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Stationary solutions and their stability for Kimura's diffusion model with intergroup selection

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Introduction

In 1983, M. Kimura proposed a diffusion model of intergroup selection in population genetics, with special regards on an altruistic allele (see [4] and [5]). After that, one of the present authors proved the existence and uniqueness of the solutions (see [8]). In this paper, we study the global behavior of the solution; the existence and the number of stationary solutions as well as their stability.

The equation is a kind of initial boundary value problem for a parabolic partial differential equation of mean field type. The unknown $U(t, dx)$ is a probability measure on the interval $0 \leq x \leq 1$ depending on time (generation) t, where the space variable x denotes the frequency of the altruistic allele. The coefficients of the equation depend on the first moment, and also on five important parameters v', v, s, m and c representing the rates of mutations in both directions, individual selection, migration and intergroup competition respectively (another parameter N, the population size of the groups, is of less importance for our analysis).

The plan of this article is as follows. In §1, we state our main results and explain briefly the model in population genetics. Our Theorem 1 is basically a reproduction of the former result in [8], while Theorem 2 gives the precise number of the stationary solutions. In Theorem 3, we discuss the convergence of the solution $U(t, dx)$ as $t \to \infty$ to a stationary solution. Theorem 4 states how the parameters in the coefficients affect this model, and in Theorem 5, we verify one of the main results of M. Kimura in [4] and [5]. The proof of the Theorems are given in the next consecutive §§2, 3 and 4. §§A and B are devoted to the review of the theory of continued fractions and to the study of a certain function $F(y)$ which is represented by continued fraction.

As in [8], we make a systematic use of the moment sequences of the solutions. Those of the stationary solutions satisfy a recurrence equation of the three consecutive terms, so that we can make use of continued fractions (see §A). It turns out by

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Lemma 1.1 that the stationary solutions are in one to one correspondence with the fixed points of the auxiliary function $F(y)$. The study of this function $F(y)$ in §§B and 2 enables us to enumerate all the stationary solutions (Theorem 2). At the same time, we can give the explicit formulas for the stationary solutions (see Lemma 2.1).

We need more delicate arguments for the study of the behavior of $U(t, dx)$ as $t\to\infty$. The crucial idea of the proof is a comparison lemma on moment sequences of the solutions of elliptic equations of mean field type, which are derived by a difference analogue of the parabolic equation (Lemma 1.2). In the cases where there are two stationary solutions, we study some properties of the principal eigenvalues and the corresponding eigenfunctions for our elliptic equations, and exploit them to prove the stability of stationary solutions.

Finally we note that T. Shiga proved the existence and uniqueness for a class of non-linear diffusion equations including the present equations as well as the multi-dimensional ones (see [7]).

The authors of the present paper would like to say that the relevant diffusion model of M. Kimura has a quite beautiful and interesting structure from the view-points of the theory of partial differential equations and of the probability theory. The authors are indebted to this structure for their analysis of the model.

§1. Problems and results.

Problem (K) or the diffusion model of Kimura. *Let N, v', v, s, m and c be given real numbers satisfying*

$$N>0, \quad v'\geq 0, \quad v\geq 0, \quad s>0, \quad m>0 \quad and \quad 0\leq c\leq v+m. \tag{1}$$

Let also $\Phi(dx)$ be a given probability measure on $[0, 1]$. *A measure valued function $U(t, dx)$ defined on the half line $0\leq t<+\infty$ is called a solution for the diffusion model of Kimura, if it satisfies the following conditions* (K.1), (K.2) *and* (K.3):

(K.1) *for each $t\in[0, +\infty)$, $U(t, dx)$ is a probability measure on the closed interval $0\leq x\leq 1$;*

(K.2) *for any continuous function $f(x)$ on* $[0, 1]$,

$$\lim_{t\to 0}\int_{0}^{1+} f(x)U(t, dx)=\int_{0-}^{1+} f(x)\Phi(dx);$$

(K.3) *for any function $f(x)$ of class C^2 on* $[0, 1]$,

$$\begin{aligned}\frac{d}{dt}\int_{0-}^{1+} f(x)U(t, dx)=\int_{0-}^{1+}\Big[&\frac{x(1-x)}{4N}f''(x)+\{v'(1-x)-vx-sx(1-x)\\ &+m(\bar{x}(t)-x)\}f'(x)+c(x-\bar{x}(t))f(x)\Big]U(t, dx), \quad t\in(0, +\infty),\end{aligned} \tag{2}$$

$$where \quad \bar{x}(t)=\int_{0-}^{1+} xU(t, dx).$$

If $\Phi(dx)$ and $U(t, dx)$ have density functions $\phi(x)$ and $u(t, x)$ respectively (that is, $\Phi(dx)=\phi(x)dx$ and $U(t, dx)=u(t, x)dx$), then the problem (K) is reduced to an initial boundary value problem for a partial differential equation of parabolic type (see M. Kimura [4], [5], and N. Shimakura [8]).

For a solution $U(t, dx)$ of the problem (K), we define the moment sequence $\{M_n(t)\}_{n=0}^{\infty}$ of $U(t, dx)$ by

$$M_n(t)=\int_{0-}^{1+} x^n U(t, dx) \qquad \text{for} \quad t\geq 0 \quad \text{and} \quad n=0, 1, 2,\dots \tag{3}$$

Moment equation (M). *A sequence of real numbers* $\{M_n(t)\}_{n=0}^{\infty}$ *defined on* $t\in[0, +\infty)$ *is called a solution of the moment equation for the diffusion model of Kimura, if it satisfies the following conditions* (M.1), (M.2) *and* (M.3):

(M.1) *for each* $t\in[0, +\infty)$, $M_0(t)=1$ *and* $\{M_n(t)\}_{n=0}^{\infty}$ *is completely monotone, that is*

$$\sum_{p=0}^{n}(-1)^p\binom{n}{p}M_{k+p}(t)\geq 0, \qquad \text{for} \quad k, n=0, 1, 2,\dots ;$$

(M.2) $$\lim_{t\downarrow 0} M_n(t)=\int_{0-}^{1+} x^n\Phi(dx) \qquad \text{for } n=0, 1, 2,\dots ;$$

(M.3) *it satisfies the system of differential equations*

$$\frac{d}{dt}M_n(t)=(ns+c)M_{n+1}(t)+n\left(\frac{n-1}{4N}+v'+mM_1(t)\right)M_{n-1}(t)$$

$$-\left\{n\left(\frac{n-1}{4N}+v'+v+s+m\right)+cM_1(t)\right\}M_n(t), \tag{4}$$

for $t\in(0, +\infty)$ *and* $n=1, 2, 3,\dots$

If $U(t, dx)$ is a solution of the problem (K), then its moment sequence $\{M_n(t)\}_{n=0}^{\infty}$ is a solution of the problem (M) (with $\bar{x}(t)=M_1(t)$) and vice versa. Therefore, the two problems (K) and (M) are equivalent (see Chapter III of D. V. Widder [10] for the general theory on moment sequences).

The following theorem is proved in [8] in the case where $\Phi(dx)$ has a density function. The proof for the general case will be given at the end of this section.

Theorem 1 (*Existence and uniqueness of the solution*). *Assume that the parameters* (N, v', v, s, m, c) *satisfy the condition* (1) *except for the restriction on* c. *Then, for each probability measure* $\Phi(dx)$ *on* $[0, 1]$, *there exists one and only one solution* $U(t, dx)$ *of the problem* (K).

In this memoir, we shall study the stationary solutions for the model of Kimura and also investigate the convergence of the solution of the problem (K) to a stationary solution as time t tends to infinity.

Stationary problem (SK). *A measure* $U(dx)$ *is called a stationary solution for the model of Kimura, if it satisfies the following conditions* (SK.1) *and* (SK.2):

(SK.1) $U(dx)$ *is a probability measure on* $[0, 1]$;

(SK.2) *for any function* $f(x)$ *of class* C^2 *on* $[0, 1]$,

(5) $$\int_{0-}^{1+}\Big[\frac{x(1-x)}{4N}f''(x)+\{v'(1-x)-vx-sx(1-x)+m(\bar{x}-x)\}f'(x)$$
$$+c(x-\bar{x})f(x)\Big]U(dx)=0, \quad \textit{where} \quad \bar{x}=\int_{0-}^{1+}xU(dx).$$

For a stationary solution $U(dx)$, we denote its moment sequence by $\{M_n\}_{n=0}^{\infty}$. Then $\{M_n\}_{n=0}^{\infty}$ is a solution of the following problem (SM), and the two problems (SK) and (SM) are equivalent.

Stationary moment equation (SM). *A sequence of real numbers* $\{M_n\}_{n=0}^{\infty}$ *is called a solution of the stationary moment equation for the diffusion model of Kimura, if it satisfies the following conditions* (SM.1) *and* (SM.2):
(SM.1) $\{M_n\}_{n=0}^{\infty}$ *is completely monotone with* $M_0=1$;
(SM.2) *it satisfies the recurrence equation*

(6) $$(ns+c)M_{n+1}=\Big\{n\Big(\frac{n-1}{4N}+v'+v+s+m\Big)+cM_1\Big\}M_n$$
$$-n\Big(\frac{n-1}{4N}+v'+mM_1\Big)M_{n-1} \quad \textit{for} \quad n=1, 2, 3, .$$

We first note that two unit distributions can be stationary solutions. Let δ_0 (δ_1) be the probability measure defined by

$$\int_{0-}^{1+}f(x)\delta_0(dx)=f(0) \quad (\text{resp.} \quad \int_{0-}^{1+}f(x)\delta_1(dx)=f(1))$$

for all continuous functions $f(x)$ on $[0, 1]$. Then it follows that

(7) $$\delta_0 \textit{ is a stationary solution if and only if } v'=0,$$
$$\delta_1 \textit{ is a stationary solution if and only if } v=0.$$

Any other stationary solution $U(dx)$ has a density function $u(x)$ (that is, $U(dx)=u(x)dx$), which is obtained as a solution of a boundary value problem for an ordinary differential equation (see §2 below). We call the density $u(x)$ as well as the measure $U(dx)=u(x)dx$ itself a *stationary* L^1*-solution.*

Let us now enumerate all the stationary solutions. For this purpose we should divide the problem into seven cases:

Case (a) $v'>0$, $v>0$;

Case (b) $v'=0$, $v>0$ *and* $L_0\leq 0$, *where*

(8) $L_0=L_0(N, v, s, m, c)$

$$=\frac{m}{v+m}\,{}_1F_1\Big(\frac{c}{s}+1, 4N(v+m)+1; -4Ns\Big)-{}_1F_1\Big(\frac{c}{s}, 4N(v+m); -4Ns\Big)$$

and ${}_1F_1$ *stands for the hypergeometric series of Kummer* (*see* (11) *of* §B *below*);

Case (c) $v'>0$, $v=0$ *and* $L_1\leq 0$, *where*

$$L_1 = L_1(N, v', s, m, c) \tag{9}$$
$$= \frac{m}{v'+m}\, {}_1F_1\Big(\frac{c}{s}+1,\ 4N(v'+m)+1;\ 4Ns\Big) - {}_1F_1\Big(\frac{c}{s},\ 4N(v'+m);\ 4Ns\Big);$$

Case (d) $v'=0$, $v>0$ *and* $L_0>0$;

Case (e) $v'>0$, $v=0$ *and* $L_1>0$;

Case (f) $v'=v=0$ *and* $c \neq 4Nsm$;

Case (g) $v'=v=0$ *and* $c=4Nsm$.

Theorem 2. (*Number of stationary solutions*).

(i) *There is one and only one stationary solution in the cases* (a), (b) *and* (c):
(a) L^1-*solution*, (b) δ_0, (c) δ_1.

(ii) *There are exactly two stationary solutions in the cases* (d), (e) *and* (f):
(d) L^1-*solution and* δ_0, (e) L^1-*solution and* δ_1, (f) δ_0 *and* δ_1.

(iii) *In the case* (g), *there are infinitely many stationary solutions. More precisely, for each* $y \in [0, 1]$, *there exists a stationary solution* $U_y(dx)$ *with* $\bar{x} = M_1 = y$. $U_0=\delta_0$, $U_1=\delta_1$ *and* U_y *is an* L^1-*solution if* $y \in (0, 1)$.

In some special cases, the L^1-solutions can be easily found by integration of ordinary differential equation. In the case (g), $U_y(dx)$ for $y \in (0, 1)$ has the density

$$u_y(x) = \frac{\Gamma(4Nm)}{\Gamma(4Nmy)\Gamma(4Nm(1-y))}\, x^{4Nmy-1}(1-x)^{4Nm(1-y)-1}. \tag{10}$$

Also if $c=4Ns(v'+v+m)$ in the case (a), the first moment is equal to $\xi = v'/(v'+v)$ and the density function is given by

$$u(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha\xi)\Gamma(\alpha(1-\xi))}\, x^{\alpha\xi-1}(1-x)^{\alpha(1-\xi)-1} \quad \text{with} \quad \alpha = c/s. \tag{11}$$

The stationary L^1-solutions in the general case will be given in §2 below.

Let us now explain the idea of our proof of Theorem 2. As is noted above, the problem (SK) is equivalent to (SM). If we put

$$P=4Nv', \quad Q=4Nv, \quad S=4Ns, \quad M=4Nm \quad \text{and} \quad C=4Nc, \tag{12}$$

then the equation (6) can be rewritten as

$$M_{n+1} = b_n(M_1)M_n - a_n(M_1)M_{n-1} \qquad \text{for} \quad n=1, 2, 3, \dots, \tag{13}$$

where a_n's and b_n's are defined by (1) of §B. Regarding M_1 as a parameter y running over the interval [0, 1], we introduce an auxiliary function

$$F(y) = \frac{a_1(y)}{b_1(y)} - \frac{a_2(y)}{b_2(y)} - \cdots - \frac{a_n(y)}{b_n(y)} - \cdots, \tag{14}$$

where the right hand side is a continued fraction. The following is the key lemma for our proof of Theorem 2:

Lemma 1.1. (*First moment of the stationary solution*)
(i) *Let $U(dx)$ be a stationary solution and $\bar{x}$ its first moment. Then $\bar{x}$ is a fixed point of the mapping $y \to F(y)$, that is, it is a solution of the equation*

$$y=F(y) \quad and \quad 0 \leq y \leq 1. \tag{15}$$

(ii) *Conversely, if $\bar{x}$ is a solution of the equation* (15), *then there exists one and only one stationary solution $U(dx)$ whose first moment is equal to $\bar{x}$. Further, if $\bar{x}$ is contained in* (0, 1), *then $U(dx)$ is a stationary L^1-solution.*
(iii) *$y=0$ (or $y=1$) is a solution of* (15) *if and only if $v'=0$ (resp. $v=0$), and it corresponds to δ_0 (resp. δ_1).*
(iv) *Except for the case* (g), *the equation* (15) *has at most one solution $\bar{x}$ in the open interval* (0, 1). *Every such solution $\bar{x}$ (if exists) satisfies the inequality $F'(\bar{x})<1$.*

Remark. A more detailed argument will permit us to remove the restriction $c \leq v+m$ in Theorem 2 (we keep all the other assumptions in (1) unchanged). In fact, if $c>v+m$, we choose a natural number n such that $c \leq (n+1)(v+m)$ and exploit the equation

$$F_{n+1}(y)=b_n(y)-\left(\frac{a_n(y)}{b_{n-1}(y)}-\cdots-\frac{a_2(y)}{b_1(y)}-\frac{a_1(y)}{y}\right) \quad \text{and} \quad 0 \leq y \leq 1. \tag{15'}$$

Here $F_{n+1}(y)$ is the function defined in (3) of §B, which converges in power series sense if $0 \leq c \leq (n+1)(v+m)$. The equation (15') plays the same role as (15). Therefore, if we list up all the solutions of (15'), then we could enumerate all the stationary solutions. Thus the restriction $c \leq v+m$ is not essential for Theorem 2. But, in this paper, we assume this because it makes the situation much simpler.

Let us now turn to the covergence of the solution of the problem (K) to a stationary solution as t tends to infinity.

Let $\mathscr{M}$ be the space of all finite measures on [0, 1] with weak topology. For each $U \in \mathscr{M}$, we set

$$M_n(U)=\int_{0-}^{1+} x^n U(dx), \quad n=0, 1, 2,\ldots$$

Since the set of all polynomials are dense in the space of continuous functions on [0, 1], a solution $U(t, dx)$ of the problem (K) converges weakly to $U(dx)$ if and only if

$$\lim_{t\to+\infty} M_n(U(t,\cdot))=M_n(U) \qquad \text{for every} \quad n=0, 1, 2,\ldots \tag{16}$$

Let also $\mathscr{P}$ be the subspace of $\mathscr{M}$ which consists of all probability measures on [0, 1]. We define two subsets of $\mathscr{P}$; $\mathscr{P}_*=\mathscr{P}_*(M, Q)$ is the set of all $\Phi \in \mathscr{P}$ such that

$$\liminf_{n\to+\infty} M_n(\Phi)n^\beta>0 \qquad \text{for some} \quad \beta<4N(v+m), \tag{17}$$

and $\mathscr{P}^*$ is the set of all $\Phi \in \mathscr{P}$ such that

(18) $$\limsup_{n\to+\infty} M_n(\Phi)n^{\varepsilon} < +\infty \qquad \text{for some} \quad \varepsilon>0.$$

For example, δ_a (the unit distribution with the support at $x=a$) belongs to $\mathscr{P}^*$ but not to $\mathscr{P}_*$ if $0\leq a<1$, while δ_1 to $\mathscr{P}_*$ but not to $\mathscr{P}^*$. Notice further that both of the weak closures of $\mathscr{P}_*$ and $\mathscr{P}^*$ are equal to the whole space $\mathscr{P}$.

Theorem 3. (*Stability of stationary solutions*)
(i) *In the cases* (a), (b) *and* (c), *under the assumption* $m\geq c$, *any solution of the problem* (K) *converges weakly to the stationary solution as* $t\to+\infty$.
(ii) *In the case* (d) ((e)), *under the assumption* $m\geq c$, *every solution of the problem* (K) *with the initial measure* $\Phi\in\mathscr{P}_*$ (*resp.* $\Phi\in\mathscr{P}^*$) *remains in* $\mathscr{P}_*$ (*resp.* $\mathscr{P}^*$) *at any time* t, *and converges weakly to the* L^1-*solution as* $t\to+\infty$.
(iii) *In the case* (f), *any solution of the problem* (K) *converges weakly, as* $t\to+\infty$, *to* δ_0 *if* $c<4Nsm$ (*to* δ_1 *if* $c>4Nsm$) *unless the initial measure is* δ_1 (*resp.* δ_0).

Remark. In the above theorem, we could state no results for the case (g). But we can specify the limit (if exists) of $U(t, dx)$ as $t\to+\infty$. For each $\Phi\in\mathscr{P}$, we can uniquely find a solution η of the equation

(19) $$\int_{0-}^{1+} e^{Sx}\Phi(dx) = {}_1F_1(M\eta, M; S) \quad \left(=\int_{0-}^{1+} e^{Sx}U_\eta(dx)\right) \quad \text{and} \quad 0\leq\eta\leq1.$$

Then, as $t\to+\infty$, either $U(t, dx)$ converges to $U_\eta(dx)$ or it has no limit. (Indeed, in this case, the function $H(t)$ in (59) of §3 is constant and the possible limit measure $U(dx)$ is uniquely determined via (19)). Unfortunately, we can find neither a proof of convergence nor counterexamples.

The assertions of Theorems 2 and 3 are summarized as follows except for the case (g). There exist one or two stationary solutions. If there is one, then the stationary solution is stable. If there are two and one of them is an L^1-solution, then the L^1-solution is stable and the other one is a unit measure and unstable. δ_0 is stable if $v'=0$ and c is sufficiently small, while δ_1 is stable if $v=0$ and c is sufficiently large. The relation $L_0(N, v, s, m, c)=0$ (resp. $L_1(N, v', s, m, c)=0$) is the bifurcation equation of the stationary solution from δ_0 (resp. δ_1).

Our key lemma for the proof of Theorem 3 is concerned with the equation

(20) $$\int_{0-}^{1+}\Big[\frac{x(1-x)}{4N}f''(x)+\{v'(1-x)-vx-sx(1-x)+m(\bar{x}-x)\}f'(x) + \{c(x-\bar{x})-\lambda\}f(x)\Big]U(dx) = -\lambda\int_{0-}^{1+} f(x)\Phi(dx)$$
$$\text{where} \quad \bar{x}=\int_{0-}^{1+} xU(dx),$$

for all functions f of class C^2 on [0, 1]. As will be seen in Lemma 3.2 below, if $\lambda>c$ and $m\geq c$, then the equation (20) has at least one solution in $\mathscr{P}$. Further the set of means of the solutions of (20) in $\mathscr{P}$ is a non-empty closed set in [0, 1], possibly a singleton, and we can specify a solution $U(dx; \lambda, \Phi)$ of the equation (20) in $\mathscr{P}$

with the smallest (or the largest if necessary) mean.

Lemma 1.2. (*Comparison of moment sequences*). *Let* $\lambda > c$, $m \geq c$ *and* Φ_1, $\Phi_2 \in \mathscr{P}$, *and assume that*

$$M_n(\Phi_1) \leq M_n(\Phi_2), \quad n=0, 1, 2,... \tag{21}$$

Then it holds that

$$M_n(U(\cdot\,; \lambda, \Phi_1)) \leq M_n(U(\cdot\,; \lambda, \Phi_2)), \quad n=0, 1, 2,... \tag{22}$$

The next theorem tells us how the parameters (N, v', v, s, m, c) affect the model of Kimura.

Theorem 4. (*Dependence on parameters*) *Except for the case* (g), *let* $\{M_n\}_{n=0}^{\infty}$ *be the moment sequence of the stationary* L^1*-solution, and assume that* $0 \leq c \leq 2m$. *Then we have*

$$\begin{aligned} &\frac{\partial}{\partial N} M_n < 0, \quad \frac{\partial}{\partial v'} M_n > 0, \quad \frac{\partial}{\partial v} M_n < 0, \\ &\frac{\partial}{\partial s} M_n < 0, \quad \frac{\partial}{\partial m} M_n < 0 \text{ and } \frac{\partial}{\partial c} M_n > 0, \end{aligned} \qquad \text{for every } n \geq 1. \tag{23}$$

The following theorem is one of the most interesting discoveries of M. Kimura for this model:

Theorem 5. (*Numerical result of Kimura* [4], [5]) *Assume the cases* (a), (*b*) *or* (c) *with* $0 \leq c \leq m$, *and fix* N, s *and* m. *Denote by* $\bar{x}(t)$ *the first moment of a solution of the problem* (K).

(i) *Take a* c_0 *such that* $0 \leq c_0 < 4Nsm$ *and* $c_0 \leq m$. *Then, for any positive number* ε, *there exists a positive number* η *such that*

$$0 \leq \lim_{t \to +\infty} \bar{x}(t) < \varepsilon \qquad \text{for all} \quad 0 \leq c \leq c_0 \quad \text{and} \quad v' < \eta.$$

(ii) *Take a* c_1 *such that* $4Nsm < c_1 \leq m$. *Then, for any positive number* ε, *there exists a positive number* η *such that*

$$1 - \varepsilon < \lim_{t \to +\infty} \bar{x}(t) \leq 1 \qquad \text{for all} \quad c_1 \leq c \quad \text{and} \quad v < \eta.$$

Following M. Kimura [4] and [5], we now explain the diffusion model (K) in terms of population genetics. Consider a hypothetical population (species) consisting of an infinite number of competing subgroups (demes). Each of demes is assumed to have an equal effective size N independently of time (generation) t.

Let us look at a pair of alleles A and A' at a particular gene locus, and denote by x $(0 \leq x \leq 1)$ the frequency of A' in a deme. We consider the frequency distribution of x among the entire collection of demes making up the species. Let $U(t, dx)$ be the probability measure representing the distribution of x, that is, for a Borel set E in $[0, 1]$, the value $U(t, E)$ is equal to the fraction of demes whose frequency of A' at time t is in E.

In each generation, mutation occurs from A to A' at the rate of v' and from A' to A at the rate of v. A' is assumed to have selective disadvantage s (>0) relative to A. Migration is assumed to occur in the following way: each deme contributes emigrants to the entire gene pool of the species at the rate m (>0) and receives immigrants from that pool at the same rate. $\bar{x}(t)$ represents the average frequency of A' in the entire species.

Moreover, we also assume the effect of interdeme selection. Denote by c (>0) the coefficient of interdeme selection. Then cx represents the rate at which the number of demes belonging to the gene frequency class x changes through interdeme competition. Thus, the allele A' has disadvantage in individual selection but advantage in interdeme selection. So, we refer to A' as an "*altruistic allele*".

Theorem 4 above shows that the increase of everyone of the parameters N, v, s and m is disadvantageous to A', while the increase of v' and c is advantageous to A'.

M. Kimura has introduced in [4] and [5] a simple indicator

$$D=(c/m)-4Ns \tag{24}$$

to compare the effects of individual selection s and interdeme selection c. Assuming that the mutation rates v' and v are very small and negligible, M. Kimura states that $\bar{x}$ (the mean of x in the stationary state) is very close to 1 if $D>0$ and to 0 if $D<0$ (see Theorem 5 above). And he concludes as follows: if $D>0$, then the interdeme competition prevails over the individual selection and the altruistic allele A' predominates. If, on the contrary, $D<0$, then A' becomes rare and cannot be established in the species. In this memoir, we have justified his numerical result by proving Theorems 3 and 5.

We close this section with the following

Proof of Theorem 1. Let $\Phi(dx)$ be any given probability measure on $[0, 1]$. We have to prove the existence and uniqueness of the solution $U(t, dx)$ of problem (K) which satisfies $U(0+, dx)=\Phi(dx)$.

Take a function $\alpha(x)$ of class C^∞ such that

$$\alpha(x)=0 \quad \text{if} \quad |x|\geq 1, \quad \alpha(x)>0 \quad \text{if} \quad |x|<1 \quad \text{and} \quad \int_{-1}^{1}\alpha(x)dx=1.$$

For any positive number ε, we put

$$\alpha_\varepsilon(x)=\alpha(x/\varepsilon)/\varepsilon, \quad K_\varepsilon=\left\{\int_0^1 dx\int_{0-}^{1+}\alpha_\varepsilon(x-y)\Phi(dy)\right\}^{-1},$$

$$\phi_\varepsilon(x)=K_\varepsilon\int_{0-}^{1+}\alpha_\varepsilon(x-y)\Phi(dy) \qquad \text{for} \quad x\in[0, 1]$$

$$\text{and} \quad \Phi_\varepsilon(dx)=\phi_\varepsilon(x)dx.$$

Then $\Phi_\varepsilon(dx)$ is a probability measure on $[0, 1]$ with the density function $\phi_\varepsilon(x)$ and converges weakly to $\Phi(dx)$ as $\varepsilon\downarrow 0$. By the Theorem in §3 of [8], there exists one and only one solution $U_\varepsilon(t, dx)$ (with density function) of the problem (K) satisfying

$U_\varepsilon(0+, dx)=\Phi_\varepsilon(dx)$. Let $\{M_n\}_{n=0}^\infty$, $\{M_n^{(\varepsilon)}\}_{n=0}^\infty$ and $\{M_n^{(\varepsilon)}(t)\}_{n=0}^\infty$ be the moment sequences of $\Phi(dx)$, $\Phi_\varepsilon(dx)$ and $U_\varepsilon(t, dx)$ respectively. The weak convergence of $\Phi_\varepsilon(dx)$ to $\Phi(dx)$ implies

$$\lim_{\varepsilon\downarrow 0} M_n^{(\varepsilon)}=M_n \qquad \text{for every} \quad n\geq 0.$$

Hence by the Lemma in §5 of [8], for any $n\geq 0$ and $t>0$, $M_n^{(\varepsilon)}(t)$ tends to a limit as $\varepsilon\downarrow 0$, which we denote by $M_n(t)$. Then the sequence $\{M_n(t)\}_{n=0}^\infty$ is a solution of the problem (M) satisfying $M_n(0+)=M_n$ for all $n\geq 0$. This sequence $\{M_n(t)\}_{n=0}^\infty$ determines the solution $U(t, dx)$ of the problem (K) with $U(0+, dx)=\Phi(dx)$. Consequently, we have proved the existence of a solution.

The uniqueness also follows from the Lemma in §5 of [8].

Theorem 1 is proved.

§ 2. Proof of Theorem 2.

To prove Theorem 2, it suffices to prove Lemma 1.1 and apply it to the cases (a), (b),..., (g).

In this and the next sections we use the letters P, Q, S, M and C to denote

$$(1) \qquad P=4Nv',\quad Q=4Nv,\quad S=4Ns,\quad M=4Nm \quad\text{and}\quad C=4Nc.$$

As in §B, we assume that

$$(\text{B.1}) \qquad P\geq 0,\quad Q\geq 0,\quad S>0,\quad M>0 \quad\text{and}\quad 0\leq C\leq Q+M.$$

Then by Lemma B.1, the continued fraction

$$(2) \qquad F(y)=\frac{a_1(y)}{b_1(y)}-\frac{a_2(y)}{b_2(y)}-\cdots-\frac{a_n(y)}{b_n(y)}-\cdots$$

converges (in power series sense) at every point $y\in[0, 1]$. We make use of the results in §§A and B on continued fractions, especially on this function $F(y)$.

The stationary problem (SK) is equivalent to the stationary moment problem (SM). The latter is to find a completely monotone sequence $\{M_n\}_{n=0}^\infty$ with $M_0=1$ which satisfies

$$(3) \qquad M_{n+1}=b_n(M_1)M_n-a_n(M_1)M_{n-1} \qquad \text{for} \quad n=1, 2, 3,\ldots$$

Notice that the first moment $M_1=\bar{x}$ uniquely determines the whole sequence $\{M_n\}_{n=0}^\infty$ through (3). Hence the uniqueness part in Lemma 1.1 (ii) is clear.

Proof of Lemma 1.1 (iii). The assertions in this part are direct consequences of those in Lemma B.1 (ii) and (iii).

Lemma 1.1 (iii) being now established, for the proof of the Lemma, we have only to deal with the solutions $\bar{x}$ of the equation

$$(4) \qquad y=F(y) \quad\text{and}\quad 0<y<1$$

(see (15) of §1).

Proof of Lemma 1.1 (i). Let $U(dx)$ be a stationary solution (other than δ_0 and δ_1) and $\{M_n\}_{n=0}^{\infty}$ be its moment sequence. Then the sequence $\{R_n\}_{n=1}^{\infty}$ defined by $R_n = M_n/M_{n-1}$, $n=1, 2, 3,\dots$, is nondecreasing and satisfies $0 < R_n \le 1$, $n=1, 2, 3,\dots$. Putting $\bar{x} = M_1$, we have $R_n = a_n(\bar{x})/(b_n(\bar{x}) - R_{n+1})$ from the equation (3) and so

$$\bar{x} = R_1 = \frac{a_1(\bar{x})}{b_1(\bar{x})} - \frac{a_2(\bar{x})}{b_2(\bar{x})} - \cdots - \frac{a_n(\bar{x})}{b_n(\bar{x}) - R_{n+1}}.$$

Hence, applying Lemma A.2 with $a'_k = a_k = a_k(\bar{x})$ for $1 \le k \le n$, $b'_k = b_k = b_k(\bar{x})$ for $1 \le k \le n-1$, $b'_n = b_n(\bar{x}) - R_{n+1}$ and $b_n = b_n(\bar{x})$, we obtain

$$\left| \bar{x} - \frac{a_1(\bar{x})}{b_1(\bar{x})} - \frac{a_2(\bar{x})}{b_2(\bar{x})} - \cdots - \frac{a_n(\bar{x})}{b_n(\bar{x})} \right| \le \frac{R_{n+1}}{b_n(\bar{x}) - R_{n+1}} \prod_{k=1}^{n-1} \frac{1}{b_k(\bar{x}) - 1}.$$

The right hand side of the above inequality tends to 0 as $n \to +\infty$ and we have $\bar{x} = F(\bar{x})$. The assertion (i) is proved.

For the proof of Lemma 1.1 (ii) and (iv), we introduce a linear ordinary differential operator L_y with parameter y:

$$L_y[f](x) = \frac{d}{dx} B_y[f](x) + C(x-y)f(x), \tag{5}$$

where

$$B_y[f](x) = \left(\frac{d}{dx} + S\right)\{x(1-x)f\}(x) \tag{6}$$
$$\qquad -(P+My)(1-x)f(x) + (Q+M-My)xf(x).$$

If $u(x)$ is the density function of a stationary solution $U(dx)$ (that is, $U(dx) = u(x)dx$), then $u(x)$ satisfies the following conditions (7), (8) and (9) and vice versa:

$$u(x) \ge 0 \quad \text{in} \quad (0, 1) \quad \text{and} \quad \int_0^1 u(x)dx = 1; \tag{7}$$

$$L_{\bar{x}}[u] = 0 \quad \text{in} \quad (0, 1), \quad \text{where} \quad \bar{x} = \int_0^1 xu(x)dx; \tag{8}$$

$$B_{\bar{x}}[u](x) \longrightarrow 0 \quad \text{as either} \quad x \downarrow 0 \quad \text{or} \quad x \uparrow 1. \tag{9}$$

We call such a function $u(x)$ a stationary L^1-solution as in §1.

Let ${}^tL_y$ be the transposed operator of L_y:

$${}^tL_y[g](x) = x(1-x)(g'' - Sg')(x) + (P+My)(1-x)g'(x) \tag{10}$$
$$\qquad -(Q+M-My)xg'(x) + C(x-y)g(x).$$

For each $y \in (0, 1)$, we put

$$w(x, y) = x^{P+My-1}(1-x)^{Q+M(1-y)-1}e^{-Sx} \quad \text{for} \quad x \in (0, 1). \tag{11}$$

Then we have

$$B_y[w(\cdot, y)f](x)=x(1-x)w(x, y)f'(x) \tag{12}$$

for any function f of class C^1 on $[0, 1]$. So, $B_y[w(\cdot, y)f](x)$ for such function f tends to 0 as either $x \downarrow 0$ or $x \uparrow 1$, provided $0<y<1$. Furthermore, if f is of class C^2 on $[0, 1]$, then we have

$$L_y[w(\cdot, y)f](x)=w(x, y){}^tL_y[f](x). \tag{13}$$

Therefore, to solve the equation $L_y[u]=0$, it suffices to solve the equation ${}^tL_y[\hat{u}]=0$ and put $u(x)=w(x, y)\hat{u}(x)$.

Lemma 2.1 (*Particular solution*). *For each* $y\in(0, 1)$, *define*

$$\hat{u}(x, y)=1+\sum_{n=1}^{\infty}\frac{Cx^n}{nS+C}\prod_{k=1}^{n}\frac{F_k(y)}{a_k(y)}, \qquad x\in[0, 1], \tag{14}$$

where $F_k(y)$*'s are those in* (3) *of* §B. *Then,* $\hat{u}(x, y)$ *is of class* C^∞ *with respect to* x *on* $[0, 1]$ *and satisfies*

$${}^tL_y[\hat{u}(\cdot, y)](x)=C\{F(y)-y\}. \tag{15}$$

Proof. Since $0\leq F_k(y)\leq 1$ for all k by Lemma B.1 and since $a_k(y)$ tends to infinity as $k\to+\infty$ uniformly in y, it holds that $\hat{u}(x, y)$ converges and is of class C^∞ with respect to x. Term by term differentiation of $\hat{u}$ yields (15), where we use

$${}^tL_y[x^n]=(nS+C)\{a_n(y)x^{n-1}+x^{n+1}-b_n(y)x^n\} \qquad \text{for} \quad n\geq 0, \tag{16}$$

with conventions $a_0(y)=0$ and $b_0(y)=y$ (see also Lemma A.3).
Lemma 2.1 is proved.

Proof of Lemma 1.1 (ii). Assume first that $0<C\leq Q+M$ and let $\bar{x}$ be a solution of (4). Then, the function

$$u_{\bar{x}}(x)=\frac{1}{K}w(x, \bar{x})\hat{u}(x, \bar{x}) \quad \text{with} \quad K=\int_0^1 w(x, \bar{x})\hat{u}(x, \bar{x})dx \tag{17}$$

satisfies the conditions (7), (8) and (9). Therefore, $u_{\bar{x}}(x)$ is a stationary L^1-solution.

Assume next $C=0$. Since ${}^tL_y[1]=0$ in this case, we have $L_y[w(\cdot, y)]=0$. The function $w(x, y)$ in x is non-negative, belongs to $L^1(0, 1)$ and satisfies the boundary condition provided $y\in(0, 1)$ (see (12)). Therefore, a constant multiple of $w(x, \bar{x})$ is a stationary L^1-solution if and only if $\bar{x}\in(0, 1)$ and

$$\int_0^1 xw(x, \bar{x})dx=\bar{x}\int_0^1 w(x, \bar{x})dx. \tag{18}$$

On the other hand, we can show by power series expansion of $e^{S(1-x)}$ that

$$e^S\int_0^1 w(x, y)dx=\frac{\Gamma(P+My)\Gamma(Q+M-My)}{\Gamma(P+Q+M)}\,{}_1F_1(Q+M-My, P+Q+M; S),$$

$$e^S\int_0^1 xw(x, y)dx = \frac{\Gamma(P+My+1)\Gamma(Q+M-My)}{\Gamma(P+Q+M+1)}$$

$$\times {}_1F_1(Q+M-My,\ P+Q+M+1;\ S).$$

So, by (15) of §B, we have

$$\int_0^1 xw(x, y)dx/\int_0^1 w(x, y)dx = F(y). \tag{19}$$

Consequently, (18) holds if and only if $\bar{x}$ is a solution of (4). The assertion (ii) is proved.

Proof of Lemma 1.1 (iv). Assume first that $0<C\leq Q+M$. It suffices to prove that

$$\frac{\partial F}{\partial y}(\bar{x})<1 \quad \text{whenever} \quad \bar{x} \quad \text{solves} \quad (4). \tag{20}$$

Differentiating both sides of (15) with respect to y, we have

$${}^tL_y\left[\frac{\partial \hat{u}}{\partial y}\right] + M\frac{\partial \hat{u}}{\partial x} - C\hat{u} = C\left\{\frac{\partial F}{\partial y}(y)-1\right\}.$$

Substituting $y=\bar{x}$, multiplying both sides by $w(x, \bar{x})\hat{u}(x, \bar{x})$ and integrating them over $0<x<1$, we obtain

$$CK\left\{1-\frac{\partial F}{\partial y}(\bar{x})\right\} = \int_0^1 w\hat{u}\left(C\hat{u}-M\frac{\partial \hat{u}}{\partial x}\right)dx, \tag{21}$$

where $\hat{u}$ and w stand for $\hat{u}(x, \bar{x})$ and $w(x, \bar{x})$ respectively. Here we used the fact

$$\int_0^1 w\hat{u}\,{}^tL_{\bar{x}}\left[\frac{\partial \hat{u}}{\partial y}\right]dx = \int_0^1 \frac{\partial \hat{u}}{\partial y} L_{\bar{x}}[w\hat{u}]dx = 0.$$

To show (20), it remains to prove that

$$0<M\frac{\partial \hat{u}}{\partial x}<C\hat{u} \qquad \text{for} \quad x\in(0, 1). \tag{22}$$

Let $\xi(x)=\frac{\partial \hat{u}}{\partial x}/\hat{u}$. Then it is positive and of class C^∞ on [0, 1]. Further it satisfies the equation

$$\frac{d\xi}{dx} = S\xi-\xi^2-\frac{(P+M\bar{x})\xi-C\bar{x}}{x}+\frac{(Q+M-M\bar{x})\xi-C(1-\bar{x})}{1-x}. \tag{23}$$

By the continuity of $d\xi/dx$ at $x=0$ and $x=1$, we have from (23) that

$$\xi(0)=\frac{C\bar{x}}{P+M\bar{x}}\leq\frac{C}{M} \quad \text{and} \quad \xi(1)=\frac{C-C\bar{x}}{Q+M-M\bar{x}}\leq\frac{C}{M}.$$

Suppose that the maximum of ξ on [0, 1] is not smaller than C/M and that the maximum is attained at some interior point $x_0\in(0, 1)$. Then it holds that $\xi(x_0)\geq$

C/M, $(d\xi/dx)(x_0)=0$ and $(d^2\xi/dx^2)(x_0)\leq 0$. But a computation with the aid of the first two inequalities shows that

$$\frac{d^2\xi}{dx^2}(x_0)=\frac{(P+M\bar{x})\xi-C\bar{x}}{x_0{}^2}+\frac{(Q+M-M\bar{x})\xi-C(1-\bar{x})}{(1-x_0)^2}\geq 0.$$

Therefore, $(d^2\xi/dx^2)(x_0)$ must be zero. This is possible only if $P=Q=C-SM=0$, that is, in the excluded case (g). Thus, we have proved (22). Now the desired (20) follows from (21) and (22).

We next turn to the case $C=0$. Since $F(0)\geq 0$ and $F(1)\leq 1$, it suffices to show that $F(y)$ is of class C^2 and that $F''(x)=(\partial^2F/\partial y^2)(x)$ is positive for $y\in(0, 1)$.

By (6) and (7) of §B, $F'_n=\partial F_n/\partial y$ is positive for any $n\geq 1$ because $d_{n,y}=M/S>0$ in our case $C=0$. Next, differentiating twice both sides of $F_{n+1}(y)=b_n(y)-a_n(y)/F_n(y)$, we have

$$F''_n=\frac{F_n}{a_n}(2F'_nF'_{n+1}+F_nF''_{n+1}).$$

Therefore, it follows that

$$F''_n=\sum_{n=1}^{\infty}\frac{2}{F_n}F'_nF'_{n+1}\prod_{k=1}^{n}(F_k^2/a_k),$$

the right hand side being convergent and positive. Thus, we have the the desired conclusion. The assertion (iv) is verified.

The proof of Lemma 1.1 is now complete.

We note that, in the case (g), our continued fraction $F(y)$ is identically equal to y by Lemma B.3. Therefore, $\hat{u}(x, y)=e^{Sx}$ by integration of the equation (15) and $\xi(x)$ is identically equal to S. The solution $u_{\bar{x}}(x)$ defined by (17) for arbitrary y in place of $\bar{x}$ is nothing but $u_y(x)$ in (10) of §1.

Proof of Theorem 2. The assertion (iii) (for the case (g)) is already proved, because we have given the precise form of the stationary solutions (see (10) of §1). Applying Lemma 1.1 to each of the other cases, we will enumerate all the fix points of the function F in the closed interval [0, 1].

Case (f) where $P=Q=0$ and $C\neq SM$. By Lemma B.3 (ii) and (iii), F has only two fixed points $\bar{x}=0$ and $\bar{x}=1$ giving the stationary solutions δ_0 and δ_1 respectively.

Case (a) where $P>0$ and $Q>0$. By Lemma B.1 (ii) and (iii), $F(0)>0$ and $F(1)<1$. Therefore, by Lemma 1.1 (iv), there exists one and only one fixed point of F in the open interval (0, 1), which gives a stationary L^1-solution.

Cases (b) *and* (d) where $P=0$ and $Q>0$. In this case, we have $F(0)=0$ and $F(1)<1$. Let us make use of the formula (10) of §B for $F'(0)=\dfrac{\partial F}{\partial y}(0)$. At first, it should be noticed that ${}_1F_1(C/S, Q+M; -S)$ is positive because of $0\leq C\leq Q+M$ (the proof is not difficult if we expand this function in power series in S). Comparing $F'(0)$ with

$$L_0=\frac{M}{Q+M}\,{}_1F_1\Big(\frac{C}{S}+1, Q+M+1; -S\Big)-{}_1F_1\Big(\frac{C}{S}, Q+M; -S\Big),$$

we see that $F'(0)<1$ ($=1$, >1) if and only if $L_0<0$ (resp. $=0$, >0).

If $L_0>0$ (Case of (d)), then $F'(0)>1$. So, there exists one and only one fixed point of F in the open interval (0, 1), while $\bar{x}=0$ is another fixed point. Therefore, we have exactly two stationary solutions, an L^1-solution and δ_0.

If $L_0<0$ (subcase of (b)), $F'(0)<1$. Then, $\bar{x}=0$ is the only fixed point of F in the closed interval [0, 1]. In fact, if there exists a fixed point in (0, 1), we should have $F'(\bar{x})\geq 1$ at the minimum $\bar{x}$ in (0, 1). This contradicts to the assertion $F'(\bar{x})<1$ in Lemma 1.1 (iv). Therefore, δ_0 is the only stationary solution in this subcase.

In order to deal with the remainder subcase $L_0=0$ of (b), we will prove that

(24) $$\frac{\partial^2 F}{\partial Q\partial y}<0 \quad \text{at} \quad y=0 \quad \text{and} \quad Q>0 \quad \text{if} \quad P=0.$$

Emphasizing the dependence of F on Q, we denote $F(y)=F(y, Q)$. As is shown in the proof of (10) in §B, we have

$$\frac{\partial F}{\partial y}(0, Q)=\frac{a_1'(0)}{b_1(0)}-\frac{a_2(0)}{b_2(0)}-\cdots-\frac{a_n(0)}{b_n(0)}-\cdots,$$

where $a_1'(0)=(\partial a_1/\partial y)(0)$. So $(\partial^2 F/\partial Q\partial y)(0, Q)$ is calculated quite analogously to $\partial F/\partial Q$ is and we obtain (24) (see (6) and (7) in §B where $d_{n,Q}<0$ for any n).

Now assume that $L_0=0\left(\frac{\partial F}{\partial y}(0, Q)=1\right)$. Suppose moreover that there exists a point $y_1\in(0, 1)$ at which $F(y_1, Q)\geq y_1$. Then there exists a fixed point $\bar{x}$ of F such that $y_1\leq\bar{x}<1$. Since $\frac{\partial F}{\partial y}(\bar{x}, Q)<1$, there exists a $y_0\in(0, \bar{x})$ at which $F(y_0, Q)>y_0$. By continuity of F on Q, we find a Q' which is larger than but close to Q such that $F(y_0, Q')>y_0$. Combining (24) with $\frac{\partial F}{\partial y}(0, Q)=1$, we have $\frac{\partial F}{\partial y}(0, Q')<1$ which means that $L_0<0$ if Q is replaced by Q'. From what we have shown just above, it then follows that $F(y, Q')<y$ for $y\in(0, 1]$ contradicting the inequality $F(y_0, Q')>y_0$. Hence $F(y, Q)<y$ for all $y\in(0, 1]$ and $\bar{x}=0$ is the only fixed point of F in the interval [0, 1]. This means that δ_0 is the only stationary solution also in the subcase $L_0=0$.

We have finished the proof for the cases (b) and (d).

Cases (*c*) *and* (*e*) where $P>0$ and $Q=0$. By (9) of §B and (9) of §1, we have

(25) $$\frac{\partial F}{\partial y}(1)=1+L_1.$$

If $L_1\neq 0$, the reasoning is the same as in the previous cases (b) and (d). If $L_1<0$ (subcase of (c)), then $\bar{x}=1$ is the only fixed point of F in the closed interval [0, 1] and so δ_1 is the only stationary solution. If $L_1>0$ (Case (e)), then we have exactly two fixed points of F, one in the open interval (0, 1) and the other equal to $\bar{x}=1$. So we have two stationary solutions, an L^1-solution and δ_1.

We now treat the remaining subcase of (c) where $L_1=0$. Let us write $F(y)=F(y, P)$ regarding P as a distinguished parameter. We will show that

(26) $$\frac{\partial^2 F}{\partial P\partial y}(1, P_0)<0 \quad \text{if} \quad P_0>0, Q=0 \quad \text{and} \quad \frac{\partial F}{\partial y}(1, P_0)=1.$$

Suppose first that $C=0$. Then by (9) of §B,

$$\frac{\partial F}{\partial y}(1, P)=\frac{M}{P+M}{}_1F_1(1, P+M+1; S)=\sum_{n=0}^{\infty} MS^n\prod_{k=0}^{n}\frac{1}{P+M+k}$$

which is obviously decreasing in P. So (26) holds.

Suppose next that $0<C\leq M$. Putting $\alpha=C/S$ and $\gamma=P+M$, we have by definition that

$$L_1=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\gamma)S^n}{\Gamma(\alpha+1)\Gamma(\gamma+n+1)n!}(M\alpha+Mn-\alpha\gamma-\alpha n).$$

If $\alpha\geq M$, then the last factor $M\alpha+Mn-\alpha\gamma-\alpha n$ is negative for all $n\geq 1$ and $L_1<0$. Thus we may assume $0<\alpha<M$ in our subcase $L_1=0$. In the sequel, we will make use of the integral representation

$${}_1F_1(\alpha, \gamma; S)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 g(r)dr,$$

$${}_1F_1(\alpha+1, \gamma+1; S)=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma-\alpha)}\int_0^1 rg(r)dr,$$

$$\text{where}\quad g(r)=r^{\alpha-1}(1-r)^{\gamma-\alpha-1}e^{Sr}.$$

The above formulas can be obtained by power series expansion of e^{Sr}. Note that the function $h(P)=\Gamma(\alpha+1)\Gamma(\gamma-\alpha)L_1/M\Gamma(\gamma)$ is written as

$$h(P)=\int_0^1 g(r)(r-\rho)dr \quad\text{with}\quad \rho=\alpha/M.$$

If $L_1=0$ at some point $P=P_0$, then $h(P_0)=0$ and

$$\frac{\partial h}{\partial P}(P_0)=\int_0^1 g(r)(r-\rho)\log(1-r)dr=\int_0^1 g(r)(r-\rho)\log\frac{1-r}{1-\rho}dr.$$

The function $(r-\rho)\log\frac{1-r}{1-\rho}$ is negative for $r\in(0, 1)$ except at $r=\rho$. Therefore, $\frac{\partial h}{\partial P}(P_0)$ is negative and (26) follows from (25) and the definition of $h(P)$.

Now, due to (26), we have the only fixed point $\bar{x}=1$ of F in the closed interval $[0, 1]$ and δ_1 is the only stationary solution, provided $L_1=0$.

We finished the proof of Theorem 2 for the cases (c) and (e).

§3. Proof of Theorem 3.

In this section we will prove Theorem 3. Throughout this section except for the Proof of Theorem 3 (iii) we assume that $C\leq M$.

For each $y\in[0, 1]$, we introduce a differential operator $\mathscr{D}_y$, which is a part of the operator ${}^tL_y$ in §2;

(1) $\mathscr{D}_y f(x)=x(1-x)f''(x)+\{P(1-x)-Qx-Sx(1-x)+M(y-x)\}f'(x), \quad x\in(0, 1).$

This is a diffusion operator and the corresponding scale function $s_y(x)$ and the speed measure function $m_y(x)$ are given by

$$(2)\quad \begin{aligned} s_y(x) &= \int_{1/2}^{x} u^{-p}(1-u)^{-q}e^{Su}du, \qquad x\in(0,\,1),\\ m_y(x) &= \int_{1/2}^{x} u^{p-1}(1-u)^{q-1}e^{-Su}du, \quad x\in(0,\,1) \end{aligned}$$

respectively, where

$$(3)\qquad p=p_y=P+My, \quad q=q_y=Q+M(1-y).$$

Note that these functions are chosen so that they represent $\mathscr{D}_y$ as $\mathscr{D}_y=d^2/dm_y ds_y$. The classification of the boundaries in Feller's sense is as follows (see [2], p. 130);
(i) the boundary 0 is exit and non-entrance (regular, entrance and non-exit) if $p_y=0$ (resp. $0<p_y<1$, $1\le p_y$),
(ii) the boundary 1 is exit and non-entrance (regular, entrance and non-exit) if $q_y=0$ (resp. $0<q_y<1$, $1\le q_y$).

Further, the equations (2) and (5) in §1 are rewritten as

$$(4)\quad \begin{aligned} 4N\frac{d}{dt}\int_{0-}^{1+} f(x)U(t,\,dx) &= \int_{0-}^{1+}[\mathscr{D}_{\bar{x}(t)}f(x)+C(x-\bar{x}(t))f(x)]U(t,\,dx),\\ &t\in(0,\,+\infty), \quad \text{where} \quad \bar{x}(t)=\int_{0-}^{1+} xU(t,\,dx) \end{aligned}$$

and

$$(5)\quad \int_{0-}^{1+}[\mathscr{D}_{\bar{x}}f(x)+C(x-\bar{x})f(x)]U(dx)=0, \quad \text{where} \quad \bar{x}=\int_{0-}^{1+} xU(dx)$$

respectively, for all functions f of class C^2 on [0, 1]. Notice that the equation (4) corresponds to a diffusion process with reflecting boundary condition if the relevant boundary is regular. The difference equation we will make use of for the study of (4) is the following

$$(6)\quad \begin{aligned} &\int_{0-}^{1+}[\mathscr{D}_{\bar{x}(k+1)}f(x)+C(x-\bar{x}(k+1))f(x)]U(k+1,\,dx)\\ &\qquad =\frac{4N}{h}\left\{\int_{0-}^{1+} f(x)U(k+1,\,dx)-\int_{0-}^{1+} f(x)\,U(k,\,dx)\right\}.\\ &\qquad \text{with} \quad \bar{x}(k+1)=\int_{0-}^{1+} xU(k+1,\,dx), \quad k=0,\,1,\,2,\dots, \end{aligned}$$

where h is the time mesh. The equation (6) is reduced to (20) in §1, that is

$$(7)\quad \begin{aligned} &\int_{0-}^{1+}[\mathscr{D}_{\bar{x}}f(x)+\{C(x-\bar{x})-\Lambda\}f(x)]U(dx)=-\Lambda\int_{0-}^{1+} f(x)\Phi(dx)\\ &\qquad\qquad \text{with} \quad \bar{x}=\int_{0-}^{1+} xU(dx), \end{aligned}$$

for all functions f of class C^2 on [0, 1], where $\Lambda=4N/h$ as in (1) of §2. To solve the equation (7) we consider, for each y, $z\in[0,\,1]$, an auxiliary equation

(8) $$\int_{0-}^{1+}[\mathscr{D}_y f(x)+\{C(x-z)-\Lambda\}f(x)]\,U(dx)=-\Lambda\int_{0-}^{1+}f(x)\Phi(dx),$$

for all functions f of class C^2 on $[0, 1]$. We also need, for each $y,\ z\in[0, 1]$, the operator $\mathscr{G}_{y,z}$ on the space of all real sequences $\gamma=\{\gamma_n\}_{n=0}^{\infty}$ defined by

(9) $$\begin{aligned}(\mathscr{G}_{y,z}\gamma)_n=(nS+C)\gamma_{n+1}+n(n-1+P+My)\gamma_{n-1}\\ -\{n(n-1+P+Q+M+S)+Cz\}\gamma_n,\quad n=0,1,2,\dots,\end{aligned}$$

where $\gamma_{-1}=0$ by convention. We note that a measure $U(dx)$ on $[0, 1]$ solves the equation (8), if and only if its moment sequence $\gamma_n=M_n(U)$, $n=0, 1, 2,\dots$, is completely monotone and solves the equation

(10) $$(\mathscr{G}_{y,z}\gamma)_n-\Lambda\gamma_n=-\Lambda M_n(\Phi),\quad n=0,1,2,\dots$$

Lemma 3.1. *Let $\Lambda>C$, $y,\ z\in[0, 1]$ and $\Phi\in\mathscr{P}$. Then the equation* (8) *admits a unique solution $U_{y,z}(dx)=U_{y,z}(dx;\Lambda,\Phi)$ in $\mathscr{M}$. It satisfies*

(11) $$U_{y,z}([0,1];\Lambda,\Phi)\le\Lambda/(\Lambda-C),$$

and the sequence $\gamma_n=\gamma_n(y,z;\Lambda,\Phi)=M_n(U_{y,z})$ is a unique bounded solution of (10). *Further, the function $\gamma_n(y,z;\Lambda,\Phi)$ is continuous in $y,\ z\in[0, 1]$.*

Proof. It is well known that there exists a positive increasing (decreasing) solution g_1 (resp. g_2) of the equation

$$\mathscr{D}_y g(x)+C(x-z)g(x)=\Lambda g(x),\quad x\in(0,1).$$

The function g_1 (resp. g_2) is unique up to a constant multiple either if the boundary 0 (resp. the boundary 1) is non-regular, or if we set a boundary condition $\lim_{x\downarrow 0}g_1^+(x)=0$ (resp. $\lim_{x\uparrow 1}g_2^+(x)=0$) when the boundary is regular (see [2], §4.6 or [6] e.g.). Further, it holds that $g_1(0)=0$ ($g_2(1)=0$) if and only if $P=y=0$ (resp. $Q=1-y=0$). In the above, the symbol $g^+(x)$ stands for the right derivative of $g(x)$ with respect to the scale function s_y; $g^+(x)=\lim_{\varepsilon\downarrow 0}\{g(x+\varepsilon)-g(x)\}/\{s_y(x+\varepsilon)-s_y(x)\}=g'(x)x^p(1-x)^q e^{-Sx}$.

In our case, $g_1(x)$ ($g_2(x)$) has a Taylor expansion at $x=0$ (resp. $x=1$) with the radius of convergence 1. Indeed, setting

$$g_1(x)=\sum_{n=0}^{\infty}G_n x^n\quad(\text{resp. } g_2(x)=\sum_{n=0}^{\infty}\hat{G}_n(1-x)^n),$$

we see that $\{G_n\}_{n=0}^{\infty}$ and $\{\hat{G}_n\}_{n=0}^{\infty}$ satisfy recurrence equations. Thus we can determine them inductively if we start with

$$G_1=\hat{G}_1=1,\quad G_0=p_y/(\Lambda+Cz),\quad \hat{G}_0=q_y/(\Lambda-C+Cz).$$

Actually we have a more precise formula for $\{G_n\}_{n=0}^{\infty}$. Let

$$a_n=\frac{n(n-1+P+My)}{nS+C},\quad b_n=\frac{n(n-1+P+Q+M+S)+Cz+\Lambda}{nS+C},\quad n=1,2,\dots,$$

and $\{\xi_n^{(1)}\}_{n=0}^{\infty}$ and $\{\xi_n^{(2)}\}_{n=0}^{\infty}$ be those in Lemma A.3 (see also (35) and (36) of §A). Then we have

$$G_n = \frac{S+C}{(nS+C)a_2\cdots a_n}\left\{\frac{\Lambda+Cz-CF_1}{\Lambda+Cz}\xi_n^{(1)} + \frac{C}{\Lambda+Cz}\xi_n^{(2)}\right\} \qquad \text{for} \quad n\geq 2.$$

This formula shows that $G_n>0$ for all $n\geq 1$ if $\Lambda>C$.

Define as usual the Green function $G(x, w)$, x, $w\in(0, 1)$ by

$$G(x, w)=G(w, x)=B^{-1}g_1(x)g_2(w), \quad x\leq w, \quad x, w\in(0, 1),$$

where $B=g_1^+(1/2)g_2(1/2)-g_1(1/2)g_2^+(1/2)$. We also set $u_{y,z}(x)=\Lambda\int_{0-}^{1+}G(x, w)\Phi(dw)$. Notice that the function $u(x)=u_{y,z}(x)$ satisfies

$$\int_0^1[\mathscr{D}_y f(x)+\{C(x-z)-\Lambda\}f(x)]u(x)dm_y(x)$$

$$=[f^+(x)u(x)]_{0+}^{1-}-[f(x)u^+(x)]_{0+}^{1-}-\Lambda\int_{0+}^{1-}f(x)\Phi(dx),$$

for all functions f of class C^2 on $[0, 1]$.

Assume now that $P+y>0$ and $Q+1-y>0$. Then we have $B=-(g_1g_2^+)(0+)=(g_1^+g_2)(1-)$, and the measure $U_{y,z}(dx)=u_{y,z}(x)dm_y(x)$ satisfies (8) and (11). Hence $\gamma_n=M_n(U_{y,z})$ is a bounded solution of (10).

Assume next that $P=y=0$. In this case, it follows that $B=(g_1^+g_2)(0+)=(g_1^+g_2)(1-)$. Hence we can see that the measure

$$U_{0,z}(dx)=u_{0,z}(x)dm_0(x)+A_0(\Phi)\delta_0(dx)$$

$$\text{with} \quad A_0(\Phi)=\Lambda g_1^+(0)\int_{0-}^{1+}g_2(x)\Phi(dx)/B(Cz+\Lambda)$$

satisfies (11) and the sequence $\gamma_n=M_n(U_{0,z})$ is a bounded solution of (10). Hence $U_{0,z}(dx)$ solves (8).

Similarly, in the case of $Q=1-y=0$, it holds that $B=-(g_1g_2^+)(0+)=-(g_1g_2^+)(1-)$. Thus we see that the measure

$$U_{1,z}(dx)=u_{1,z}(x)dm_1(x)+A_1(\Phi)\delta_1(dx)$$

$$\text{with} \quad A_1(\Phi)=-\Lambda g_2^+(1)\int_{0-}^{1+}g_1(x)\Phi(dx)/B(Cz+\Lambda-C)$$

satisfies the desired conditions.

We will now prove that a solution of (8) is unique. For this, it suffices to show that a bounded solution of (10) is unique. Assume that there are two bounded solutions $\gamma_n^{(1)}$ and $\gamma_n^{(2)}$. Then the difference $\gamma_n=\gamma_n^{(1)}-\gamma_n^{(2)}$ satisfies the equation

(12) $$(\mathscr{G}_{y,z}\gamma)_n-\Lambda\gamma_n=0, \quad n=0, 1, 2,\ldots$$

Let $\{\xi_n^{(1)}\}_{n=0}^{\infty}$ and $\{\xi_n^{(2)}\}_{n=0}^{\infty}$ be those in the above. Then we can express γ_n as $\gamma_n=A_1\xi_n^{(1)}+A_2\xi_n^{(2)}$ for some constants A_1 and A_2. Since γ_n is bounded, A_1 must be zero by Lemma A.3. Hence the equation (12) for $n=0$ is reduced to

$$A_2\{C(\xi_1^{(2)}-\xi_0^{(2)})-(\Lambda-Cz)\xi_0^{(2)}\}=0.$$

Since the factor in the parentheses is negative, we have $A_2=0$ and $\gamma_n^{(1)}-\gamma_n^{(2)}=0$.

To show the continuity of $\gamma_n(y, z; \Lambda, \Phi)$ in $y, z\in[0, 1]$, take $y_0, z_0\in[0, 1]$ and $\{(y_k, z_k)\}$ such that $\lim_{k\to+\infty}(y_k, z_k)=(y_0, z_0)$. Since the system $\{U_{y_k,z_k}(dx)\}$ is tight in $\mathscr{M}$ (relatively compact in $\mathscr{M}$ with the weak topology), we can choose its sub-sequence which tends to a $U(dx)\in\mathscr{M}$. Then $\gamma_n=M_n(U)$ is a bounded solution of (10) with $y=y_0$ and $z=z_0$. Hence due to the uniqueness, we have $\gamma_n=\gamma_n(y_0, z_0; \Lambda, \Phi)$, which implies $U(dx)=U_{y_0,z_0}(dx; \Lambda, \Phi)$. Since $\{(y_k, z_k)\}$ is arbitrary, the continuity follows. The Lemma is proved.

Lemma 3.2. *Let* $\Lambda>C$ *and* $\Phi\in\mathscr{P}$. *Then the equation* (7) *has at least one solution in* $\mathscr{P}$. *The set of means of the solutions of* (7) *is a non-empty closed set in* $[0, 1]$, *possibly a singleton. Further it is included in* $(0, 1]$ (*resp.* $[0, 1)$) *unless* $P=0$ *and* $\Phi=\delta_0$ ($Q=0$ *and* $\Phi=\delta_1$).

Proof. Notice first that the equation (10) for $n=0$ is rewritten as

(13) $C\{\gamma_1(y, z; \Lambda, \Phi)-z\gamma_0(y, z; \Lambda, \Phi)\}-\Lambda\gamma_0(y, z; \Lambda, \Phi)=-\Lambda, \quad y, z\in[0, 1].$

Let E_0 be the set of solutions of the equation

(14) $$\gamma_0(y, y; \Lambda, \Phi)=1, \quad 0\le y\le 1$$

and E_1 be that of solutions of

(15) $$\gamma_1(y, y; \Lambda, \Phi)=y, \quad 0\le y\le 1.$$

We also denote $E=E_0\cap E_1$. By the continuity of $\gamma_n(y, z; \Lambda, \Phi)$ in $y, z\in[0, 1]$, the set E is closed.

Suppose now that $C>0$. It then follows from (13) and

$$\gamma_1(0, 0; \Lambda, \Phi)=\int_0^1 xu_{0,0}(x; \Lambda, \Phi)dm_y(x)\ge 0$$

$$(\gamma_0(1, 1; \Lambda, \Phi)-\gamma_1(1, 1; \Lambda, \Phi)=\int_0^1(1-x)u_{1,1}(x; \Lambda, \Phi)dm_y(x)\ge 0)$$

that $\gamma_0(0, 0; \Lambda, \Phi)\ge 1$ (resp. $\gamma_0(1, 1; \Lambda, \Phi)\le 1$). Hence we can see that the set E_0 is not empty. Since $E_0\subset E_1$ in this case, this assures that E is non-empty. Also in the case of $C=0$, the set E is non-empty, because of the relations $E=E_1\subset E_0=[0, 1]$, $\gamma_1(0, 0; \Lambda, \Phi)\ge 0$ and $\gamma_1(1, 1; \Lambda, \Phi)\le 1$ for this case.

Now, if $P>0$ or $\Phi\neq\delta_0$, then $g_1(0)>0$ or $\Phi((0, 1])>0$ respectively. Hence it follows that $u_{0,0}(x; \Lambda, \Phi)>0$ and $\gamma_1(0, 0; \Lambda, \Phi)>0$. Thus we see that the set E is included in $(0, 1]$ in both of the cases $C=0$ and $C>0$. By the same way, if $Q>0$ or $\Phi\neq\delta_1$, then $\gamma_0(1, 1; \Lambda, \Phi)-\gamma_1(1, 1; \Lambda, \Phi)>0$. Hence, the set E is included in $[0, 1)$ for $C\ge 0$.

Finally, for each $\bar{x}\in E$, the measure

$$U(dx)=U_{\bar{x},\bar{x}}(dx; \Lambda, \Phi)$$

is a solution of the equation (7) in $\mathscr{P}$, and conversely, the mean $\bar{x}$ of each solution of the equation (7) in $\mathscr{P}$ belongs to the set E. Hence the assertion (3) follows. The Lemma is proved.

In the sequel we denote a solution of the equation (7) with the smallest mean by $U(dx; \Lambda, \Phi)$ and its mean by $\bar{x}(\Lambda, \Phi)$. Notice that $\bar{x}(\Lambda, \Phi)$ coincides with the smallest solution of (15).

Proof of Lemma 1.2. Assume that (21) in §1 holds. We will first show that

$$\bar{x}(\Lambda, \Phi_1) \leq \bar{x}(\Lambda, \Phi_2). \tag{16}$$

For each $y, z \in [0, 1]$, let $\gamma_n^{(i)} = \gamma_n(y, z; \Lambda, \Phi_i)$, $n = 0, 1, 2, \ldots$, $i = 1, 2$. Then it follows from the equation (10) that

$$(\mathscr{G}_{y,z}(\gamma^{(2)} - \gamma^{(1)}))_n - \Lambda(\gamma_n^{(2)} - \gamma_n^{(1)}) = -\Lambda(M_n(\Phi_2) - M_n(\Phi_1)), \quad n = 0, 1, 2, \ldots$$

The right hand sides of the above formulas are non-positive for all $n = 0, 1, 2, \ldots$. Further $\lim_{n \to +\infty}(\gamma_n^{(2)} - \gamma_n^{(1)}) = 0$ if $q_y > 0$, and $= A_1(\Phi_2) - A_1(\Phi_1) \geq 0$ if $q_y = 0$ since g_1 has a Taylor expansion at 0 with positive coefficients. Hence, by Lemma A.4, we obtain $(\gamma^{(1)} - \gamma^{(1)})_n \geq 0$, $n = 0, 1, 2, \ldots$, that is,

$$\gamma_n(y, z; \Lambda, \Phi_1) \leq \gamma_n(y, z; \Lambda, \Phi_2), \quad n = 0, 1, 2, \ldots, \quad y, z \in [0, 1]. \tag{17}$$

Especially, it holds that $\gamma_0(y, y; \Lambda, \Phi_1) \leq \gamma_0(y, y; \Lambda, \Phi_2)$ and $\gamma_1(y, y; \Lambda, \Phi_1) \leq \gamma_1(y, y; \Lambda, \Phi_2)$ for all $y \in [0, 1]$, which proves (16).

For the proof of (22) in §1, let $\bar{x}^{(i)} = \bar{x}(\Lambda, \Phi_i)$, $i = 1, 2$ and $\gamma_n^{(i)} = \gamma_n(\bar{x}^{(i)}, \bar{x}^{(i)}; \Lambda, \Phi_i)$, $n = 0, 1, 2, \ldots$, $i = 1, 2$. Then, by means of (10), the sequences $\{\gamma_n^{(i)}\}_{n=0}^{\infty}$, $i = 1, 2$ satisfy the equation

$$\begin{aligned} &(\mathscr{G}_{\bar{x}^{(i)}, \bar{x}^{(i)}}\gamma^{(i)})_n - \Lambda\gamma_n^{(i)} = -\Lambda M_n(\Phi_i), \quad n = 1, 2, \ldots, \\ &\gamma_0^{(i)} = 1, \qquad\qquad i = 1, 2. \end{aligned} \tag{18}$$

Subtracting both sides of (18) for $i = 1$ from the corresponding ones for $i = 2$, we obtain

$$\begin{aligned} &(\mathscr{G}_{\bar{x}^{(2)}, \bar{x}^{(2)}}(\gamma^{(2)} - \gamma^{(1)}))_n - \Lambda(\gamma^{(2)} - \gamma^{(1)})_n \\ &= -(\bar{x}^{(2)} - \bar{x}^{(1)})\{nM\gamma_{n-1}^{(1)} - C\gamma_n^{(1)}\} - \Lambda\{M_n(\Phi_2) - M_n(\Phi_1)\}, \quad n = 1, 2, \ldots, \\ &\gamma_0^{(2)} - \gamma_0^{(1)} = 0. \end{aligned} \tag{19}$$

Since the sequence $\{\gamma_n^{(1)}\}_{n=0}^{\infty}$ is non-increasing in n, the inequality (16) and our assumption $M \geq C$ imply that the right hand side of (19) is non-positive for $n = 1, 2, \ldots$. Further, $\lim_{n \to +\infty}(\gamma_n^{(2)} - \gamma_n^{(1)}) \geq 0$. Hence, due to Lemma A.4 again, we arrive at the inequalities $(\gamma^{(2)} - \gamma^{(1)})_n \geq 0$, $n = 0, 1, 2, \ldots$, whence the desired (22) in §1 follows. The Lemma is proved.

Proof of Theorem 3 (i). *Step* 1. Let $\Lambda = 4N/h > C$. For each $\Phi \in \mathscr{P}$, we inductively define a sequence of measures $\{U(k, dx; \Lambda, \Phi)\}_{k=0}^{\infty}$ by

$$(20)\qquad \begin{aligned} &U(0, dx; \Lambda, \Phi)=\Phi(dx),\\ &U(k+1, dx; \Lambda, \Phi)=U(dx; \Lambda, U(k,\cdot\,; \Lambda, \Phi)), \quad k=0, 1, 2,\dots \end{aligned}$$

Then it is clear that the sequence $\{U(k, dx; \Lambda, \Phi)\}_{k=0}^{\infty}$ satisfies the system of equations (6). We define next a system of measures $\{U_\Lambda(t, dx; \Phi)\}_{t\geq 0}$ by

$$(21)\qquad \begin{aligned} U_\Lambda(t, dx; \Phi)=\Lambda\{(k+1)h-t\}U(k, dx; \Lambda, \Phi)+\Lambda\{t-kh\}U(k+1, dx; \Lambda, \Phi),\\ \text{for}\quad t\in[kh, (k+1)h), \quad k=0, 1, 2,\dots. \end{aligned}$$

Then, by the exactly same way as in Shimakura [8; §6], we can see that

$$(22)\qquad \lim_{\Lambda\to+\infty} U_\Lambda(t, dx; \Phi)=U(t, dx; \Phi), \text{ weakly for each } t\geq 0,$$

where $U(t, dx; \Phi)$ is the solution of (4) with the initial measure $\Phi(dx)$. Note that the proof of (22) is based on the fact that the system of moments

$$\{M_n(U_\Lambda(t,\cdot\,; \Phi));\ 0\leq t\leq T\}_{\Lambda>C}$$

forms an Ascoli-Arzelà sequence for all $n=0, 1, 2,\dots$ and $T>0$.

Step 2. Fix a $\Phi\in\mathscr{P}$. By Lemma 1.2, we can easily show by induction that

$$\begin{aligned} &M_n(U(k,\cdot\,; \Lambda, \delta_0))\leq M_n(U(k,\cdot\,; \Lambda, \Phi))\leq M_n(U(k,\cdot\,; \Lambda, \delta_1)),\\ &M_n(U(k+1,\cdot\,; \Lambda, \delta_0))\geq M_n(U(k,\cdot\,; \Lambda, \delta_0)),\\ &M_n(U(k+1,\cdot\,; \Lambda, \delta_1))\leq M_n(U(k,\cdot\,; \Lambda, \delta_1)), \quad k, n=0, 1, 2,\dots, \end{aligned}$$

for all $\Lambda>C$. Hence, due to (21) and (22), the sequence $\{M_n(U(t,\cdot\,; \delta_0))\}_{t\geq 0}$ ($\{M_n(U(t,\cdot\,; \delta_1))\}_{t\geq 0}$) is non-decreasing (resp. non-increasing) in t for each $n=0, 1, 2,\dots$ and it holds that

$$(23)\qquad M_n(U(t,\cdot\,; \delta_0))\leq M_n(U(t,\cdot\,; \Phi))\leq M_n(U(t,\cdot\,; \delta_1)), \quad n=0, 1, 2,\dots, \quad t\geq 0.$$

Therefore there exist the limits

$$M_n^{(0)}=\lim_{t\to+\infty} M_n(U(t,\cdot\,; \delta_0)), \quad M_n^{(1)}=\lim_{t\to+\infty} M_n(U(t,\cdot\,; \delta_1)), \quad n=0, 1, 2,\dots,$$

and

$$\lim_{t\to+\infty}\frac{d}{dt} M_n(U(t,\,\cdot\,; \delta_0))=\lim_{t\to+\infty}\frac{d}{dt} M_n(U(t,\,\cdot\,; \delta_1))=0, \quad n=0, 1, 2,\dots$$

This with the equation (4) implies that both of $\{M_n^{(0)}\}_{n=0}^{\infty}$ and $\{M_n^{(1)}\}_{n=0}^{\infty}$ satisfy (SM.1) and (SM.2) in §1. But due to Theorem 2, the solution of (SM.1) and (SM.2) in §1 is unique. Hence we have

$$M_n\equiv M_n^{(0)}=M_n^{(1)}, \quad n=0, 1, 2,\dots$$

This with (23) shows that

$$(24)\qquad M_n=\lim_{t\to+\infty} M_n(U(t,\cdot\,; \Phi)), \quad n=0, 1, 2,\dots$$

Thus, by the continuity theorem of moments, we obtain

(25) $$U(dx)=\lim_{t\to+\infty} U(t, dx; \Phi), \quad \text{weakly},$$

where $U(dx)$ is the unique stationary solution of the problem (SK).

Theorem 3 (i) is proved.

We next turn to the proof of Theorem 3 (ii). Until the end of the proof of Lemma 3.4, we assume $C>0$. Let $y\in(0, 1)$. Then, due to the classification of boundaries at the beginning of this section, both of the boundaries 0 and 1 of the the diffusion process corresponding to the differential operator $\mathscr{D}_y$ are regular or 'entrace and non-exit'. Hence the Green operators corresponding to the diffusion operator $\mathscr{D}_y+CxI$ (with the boundary condition $u^+(0+)=0$ (or $u^+(1-)=0$) if the boundary is regular) are compact operators on the Hilbert space $L^2([0, 1], dm_y)$ and the generator $\mathscr{D}_y+CxI$ admits the maximal eigenvalue $C\alpha(y)$ and the corresponding positive eigenfunction $\varphi_y(x)$ in $L^2([0, 1], dm_y)$. Since $0<m_y(1)-m_y(0)<+\infty$, the eigenfunction $\varphi_y(x)$ also belongs to $L^1([0, 1], dm_y)$. Thus we can normalize it so that

(26) $$\int_0^1 \varphi_y(x)dm_y(x)=1,$$

which makes the measure $\Phi_y(dx)\equiv\varphi_y(x)dm_y(x)$ a probability measure.

We will later show that, in the case of (d) ((e)), $\Phi\in\mathscr{P}_*$ (resp. $\Phi\in\mathscr{P}^*$) implies the relations $M_n(\Phi)\geq M_n(\Phi_y)$ (resp. $M_n(\Phi)\leq M_n(\Phi_y)$), $n=0, 1, 2,\dots$ for some $y\in(0, \bar{x})$ (resp. $(\bar{x}, 1)$) (see Lemma 3.5). Further, we will show that the moment sequences $M_n(U(t,\cdot\,; \Phi_y))$, $n=1, 2,\dots$ are non-decreasing (non-increasing) in t for $y\in(0, \bar{x})$ (resp. $(\bar{x}, 1)$) in the case (d) (resp. (e)).

Now by virtue of the symmetry of $\mathscr{D}_y$ on the space of $L^2([0, 1], dm_y)$ (see the argument in the proof of Lemma 3.1 for details), it holds that

(27) $$\int_{0-}^{1+} [\mathscr{D}_y f(x)+C\{x-\alpha(y)\}f(x)]\Phi_y(dx)=0$$

for all functions f of class C^2 on $[0, 1]$. Hence the sequence $\gamma_n=\gamma_n(y)=M_n(\Phi_y)$, $n=0, 1, 2,\dots$ satisfies the equation

(28) $$(\mathscr{G}_{y,\alpha(y)}\gamma)_n=0, \quad n=0, 1, 2,\dots$$

Note that the equation (28) for $n=0$ is reduced to

(29) $$\alpha(y)=\gamma_1(y)=\int_0^1 x\varphi_y(x)dm_y(x)\in(0, 1),$$

and the equation (28) is equivalent to

(30) $$\gamma_{n+1}=b_n(\alpha(y))\gamma_n-a_n(y)\gamma_{n-1}, \quad n=0, 1, 2,\dots,$$

where a_n's and b_n's are given in (1) of §B. Motivated by this fact, we consider the difference equation

(31) $$\gamma_{n+1}=b_n(z)\gamma_n-a_n(y)\gamma_{n-1}, \quad n=0, 1, 2,\dots,$$

and the continued fractions

(32) $$H_n(y, z)=\frac{a_n(y)}{b_n(z)}-\frac{a_{n+1}(y)}{b_{n+1}(z)}-\cdots-\frac{a_{n+k}(y)}{b_{n+k}(z)}-\cdots$$

for all $n=1, 2, 3,\dots$, and $y, z\in(0, 1)$.

For each $y\in(0, 1)$, we can find an integer $n_0(\geq 1)$ such that

(33) $$b_n(z)\geq b_n(0)\geq a_n(y)+1 \qquad \text{for all} \quad z\in(0, 1) \quad \text{and} \quad n\geq n_0.$$

Hence, by virtue of Lemma A.1, the continued fraction $H_n(y, z)$ converges in power series sense for all $z\in(0, 1)$ and $n\geq n_0$. We define inductively a sequence $\{z_*^{(n)}(y)\}_{n=1}^{\infty}$ by

(34) $$\begin{aligned} &z_*^{(n)}(y)=0, && n\geq n_0,\\ &z_*^{(n)}(y)=\inf\{z\in(z_*^{(n+1)}(y), 1)\colon b_n(z)>H_{n+1}(y, z)\}, && n=n_0-1, n_0-2,\dots, 1. \end{aligned}$$

Since $b_n(z)$ is increasing in z, we can show by induction that $H_n(y, z)$ converges in power series sense for $z\in(z_*^{(n)}(y), 1)$ and decreasing in z there, which justifies the above definition. Clearly, $1\geq z_*^{(1)}(y)\geq z_*^{(2)}(y)\geq\cdots\geq z_*^{(n)}(y)\geq\cdots\geq 0$.

Lemma 3.3. (1) *The function $z_*^{(1)}(y)$ is non-decreasing in y. Further, for each $y\in(0, 1)$ and $n=1, 2,\dots$, the continued fraction $H_n(w, z)$ converges in power series sense for $(w, z)\in(0, y]\times(z_*^{(1)}(y), 1]$. It is increasing in w and decreasing in z on the rectangle.*
(2) *If the equation* (31) *has a non-increasing positive solution $\gamma=\{\gamma_n\}_{n=0}^{\infty}$ for some $y, z\in(0, 1)$, then $z>z_*^{(1)}(y)$ and $\gamma_n/\gamma_{n-1}=H_n(y, z)$, $n=1, 2,\dots$*
(3) *It holds that*

(35) $$z_*^{(1)}(y)<\alpha(y)\wedge y, \quad y\in(0, 1),$$

(36) $$H_n(y, \alpha(y))=M_n(\Phi_y)/M_{n-1}(\Phi_y), \quad n=1, 2,\dots, \quad y\in(0, 1),$$

where $a\wedge b$ stands for $\min\{a, b\}$.

Proof. (1) Noting that $a_n(y)$ and $b_n(z)$ are increasing in y and z respectively, one can easily prove the assertions by induction from the arguments just above the Lemma.
(2) Let $\gamma=\{\gamma_n\}_{n=0}^{\infty}$ be a non-increasing positive solution of (31), and set $h_n=\gamma_n/\gamma_{n-1}$, $n=1, 2,\dots$ (see Lemma A.3). Then by the exactly same way as in the proof of Lemma 1.1 in §2, it follows from (33) that $h_n=H_n(y, z)$ for all $n\geq n_0$. Further, by the relations $h_n=a_n(y)/(b_n(z)-h_{n+1})$ and $0<h_n\leq 1<+\infty$, we can show that $b_n(z)>h_{n+1}$ for $n=1, 2,\dots, n_0-1$. This proves the assertion.
(3) It follows from the proof of Lemma B.1 that $b_n(y)>F_{n+1}(y)$ for $n=1, 2,\dots$, and $y\in(0, 1)$. Since $F_n(y)=H_n(y, y)$, $n=1, 2,\dots$, we then have by induction that

$$b_n(z)>H_{n+1}(y, z) \qquad \text{for all} \quad z\geq y \quad \text{and} \quad n=n_0-1, n_0-2,\dots, 1.$$

Hence $z_*^{(1)}(y)<y$. On the other hand, due to (30), the assertion (2) applied to the sequence $\gamma_n=M_n(\Phi_y)$ assures $z_*^{(1)}(y)<\alpha(y)$. Hence the inequality (35) follows. The relation (36) is now clear from the assertion (2) and the equation (30) again.

The Lemma is proved.

Corollary 3.1. *For each* $y\in(0,1)$, *the eigenvalue* $\alpha(y)$ *is the unique solution* z *of the equation*

$$H_1(y,z)=z,\quad z_*^{(1)}(y)<z\le 1. \tag{37}$$

Proof. It is clear from (29) and (36) for $n=1$ that the eigenvalue $\alpha(y)$ solves the equation (37). On the other hand, a solution of (37) is unique, since $H_1(y,z)$ is decreasing in $z\in(z_*^{(1)}(y),1)$.

Lemma 3.4. (1) *The eigenvalue* $\alpha(y)$ *is increasing in* $y\in(0,1)$.
(2) *An* $\bar{x}\in(0,1)$ *is a solution of the equation* (15) *in* §1 *if and only if it solves the equation*

$$\alpha(y)=y,\quad y\in(0,1). \tag{38}$$

Further, for each such solution $\bar{x}\in(0,1)$, *it holds that* $F'(\bar{x})<1$ *if and only if* $\alpha'(\bar{x})<1$. *Hence, for each* $y\in(0,1)$, $F(y)<y$ $(=y,\ >y)$ *if and only if* $\alpha(y)<y$ *(resp.* $=y,\ >y$*).*
(3) *If* $P=0$ $(Q=0)$, *then*

$$\lim_{y\downarrow 0}\alpha(y)=0\quad (\textit{resp.}\ \lim_{y\uparrow 1}\alpha(y)=1). \tag{39}$$

Further, it holds that

$$\lim_{y\downarrow 0}\Phi_y(dx)=\delta_0(dx)\quad (\textit{resp.}\ \lim_{y\uparrow 1}\Phi_y(dx)=\delta_1(dx)). \tag{40}$$

Proof. (1) Suppose that $0<y_1<y_2<1$ and $\alpha(y_1)\ge\alpha(y_2)$. Then, from Lemma 3.3 (1) and Corollary 3.1, we have

$$\alpha(y_1)=H(y_1,\alpha(y_1))<H(y_2,\alpha(y_1))\le H(y_2,\alpha(y_2))=\alpha(y_2)$$

which contradicts $\alpha(y_1)\ge\alpha(y_2)$. Hence the assertion (1) follows.
(2) The first assertion is easily seen from Corollary 3.1 and the relation $F(y)=H_1(y, y)$. For the second assertion, differentiate the relation $\alpha(y)=H_1(y,\alpha(y))$ and put $y=\bar{x}$ to obtain

$$\alpha'(\bar{x})\left\{1-\frac{\partial}{\partial z}H_1(\bar{x},\bar{x})\right\}=\frac{\partial}{\partial y}H_1(\bar{x},\bar{x}). \tag{41}$$

On the other hand, from the relation $F(y)=H_1(y,y)$, we obtain

$$F'(\bar{x})=\frac{\partial}{\partial y}H_1(\bar{x},\bar{x})+\frac{\partial}{\partial z}H_1(\bar{x},\bar{x}).$$

Hence it follows that

$$(\alpha'(\bar{x})-1)\left\{1-\frac{\partial}{\partial z}H_1(\bar{x},\bar{x})\right\}=F'(\bar{x})-1, \tag{42}$$

which with the relation $\partial H_1(\bar{x}, \bar{x})/\partial z<0$ derives the conclusion.

The last assertion is a direct consequence of the first and the second ones.

(3) First we assume $P=0$. Due to the assumption $M\geq C$, we can find an $\varepsilon>0$ such that

$$(43) \qquad a_2(\varepsilon)<b_2(0)-1$$

and the inequalities in (33) holds for all $y\in(0, \varepsilon)$, $z\in(0, 1)$ and $n=3, 4,\dots$. Hence, by virtue of Lemma A.1, the continued fractions $H_2(y, z)$ and $H_3(y, z)$ converge in power series sense for all $(y, z)\in(0, \varepsilon)\times(0, 1)$. Further, it holds that $H_3(y, z)\leq 1$, $(y, z)\in(0, \varepsilon)\times(0, 1)$ and so

$$(44) \qquad 0<H_2(y, z)\leq \frac{a_2(\varepsilon)}{b_2(0)-1}<1, \quad (y, z)\in(0, \varepsilon)\times(0, 1).$$

But $b_1(z)\geq b_1(0)\geq 1$. Hence we have the convergence of $H_1(y, z)$ for $(y, z)\in(0, \varepsilon)\times(0, 1)$ and

$$(45) \quad 0<H_1(y, z)\leq \frac{(b_2(0)-1)My}{(S+C)(b_2(0)-1-a_2(\varepsilon))}<+\infty, \quad (y, z)\in(0, \varepsilon)\times(0, 1).$$

This with the relation $\alpha(y)=H_1(y, \alpha(y))$ implies $\lim_{y\downarrow 0}\alpha(y)=0$, proving (39). Combining (39) with (30), we inductively obtain that

$$\lim_{y\downarrow 0} M_n(\Phi_y)=0, \quad n=1, 2,\dots$$

Hence the formula (40) follows.

We next turn to the case of $Q=0$. Notice first that $z_*^{(1)}(y)<1$ and

$$1\geq\alpha(y)=H_1(y, \alpha(y))\geq H_1(y, 1), \quad y\in(0, 1).$$

On the other hand, we have $H_1(y, 1)\uparrow F(1)$ as $y\uparrow 1$, and $F(1)=1$ by Lemma B.1 (iii). Hence the relation $\lim_{y\uparrow 1}\alpha(y)=1$ follows. This with (30) implies

$$\lim_{y\uparrow 1} M_n(\Phi_y)=1, \quad n=1, 2,\dots$$

inductively, proving the relation (40) for this case.

The Lemma is proved.

In the case of $C=0$, the maximal eigenvalue of the generator $\mathscr{D}_y$ is equal to 0 and the corresponding eigenfunction $\varphi_y(x)$ is just a positive constant. Hence we can not fix the function $\alpha(y)$ uniquely. In the following, we set

$$(46) \qquad \alpha(y)=F(y), \quad y\in[0, 1] \quad \text{whenever} \quad C=0.$$

Notice that the formulas from (27) to (30) are also valid in the case of $C=0$ with the convention (46).

Lemma 3.5. *It holds that*

$$(47) \qquad \limsup_{n\to+\infty}\sup_{y\in(0,1)} M_n(\Phi_y)n^{Q+M(1-y)}<\infty,$$

(48) $$\liminf_{n\to+\infty}\ \inf_{y\in(1/2,1)} M_n(\Phi_y)n^{Q+M(1-y)}>0.$$

Proof. By the same way as in the Proof (ii) and (iv) of Lemma 2.1, we can write down the eigenfunction $\varphi_y(x)$ explicitly:

(49) $$\varphi_y(x)=g_y(x)/\int_0^1 g_y(x)dm_y(x), \quad x\in[0, 1],$$

where

(50) $$g_y(x)=1+\sum_{j=1}^{\infty}\frac{Cx^j}{jS+C}\prod_{k=1}^{j}\frac{H_k(y, \alpha(y))}{a_k(y)}.$$

Since $g_y(x)$ is increasing in x, we have

(51) $$1\le g_y(x)\le 1+\frac{\alpha(y)}{a_1(y)}\sum_{j=1}^{\infty}\frac{C}{jS+C}\prod_{k=2}^{j}\frac{H_k(y, \alpha(y))}{a_k(y)},$$

where $\prod_{k=2}^{1} H_k/a_k=1$ by convention. Further it holds that

$$\sup_{y\in(0,1)}\sum_{j=1}^{\infty}\frac{C}{jS+C}\prod_{k=2}^{j}\frac{H_k(y, \alpha(y))}{a_k(y)}<\infty.$$

Hence we can find a positive constant K such that

(52) $$1\le g_y(x)\le 1+K\alpha(y)/(P+My), \quad x\in[0, 1], \quad y\in(0, 1).$$

This with (2) implies that

$$\begin{aligned}M_n(\Phi_y)&\le\left(1+\frac{K\alpha(y)}{P+My}\right)\int_0^1 x^n dm_y(x)/\int_0^1 dm_y(x)\\ &\le\left(1+\frac{K\alpha(y)}{P+My}\right)e^S\frac{\Gamma(P+n+My)\Gamma(P+Q+M)}{\Gamma(P+Q+M+n)\Gamma(P+My)}.\end{aligned}$$

Now by the well known formula $\lim_{n\to+\infty}n^{-a}\Gamma(n+a)/\Gamma(n)=1$ for the gamma function (see [1], 1.18.(5) e.g.), we obtain (47).

Similarly, we have the inequality

$$M_n(\Phi_y)\ge\left(1+\frac{K\alpha(y)}{P+My}\right)^{-1}e^{-S}\frac{\Gamma(P+n+My)\Gamma(P+Q+M)}{\Gamma(P+Q+M+n)\Gamma(P+My)},$$

which implies (48).

The Lemma is proved.

Proof of Theorem 3 (ii). Since the following proof for the case (d) is also valid for the case (e) with minor changes, we will only prove the assertion in the case of (d). Thus we assume in the sequel that

$$P=0, \quad Q>0, \quad M\ge C \quad \text{and} \quad L_0>0.$$

By Theorem 2, we have a unique L^1-solution $U(dx)$ with the mean $\bar{x}\in(0, 1)$. Further it holds that $\alpha(y)>y$ for all $y\in(0, \bar{x})$ by Lemma 3.4. In the following proof, we denote a solution of the equation (7) with the largest mean by $U(dx; \Lambda, \Phi)$ and its mean by $\bar{x}(\Lambda, \Phi)$, same symbols as those for the smallest one which are defined just after Lemma 3.2. Notice that all the parallel arguments there are also valid for these values.

Step 1. In this step, we will show the inequalities

$$(53) \qquad M_n(U(\cdot\,; \Lambda, \Phi_y)) \geq M_n(\Phi_y), \quad n=0, 1, 2,\dots, \quad y \in (0, \bar{x}),$$

for each $\Lambda > C$.

Let $y \in (0, \bar{x})$ and $\gamma_n^{(1)}(y) = M_n(\Phi_y)$, $n=0, 1, 2,\dots$ Then it follows from (28) that

$$(54) \qquad (\mathscr{G}_{y,\alpha(y)}\gamma^{(1)})_n - \Lambda\gamma_n^{(1)} = -\Lambda M_n(\Phi_y), \quad n=0, 1, 2,\dots$$

On the other hand, the sequence $\gamma_n^{(2)} = \gamma_n(\alpha(y), \alpha(y); \Phi_y)$ (see Lemma 3.1 for the definition of $\gamma_n(y, z; \Lambda, \Phi)$) satisfies

$$(55) \qquad (\mathscr{G}_{\alpha(y),\alpha(y)}\gamma^{(2)})_n - \Lambda\gamma_n^{(2)} = -\Lambda M_n(\Phi_y), \quad n=0, 1, 2,\dots$$

Subtracting both sides of (54) from those of (55), we then arrive at

$$(\mathscr{G}_{\alpha(y),\alpha(y)}(\gamma^{(2)} - \gamma^{(1)}))_n - \Lambda(\gamma^{(2)} - \gamma^{(1)})_n = -nM(\alpha(y) - y)\gamma_{n-1}^{(1)}, \quad n=0, 1, 2,\dots,$$

with the relations $\lim_{n\to+\infty} \gamma_n^{(i)} = 0$, $i=1, 2$. Hence, due to the maximum principle Lemma A.4, we obtain

$$\gamma_n(\alpha(y), \alpha(y); \Lambda, \Phi_y) \geq M_n(\Phi_y), \quad n=0, 1, 2,\dots,$$

especially

$$\gamma_1(\alpha(y), \alpha(y); \Lambda, \Phi_y) \geq M_1(\Phi_y) = \alpha(y) > y.$$

Therefore, the largest solution $\bar{x}(\Lambda, \Phi)$ of the equation (15) is greater than $\alpha(y)$ by the arguments there. Now using the maximum principle again, we can easily obtain our conclusion (53).

Step 2. In this step, we will show the formula (25) where $U(dx)$ is the unique stationary L^1-solution of the problem (SK).

By Lemma 3.2, we have $\bar{x}(\Lambda, \delta_1) = M_1(U(\cdot\,; \Lambda, \delta_1)) < 1$, whence the measure $U(dx; \Lambda, \delta_1)$ is different from $\delta_1(dx)$. This with the inequalities (53) admits us to trace the arguments in Proof of Theorem 3 (i) to obtain

$$(56) \quad M_n(U) = \lim_{t\to+\infty} M_n(U(t, \cdot\,; \delta_1)) = \lim_{t\to+\infty} M_n(U(t, \cdot\,; \Phi_y)),$$

$$n=0, 1, 2,\dots, \quad y \in (0, \bar{x}).$$

Now take a $\Phi \in \mathscr{P}_*$. Then, comparing (17) in §1 with (47), we can find an $\varepsilon \in (0, \bar{x})$ and an integer $n_0 (\geq 1)$ such that

$$(57) \qquad M_n(\Phi) \geq M_n(\Phi_y), \quad y \in (0, \varepsilon), \quad n=n_0+1, n_0+2,\dots$$

Further, since $\Phi \neq \delta_0$, the formula (40) enables us to find a $y \in (0, \varepsilon)$ such that

$$(58) \qquad \min_{1\leq n\leq n_0} M_n(\Phi) \geq \max_{1\leq n\leq n_0} M_n(\Phi_y).$$

Now the inequalities (57) and (58) with the arguments in Proof of Theorem 3 (i) imply

$$M_n(U(t,\cdot\,;\,\Phi_y))\le M_n(U(t,\cdot\,;\,\Phi))\le M_n(U(t,\cdot\,;\,\delta_1)),$$

$$n=0,\,1,\,2,\ldots,\quad t\ge 0.$$

This and (56) prove the desired (25).

Theorem 3 (ii) is proved.

Proof of Theorem 3 (iii).

(59) $$H(t)=\int_{0-}^{1+}e^{Sx}U(t,\,dx)=\sum_{n+0}^{\infty}\frac{S^n}{n!}M_n(t).$$

Applying (2) of §1 to the function $f(x)=e^{Sx}$, we then have

(60) $$\frac{d}{dt}H(t)=\frac{C-SM}{4N}K(t),\quad\text{where}\quad K(t)=\int_{0-}^{1+}(x-\bar{x}(t))e^{Sx}U(t,\,dx).$$

Notice that the relations $e^{Sx}-1\le(e^S-1)x$ for $0\le x\le 1$ and $M_n(t)\ge\bar{x}(t)^n$ for $n=0$, 1, 2,... imply

(61) $$H(t)-1\le(e^S-1)\bar{x}(t),$$

(62) $$\bar{x}(t)\le(1/S)\log H(t).$$

In order to estimate the variance $M_2(t)-\bar{x}(t)^2$ by means of the function $K(t)$, we rewrite $K(t)$ as

$$K(t)=\frac{1}{2}\int_{0-}^{1+}\int_{0-}^{1+}(x-y)(e^{Sx}-e^{Sy})U(t,\,dx)U(t,\,dy).$$

Since $S(x-y)^2\le(x-y)(e^{Sx}-e^{Sy})\le Se^S(x-y)^2$ for $0\le x,\,y\le 1$, we have

(63) $$e^{-S}K(t)\le S\{M_2(t)-\bar{x}(t)^2\}\le K(t).$$

By the relation (60), we see that $K(t)$ is integrable over the interval $(0,\,+\infty)$, because $C\ne SM$ by our assumption and $H(t)$ is bounded. Therefore, by (63), the variance $M_2(t)-\bar{x}(t)^2$ is integrable. Furthermore, by the equation (4) of §1 (with $n=1$), we obtain

$$4N\frac{d\bar{x}(t)}{dt}=(S+C)(M_2(t)-\bar{x}(t)^2)-S\bar{x}(t)(1-\bar{x}(t)),$$

which yields

(64) $$\int_0^{+\infty}\bar{x}(t)(1-\bar{x}(t))dt<+\infty.$$

Suppose now that $C>SM$. Then $H(t)$ is non-decreasing by (60) and $H(0)>1$ unless the initial measure is equal to δ_0. Hence, by (61), the greatest lower bound of the mean $\bar{x}(t)$ is positive and, by (64), one has $\int_0^{+\infty}(1-\bar{x}(t))dt<+\infty$. Since the derivative $d\bar{x}(t)/dt$ is bounded, this implies that $\bar{x}(t)$ tends to 1 as $t\to+\infty$. Thus, due to the inequality $M_n(t)\geq\bar{x}(t)^n$ again, we see that $M_n(t)$ also tends to 1 for every $n\geq 1$. This means that the solution $U(t, dx)$ converges to δ_1 as $t\to+\infty$.

Suppose, on the contrary, that $C<SM$. In this case, $H(t)$ is non-increasing. Further $H(0)<e^S$ unless the initial measure is equal to δ_1. So the least upper bound of the mean $\bar{x}(t)$ is less than 1 by (62). Therefore we have $\int_0^{+\infty}\bar{x}(t)dt<+\infty$, which implies that $\bar{x}(t)$ tends to 0 as $t\to+\infty$. Hence, using the inequality $M_n(t)\leq\bar{x}(t)$, we see that $M_n(t)$ tends to 0 as $t\to+\infty$ for every $n\geq 1$. This ensures that the solution $U(t, dx)$ converges weakly to δ_0 as $t\to+\infty$.

The assertion (iii) of Theorem 3 is proved.

§ 4. Proof of Theorems 4 and 5.

Proof of Theorem 4. Let $U(dx)$ and $\{M_n\}_{n=0}^{\infty}$ be the stationary L^1-solution and its moment sequence. By Lemma 1.1, we have $0<M_1=\bar{x}=F(\bar{x})<1$ and $(\partial F/\partial y)(\bar{x})<1$. Moreover, the density function of $U(dx)$ is positive in (0, 1) as is shown in §2. Therefore, by Schwarz' inequality $M_n^2<M_{n-1}M_{n+1}$ and the relation $M_n=\prod_{k=1}^n F_k(\bar{x})$, we see that

$$(1)\qquad 0<\bar{x}\leq F_n(\bar{x})<F_{n+1}(\bar{x})<1 \qquad \text{for all} \quad n\geq 1.$$

This implies that

$$(2)\qquad \begin{matrix} d_{n,P}>0, \quad d_{n,Q}<0, \quad d_{n,S}<0, \\ d_{n,M}<0 \quad (\text{except } d_{1,M}=0), \quad d_{n,C}>0, \end{matrix} \qquad \text{for all } n\geq 1$$

at $y=\bar{x}$, where $d_{n,u}$'s are those defined by (7) in §B. Moreover, rewriting

$$a_n(y)=n\left(\frac{n-1}{4N}+v'+my\right)/(ns+c),$$

$$b_n(y)=\left\{n\left(\frac{n-1}{4N}+v'+v+s+m\right)+cy\right\}/(ns+c),$$

and defining $d_{n,N}=\partial a_n/\partial N-F_n\partial b_n/\partial N$, we have

$$(3)\qquad d_{1,N}=0 \quad \text{and} \quad d_{n,N}<0 \qquad \text{for} \quad n\geq 2.$$

Now, (1), (2) and (3) above with (6) in §B imply

$$(4)\qquad \begin{matrix} \frac{\partial}{\partial N}F_n<0, \quad \frac{\partial}{\partial v'}F_n>0, \quad \frac{\partial}{\partial v}F_n<0, \\ \frac{\partial}{\partial s}F_n<0, \quad \frac{\partial}{\partial m}F_n<0, \quad \frac{\partial}{\partial c}F_n>0, \end{matrix} \qquad \text{for all } n\geq 1$$

at $y=\bar{x}$ (note that $P=4Nv'$, $Q=4Nv$, $S=4Ns$, etc.).

Let us first show that the inequalities (23) in §1 holds for $n=1$. Let u denote one of the parameters N, v', v, s, m and c. Then, by the relation $\bar{x}=F(\bar{x})$ we have

$$\frac{\partial \bar{x}}{\partial u} = \frac{\partial F}{\partial u} \Big/ \left(1 - \frac{\partial F}{\partial y}(\bar{x})\right).$$

So the sign of $\partial \bar{x}/\partial u$ coincides with that of $\partial F/\partial u$. Therefore, by means of (4) for $n=1$, we see that the inequalities (23) in the Theorem hold for $n=1$. In order to verify (23) for $n \geq 2$, we put

$$\frac{\delta}{\delta u} F_n = \frac{\partial}{\partial u} \{F_n(\bar{x})\} = \left(\frac{\partial}{\partial u} F_n + \frac{\partial \bar{x}}{\partial u} \frac{\partial}{\partial y} F_n\right)\Big|_{y=\bar{x}}.$$

We will show that

$$(5) \qquad \begin{matrix} \frac{\delta}{\delta N} F_n < 0, & \frac{\delta}{\delta v'} F_n > 0, & \frac{\delta}{\delta v} F_n < 0, \\ \frac{\delta}{\delta s} F_n < 0, & \frac{\delta}{\delta m} F_n < 0, & \frac{\delta}{\delta c} F_n > 0, \end{matrix} \qquad \text{for all } n \geq 2.$$

Analogously to (6) of §B, we can prove that

$$\frac{\delta}{\delta u} F_n = \frac{F_n}{a_n} \left\{ e_{n,u} + \sum_{k=1}^{\infty} e_{n+k,u} \prod_{j=1}^{k} \frac{F_{n+j-1} F_{n+j}}{a_{n+j}} \right\},$$

$$\text{where} \quad e_{n,u} = d_{n,u} + \frac{\partial \bar{x}}{\partial u} d_{n,y}.$$

Since $d_{n,y} > 0$ for all n (because of $C \leq 2M$) and since the sign of $d_{n,u}$ coincides with that of $\partial \bar{x}/\partial u$ for $n \geq 2$, the sign of $e_{n,u}$ is the same as that of $\partial \bar{x}/\partial u$. Therefore (5) hold for $n \geq 2$. Finally, by the relation

$$\frac{1}{M_n} \frac{\partial}{\partial u} M_n = \sum_{k=1}^{n} \frac{1}{F_k} \frac{\delta}{\delta u} F_k,$$

we have the inequalities (23) in the Theorem also for $n \geq 2$.

Theorem 4 is proved.

Proof of Theorem 5. Let us begin with the assertion (i). Choose c_0 and c as in the Theorem and let $U(dx)$ be the stationary solution $U(dx)$ with the first moment $\bar{x}$. Assuming the case (a), we will show that $\bar{x} < \varepsilon/2$ whenever $v' \geq 0$ is sufficiently small (notice that $\bar{x} = 0$ in the case (b)).

Denote $F(y)$ by $F(y; v', v, c)$ and $\bar{x}$ by $\bar{x}(v', v, c)$ to distinguish the parameters v', v and c from the others. By Theorem 4, we have $\bar{x}(v', v, c) \leq \bar{x}(v', v, c_0) \leq \bar{x}(v', 0, c_0)$. Moreover, by continuity of F in v', $F(y; v', 0, c_0)$ tends (uniformly in y) to $F(y; 0, 0, c_0)$ as $v' \downarrow 0$ and $F(y; 0, 0, c_0) < y$ for $y \in (0, 1)$ due to Lemma B.3. So, there exists a positive number η such that $\bar{x}(v', 0, c_0) < \varepsilon/2$ if $v' < \eta$, that is $\bar{x}(v', v, c) < \varepsilon/2$ if $v' < \eta$. On the other hand, by Theorem 3, $U(t, dx)$ converges weakly to $U(dx)$ as $t \to +\infty$. So the assertion (i) of the Theorem follows.

For the proof of the assertion (ii), we can proceed quite similarly as above. Noticing that $\bar{x} = 1$ in the case (c), we assume the case (a). If c_1 and c are chosen as in the Theorem, then we have $\bar{x} = \bar{x}(v', v, c) \geq \bar{x}(v', v, c_1) \geq \bar{x}(0, v, c_1)$. Further $\bar{x}(0, v, c_1)$ tends to 1 as $v \downarrow 0$ because $F(y; 0, 0, c_1) > y$ for $y \in (0, 1)$. Therefore, $\bar{x}$

is sufficiently close to 1 if $v \geq 0$ is small enough. Moreover, $\bar{x}(t)$ tends to $\bar{x}$ as $t \to +\infty$. So the assertion (ii) follows.

Now, Theorem 5 is proved.

§ A. Continued fraction and recurrence equation.

We shall start with fixing the notion of convergence in power series sense of a continued fraction

$$(1) \qquad S = \frac{a_1}{b_1} - \frac{a_2}{b_2} - \cdots - \frac{a_n}{b_n} - \cdots,$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of real or complex numbers. Let us first deal with the special one

$$(2) \qquad T = T(z_1, \dots, z_n, \dots) = \frac{z_1}{1} - \frac{z_2}{1} - \cdots - \frac{z_n}{1} - \cdots.$$

Given a sequence of indeterminates $\{z_n\}_{n=1}^{\infty}$, define the polynomials $A_n = A_n(z_1, \dots, z_n)$ and $B_n = B_n(z_1, \dots, z_n)$ successively by

$$(3) \qquad \begin{aligned} &A_1 = A_2 = z_1, \quad B_1 = 1, \quad B_2 = 1 - z_2, \\ &A_{n+1} = A_n - z_{n+1} A_{n-1} \quad \text{and} \quad B_{n+1} = B_n - z_{n+1} B_{n-1} \qquad \text{for} \quad n \geq 2. \end{aligned}$$

For each $n \geq 1$, we shall call the meromorphic function

$$(4) \qquad T_n(z_1, \dots, z_n) = A_n / B_n$$

the n-th approximant of T. Then, it follows that

$$(5) \qquad \begin{aligned} &T_1(z_1) = z_1, \quad T_2(z_1, z_2) = z_1/(1 - z_2), \\ &T_{n+1}(z_1, \dots, z_{n+1}) = T_n(z_1, \dots, z_{n-1}, z_n/(1 - z_{n+1})), \end{aligned}$$

which proves

$$(6) \qquad T_1(z_1) = z_1, \quad T_n(z_1, \dots, z_n) = \frac{z_1}{1} - \frac{z_2}{1} - \cdots - \frac{z_n}{1} \quad \text{for } n \geq 2.$$

Hence the meromorphic function T_n is analytic at $(z_1, \dots, z_n) = (0, \dots, 0)$ and the coefficients in its expansion are nonnegative integers. Further, we have from (5) that

$$(7) \qquad T_{n+1}(z_1, \dots, z_{n+1}) = T_n(z_1, \dots, z_n) + z_{n+1} R_{n+1}(z_1, \dots, z_{n+1}),$$

where R_n is a function being analytic at $(z_1, \dots, z_{n+1}) = (0, \dots, 0)$ with nonnegative integer coefficients in its expansion. Actually, the relation (7) is nothing but the usual difference formula

$$(8) \qquad T_{n+1}(z_1, \dots, z_{n+1}) - T_n(z_1, \dots, z_n) = \frac{z_1 \cdots z_{n+1}}{B_n B_{n+1}} \qquad \text{for} \quad n \geq 1,$$

and $(z_1, \dots, z_{n+1}) = (0, \dots, 0)$ is a zero point of R_{n+1} of order n. From (7), we have

(9) $$T_n(z_1,\dots, z_n)= \sum_{j=1}^{n} z_j R_j(z_1,\dots, z_j),$$

with convention $R_1=1$. Denote also the homogeneous part of degree k in the expansion of T_n (resp. R_n) by $H_{n,k}(R_{n,k})$. Then $R_{n,k}=0$ for $0\le k\le n-2$, and the expression (9) is expanded as

$$T_n(z_1,\dots, z_n)= \sum_{k=1}^{n}\sum_{j=1}^{k} z_j R_{j,k-1}(z_1,\dots, z_j) + \sum_{k=n+1}^{\infty}\sum_{j=1}^{n} z_j R_{j,k-1}(z_1,\dots, z_j), \quad n=1, 2,\dots$$

Hence, for each $k=1, 2,\dots$, the polynomials $H_{n,k}$, $n\ge k$ are the same and equal to

$$H_k= \sum_{j=1}^{k} z_j R_{j,k-1}(z_1,\dots, z_j), \qquad \text{for} \quad n\ge k.$$

Definition 1. *We say that the continued fraction*

$$T(z_1,\dots, z_n,\dots)= \frac{z_1}{1} - \frac{z_2}{1} - \cdots - \frac{z_n}{1} - \cdots$$

converges in power series sense at $(z_1^0,\dots, z_n^0,\dots)$ *if and only if*

(10) $$\sum_{k=1}^{\infty} H_k(|z_1^0|,\dots, |z_k^0|)<\infty .$$

In this case, we define $T(z_1^0,\dots, z_n^0,\dots)$ *by*

(11) $$T(z_1^0,\dots, z_n^0,\dots)= \sum_{k=1}^{\infty} H_k(z_1^0,\dots, z_k^0).$$

Notice that the condition (10) coincides with

$$\sum_{k=1}^{\infty}\sum_{j=1}^{k} |z_j^0| R_{j,k-1}(|z_1^0|,\dots, |z_j^0|)<\infty$$

and, in this case, the formula (11) is equivalent to

(12) $$T(z_1^0,\dots, z_n^0,\dots)= \sum_{k=1}^{\infty}\sum_{j=1}^{k} z_j^0 R_{j,k-1}(z_1^0,\dots, z_k^0),$$

which implies

(13) $$T(z_1^0,\dots, z_n^0,\dots)= \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} z_j^0 R_{j,k-1}(z_1^0,\dots, z_k^0)=\lim_{n\to\infty} T_n(z_1^0,\dots, z_n^0).$$

We now turn to a general continued fraction S in (1). Two continued fractions

(14) $$S= \frac{a_1}{b_1} - \frac{a_2}{b_2} - \cdots - \frac{a_n}{b_n} - \cdots \quad \text{and} \quad S'= \frac{a'_1}{b'_1} - \frac{a'_2}{b'_2} - \cdots - \frac{a'_n}{b'_n} - \cdots$$

are said to be equivalent if and only if there exists a sequence of non-zero numbers $\{r_n\}_{n=0}^{\infty}$ with $r_0=1$ such that

(15) $$a'_n = r_{n-1} r_n a_n \quad \text{and} \quad b'_n = r_n b_n \qquad \text{for every} \quad n \geq 1.$$

Notice that, in this case, the n-th approximants

(16) $$S_n = \frac{a_1}{b_1} - \frac{a_2}{b_2} - \cdots - \frac{a_n}{b_n} \quad \text{and} \quad S'_n = \frac{a'_1}{b'_1} - \frac{a'_2}{b'_2} - \cdots - \frac{a'_n}{b'_n}$$

of S and S' coincide with each other for all $n=1, 2,\dots$. Taking now $r_n = 1/b_n$ for $n \geq 1$, we see that $S_n = T_n(z_1^0,\dots, z_n^0)$ for all $n=1, 2,\dots$ and the continued fraction S is equivalent to $T(z_1^0,\dots, z_n^0,\dots)$, where

(17) $$z_1^0 = a_1/b_1 \quad \text{and} \quad z_n^0 = a_n/(b_{n-1} b_n) \qquad \text{fro} \quad n \geq 2.$$

Thus, we can define the continued fraction S in the following way:

Definition 2. *We say that the continued fraction*

$$S = \frac{a_1}{b_1} - \frac{a_2}{b_2} - \cdots - \frac{a_n}{b_n} - \cdots$$

converges in power series sense if and only if the continued fraction $T(z_1,\dots, z_n,\dots)$ *converges in power series sense at* $(z_1^0,\dots, z_n^0,\dots)$, *where* z_n^0*'s are given in* (17). *In this case, we define* S *by*

(18) $$S = T(z_1^0,\dots, z_n^0,\dots) \quad (= \lim_{n \to +\infty} S_n).$$

Now, we assume that a_n's are non-negative and b_n's positive. Let us introduce the following sufficient condition for S to converge in power series sense:

(A.1) $$a_1 \geq 0, \quad b_1 > 0 \quad \text{and} \quad b_1(b_2 - 1) > a_2;$$

(A.2) $$a_n \geq 0 \quad \text{and} \quad b_n \geq a_n + 1 \qquad \text{for all} \quad n \geq 2.$$

The following lemma is a slight modificatin of the theorems of Pringsheim and Perron (cf. W. B. Jones and W. J. Thron [3], pp. 92–93).

Lemma A.1. *Assume* (A.1) *and* (A.2). *Then,*

(i) *the continued fraction* S *converges in power series sense and* $S \geq 0$ ($S=0$ *if and only if* $a_1 = 0$);

(ii) *if moreover* $b_1 \geq a_1 + 1$, *then we have* $S \leq 1$ ($S=1$ *if and only if* $b_n = a_n + 1$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} \prod_{k=1}^{n} a_k = +\infty$).

Proof. Put

(19) $$r_n = a_n/(b_n - 1) \quad \text{and} \quad s_n = r_n/b_{n-1} \qquad \text{for} \quad n \geq 2,$$

with conventions $r_n = s_n = 0$ if $a_n = 0$. Then the condition (A.2) implies

(20) $$1 \geq r_n \geq a_n/(b_n - r_{n+1}) \qquad \text{for} \quad n \geq 2,$$

or equivalently, $1/b_{n-1} \geq s_n \geq z_n^0/(1 - s_{n+1})$ for $n \geq 2$. So, $1 > s_n$ for $n \geq 3$. Further (A.1) assures $1 > s_2$. Therefore,

(21) $$1>s_n\geq z_n^0/(1-s_{n+1}) \quad \text{for all} \quad n\geq 2.$$

Repeated use of this yields

$$\frac{z_1^0}{1-s_2}\geq \frac{z_1^0}{1} - \frac{z_2^0}{1} - \cdots - \frac{z_n^0}{1} - \frac{s_{n+1}}{1}\geq T_n(z_1^0,\ldots,z_n^0)\geq z_1^0$$

for $n\geq 2$. Hence the sequence $\{T_n(z_1^0,\ldots, z_n^0)\}_{n=1}^{\infty}$ is increasing and bounded to the above by $z_1^0/(1-s_2)$. Consequently, the continued fraction S converges in power series sense and satisfies $0\leq z_1^0\leq S\leq z_1^0/(1-s_2)$. It is clear that $S=0$ implies $a_1=0$ and vice versa.

(ii) Suppose moreover $b_1\geq a_1+1$. Then we have $z_1^0\leq 1-s_2$, or $S\leq 1$. If $b_n>a_n+1$ for some n, then we have $S<1$. Therefore, $S=1$, implies $b_n=a_n+1$ for all $n\geq 1$. Suppose now that $b_n=a_n+1$ for all $n\geq 1$. Then we have

$$\frac{1}{1-S}=1+R, \quad \text{where} \quad R=\sum_{n=1}^{\infty}\prod_{k=1}^{n}a_k$$

(see W. B. Jones and W. J. Thron [3], (2.3.28) in p. 37). Hence, $S=1$ if and only if $R=+\infty$. The Lemma is proved.

We next estimate the difference of two continued fractions S and S' in (14), where both of the pairs $\{a_n\}$, $\{b_n\}$ and $\{a'_n\}$, $\{b'_n\}$ satisfy the condition (A.1) and that in Lemma A.1 (ii). Fix an n (≥ 1) and set

$$S_k^{(n)}=S_{k,k}^{(n)}=\frac{a_k}{b_k} - \frac{a_{k+1}}{b_{k+1}} - \cdots - \frac{a_n}{b_n} \qquad \text{for} \quad 1\leq k\leq n,\ S_{n+1}^{(n)}=0,$$

$$S_{j,k}^{(n)}=\frac{a'_j}{b'_j} - \frac{a'_{j+1}}{b'_{j+1}} - \cdots - \frac{a'_{k-1}}{b'_{k-1}-S_k^{(n)}} \qquad \text{for} \quad 1\leq j<k\leq n+1,$$

$$\delta_k^{(n)}=S_{k,k+1}^{(n)}-S_{k,k}^{(n)} \qquad \text{for} \quad 1\leq k\leq n.$$

Notice that $S_n=S_1^{(n)}$ and $S'_n=S_{1,n+1}^{(n)}$. Further, as in the proof of Lemma A.1, we have

(22) $$0\leq S_{k,k}^{(n)}\leq 1,\quad 0\leq S_{j,k}^{(n)}\leq \frac{a'_j}{b'_j-1}\leq 1 \qquad \text{for} \quad 1\leq j\leq k\leq n.$$

Lemma A.2. (i) *For each* $1\leq k\leq n$, *it holds that*

(23) $$\delta_k^{(n)}=\{a'_k-a_k-(b'_k-b_k)S_k^{(n)}\}/(b'_k-S_{k+1}^{(n)}).$$

(ii) *Under the above assumptions, it holds that*

(24) $$S'_n-S_n=\delta_1^{(n)}+\sum_{k=2}^{n}\delta_k^{(n)}\prod_{j=1}^{k-1}S_{j,k}^{(n)}S_{j,k+1}^{(n)}/a'_j,$$

(25) $$|S'_n-S_n|\leq|\delta_1^{(n)}|+\sum_{k=2}^{n}|\delta_k^{(n)}|\prod_{j=1}^{k-1}1/(b'_j-1).$$

Proof. (i) The formula (23) is a direct consequence of the relations

$$S^{(n)}_{k,k+1}=\frac{a'_k}{b'_k-S^{(n)}_{k+1}}, \quad S^{(n)}_k=\frac{a_k}{b_k-S^{(n)}_{k+1}}.$$

(ii) Noticing that

$$S^{(n)}_{j,k+1}=\frac{a'_j}{b'_j-S^{(n)}_{j+1,k+1}}, \quad S^{(n)}_{j,k}=\frac{a'_j}{b'_j-S^{(n)}_{j+1,k}}, \quad 1\le j<k,$$

we have

$$S^{(n)}_{j,k+1}-S^{(n)}_{j,k}=S^{(n)}_{j,k}S^{(n)}_{j,k+1}(S^{(n)}_{j+1,k+1}-S^{(n)}_{j+1,k})/a'_j, \quad 1\le j\le k,$$

which implies

$$S^{(n)}_{1,k+1}-S^{(n)}_{1,k}=\delta^{(n)}_k\prod_{j=1}^{k-1}S^{(n)}_{j,k}S^{(n)}_{j,k+1}/a'_j.$$

Combining this with the relation $S'_n-S_n=\sum_{k=1}^{n}(S^{(n)}_{1,k+1}-S^{(n)}_{1,k})$, we obtain (24). The estimate (25) is now clear from (22) and (24). The Lemma is proved.

We now turn to the recurrence equation

$$\xi_{n+1}=b_n\xi_n-a_n\xi_{n-1} \qquad \text{for} \quad n=1, 2, 3,... \tag{26}$$

A sequence $\{\xi_n\}_{n=0}^{\infty}$ of real numbers is called a *solution of the equation* (26) if it satisfies (26). A solution is uniquely determined by the two values ξ_0 and ξ_1. Two solutions $\{\xi_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ are said to be *linearly independent* if $\xi_0\eta_1-\xi_1\eta_0\neq 0$.

We now specify the linearly independent solutions $\{\xi^{(1)}_n\}_{n=0}^{\infty}$ and $\{\xi^{(2)}_n\}_{n=0}^{\infty}$ of (26) by

$$\begin{aligned}&\xi^{(1)}_0=0 \quad\text{and}\quad \xi^{(1)}_1=1,\\ &\xi^{(2)}_0=1 \quad\text{and}\quad \xi^{(2)}_1=S.\end{aligned} \tag{27}$$

Note that any solution $\{\xi_n\}_{n=0}^{\infty}$ of (26) is represented as

$$\xi_n=(\xi_1-\xi_0S)\xi^{(1)}_n+\xi_0\xi^{(2)}_n \qquad \text{for all} \quad n\ge 1. \tag{28}$$

The solution $\{\xi^{(2)}_n\}_{n=0}^{\infty}$ coincides with the minimal solution of (26) in the book of W. B. Jones and Thron [3] (see pp. 163–164). The next lemma is related to the theorem of Pincherle (*ibid.*).

Lemma A.3. (i) *Assume* (A.1) *and* (A.2). *Then we have*

$$0\le\xi^{(2)}_{n+1}\le\xi^{(2)}_n \qquad \textit{for all} \quad n\ge 1, \tag{29}$$

$$0<\xi^{(1)}_n\le\xi^{(1)}_{n+1} \qquad \textit{for all} \quad n\ge 2. \tag{30}$$

(ii) *Assume* (A.1), (A.2) *and*

$$\sum_{n=1}^{\infty}\prod_{k=1}^{n}a_k=+\infty. \tag{A.3}$$

Then,

(31) $$\lim_{n\to+\infty} \xi_n^{(1)} = +\infty.$$

Proof. Put

(32) $$F_n = \frac{a_n}{b_n} - \frac{a_{n+1}}{b_{n+1}} - \cdots - \frac{a_{n+k}}{b_{n+k}} - \cdots \quad \text{for} \quad n=1,2,3,\dots$$

Obviously, $F_1 = S$. Analogously to the proof of Lemma A.1, we can show that all the continued fractions F_n's converge in power series sense and that

(33) $$0 \le F_n \le r_n \le 1 \qquad \text{for all} \quad n \ge 2$$

(see (19) and (20)). Moreover, we have

(34) $$F_n = a_n/(b_n - F_{n+1}) \qquad \text{for all} \quad n \ge 1.$$

So we can make a solution $\{\eta_n\}_{n=0}^{\infty}$ of (26) by setting $\eta_0 = 1$ and $\eta_n = F_n \eta_{n-1}$ for $n=1, 2, \dots$. It satisfies $0 \le \eta_{n+1} \le \eta_n$ for all $n \ge 1$ due to (33). Since $\eta_n = \xi_n^{(2)}$ for $n=0$ and $n=1$, we have $\{\eta_n\}_{n=0}^{\infty} = \{\xi_n^{(2)}\}_{n=0}^{\infty}$. Thus we see (29) and that

(35) $$\xi_0^{(2)} = 1 \quad \text{and} \quad \xi_n^{(2)} = \prod_{k=1}^{n} F_k \qquad \text{for all} \quad n \ge 1.$$

Let us prove (30). By definition, we have $\xi_2^{(1)} = b_1$ and $\xi_3^{(1)} = b_1 b_2 - a_2$, so $\xi_3^{(1)} > \xi_2^{(1)} > 0$ because of (A.1). Suppose now $n \ge 3$. Assuming $\xi_n^{(1)} \ge \xi_{n-1}^{(1)} \ge 0$, we have

$$\xi_{n+1}^{(1)} - \xi_n^{(1)} = (b_n - 1 - a_n)\xi_n^{(1)} + a_n(\xi_n^{(1)} - \xi_{n-1}^{(1)}) \ge 0.$$

Therefore, $\{\xi_n^{(1)}\}_{n=2}^{\infty}$ is non-decreasing, and (30) follows.

It remains to prove the assertion (ii). Using the above formula, we can show by induction that

$$\xi_{n+1}^{(1)} - \xi_n^{(1)} \ge \prod_{k=1}^{n} a_k \qquad \text{for} \quad n \ge 1.$$

Therefore, the assumption (A.3) implies (31).

The Lemma is proved.

We note that, under the assumptions (A.1), (A.2), it also holds that

(36) $$\xi_n^{(1)} = \sum_{k=1}^{n} \prod_{r=1}^{k-1} (b_r - F_{r+1}) \Bigl(\prod_{s=k+1}^{n} F_s \Bigr) \qquad \text{for all} \quad n \ge 1,$$

with conventions $\prod_{r=1}^{0} (b_r - F_{r+1}) = \prod_{s=n+1}^{n} F_s = 1$.

We close this section by introducing a maximum principle lemma in recurrence equations. Let $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$ and $\{\nu_n\}_{n=0}^{\infty}$ be non-negative sequences with

(37) $$\lambda_0 + \mu_0 + \nu_0 > 0 \quad \text{and} \quad \lambda_n,\ \mu_n > 0, \quad n=1, 2, \dots$$

An operator $\mathscr{G}$ on the space of all sequences $\xi = \{\xi_n\}_{n=0}^{\infty}$ is defined by

$$(\mathscr{G}\xi)_n = \mu_n \xi_{n-1} + \lambda_n \xi_{n+1} - (\lambda_n + \mu_n + \nu_n)\xi_n, \quad n=0, 1, 2, \dots,$$

where $\xi_{-1}=0$ by convention. Note that the recurrence equation (26) can be rewritten as $(\mathscr{G}\xi)_n=0$, $n=1, 2,\dots$, with an appropriate choice of coefficients.

The following lemma is well known for the readers who are familiar to the theory of birth and death processes or of one-dimensional generalized diffusion processes (gap processes). But we will give it here for completeness.

Lemma A.4. *Let a sequence* $\xi=\{\xi_n\}_{n=0}^{\infty}$ *satisfy*

$$(38)\qquad (\mathscr{G}\xi)_n\leq 0,\quad n=0, 1, 2,\dots \quad \textit{and} \quad \liminf_{n\to+\infty}\xi_n\geq 0.$$

Then it holds that

$$(39)\qquad \xi_n\geq 0,\quad n=0, 1, 2,\dots$$

Proof. First we assume $\lambda_0=0$. Then it follows from (37) and (38) for $n=0$ that $\xi_0\geq 0$. Suppose now that the inequalities (39) do not hold. Then we can find a positive integer n_0 such that

$$(40)\qquad \xi_{n_0}<0,\quad \xi_{n_0}\leq\xi_{n_0-1}\wedge\xi_{n_0+1}\quad \text{and}\quad \xi_{n_0}<\xi_{n_0-1}\vee\xi_{n_0+1},$$

where $a\wedge b$ ($a\vee b$) stands for $\min\{a, b\}$ (resp. $\max\{a, b\}$). The second and the third inequalities of (40) combined with (38) imply $-\nu_{n_0}\xi_{n_0}<0$, contradicting the first inequality of (40).

We next assume that $\lambda_0>0$. In this case, we have $\xi_0\geq\xi_1\wedge 0$. Indeed, if $\xi_0<\xi_1\wedge 0$, then it follows that

$$0<\lambda_0(\xi_1-\xi_0)\leq(\mu_0+\nu_0)\xi_0\leq 0,$$

which is a self-contradiction. Now the rest of the proof is the same as that for the case $\lambda_0=0$ in the above. The Lemma is proved.

§B. Properties of the function F(y).

Let y be a variable running over the closed interval [0, 1]. Let P, Q, S, M and C be real parameters. Throughout this section (possibly except in Lemma B.3), we assume that

$$(\text{B.1})\qquad P\geq 0,\quad Q\geq 0,\quad S>0,\quad M>0\quad \textit{and}\quad 0\leq C\leq Q+M.$$

We put

$$(1)\qquad \begin{aligned} a_n(y)&=n(n-1+P+My)/(nS+C),\\ b_n(y)&=\{n(n-1+P+Q+S+M)+Cy\}/(nS+C),\end{aligned}\qquad \text{for}\quad n\geq 1.$$

In this section, we consider the continued fraction

$$(2)\qquad F(y)=\frac{a_1(y)}{b_1(y)}\,-\,\frac{a_2(y)}{b_2(y)}\,-\cdots-\,\frac{a_n(y)}{b_n(y)}\,-\cdots$$

as a function of variable y with parameters (P, Q, S, M, C). We also deal with the functions

(3) $$F_n(y)=\frac{a_n(y)}{b_n(y)}\;-\;\frac{a_{n+1}(y)}{b_{n+1}(y)}\;-\cdots-\;\frac{a_{n+k}(y)}{b_{n+k}(y)}\;-\cdots$$

for $n=1, 2, 3,\ldots$ Obviously $F_1(y)=F(y)$ and we have

(4) $$F_n(y)=a_n(y)/(b_n(y)-F_{n+1}(y)) \qquad \text{for} \quad n\geq 1,$$

whenever the both sides make sense.

Lemma B.1. (i) *For each n, the continued fraction $F_n(y)$ converges in power series sense at any point y in* [0, 1] *and satisfies*

(5) $$0\leq F_n(y)\leq 1 \qquad \textit{for} \quad n\geq 1;$$

(ii) $F(y)=0$ *if and only if* $P=0$ *and* $y=0$;
(iii) $F(y)=1$ *if and only if* $Q=0$ *and* $y=1$.

Proof. Notice first that $a_1(y)\geq 0$ ($a_1(y)=0$ if and only if $P=y=0$), $a_n(y)>0$ and $b_n(y)>1$ for all $n\geq 2$ and y in [0, 1]. Since

$$(nS+C)(b_n(y)-1-a_n(y))=nQy+(nQ+nM-C)(1-y),$$

we always have $b_n(y)\geq a_n(y)+1$ due to the assumption (B.1). Moreover,
either (*a*) $b_n(y)>1$ and $b_{n+1}(y)\geq a_{n+1}(y)+1$
or (b) $b_n(y)\geq 1$ and $b_{n+1}(y)>a_{n+1}(y)+1$
holds for any n and y. Indeed, (a) holds if $0<y\leq 1$ and (b) holds if $0\leq y<1$. Applying (ii) of Lemma A.1 to the $F_n(y)$, we see that the assertion (i) holds.

(ii) and (iii) follow from the assertions in brackets of Lemma A.1.

Remark. Actually $F(y)$ converges under a more relaxed condition on C, for example, $0\leq C\leq \mathrm{Min}\,(P+Q+M, 2(Q+M))$. But in this case, the value of $F(y)$ may exceed 1.

Lemma B.2. *For each* $n\geq 1$, $F_n(y)$ *is a function of class* C^1 *in* 6 *variables* (y, P, Q, S, M, C) *in the region given by* $0\leq y\leq 1$ *and* (B.1).

Proof. Let u be one of y, P, Q, S, M and C. Differentiating both sides of (4), we have

$$\frac{\partial}{\partial u}F_n=(F_n/a_n)\left(\frac{\partial}{\partial u}a_n-F_n\frac{\partial}{\partial u}b_n+F_n\frac{\partial}{\partial u}F_{n+1}\right).$$

By repeated use of this, we obtain

(6) $$\frac{\partial}{\partial u}F_n=\sum_{k=n}^{\infty}\frac{d_{k,u}}{F_k}\prod_{j=n}^{k}\frac{F_j^2}{a_j}, \quad \text{where } d_{n,u}=\frac{\partial}{\partial u}a_n-F_n\frac{\partial}{\partial u}b_n.$$

A computation shows that

(7) $$\begin{aligned} &d_{n,y}=(nM-CF_n)/(nS+C), && d_{n,P}=n(1-F_n)/(nS+C),\\ &d_{n,Q}=-nF_n/(nS+C), && d_{n,S}=-nF_n(1-F_{n+1})/(nS+C),\\ &d_{n,M}=n(y-F_n)/(nS+C), && d_{n,C}=F_n(F_{n+1}-y)/(nS+C). \end{aligned}$$

At each point (y, P, Q, S, M, C), there exists a positive number $L=L(y, P, Q, S, M, C)$ such that $|d_{n,u}|\leq L$ for $n\geq 1$. Therefore, the series on the right hand side of (6) is absolutely convergent, because (5) holds and a_n tends to infinity as $n\to+\infty$.

It should be noted that, for $n=1$, the formula (6) is reduced to

$$\frac{\partial F}{\partial u}=(d_{1,u}/d_{1,y})\frac{\partial F}{\partial y} \quad \text{at} \quad y=P=0.$$

The derivative $\frac{\partial F}{\partial y}$ at $y=P=0$ will be calculated in (10) below.

The Lemma is proved.

We have the following formula for $\frac{\partial F}{\partial y}$:

If $P>0$ or $0<y\leq 1$, then

$$\text{(8)} \qquad \frac{\partial F}{\partial y}(y)=\sum_{n=1}^{\infty}\frac{\Gamma((C/S)+n)\Gamma(P+My)S^{n-1}}{\Gamma((C/S)+1)\Gamma(P+My+n)(n-1)!} \times\left(M-\frac{C}{n}F_n\right)\prod_{k=1}^{n}(F_{k-1}F_k),$$

where we put $F_0=1$ by convention.

If $Q=0$, then

$$\text{(9)} \qquad \frac{\partial F}{\partial y}(1)=1-{}_1F_1(C/S,\ P+M;\ S)+\frac{M}{P+M}\,{}_1F_1((C/S)+1,\ P+M+1;\ S)$$

If $P=0$, then

$$\text{(10)} \qquad \frac{\partial F}{\partial y}(0)=\frac{M}{Q+M}\,{}_1F_1((C/S)+1,\ Q+M+1;\ -S)/{}_1F_1(C/S,\ Q+M;\ -S).$$

In (9) and (10), ${}_1F_1$ stands for the confluent hypergeometric series of Kummer (see [1], Chapter 6);

$$\text{(11)} \qquad {}_1F_1(\alpha,\ \gamma;\ t)=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\gamma)t^n}{\Gamma(\alpha)\Gamma(\gamma+n)n!}.$$

Proofs of (8), (9) *and* (10). The formula (8) follows at once from (1) and (6). Substituting $Q=0$ and $y=1$ into (8), we obtain (9).

Let us prove (10). If $P=0$, then $\frac{\partial F}{\partial y}(0)$ is equal to the continued fraction in (2) with $y=0$ but with a_1 replaced by $(da_1/dy)(0)$. So we have

$$\frac{\partial F}{\partial y}(0)=\frac{M}{Q+M}g(C/S,\ Q+M;\ -S),$$

where

$$\text{(12)} \qquad g(\alpha,\ \gamma;\ t)=\frac{\gamma}{\gamma-t}\,+\,\frac{(\alpha+1)t}{\gamma+1-t}\,+\,\frac{(\alpha+2)t}{\gamma+2-t}\,+\cdots+\,\frac{(\alpha+n)t}{\gamma+n-t}\,+\cdots.$$

Note that the function g satisfies the functional equation

$$g(\alpha, \gamma; t)=\gamma/\left(\gamma-t+\frac{\alpha+1}{\gamma+1}tg(\alpha+1, \gamma+1; t)\right). \tag{13}$$

A formal power series in t whose coefficients are rational functions in (α, γ) is unique if it satisfies (13). In addition, it is easy to verify that ${}_1F_1(\alpha+1, \gamma+1; t)/{}_1F_1(\alpha, \gamma; t)$ does so. Therefore, we have

$$g(\alpha, \gamma; t)={}_1F_1(\alpha+1, \gamma+1; t)/{}_1F_1(\alpha, \gamma; t) \tag{14}$$

The formula (10) is proved.

A further application of (14) is the following:

$$F(y)=\frac{P+My}{P+Q+M}\frac{{}_1F_1(Q+M(1-y), P+Q+M+1; S)}{{}_1F_1(Q+M(1-y), P+Q+M; S)} \quad \text{if} \quad C=0. \tag{15}$$

Proof. Putting $\alpha=P+My$ and $\beta=P+Q+M$, we have

$$a_n(y)=(n-1+\alpha)/S \quad \text{and} \quad b_n(y)=(n-1+\beta+S)/S.$$

Therefore, $F(y)=(\alpha/\beta)g(\alpha, \beta; -S)$. This and Kummer's identity

$${}_1F_1(\alpha, \beta; -t)=e^{-t}{}_1F_1(\beta-\alpha, \beta; t)$$

imply (15).

Lemma B.3. *Assume that $P=Q=0$, $S>0$, $M>0$ and $C\geq 0$ (C can be larger than M in this lemma).*
(i) *If $C=SM$, then $F(y)$ converges in power series sense and $F(y)=y$ for all y in the interval $[0, +\infty)$;*
(ii) *if $C<SM$, then $F(y)$ converges in power series sense for all y in $[0, 1]$ and $0<F(y)<y$ for all y in $(0, 1)$;*
(iii) *if $C>SM$, $0<y<1$ and if $F(y)$ converges in power series sense at y, then $F(y)>y$.*

Proof. As in (9) of §A, let us define z_n's by

(16) $z_1=a_1(y)/b_1(y)$ and $z_n=a_n(y)/(b_{n-1}(y)b_n(y))$ for $n\geq 2$.

To prove (i), suppose $C=SM$ and put

$$A_n=\frac{n(n-1+M)}{S(n+My)} \quad \text{and} \quad r_n=\frac{n+M}{n+My} \quad \text{for } n\geq 1.$$

Then, $r_1a_1(y)=yA_1$, $r_{n-1}r_na_n(y)=A_n$ for $n\geq 2$ and $r_nb_n(y)=1+A_n$ for $n\geq 1$. Hence, $F(y)/y$ is equivalent to

$$\frac{A_1}{1+A_1} - \frac{A_2}{1+A_2} - \cdots - \frac{A_n}{1+A_n} - \cdots.$$

The last continued fraction is convergent in power series sense at any $y\in[0, +\infty)$ and is equal to 1, because $A_n>0$ for all $n\geq 1$ and $\sum_{n=1}^{\infty}\prod_{k=1}^{n}A_k=+\infty$ (see Lemma

A.1 (ii)). Therefore, $F(y)$ is identically equal to y in $[0, +\infty)$.

To prove (ii) and (iii), we first assume that $C>0$. Fix S and M, and denote $F(y)=F(y, C)$ and $CF(t/C, C)=\bar{F}(t, C)$. Then we have

$$\bar{F}(t, C)= \frac{\bar{z}_1(t, C)}{1} - \frac{\bar{z}_2(t, C)}{1} - \cdots - \frac{\bar{z}_n(t, C)}{1} - \cdots,$$

where $\bar{z}_1(t, C)=Cz_1(t/C, C)$ and $\bar{z}_n(t, C)=z_n(t/ C, C)$ for $n\geq 2$. $\bar{z}_1$ is independent of C and positive for $t>0$. Further, it holds that

$$\text{(17)} \qquad \bar{z}_n(t, C)-\bar{z}_n(t, SM)=K_n(t)(C-SM)(C-t) \qquad \text{for} \quad n\geq 2,$$

where $K_n(t)$ is positive for $0\leq t<+\infty$.

If $0<C<SM$, then $0<\bar{z}_n(t, C)<\bar{z}_n(t, SM)$ for all $n\geq 2$ and $0<t<C$. Therefore, the convergence of $\bar{F}(t, SM)$ in power series sense implies that of $\bar{F}(t, C)$ for $0\leq t\leq C$ and the strict inequalities $0<\bar{F}(t, C)<\bar{F}(t, SM)$ for $0<t<C$. Since (i) implies $\bar{F}(t, SM)=t$, this assures $0<\bar{F}(t, C)<t$ for $0<t<C$, or $0<F(y, C)<y$ for $0<y<1$.

Suppose next that $C>SM$. By (17), $\bar{z}_n(t, C)>\bar{z}_n(t, SM)>0$ for all $n\geq 2$ and $0<t<C$. Therefore, if $\bar{F}(t, C)$ converges in power series sense, then we have $\bar{F}(t, C)>\bar{F}(t, SM)=t$, or $F(y, C)>y$ for $0<y<1$.

It ramains the case where $C=0$. In this case we have $C<SM$ automatically and, by (15), it suffices to verify the inequality

$$\text{(18)} \qquad {}_1F_1(\alpha, \gamma+1; t)<{}_1F_1(\alpha, \gamma; t) \qquad \text{for} \quad \alpha>0, \quad \gamma>0 \quad \text{and} \quad t>0.$$

In the power series expansions (see (11)), the coefficients of t^n for ${}_1F_1(\alpha, \gamma+1; t)$ is positive and smaller than the corresponding one for ${}_1F_1(\alpha, \gamma; t)$ for every $n\geq 1$. So (18) holds.

The Lemma is proved.

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