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On polynomial generators for the generalized homology of BSU

Dedicated to Professor Hirosi Toda on his 60th birthday

By

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§ 0. Introduction.

Let BSU be the classifying space of the infinite special unitary group SU. Let E be a complex oriented theory. Then E_*BSU is a subring of E_*BU. (See section 2.) In [8], S.O. Kochman determines the generators for the polynomial ring H_*BSU. (See also [6] and [7].) A. Baker gives also polynomial generators for E_*BSU in [3] by use of a geometrical construction yielding elements in the homology of $BSU(3)$. (See also [4].)

In this note, we give polynomial generators for E_*BSU in the words of E_*BU by a simple algebraic method.

In section 1, we study the Gysin sequence of an S^1-bundle $BSU \to BU$.

In section 2, we introduce some algebraic notations and define $p^E_{i,j} \in E_{2(i+j)}\ BSU$ as the coefficient of some formal power series. By the result of [6], one can easily show that linear combinations of $p^E_{i,j}$ are polynomial generators for H_*BSU. Then the Atiyah-Hirzebruch spectral sequence says that linear combinations of $p^E_{i,j}$ are polynomial generators for E_*BSU.

In section 3, we give a geometrical proof of the Proposition 2.3 which is the key for our main result.

§ 1. The Gysin sequence.

Let i: $SU \to U$ and j: $U(1) \to U$ be the usual inclusions. Let $B\det$: $BU \to BU(1)$ be the map induced from the determinant map det: $U \to U(1)$. Then the composition $B\det \circ Bj$ is an identity map. The map Bi: $BSU \to BU$ is a S^1-bundle and is the inclusion of the homotopy fibre of $B\det$.

Then we have a Gysin sequence

$$(1.1) \qquad \cdots \to E_*BSU \to E_*BU \xrightarrow{d} E_{*-2}BU \to E_{*-1}BSU \to \cdots$$

In the case of the ordinary homlogy, H_*BSU is a polynomial ring with the even dimensional generators. (See Adams [2].) So (1.1) splits as the short exact

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sequences

(1.2) $$0 \to H_{2*}BSU \to H_{2*}BU \to H_{2*-2}BU \to 0\,.$$

The first thing to do is to study the homomorphism d. Let us recall the structure theorem of E_*BU. (See Adams [1].)

Let $x^E \in E^2BU(1)$ be the Euler class of E.

Theorem 1.3.
(i) *$E_*BU(1)$ is a free E_*pt-module generated by $\beta_0^E, \beta_1^E, \cdots, \beta_n^E, \cdots$ where β_i^E is the dual of $(x^E)^i$.*
(ii) $E_*BU = E_*pt[\beta_1^E, \beta_2^E, \cdots, \beta_n^E, \cdots]$ *where* $\beta_i^E = Bj_*\beta_i^E$.
(iii) $\phi\beta_n^E = \sum_{i+j=n} \beta_i^E \otimes \beta_j^E$.

We often omit the superscript E for the simplicity.

Let $\alpha \in E_*BU$ and $y \in E^*BU$. Then by the definition of the Gysin sequence, we obtain an equality

(1.4) $$\langle \mathrm{d}\alpha, y\rangle = \langle \alpha, ty\rangle$$

where t is the Thom class of the complex line bundle which is classified by $B\det\colon BU \to BU(1)$.

Let $\mu^E(X, Y) = \sum a_{i,j}^E X^i Y^j$ be the formal group of E. Then we have the following proposition.

Proposition 1.5.
(i) $\mathrm{d}\beta_n = \beta_{n-1}$ *for* $n > 0$.
(ii) $\mathrm{d}(ab) = \sum a_{i,j}\mathrm{d}^i a \cdot \mathrm{d}^j b$ *for* $a, b \in E_*BU$ *where* $\mathrm{d}^0 = id$.

Proof. Let $\omega\colon BU \times BU \to BU$ be the map induced from the Whitney sum and $m\colon BU(1) \times BU(1) \to BU(1)$ the map induced from the tensor product of the line bundles. We consider BU and $BU(1)$ as H-spaces by these maps. Since $B\det$ is an H-map,

$$\omega^*(B\det^* x^E) = \mu^E(t \otimes 1, 1 \otimes t)\,.$$

By the duality, we get

$$\langle \beta_i, t^j\rangle = \langle Bj_*\beta_i, t^j\rangle = \langle \beta_i, Bj^* t^j\rangle = \langle \beta_i, (x^E)^j\rangle = \delta_{i,j}\,.$$

So $\langle \beta_n, ty\rangle = \langle \phi\beta_n, t \otimes y\rangle = \langle \beta_{n-1}, y\rangle$. Thus (i) is proved.

Put $\omega^* y = \sum y' \otimes y''$. Then we have the following equality

$$\begin{aligned}\langle ab, ty\rangle &= \langle a \otimes b, \omega^*(ty)\rangle = \langle a \otimes b, \mu^E(t \otimes 1, 1 \otimes t) \cdot \omega y^*\rangle \\ &= \sum\sum a_{i,j}\langle a, t^i y'\rangle\langle b, t^j y''\rangle \\ &= \sum\sum a_{i,j}\langle \mathrm{d}^i a, y'\rangle\langle \mathrm{d}^j b, y''\rangle \\ &= \sum a_{i,j}\langle \mathrm{d}^i a \otimes \mathrm{d}^j b, \sum y' \otimes y''\rangle = \langle \sum a_{i,j}\mathrm{d}^i a \cdot \mathrm{d}^j b, y\rangle\,.\end{aligned}$$

Example. In the case of the complex K-theory, let $t\in K_2(pt)\cong \boldsymbol{Z}$ be the generator such that $\mu^K(X, Y)=X+Y+tXY$. So we obtain

$$\mathrm{d}\beta_1=1,\ \mathrm{d}\beta_2=\beta_1 \quad\text{and}\quad \mathrm{d}(2\beta_2-(\beta_1)^2+t\beta_1)=0\,.$$

§ 2. Polynomial generators.

Let R be a (graded) commutative ring with a unity $1\in R$ and A a (graded) commutative and unitary R-algebra. Let $A[[X_1, X_2, \cdots, X_n]]$ be the ring of formal power series in indeterminants $X_1, X_2, \cdots, X_n$ over A ($\deg X_i=-2$). Let $f\colon A\to B$ be an R-module homomorphism. Then f is extended naturally to the R-module homomorphism

$$f\colon A[[X_1, X_2, \cdots, X_n]] \to B[[X_1, X_2, \cdots, X_n]]\,.$$

We put $R=E_*pt$ and $A=E_*BU$. Let $\beta(X)\in A[[X]]$ be $\sum_{i\geq 0}\beta_i X^i$. Then we deduce the following lemma from (1.5).

Lemma 2.1.
(i) $\mathrm{d}\beta(X)=X\beta(X)$,
(ii) $\mathrm{d}(f(X, Y)g(X, Y))=\sum a_{i,j}\mathrm{d}^i f(X, Y)\mathrm{d}^j g(X, Y)$
for $f(X, Y), g(X, Y)\in A[[X, Y]]$ *with the degree zero.*

Since $\beta(X)$ is a unit in $A[[X]]$, we can define $P(X, Y)\in A[[X, Y]]$ by the following formula

$$P(X, Y)=(\beta(X)\beta(Y))\beta(\mu^E((X, Y)))^{-1}\,. \tag{2.2}$$

Then we have the following proposition.

Proposition 2.3. $\mathrm{d}P(X, Y)=0$.

Proof. By (2.1), we have

$$\mathrm{d}(\beta(X)\beta(Y))=\sum a_{i,j}\mathrm{d}^i\beta(X)\mathrm{d}^j\beta(Y)=\sum a_{i,j}X^iY^j\beta(X)\beta(Y)\,.$$

So we have

$$\begin{aligned}\mathrm{d}P(X, Y)&=\sum a_{i,j}\mathrm{d}^i(\beta(X)\beta(Y))\mathrm{d}^j(\beta(\mu(X, Y))^{-1})\\&=\beta(X)\beta(Y)\sum a_{i,j}(\mu(X, Y))^i\mathrm{d}^j(\beta(\mu(X, Y))^{-1})\,.\end{aligned}$$

We have also the following equalities

$$\begin{aligned}0=\mathrm{d}1&=\mathrm{d}(\beta(\mu(X, Y))(\beta(\mu(X, Y))^{-1})\\&=\sum a_{i,j}\mathrm{d}^i\beta(\mu(X, Y))\mathrm{d}^j(\beta(\mu(X, Y))^{-1})\\&=\sum a_{i,j}(\mu(X, Y))^i\beta(\mu(X, Y))\mathrm{d}^j(\beta(\mu(X, Y))^{-1})\\&=\beta(\mu(X, Y))\sum a_{i,j}(\mu(X, Y))^i\mathrm{d}^j(\beta(\mu(X, Y))^{-1})\,.\end{aligned}$$

Since $\beta(\mu(X, Y))$ is a unit, $dP(X, Y)=0$.

Let $p^E_{i,j}\in E_{2(i+j)}BU$ be the coefficient of $P(X, Y)$ at X^iY^j. Since $P(0, Y)=P(X, 0)=1$,

$$P(X, Y) = 1+\sum_{i,j>0}p^E_{i,j}X^iY^j .$$

For each $n\in \boldsymbol{N}$, we put

$$\nu(n) = \text{g.c.d.}\left\{\binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n-1}\right\} .$$

Then $\nu(n)$ is p if $n=p^s$, p prime, and 1 if n is not a power of a prime. Let $\lambda_{n,1}, \lambda_{n,2}, \cdots, \lambda_{n,n-1}$ be integers such that

$$\nu(n) = \lambda_{n,1}\binom{n}{1}+\lambda_{n,2}\binom{n}{2}+\cdots+\lambda_{n,n-1}\binom{n}{n-1} .$$

We take p^E_n such that $Bi_*p^E_n=\lambda_{n,1}p^E_{n-1,1}+\lambda_{n,2}p^E_{n-2,2}+\cdots+\lambda_{n,n-1}p^E_{1,n-1}$ for $n>1$. Then we are ready to prove the main result.

Theorem 2.4. *$E_*BSU=E_*pt[p^E_2, p^E_3, \cdots, p^E_n, \cdots]$ as an E_*pt-algebra.*

Proof. First we prove the theorem in the case of $E=H$. By (2.2), one can easily show that $p_{i,j}\equiv-\binom{i+j}{i}\beta_{i+j}$ modulo decomposables. So $Bi_*p_n\equiv-\nu(n)\beta_n$ modulo decomposables. Then the theorem follows the result of Kochman [6, Theorem 3.3].

Let us consider the Atiyah-Hirzebruch spectral sequence $H_*(BSU; E_*pt)\Rightarrow E_*BSU$. Then the monomials $p_{i_1}p_{i_2}\cdots p_{i_r}$ give an E_*pt-base for the E^2-term. Since all differentials vanish, the result follows.

Remark. In the case of $E=H$, we can prove that the subalgebra generated by $\{p_{i,j}\}_{i,j>0}$ is a polynomial ring $Z[p_2, p_3, \cdots]$ by the algebraic method. (See [1] and [5].)

Let $A_{i,j}$ $(i, j>0)$ be the indeterminants. Put $F(X, Y)=1+\sum_{i,j>0}A_{i,j}X^iY^j$ and set $F(X+Y, Z)F(X, Y)-F(X, Y+Z)F(Y, Z)=\sum B_{i,j,k}X^jY^iZ^k$. Let I be the ideal of $Z[A_{i,j}; i, j>0]$ generated $B_{i,j,k}$ and $A_{i,j}-A_{j,i}$. We define L as the quotient $Z[A_{i,j}; i, j>0]/\text{I}$. Since $B_{i,j,k}\equiv\binom{i+j}{i}A_{i+j,k}-\binom{k+j}{j}A_{i,j+k}$ modulo decomposables, one can prove that each $A_{i,j}$ $(i+j=n)$ is written as a multiple of $A_n=\lambda_{n,1}A_{n-1,1}+\cdots+\lambda_{n,n-1}A_{1,n-1}$ modulo decomposables. (See Hazewinkel [5], 4.2., binomial coefficient lemma.) Let $Z[t_2, t_3, \cdots]$ be the polynomial ring generated by the variables $t_2, t_3, \cdots$ and φ: $Z[t_2, t_3, \cdots]\to L$ be the ring homomorphism defined by $\varphi(t_n)=A_n$. Then φ is an epimorphism. We define θ: $L\to A$ to be the ring homomorphism by the equality $\theta(A_{i,j})=p_{i,j}$. Clearly $\theta\circ\varphi$ is a monomorphism. Thus φ is a ring isomorphism and the result follows.

§ 3. The geometrical proof of (2.3).

Let $\tau: BU \to BU$ be the classifying map of the inverse bundle and let $c: BU(1) \to BU(1)$ be the induced map from the complex conjugation.

We can consider $E_*BU[[X]]$ as $(E \wedge BU_+)_* BU(1)_+$ where X is the image of x^E by the Boardman map $B: E_*(\ ) \to (E \wedge BU_+)_*(\ )$. $E_*BU[[X, Y]]$ is also identified with $(E \wedge BU_+)_*(BU(1)_+ \wedge BU(1)_+)$.

Then, one can easily show that $\beta(X) \in E_*BU[[X]]$ is represented by the composition

$$BU(1)_+ \xrightarrow{Bj} BU_+ = S^0 \wedge BU_+ \xrightarrow{\iota \wedge id} E \wedge BU_+$$

and $\beta(\mu(X, Y)) \in E_*BU[[X, Y]]$ is the composition of this map and $m: BU(1)_+ \wedge BU(1)_+ \to BU(1)_+$. (See Lemma 6.2. in [1], part 2.)

Then, $P(X, Y)$ in section 2 is represented by the composition

$$(\iota \wedge id) \circ \omega \circ (\tau \circ Bj \wedge \omega) \circ (m \wedge Bj \wedge Bj) \circ \Delta : BU(1)_+ \wedge BU(1)_+ \to E \wedge BU_+ .$$

where Δ is the diagonal map of $BU(1)_+ \wedge BU(1)_+$. Since $m \circ (B\det \wedge B\det) \simeq B\det \circ \omega$, $c \circ B\det \simeq B\det \circ \tau$ and $B\det \circ Bj \simeq id$, we have the following homotopies

$$\begin{aligned} &B\det \circ \omega \circ (\tau \circ Bj \wedge \omega) \circ (m \wedge Bj \wedge Bj) \circ \Delta \\ &\simeq m \circ (B\det \wedge B\det) \circ (\tau \circ Bj \wedge \omega) \circ (m \wedge Bj \wedge Bj) \circ \Delta \\ &\simeq m \circ (c \circ B\det \circ Bj \wedge B\det \circ \omega) \circ (m \wedge Bj \wedge Bj) \circ \Delta \\ &\simeq m \circ (c \wedge m \circ (B\det \wedge B\det)) \circ (m \wedge Bj \wedge Bj) \circ \Delta \simeq m \circ (c \wedge id) \circ (m \wedge m) \circ \Delta . \end{aligned}$$

Thus, $B\det \circ \omega \circ (\tau \circ Bj \wedge \omega) \circ (m \wedge Bj \wedge Bj) \circ \Delta : BU(1)_+ \wedge BU(1)_+ \to BU(1)_+$ is null-homotopic. So we have another proof of the fact that $P(X, Y)$ is the image of $Bi_*: E_*BSU[[X, Y]] \to E_*BU[[X, Y]]$.

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