of I_i for $i=1,\ldots, s$ such that, if $y_j=x_{1j}\cdots x_{sj}$, then for some positive integer n

$$I^{n+1}=(y_1,\ldots, y_{\bar{r}})I^n+(K \cap I^{n+1}).$$

Therefore $I^{n+q+1}=(y_1,\ldots, y_{\bar{r}})I^{n+q}$, which clearly gives the required result.

So suppose that $r\geq 1$ and that I contains a non-zero-divisor. Consider the following graded rings:

$$A=\bigoplus_{t\geq 0} I^t/I^{t+1} \quad \text{and} \quad G=\bigoplus_{t\geq 0} I^t/mI^t,$$

writing $A=\bigoplus_{t\geq 0} A_t$ and $G=\bigoplus_{t\geq 0} G_t$ in an obvious notation. Then G has (affine) dimension r. We now need a result of 'general position' type, which we introduce first in the following guise:

Lemma 2. *Let $V_1,\ldots, V_s$ be finite-dimensional vector spaces over the infinite field k, and let $H_1,\ldots, H_m$ be proper subspaces of the tensor product space V, where $V=V_1\otimes\cdots\otimes V_s$. Then there exist $v_j\in V_j$, $1\le j\le s$, such that $v_1\otimes\cdots\otimes v_s\notin H_1\cup\cdots\cup H_m$.*

Remarks. This result (which in fact holds even when the H_i are affine subspaces) is left as a little exercise in linear algebra. It can be given a variety of proofs[†], especially for particular instances of the base field. One such case, of interest in the present context, is where k is algebraically closed. Then the set of such $v_1\otimes\cdots\otimes v_s$ is the affine cone over the Segre variety in V, which is known to be an irreducible variety that is not contained in any linear subspace (indeed in any affine subspace) of V [2, Chapter XI, §2]. The result in this case follows easily from these facts.

We use this lemma in the following form, employing the previous notation:

Proposition 3. *Let $W_1,\ldots, W_m$ be proper subspaces of the k-vector space I/mI. Then there exist $x_i\in I_i$, $1\le i\le s$, such that*

$$y+mI\notin W_1\cup\cdots\cup W_m,$$

where $y=x_1\cdots x_s$.

Proof. Now I/mI is a k-linear homomorphic image of the tensor product space $V:=I_1/mI_1\otimes\cdots\otimes I_s/mI_s$ (over k) with $y+mI$ having $(x_1+mI_1)\otimes\cdots\otimes(x_s+mI_s)$ as a pre-image. If we assume the proposition to be false and pull back to V, we immediately contradict the 'general position' lemma.

We now return to the proof of the theorem. Let Ass $R=\{P_1,\ldots, P_\mu\}$, let $\{Q_1,\ldots, Q_\nu\}$ be the set of (homogeneous) associated primes of A which do not contain A_1, and let $J_1,\ldots, J_\omega$ be the (homogeneous) primes in G of co-rank r. Set $m=\mu+\nu+\omega$, and set

$$W_i=(P_i+mI)/mI, \qquad 1\le i\le\mu;$$

$$W_{\mu+i}=(Q_i\cap A_1+mA_1)/mA_1, \qquad 1\le i\le\nu;$$

$$\text{and} \quad W_{\mu+\nu+i}=J_i\cap G_1, \qquad 1\le i\le\omega.$$

Since I contains a non-zero-divisor, W_i is a proper k-subspace of G_1 for $i=1,\ldots,\mu$, by Nakayama's lemma; similarly, for $i=1,\ldots,\nu$, $W_{\mu+i}$ is a proper k-subspace of G_1; since $r>0$, it is clear that each $W_{\mu+\nu+i}$, $1\le i\le\omega$, is a proper k-subspace of G_1.

Therefore by the preceding proposition, by the definition of the W_i, and by the proof of [4, (22.1)], there exist $x_i\in I_i$ for $i=1,\ldots, s$, such that, setting $y=x_1\cdots x_s$,

(i) y is a non-zero-divisor in R;

(ii) $y+I^2$ is a superficial element [4, p. 71] of A in the sense that there exists a positive integer c with

† Thanks to Tom Lenagan, Terry Lyons and Allan Sinclair for energetically supplying just such a varied lot of proofs.

$$(I^n : yR) \cap I^c = I^{n-1}, \quad \text{when} \quad n > c;$$

and

(iii) $(y + mI)G$ is of co-rank strictly less than r in G.

Moreover, recall that by an Artin-Rees result [4, (3.12)] there exists a positive integer d such that

(iv) $I^n : yR \subseteq I^{n-d}$, when $n > d$, since $0 : yR = 0$ by (i) above.

By (ii) and (iv) therefore, when $n \geq c + d$,

$$I^n : yR = I^{n-1},$$

so in this case,

(v) $I^n \cap yR = yI^{n-1} \quad (n \geq c + d)$.

Now, consider the ring (R', m', k), where $R' = R/yR$ and $m' = m/yR$; further, for $i = 1, \ldots, s$, consider the ideal I'_i in R' where each $I'_i = I_i/yR$; let $r' = l(I')$, the analytic spread of I', where $I' = I'_1 \cdots I'_s$ so that $I' = I/yR$. Now the corresponding homogeneous affine ring G', where $G' = \bigoplus_{t \geq 0} I'^t/m'I'^t$, is easily seen (because of (v)) to coincide in the n^{th} graded piece with $G/\bar{y}G$ for all sufficiently large n, where $\bar{y} = y + mI$. Hence, using (iii) above, we have that

(vi) $r' = \dim G' = \dim G/\bar{y}G < r$.

By induction therefore, there exist elements x_{ij}, $j = 1, \ldots, r'$, of I_i for $i = 1, \ldots, s$ such that, setting $y_j = x_{1j} \cdots x_{sj}$ for $j = 1, \ldots, r'$, for sufficiently large n

$$I^n = (y_1, \ldots, y_{r'})I^{n-1} + (yR \cap I^n).$$

The result now follows easily from (v) and (vi).

Remarks 1. By Böger's inequality (see (1) in §2 above), $r \leq \dim R$, so the theorem above does generalize (in an optimal way) [7, Theorem 1.3].

2. Clearly the above theorem can be used to give slight variations of the remaining results in [7, §1]. The reader is referred to [7] for applications of these so-called complete reductions.

4. The theorem by Eakin and Sathaye and applications of it are given in [1]; the proof is essentially repeated by Sally [8] who remarks [*loc. cit.*, p. 38] that the ideal in question may be taken to be finitely generated, without loss of generality. This we now do, and in fact prove the result in the following mildly generalized form:

Theorem 4. *Let $G = \bigoplus_{t \geq 0} G_t$ be a homogeneous affine algebra, with $G_0 = k$ an infinite field, which is generated as a k-algebra by G_1. Suppose that there exist integers n and r, with $n \geq 1$ and $r \geq 0$, such that $\dim_k G_n < \binom{n+r}{r}$. Then there exist generic linear forms $y_1, \ldots, y_r$ in G_1 such that*

$$G_n = y_1 G_{n-1} + \cdots + y_r G_{n-1}.$$

Remarks 1. The theorem has an obvious trivial interpretation when $r = 0$.

2. Suppose that we fix a k-basis $x_1, \ldots, x_p$ of G_1. Thus if G were given by (1)

in §2, $x_1, \ldots, x_p$ would be the images in I/mI of a given finite generating set for I; so our notation is chosen so as to tie in with that used in [1] in an obvious way. By the phrase 'generic linear forms $y_1, \ldots, y_r$ in G_1' we mean a set of r linear forms

$$y_j = \sum_{i=1}^{p} \alpha_{ij} x_i \qquad \text{for} \quad j = 1, \ldots, r,$$

given by a point $\{\alpha_{ij}\} \in k^{rp}$ which moves in a Zariski-open subset of k^{rp} and moreover, in the statement of the conclusion of the theorem, the property that G_n equals $y_1 G_{n-1} + \cdots + y_r G_{n-1}$ is to hold for each of the sets of r linear forms given by the various points in the fixed Zariski-open subset of k^{rp}.

3. The equality $G_n = y_1 G_{n-1} + \cdots + y_r G_{n-1}$ is equivalent to the property that G be a finite module (i.e. be integral) over the homogeneous affine subalgebra $k[y_1, \ldots, y_r]$ with the homogeneous module generators having degrees at most $n-1$ in G. Thus Theorem 4 can be viewed as a normalization theorem in which the degrees of the generators are controlled at the price of the homogeneous system of integrity $\{y_1, \ldots, y_r\}$ possibly no longer being algebraically independent over k. (The y_i are algebraically independent over k if and only if r is the affine dimension of G.)

4. Suppose that we fix positive integers s and m and consider varying sets of s linear forms $\{z_1, \ldots, z_s\}$ in G_1, where $z_j = \sum_{i=1}^{p} \beta_{ij} x_i$ for $j = 1, \ldots, s$, as $\{\beta_{ij}\} \in k^{sp}$ moves in its ambient space. If $u = \dim_k G_{m-1}$ and $v = \dim_k G_m$ we can consider $z_1 G_{m-1} + \cdots + z_s G_{m-1}$ as the row-space of an $(su) \times v$-matrix over k, in an obvious way. The condition that $z_1 G_{m-1} + \cdots + z_s G_{m-1}$ is a *proper* subspace of G_m is given by the vanishing of the $v \times v$-minors of this matrix. Hence

if there exists a particular s-tuple $y_1^0, \ldots, y_s^0$ in G_1 such that $G_m = y_1^0 G_{m-1} + \cdots + y_s^0 G_{m-1}$, then there exists a generic set of elements $y_1, \ldots, y_s$ in G_1 such that $G_m = y_1 G_{m-1} + \cdots + y_s G_{m-1}$. (2)

Thus, in the context of Theorem 4, if suffices to concentrate (in the main) on producing a single suitable r-tuple of linear forms in G_1.

We now turn to the proof of Theorem 4, where, as is clear, we use many of the central ideas of the original proof by Eakin and Sathaye.

Proof of Theorem 4. The result is trivial if $r=0$, and it is clear from (2) above that the result is immediate if $n=1$. Thus we take $n \geq 2$ and $r \geq 1$. Now suppose that the result is false and consider a counterexample $G = \bigoplus_{t \geq 0} G_t$ in which r is minimal, and n is minimal for this given value of r.

As before, let $\{x_1, \ldots, x_p\}$ be a k-basis of G_1.

Case 1. *There exists i, with $1 \leq i \leq p$, such that*

$$\dim_k x_i G_{n-1} \geq \binom{n+r-1}{r}$$

For convenience, let $i=1$. We claim that $r \geq 2$. For if $r=1$, then

$$n+1>\dim_k G_n \geq \dim_k x_1 G_{n-1} \geq n,$$

so $G_n = x_1 G_{n-1}$. It then easily follows from (2) above that the theorem is true for G, which is a contradiction. Since x_1 is a homogeneous element, we can pass to the homogeneous factor ring $\bar{G} = G/x_1 G$, where we write $\bar{G} = \bigoplus_{t \geq 0} \bar{G}_t$. Since

$$\binom{n+r-1}{r-1} = \binom{n+r}{r} - \binom{n+r-1}{r}$$

it follows that $\dim_k \bar{G}_n < \binom{n+r-1}{r-1}$. By the minimality of r, there exist $\omega_1, \ldots, \omega_{r-1}$ in G_1 such that

$$\bar{G}_n = \bar{\omega}_1 \bar{G}_{n-1} + \cdots + \bar{\omega}_{r-1} \bar{G}_{n-1},$$

in an obvious notation. Hence

$$G_n = \omega_1 G_{n-1} + \cdots + \omega_{r-1} G_{n-1} + x_1 G_{n-1},$$

and a contradiction again follows from (2) above.

The alternative is

Case 2. *For* $i=1, \ldots, p$, $\dim_k x_i G_{n-1} < \binom{n+r-1}{r}$

For $i=1, \ldots, p$, let $K^{(i)} = \operatorname{Ann}_G x_i$, a homogeneous ideal, and let $G^{(i)} = G/K^{(i)}$. Then for each i we write $K^{(i)} = \bigoplus_{t \geq 0} K_t^{(i)}$ and $G^{(i)} = \bigoplus_{t \geq 0} G_t^{(i)}$, using the natural grading. Now for each i we have a degree 1 isomorphism $x_i G_{n-1} \approx G_{n-1}^{(i)}$ induced by multiplication by x_i, so $\dim_k G_{n-1}^{(i)} < \binom{n-1+r}{r}$. By the minimality of n for the given r, there exists for each i a non-empty Zariski-open subset of k^{rp} yielding a set of r linear forms $z_{1i}, \ldots, z_{ri}$ (say) in G_1 such that

$$G_{n-1}^{(i)} = \bar{z}_{1i} G_{n-2}^{(i)} + \cdots + \bar{z}_{ri} G_{n-2}^{(i)},$$

in an obvious notation. Then, for each i,

$$G_{n-1} = z_{1i} G_{n-2} + \cdots + z_{ri} G_{n-2} + K_{n-1}^{(i)},$$

so
$$x_i G_{n-1} \subseteq z_{1i} G_{n-1} + \cdots + z_{ri} G_{n-1}.$$

Intersecting the p Zariski-open subsets yields a non-empty Zariski-open subset of k^{rp}, independent of i, such that for the corresponding generic set of linear forms $z_1, \ldots, z_r$,

$$x_i G_{n-1} \subseteq z_1 G_{n-1} + \cdots + z_r G_{n-1}, \qquad \text{for} \quad i=1, \ldots, p.$$

Hence

$$G_n = x_1 G_{n-1} + \cdots + x_p G_{n-1}$$
$$\subseteq z_1 G_{n-1} + \cdots + z_r G_{n-1} \subseteq G_n,$$

and we obtain a contradiction.

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