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The spectrum of periodic generalized diffusion operators

By

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1. Introduction.

Some properties of the spectrum of periodic diffusion operators on the real line were recently studied by N. Ikeda, K. Kawazu and Y. Ogura [3], [5]. As was shown there, the spectrum has the same structure as that of Hill's operators. Namely, it is expressed as a countable union of closed intervals $[\mu_n^{(2)}, \mu_{n+1}^{(1)}]$, $n\geq 0$ with a sequence $-\infty<\mu_0^{(2)}<\mu_1^{(1)}\leq\mu_1^{(2)}<\cdots<\mu_n^{(1)}\leq\mu_n^{(2)}<\cdots\uparrow\infty$. But periodic *generalized* diffusion operators present a little different spectrum. M.G. Krein [6] proved that, for a class of periodic generalized diffusion operators, the spectrum is expressed either as $\bigcup_{n=0}^{\infty}[\mu_n^{(2)}, \mu_{n+1}^{(1)}]$ in terms of an infinite sequence as above, or as $\bigcup_{n=0}^{N-1}[\mu_n^{(2)}, \mu_{n+1}^{(1)}]$ in terms of a finite sequence $-\infty<\mu_0^{(2)}<\mu_1^{(1)}\leq\mu_1^{(2)}<\cdots\leq\mu_{N-1}^{(2)}<\mu_N^{(1)}<\infty$; in particular, if the operator is associated with a periodic discrete measure, that is, if the operator is reduced to a periodic second order difference operator, then the spectrum has the latter expression. These observations suggest that the spectrum of periodic generalized diffusion operators has a structure similar to that of Hill's operators or that of periodic second order difference operators according as the support of associated measure intersects a bounded interval with an infinite set or with a finite set. Our first aim is to show that this is valid for a class of periodic generalized diffusion operators containing Krein's operators as well as periodic diffusion operators.

The results for periodic diffusion operators by N. Ikeda, K. Kawazu and Y. Ogura also tell us that the spectrum consists only of the continuous spectrum and the point spectrum is empty as long as the operators are treated on the real line. On the other hand, if we deal with the operators on the half line, both the continuous spectrum and the point spectrum are nonempty in most cases. E.A. Coddington and N. Levinson have already pointed out in the book [1] that, for a boundary value problem of second order differential operator, the spectrum depends on the boundary condition, and the continuous spectrum and the point spectrum are nonempty. Further, W. Ledermann and G.E.H. Reuter [7] studied a class of birth and death processes and also obtained the analogous results for their generators which are

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second order difference operators. Their works interest us the problem how boundary conditions affect the spectrum. Our second aim is to clarify it for a class of our operators restricted to the half line $[0, \infty)$ with sticky elastic boundary conditions at 0.

Our tool is an eigenfunction expansion, known as Weyl-Stone-Titchmarsh-Kodaira theory [10]. We have also to recall the spectral theory of generalized diffusion operators due to K. Itô and H.P. McKean [4], [9]. Their arguments enable us to generalize Weyl-Stone-Titchmarsh-Kodaira theory developed for second order differential operators to our general case.

In Section 2 we will describe precise definitions and state our main results. The first theorem is related to the discriminant. We will see there that the discriminant has a finite number of zeros if and only if it is associated with a periodic difference operator. In Theorem 2 we will give the precise formulas of spectral measure density functions of our operators on the real line and in Theorem 3 their asymptotic behaviors near the points $\mu_n^{(j)}$ mentioned above. Theorems 4, 5 and 6 are devoted to our second aim. The continuous spectrum of operators restricted to the half line is independent of boundary conditions, but the spectral measure density functions are naturally affected by them (Theorem 4). The asymptotic behaviors of the density functions near the points $\mu_n^{(j)}$ can not be also independent of boundary conditions (Theorem 5). Neither can the point spectrum (Theorem 6). We will prove Theorem 1 in Section 3, Theorems 2 and 3 in Section 4, and Theorems 4, 5 and 6 in Section 5, respectively. Some typical examples will be given in Section 6.

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2. Definitions and main results.

2.1. Let s, m and k be real valued functions on the real line $\boldsymbol{R}$ satisfying the following conditions:

(2.1) s is continuous increasing,

(2.2) m is non-trivial right continuous nondecreasing,

(2.3) a) k is right continuous nondecreasing, or

b) k is right continuous, dk is absolutely continuous with respect to dm and the Radon-Nikodym density is bounded,

where dm and dk stand for the measures induced by m and k, respectively. We may assume $s(0)=m(0)=k(0)=0$ without loss of generality. We denote by $D(\mathfrak{G})$ the class of functions $u\in L^2(\boldsymbol{R}, m)$ such that there exists the derivative $u^+(x)$, of bounded variation on compact intervals of $\boldsymbol{R}$, and for some $h_u\in L^2(\boldsymbol{R}, m)$

$$\int_{a+}^{b+} h_u(x)dm(x) = u^+(b)-u^+(a)-\int_{a+}^{b+} u(x)dk(x)\,, \qquad a, b\in\boldsymbol{R}\,, \tag{2.4}$$

where $u^+(x)$ is the right derivative of $u(x)$ with respect to $s(x)$, that is,

$$u^+(x) = \lim_{h\downarrow 0}\frac{u(x+h)-u(x)}{s(x+h)-s(x)},$$

and the integral is read as

$$\int_{a+}^{b+} f(x)d\xi(x) = \begin{cases} \int_{(a,b]} f(x)d\xi(x) & \text{for } a\leq b, \\ -\int_{(b,a]} f(x)d\xi(x) & \text{for } a>b. \end{cases}$$

Then the map $\mathfrak{G}$: $u(\in D(\mathfrak{G}))\mapsto h_u$ is called a *generalized diffusion operator* on $\boldsymbol{R}$ (cf. [4]). $\mathfrak{G}$ is obviously reduced to a second order difference operator if m satisfies

(2.5) $$\#\{\mathrm{Supp}\,(dm)\cap(0, 1]\} = N<\infty,$$

and k satisfies (2.3.b).

We now assume that $\mathfrak{G}$ is periodic with period 1, that is, $\mathfrak{G}(u(\cdot+1))(x)=(\mathfrak{G}u)(x+1)$, $x\in\boldsymbol{R}$, for every u such that both u and $u(\cdot+1)$ belong to $D(\mathfrak{G})$. In the same way as in [3], we can see that this property is equivalent to the following:

(2.6) There is a positive ρ such that for every $x, y\in\boldsymbol{R}$

$$s(x+1)-s(y+1) = \rho(s(x)-s(y)),$$
$$m(x+1)-m(y+1) = \rho^{-1}(m(x)-m(y)),$$
$$k(x+1)-k(y+1) = \rho^{-1}(k(x)-k(y)).$$

$\mathfrak{G}$ is called a *periodic generalized diffusion operator* provided (2.1), (2.2), (2.3) and (2.6) are satisfied.

We next consider the restriction of $\mathfrak{G}$ to the half line $\boldsymbol{R}_+\equiv[0, \infty)$ with the sticky elastic boundary condition at 0. Let Γ be the collection of triplets $(\gamma_1, \gamma_2, \gamma_3)$ such that $\gamma_1=\gamma_2=\gamma_3-1=0$ or $\gamma_1\in\boldsymbol{R}_+$, $\gamma_2=1$, $\gamma_3\in\boldsymbol{R}$. For each $\gamma=(\gamma_1, \gamma_2, \gamma_3)\in\Gamma$ we define the measure $m^\gamma(dx)=\gamma_1\delta_{\{0\}}(dx)+\chi_{(0,\infty)}(x)dm(x)$ and denote by $D(\mathfrak{G}^\gamma)$ the set of all functions $u\in L^2(\boldsymbol{R}_+, m^\gamma)$ having the property that the derivative $u^+(x)$ exists and is of bounded variation on compact intervals of $\boldsymbol{R}_+$, and satisfying (2.4) for some $h_u\in L^2(\boldsymbol{R}_+, m^\gamma)$ and for every $a, b\in\boldsymbol{R}_+$ as well as the following boundary condition (2.7).

(2.7) $$\gamma_1 h_u(0) = \gamma_2 u^+(0)-\gamma_3 u(0).$$

$\mathfrak{G}^\gamma$ is the map $u(\in D(\mathfrak{G}^\gamma))\mapsto h_u$.

2.2. Let $\varphi_j(x, \lambda)$, $x\in\boldsymbol{R}$, $\lambda\in\boldsymbol{C}$, $j=1, 2$ be the solutions of the integral equations

(2.8)
$$\varphi_1(x, \lambda) = 1+\int_{0+}^{x+}(s(x)-s(y))\varphi_1(y, \lambda)(-\lambda dm(y)+dk(y)),$$
$$\varphi_2(x, \lambda) = s(x)+\int_{0+}^{x+}(s(x)-s(y))\varphi_2(y, \lambda)(-\lambda dm(y)+dk(y)).$$

Note that the Wronskian of φ_1 and φ_2 is equal to 1:

$$\varphi_1(x, \lambda)\varphi_2^+(x, \lambda)-\varphi_1^+(x, \lambda)\varphi_2(x, \lambda) = 1 . \tag{2.9}$$

We define the functions Δ and D by

$$\Delta(\lambda) = \varphi_1(1, \lambda)+\rho\varphi_2^+(1, \lambda) , \quad D(\lambda) = \Delta^2(\lambda)-4\rho . \tag{2.10}$$

$\Delta(\lambda)$ is called the discriminant following the terminologies in the theory of Hill's operator. We also define the following sets:

$$\begin{aligned} &S = \{\lambda\in \boldsymbol{R}: D(\lambda)\leq 0\} , &&S_* = \{\lambda\in \boldsymbol{R}: D(\lambda)<0\}, \\ &S_1 = \{\lambda\in \boldsymbol{R}: \varphi_1^+(1, \lambda) = 0\} , &&S_2 =\{\lambda\in \boldsymbol{R}: \varphi_2(1, \lambda) = 0\}. \end{aligned} \tag{2.11}$$

Now the following assertion is well known in the case of Hill's operators. Further, the result corresponding to the case of that $s(x)=x$, $k(x)\equiv 0$ and $\rho=1$ is obtained by M.G. Krein [6].

Theorem 1. *Assume* (2.5) *with some* $N\in \boldsymbol{N}$. *Then there exists a finite sequence* $-\infty=\mu_0^{(1)}<\mu_0^{(2)}<\mu_1^{(1)}\leq\mu_1^{(2)}<\cdots<\mu_n^{(1)}\leq\mu_n^{(2)}<\cdots\leq\mu_{N-1}^{(2)}<\mu_N^{(1)}<\mu_N^{(2)}=\infty$ *such that*

$$\begin{aligned} &\Delta(\lambda)>2\sqrt{\rho} && \text{if } \lambda\in(\mu_n^{(1)}, \mu_n^{(2)}), \text{ and } n \text{ is zero or even,} \\ &\Delta(\lambda)<-2\sqrt{\rho} && \text{if } \lambda\in(\mu_n^{(1)}, \mu_n^{(2)}), \text{ and } n \text{ is odd,} \\ &\Delta(\lambda) = 2\sqrt{\rho} && \text{if } \lambda=\mu_n^{(i)}\in \boldsymbol{R}, n \text{ is zero or even, and } i=1, 2, \\ &\Delta(\lambda) = -2\sqrt{\rho} && \text{if } \lambda=\mu_n^{(i)}\in \boldsymbol{R}, n \text{ is odd, and } i=1, 2. \end{aligned} \tag{2.12}$$

In the other cases, there exists an infinite sequence $-\infty=\mu_0^{(1)}<\mu_0^{(2)}<\mu_1^{(1)}\leq\mu_1^{(2)}<\cdots<\mu_n^{(1)}\leq\mu_n^{(2)}<\cdots\uparrow\infty$ *which satisfies* (2.12).

The following notations are used frequently.

$$\begin{aligned} &l = \sharp\{n\in \boldsymbol{N}: \mu_n^{(1)}<\mu_n^{(2)}\}, \\ &\lambda_{-1} = \mu_0^{(1)} = -\infty, \qquad \lambda_0 = \mu_0^{(2)} , \\ &\lambda_{2j+k} = \min\{\mu_n^{(k)}: \lambda_{2j}<\mu_n^{(1)}<\mu_n^{(2)}, n\in \boldsymbol{N}\}, \qquad j\geq 0, k=1, 2 . \end{aligned}$$

Clearly $1\leq l\leq\infty$ and λ_j's are defined for $j\in[-1, 2l]\cap \boldsymbol{Z}$. It follows from Theorem 1 that $l\leq N$ and $\lambda_{2l}=\infty$ in the case of (2.5); $\lambda_{2l}+l<\infty$ or $\lim_{j\to\infty}\lambda_j=l=\infty$ in the other cases. Namely

$$S = \bigcup_{j=0}^{l-1} [\lambda_{2j}, \lambda_{2j+1}] , \qquad \text{in the case of (2.5),} \tag{2.13}$$

$$S = \bigcup_{j=0}^{l-1} [\lambda_{2j}, \lambda_{2j+1}]\cup[\lambda_{2l}, \infty), \quad \text{or} \quad \bigcup_{j=0}^{\infty} [\lambda_{2j}, \lambda_{2j+1}] , \qquad \text{otherwise.} \tag{2.14}$$

2.3. As was mentioned in § 1, Weyl-Stone-Titchmarsh-Kodaira's theorem [10; Chapter 5] combined with the spectral theory of generalized diffusion operators due to K. Itô and H.P. McKean [4; § 4.11] is still effective for our operators. We summalize it.

For $\lambda \in \boldsymbol{C} \backslash \boldsymbol{R}$, there exit the limits $f_j(\lambda)$, $j=1, 2$:

$$
(2.15) \qquad \begin{aligned} f_1(\lambda) &= -\lim_{x\to+\infty} \varphi_1(x, \lambda)/\varphi_2(x, \lambda) = -\lim_{x\to+\infty} \varphi_1^+(x, \lambda)/\varphi_2^+(x, \lambda)\,, \\ f_2(\lambda) &= -\lim_{x\to-\infty} \varphi_1(x, \lambda)/\varphi_2(x, \lambda) = -\lim_{x\to-\infty} \varphi_1^+(x, \lambda)/\varphi_2^+(x, \lambda)\,. \end{aligned}
$$

We set

$$
\begin{aligned} f_{11}(\lambda) &= 1/(f_2(\lambda)-f_1(\lambda))\,, \\ f_{12}(\lambda) &= f_{21}(\lambda) = f_2(\lambda)/(f_2(\lambda)-f_1(\lambda))\,, \\ f_{22}(\lambda) &= f_1(\lambda) f_2(\lambda)/(f_2(\lambda)-f_1(\lambda))\,. \end{aligned}
$$

Define $\sigma_{jk}(u)$, j, $k=1, 2$ on $\boldsymbol{R}$ by

$$
(2.16) \quad \sigma_{jk}(u_2)-\sigma_{jk}(u_1) = \frac{1}{\pi} \lim_{v\downarrow 0} \int_{u_1}^{u_2} \mathcal{I}m\, f_{jk}(u+\sqrt{-1}v)du\,, \qquad u_1<u_2,\ j,\ k=1, 2\,,
$$

and denote the induced Stieltjes measures on $\boldsymbol{R}$ by $d\sigma_{jk}$. The matrix valued measure $[d\sigma_{jk}]_{j,k=1,2}$ is the spectral measure of $\mathfrak{G}$. Weyl-Stone-Titchmarsh-Kodaira theory in this case tells us that for $g \in L^2(\boldsymbol{R}, m)$,

$$
g(x) = \sum_{j,k=1,2} \int_{\boldsymbol{R}} \varphi_j(x, u) \Big(\int_{\boldsymbol{R}} \varphi_k(y, u) g(y) dm(y) \Big) d\sigma_{jk}(u)\,.
$$

We now set

$$
\begin{aligned} &\Phi_{11}(u) = 2|\varphi_2(1, u)|\,, \quad \Phi_{22}(u) = 2\rho|\varphi_1^+(1, u)|\,, \\ &\Phi_{12}(u) = \Phi_{21}(u) = \rho\varphi_2^+(1, u) - \varphi_1(1, u)\,, \\ &\Psi_{jk}(u) = \int_{0+}^{1+} \varphi_{3-j}(x, u)\varphi_{3-k}(x, u) dm(x)\,, \qquad j, k = 1, 2\,, \\ &\Psi(u) = \int_{0+}^{1+} \varphi_1^2(x, u)\, dm(x) \int_{0+}^{1+} \varphi_2^2(x, u) dm(x) - \Big(\int_{0+}^{1+} \varphi_1(x, u)\varphi_2(x, u) dm(x) \Big)^2. \end{aligned}
$$

The following theorem gives us the spectrum of $\mathfrak{G}$.

Theorem 2. *The spectrum of $\mathfrak{G}$ is continuous and coincides with S. The spectral measures $d\sigma_{jk}$, $j, k=1, 2$ are absolutely continuous with respect to the Lebesgue measure in S. The densities ρ_{jk} are continuous in $\mathring{S}$, the interior of S. More precisely they are given by*

$$
(2.17) \qquad \rho_{jk}(u) = \begin{cases} (-1)^{n(j+k)}\Phi_{jk}(u)/2\pi\sqrt{|D(u)|}\,, & \mu_n^{(2)} < u < \mu_{n+1}^{(1)},\ n \geq 0\,, \\ (-1)^{j+k}\Psi_{jk}(u)/2\pi\sqrt{\Psi(u)}, & u \in \mathring{S} \backslash S_*\,. \end{cases}
$$

The following is immediate from the first assertion of Theorem 2 combined with Theorem 1.

Corollary. *The spectrum of* $\mathfrak{G}$ *is bounded if and only if the set* Supp$(dm)\cap(0, 1]$ *is finite.*

Next we study the asymptotic behaviors of ρ_{jk} near the points $\mu_n^{(i)}$. Note that $\mu_n^{(i)}\notin\mathring{S}$ implies $\mu_n^{(i)}\notin S_1\cap S_2$ (see the proof of Theorem 3 in § 4).

Theorem 3. *Let* $\mu=\mu_n^{(i)}\in\boldsymbol{R}\setminus\mathring{S}$, $n\geq 0$, $i=1, 2$. *Then it holds as* $u\to\mu$, $u\in\mathring{S}$ *that for* $j, k=1, 2$

$$\rho_{jk}(u) = C_{jk}|u-\mu|^{\delta_{jk}}+O(|u-\mu|^{\delta_{jk}+1})\,, \tag{2.18}$$

where $C_{jk}=C_{jk}(\mu)$ *and* $\delta_{jk}=\delta_{jk}(\mu)$ *are given by*

$$C_{jk}(\mu) = \begin{cases} (-1)^{(n+i)(j+k)}\Phi_{jk}(\mu)/4\pi\rho^{1/4}\sqrt{|\Delta'(\mu)|}\,, & \mu\notin S_{3-j}\cup S_{3-k}\,, \\ \rho^{1/4}\Psi_{jk}(\mu)/2\pi\sqrt{|\Delta'(\mu)|}\,, & \mu\in S_{3-j}\cup S_{3-k}\,, \end{cases}$$

$$\delta_{jk}(\mu) = \begin{cases} -1/2\,, & \mu\notin S_{3-j}\cup S_{3-k}\,, \\ \ \ 1/2\,, & \mu\in S_{3-j}\cup S_{3-k}\,. \end{cases}$$

2.4. For each $\gamma=(\gamma_1, \gamma_2, \gamma_3)\in\Gamma$ we define the functions $\psi_j^\gamma(x, \lambda)$, $x\in\boldsymbol{R}_+$, $\lambda\in\boldsymbol{C}$, $j=1, 2$:

$$\begin{aligned} \psi_1^\gamma(x, \lambda) &= \gamma_2\varphi_1(x, \lambda)-(\gamma_1\lambda-\gamma_3)\varphi_2(x, \lambda)\,, \\ \psi_2^\gamma(x, \lambda) &= \{(\gamma_1\lambda-\gamma_3)\varphi_1(x, \lambda)+\gamma_2\varphi_2(x, \lambda)\}/\{|\gamma_1\lambda-\gamma_3|^2+\gamma_2\}\,. \end{aligned} \tag{2.19}$$

Then there exist the limits

$$h^\gamma(\lambda) = \lim_{x\to\infty}\psi_2^\gamma(x, \lambda)/\psi_1^\gamma(x, \lambda) = \lim_{x\to\infty}\psi_2^{\gamma+}(x, \lambda)/\psi_1^{\gamma+}(x, \lambda)$$

for $\lambda\in\boldsymbol{C}\setminus\boldsymbol{R}$. Obviously it holds by (2.15) that

$$h^\gamma(\lambda) = \{(\gamma_1\lambda-\gamma_3)f_1(\lambda)-\gamma_2\}\{\gamma_2 f_1(\lambda)+\gamma_1\lambda-\gamma_3\}^{-1}\{|\gamma_1\lambda-\gamma_3|^2+\gamma_2\}^{-1}\,. \tag{2.20}$$

Define the function σ^γ on $\boldsymbol{R}$ by

$$\sigma^\gamma(u_2)-\sigma^\gamma(u_1) = \frac{1}{\pi}\lim_{v\downarrow 0}\int_{u_1}^{u_2}\mathcal{Im}\, h^\gamma(u+\sqrt{-1}v)du\,, \qquad u_1<u_2\,. \tag{2.21}$$

We denote by $d\sigma^\gamma$ the induced Stieltjes measure on $\boldsymbol{R}$, which is the spectral measure of $\mathfrak{G}^\gamma$. Weyl-Stone-Titchmarsh-Kodaira theory leads us to the spectral representation

$$g(x) = \int_{\boldsymbol{R}}\psi_1^\gamma(x, u)\Big(\int_{\boldsymbol{R}_+}\psi_1^\gamma(y, u)g(y)m^\gamma(dy)\Big)d\sigma^\gamma(u)$$

for $g\in L^2(\boldsymbol{R}_+, m^\gamma)$.

Now let Σ^γ, Σ_c^γ, Σ_p^γ be the spectrum, the continuous spectrum, the point spectrum of $\mathfrak{G}^\gamma$, respectively. Notice that the residual spectrum of $\mathfrak{G}^\gamma$ is empty.

We are first concerned with the continuous spectrum.

Theorem 4. 1) $\Sigma_c^\gamma = S$ *for every* $\gamma \in \Gamma$. 2) *The spectral measure* $d\sigma^\gamma$ *is absolutely continuous with respect to the Lebesgue measure in S. The density function* ρ^γ *is positive continuous in* $\mathring{S}$, *the interior of S, and for* $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \Gamma$ *it is given by*

$$(2.22)\qquad \begin{aligned} \rho^\gamma(u) &= \frac{|\varphi_2(1, u)| \sqrt{|D(u)|}}{2\pi\{\{\psi_1^\gamma(1, u) - \gamma_2 \Delta(u)/2\}^2 + \gamma_2 |D(u)|/4\}}, && u \in S_*, \\ \rho^\gamma(u) &= \frac{\Psi_{11}(u)\sqrt{\Psi(u)}}{\pi\{\{(\gamma_1 u - \gamma_3)\Psi_{11}(u) - \gamma_2 \Psi_{12}(u)\}^2 + \gamma_2 \Psi(u)\}}, && u \in \mathring{S} \setminus S_*. \end{aligned}$$

We next observe the asymptotic behavior of $\rho^\gamma(u)$ as $u \in \mathring{S}$ tends to λ_j. It should be noted that for each $j \in [-1, 2l] \cap \boldsymbol{Z}$ there exists the limit

$$\lim_{\lambda \to \lambda_j, \lambda \in \boldsymbol{R} \setminus S} f_1(\lambda) = A_j,$$

and A_j is given by

$$A_j = \begin{cases} -\left(\int_0^a \varphi_1^{-2}(x, 0) ds(x)\right)^{-1}, & \text{if } j = -1, \text{ or (2.5) holds and } j = 2l, \\ (\Delta(\lambda_j)/2 - \varphi_1(1, \lambda_j))/\varphi_2(1, \lambda_j), & \text{if } \lambda_j \notin S_2, \\ +\infty, & \text{if } j \text{ is even and } \lambda_j \in S_2, \\ -\infty, & \text{if } j \text{ is odd and } \lambda_j \in S_2, \end{cases}$$

where $a = \inf\{x > 0 : m(x) > 0\}$ and $-c^{-1}$ is understood to be $-\infty$ for $c = 0$ (see the proof of Theorem 6 in § 5). We put

$$\tau_j^\gamma = \gamma_1 \lambda_j + \gamma_2 A_j - \gamma_3 \qquad (\in [-\infty, +\infty]),$$

for $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \Gamma$ and $j \in [-1, 2l] \subset \boldsymbol{Z}$, where $0 \cdot \infty = 0 \cdot (-\infty) = 0$.

Theorem 5. *Let* $\lambda = \lambda_j < \infty$, $j \in [0, 2l] \cap \boldsymbol{Z}$ *and* $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \Gamma$. *Then*

$$(2.23)\qquad \rho^\gamma(u) = C|u - \lambda|^\delta + O(|u - \lambda|^{\delta+1}) \quad \text{as} \quad u \to \lambda,\ u \in \mathring{S},$$

where C and δ *are constants depending on* λ *and* γ *given as follows:*

$$\begin{aligned} &C = |\Delta'(\lambda)|^{1/2}/\pi\rho^{1/4}\Psi_{11}(\lambda), \quad \delta = -1/2, && \text{if } \lambda \in S_2 \text{ and } \gamma_2 = 0; \\ &C = \rho^{1/4}\Psi_{11}(\lambda)/\pi|\Delta'(\lambda)|^{1/2}, \quad \delta = 1/2, && \text{if } \lambda \in S_2 \text{ and } \gamma_2 = 1; \\ &C = |\varphi_2(1, \lambda)|/\pi\rho^{1/4}|\Delta'(\lambda)|^{1/2}, \quad \delta = -1/2, && \text{if } \lambda \notin S_2 \text{ and } \tau_j^\gamma = 0; \\ &C = \rho^{1/4}|\varphi_2(1, \lambda)|\,|\Delta'(\lambda)|^{1/2}/\pi\{\psi_1^\gamma(1, \lambda) - \gamma_2 \Delta(\lambda)/2\}^2, \quad \delta = 1/2, && \\ & && \text{if } \lambda \notin S_2 \text{ and } \tau_j^\gamma \neq 0. \end{aligned}$$

Finally we turn to the point spectrum. As is clear from Theorem 4 with (2.13) and (2.14),

$$\Sigma_j^\gamma \subset \boldsymbol{R} \setminus S = \bigcup_{j=0}^{l} G_j, \qquad \gamma \in \Gamma,$$

where $G_j=(\lambda_{2j-1}, \lambda_{2j})$, $j\geq 0$. Noting that $A_{2j-1}\neq A_{2j}$ for every $j\in[0, l]\cap \boldsymbol{Z}$ (see the proof of Theorem 6 in § 5 below), we have more precise result. For each $j\in[0, l]\cap \boldsymbol{Z}$ we set

$$\varepsilon_j = (A_{2j-1}-A_{2j})/(\lambda_{2j}-\lambda_{2j-1}),$$

where $c/\infty=0$ in the case of $c\neq 0$.

Theorem 6. *Let $j\in[0, l]\cap \boldsymbol{Z}$ and $\gamma=(\gamma_1, \gamma_2, \gamma_3)\in\Gamma$.*
1) *Assume $A_{2j-1}<A_{2j}$. Then $\Sigma_p^\gamma\cap G_j$ consists of a single point provided $\tau_{2j-1}^\gamma<0<\tau_{2j}^\gamma$, and is empty in the other cases.*
2) *Let $A_{2j-1}>A_{2j}$. If $\gamma_1>\varepsilon_j$ and $\tau_{2j-1}^\gamma<0<\tau_{2j}^\gamma$, then $\Sigma_p^\gamma\cap G_j$ consists of two points. If $0\leq\gamma_1\leq\varepsilon_j$ and $\tau_{2j}^\gamma<0<\tau_{2j-1}^\gamma$, then $\Sigma_p^\gamma\cap G_j$ is empty. In the other cases, $\Sigma_p^\gamma\cap G_j$ consists of a single point.*

In the proof of above theorem we will also find that $-\infty\leq A_{-1}<A_0<\infty$. Hence we get immediately

Corollary. *Let λ^γ be the principal eigenvalue of $\mathfrak{G}^\gamma$, that is, $\lambda^\gamma=\min\Sigma^\gamma$. Then λ^γ belongs to Σ_p^γ if $\tau_{-1}^\gamma<0<\tau_0^\gamma$, and to Σ_c^γ otherwise.*

Further, Theorems 1, 4 and 6 assert

Corollary. *The spectrum of $\mathfrak{G}^\gamma$ is bounded if and only if the set* Supp$(dm)\cap$ (0, 1] *is finite.*

3. Properties of the discriminant.

In this section we prove Theorem 1. We sometimes denote

$$\varphi_j'(x, \lambda) = \partial\varphi_j(x, \lambda)/\partial\lambda, \quad \varphi_j''(x, \lambda) = \partial^2\varphi_j(x, \lambda)/\partial\lambda^2, \quad \text{etc.}$$

First we note the following equalities which are proved in the same way as in [2; § 2.3] or [8: § 2.1]. For $a\in\boldsymbol{R}$ and $j=1, 2$

$$\text{(3.1)}\qquad \varphi_j'(a, \lambda) = \int_{0+}^{a+}\{\varphi_{1,\lambda}(a)\varphi_{2,\lambda}(x)-\varphi_{2,\lambda}(a)\varphi_{1,\lambda}(x)\}\varphi_{j,\lambda}(x)dm(x),$$

$$\text{(3.2)}\qquad \varphi_j^{+\prime}(a, \lambda) = \int_{0+}^{a+}\{\varphi_{1,\lambda}^+(a)\varphi_{2,\lambda}(x)-\varphi_{2,\lambda}^+(a)\varphi_{1,\lambda}(x)\}\varphi_{j,\lambda}(x)dm(x),$$

$$\text{(3.3)}\qquad \varphi_j''(a, \lambda) = 2\int_{0+}^{a+}\{\varphi_{1,\lambda}(a)\varphi_{2,\lambda}(x)-\varphi_{2,\lambda}(a)\varphi_{1,\lambda}(x)\}dm(x) \times\int_{0+}^{x+}\{\varphi_{1,\lambda}(x)\varphi_{2,\lambda}(y)-\varphi_{2,\lambda}(x)\varphi_{1,\lambda}(y)\}\varphi_{j,\lambda}(y)dm(y),$$

$$\text{(3.4)}\qquad \varphi_j^{+\prime\prime}(a, \lambda) = 2\int_{0+}^{a+}\{\varphi_{1,\lambda}^+(a)\varphi_{2,\lambda}(x)-\varphi_{2,\lambda}^+(a)\varphi_{1,\lambda}(x)\}dm(x) \times\int_{0+}^{x+}\{\varphi_{1,\lambda}(x)\varphi_{2,\lambda}(y)-\varphi_{2,\lambda}(x)\varphi_{1,\lambda}(y)\}\varphi_{j,\lambda}(y)dm(y),$$

where $\varphi_{j,\lambda}(\cdot)=\varphi_j(\cdot,\lambda)$ and $\varphi_{j,\lambda}^{+}(\cdot)=\varphi_j^{+}(\cdot,\lambda)$.

By means of these equalities we can repeat the standard argument for Hill's operator to get some properties of the discriminant $\Delta(\lambda)$ (cf. [2; Chapters 1–3], [8; Chapter 2]). We list up them without proof.

Lemma 3.1. 1) *$\Delta(\lambda)$ is an entire function.* 2) *There exists a real number μ_0 such that $\Delta(\lambda)>2\sqrt{\rho}$ for $\lambda<\mu_0$.* 3) *All roots of the equation $\Delta^2(\lambda)-4\rho=0$ are real and lie in the interval (μ_0, ∞).* 4) *If $\Delta(\mu)=2\sqrt{\rho}$ and $\Delta'(\mu)\leq 0$ for some $\mu\in \boldsymbol{R}$, then $\Delta'(\lambda)<0$ for all $\lambda>\mu$ such that $\Delta(\xi)>-2\sqrt{\rho}$, $\xi\in(\mu, \lambda]$. Similarly, if $\Delta(\mu)=-2\sqrt{\rho}$ and $\Delta'(\mu)\geq 0$ for some $\mu\in\boldsymbol{R}$, then $\Delta'(\lambda)>0$ for all $\lambda>\mu$ such that $\Delta(\xi)<2\sqrt{\rho}$, $\xi\in(\mu, \lambda]$.*

We next give a remark on the zeros of the function $\varphi_2(1, \lambda)$, $\lambda\in\boldsymbol{C}$. In the following we set $I=(0, 1)$, and $I_m=\mathrm{Supp}(dm)\cap I$.

Lemma 3.2. *$\varphi_2(1, \cdot)$ is a positive constant if $I_m=\phi$. $\varphi_2(1, \cdot)$ has k zeros $-\infty<\xi_1<\xi_2<\cdots<\xi_k<\infty$ if $\sharp I_m=k$, a countable infinite number of zeros $-\infty<\xi_1<\xi_2<\cdots<\xi_n<\cdots\uparrow\infty$ otherwise.*

Proof. 1) Assume $I_m=\phi$. Then (2.8) with $x\in\bar{I}\equiv[0, 1]$ is reduced to

$$\varphi_2(x, \lambda) = s(x)+\int_{0+}^{x+}(s(x)-s(y))\varphi_2(y, \lambda)dk(y)\,.$$

Therefore $\varphi_2(1, \lambda)$ is independent of λ and the condition (2.3) yields $\varphi_2(1, \lambda)\geq s(1)>0$.

2) Let $I_m\neq\phi$. Fix a real number α such that $\varphi_1(x, \alpha)$ and $\varphi_2(x, \alpha)$ are positive increasing for $x\in(0, \infty)$. We put

$$g_1(x) = \varphi_2(x, \alpha)\,, \quad g_2(x) = \varphi_1(x, \alpha)-(\varphi_1(1, \alpha)/\varphi_2(1, \alpha))\varphi_2(x, \alpha)\,,$$
$$G(x, y) = G(y, x) = g_1(x)g_2(y)\,, \qquad x\leq y\,.$$

Note that $g_1(0)=g_2(1)=0$, and $g_1[g_2]$ is increasing [resp. decreasing] on $\bar{I}$. Define the operator G by

$$Gf(x) = \int_I G(x, y)f(y)dm(y)\,, \qquad x\in\bar{I},\ f\in L^2(I, m)\,.$$

For every $f\in L^2(I, m)$, $Gf(0)=Gf(1)=0$ and $Gf(x)$ is continuous in $x\in\bar{I}$. Further $Gf(x)$ satisfies the following

$$(3.5)\qquad Gf(x) = s(x)\int_I g_2(y)f(y)dm(y)-\int_{0+}^{x+}(s(x)-s(y))f(y)dm(y)$$
$$+\int_{0+}^{x+}(s(x)-s(y))Gf(y)(-\alpha dm(y)+dk(y))\,, \qquad x\in\bar{I}\,.$$

We will prove that $Gf(x)=0$ $m-a.e.$ $x\in I$ implies $f(x)=0$ $m-a.e.$ $x\in\bar{I}$. Fix an

$a\in I$ and put $b=\sup\{x: 0\leq x<a,\ m(x)<m(a)\}$, $c=\inf\{x: a<x\leq 1,\ m(a)<m(x)\}$. Note that $Gf(b)=Gf(c)=0$, that is,

$$g_2(b)\int_{0+}^{b+} g_1 fdm+g_1(b)\int_{b+}^{1+} g_2 fdm = g_2(c)\int_{0+}^{b+} g_1 fdm+g_1(c)\int_{b+}^{1+} g_2 fdm = 0\,.$$

It is trivial that $Gf(a)=0$ provided $b=c$. Let $b<c$. Since $g_1(b)g_2(c)<g_1(c)g_2(b)$, we get $\int_{0+}^{b+} g_1 fdm=\int_{b+}^{1+} g_2 fdm=0$, and hence

$$Gf(a) = g_2(a)\int_{0+}^{b+} g_1 fdm+g_1(a)\int_{b+}^{1+} g_2 fdm = 0\,.$$

a being arbitrary, we obtain $Gf(x)\equiv 0$ on $\bar{I}$. Differentiating both hand sides of (3.5) with respect to $s(x)$, we see that $f(x)=0$ $m-a.e.$ $x\in I$.

Obviously G induces a positive symmetric operator $\tilde{G}$ of the Hilbert-Schmidt type from $L^2(I, m)$ into itself. From the above observation $\tilde{G}$ has exactly k positive eigenvalues with multiplicity if $\sharp I_m=k$, a countable infinite number of positive eigenvalues, which have no point of accumulation except 0, otherwise.

We next verify that there is a one to one correspondence between zeros λ of $\varphi_2(1, \cdot)$ and eigenvalues β of $\tilde{G}$, and the correspondence is given by $\lambda=\alpha+1/\beta$. Assume $\varphi_2(1, \lambda)=0$. Since $\varphi_2(1, \alpha)>0$, we have $\lambda\neq\alpha$. By the definition of the operator G and by (2.8), we get that $G\varphi_{2,\lambda}(x)=\varphi_{2,\lambda}(x)/(\lambda-\alpha)$, $x\in\bar{I}$, where $\varphi_{2,\lambda}(x)=\varphi_2(x, \lambda)$. This means that $1/(\lambda-\alpha)$ is an eigenvalue of $\tilde{G}$. Conversely, let β be an eigenvalue of $\tilde{G}$ and $\tilde{f}$ an eigenfunction correspopnding to it. It should be noted that $\tilde{f}$ has a continuous version f on $\bar{I}$ satisfying $Gf(x)=\beta f(x)$, $x\in\bar{I}$. By virtue of (3.5), $h(x)\equiv\beta f(x)/\int_I g_2 fdm$ satisfies the integral equation

$$h(x) = s(x)+\int_{0+}^{x+}(s(x)-s(y))h(y)(-(\alpha+1/\beta)dm(y)+dk(y))\,.$$

This integral equation has the unique solution. By (2.8) the solution is identical with $\varphi_2(x, \alpha+1/\beta)$. It follows from $h(1)=0$ that $\varphi_2(1, \alpha+1/\beta)=0$, i.e. $\alpha+1/\beta$ is a zero of $\varphi_2(1, \cdot)$.

If $\varphi_2(1, \lambda)=0$, then by (2.9) $\varphi_1(1, \lambda)\varphi_2^+(1, \lambda)=1$, and hence by (3.1),

$$\varphi_2'(1, \lambda) = \varphi_1(1, \lambda)\int_{0+}^{1+}\varphi_2^2(x, \lambda)dm(x)\neq 0\,, \tag{3.6}$$

from which zeros of $\varphi_2(1, \lambda)$ are simple. Thus we get the assertion of the lemma.

q.e.d.

Now we are ready to give

Proof of Theorem 1. We first note that $\lim_{\lambda\downarrow-\infty}\Delta(\lambda)=\infty$ and $0<\lim_{\lambda\downarrow-\infty}\varphi_2(1, \lambda)\leq\infty$. Also by (2.9), (3.1), (3.2) and (3.6)

$$\varphi_2(1,\lambda)\Delta'(\lambda) = -\int_{0+}^{1+}\{\varphi_2(1,\lambda)\varphi_1(x,\lambda)-(\varphi_1(1,\lambda)-\rho\varphi_2^+(1,\lambda))\varphi_2(x,\lambda)/2\}^2 dm(x)$$
$$+(D(\lambda)/4)\int_{0+}^{1+}\varphi_2^2(x,\lambda)dm(x)<0\,,\qquad \text{if}\quad |\Delta(\lambda)|<2\sqrt{\rho}\,;$$
$$\varphi_2'(1,\lambda)\Delta(\lambda) = (\varphi_1^2(1,\lambda)+\rho)\int_{0+}^{1+}\varphi_2^2(x,\lambda)dm(x)>0\,,\quad \text{if}\quad \varphi_2(1,\lambda)=0\,.$$

If $\mathrm{Supp}(dm)\cap(0, 1]$ contains a countable infinite subset, then the assertion of the theorem follows immediately from Lemmas 3.1 and 3.2. When $m(x)$ satisfies (2.5) for some $N\in \boldsymbol{N}$, by means of (2.8) we get easily that

$$(3.7)\qquad \begin{aligned}&(-1)^N(d^N/d\lambda^N)\varphi_j(1,\lambda)\geq 0\,,\quad (d^k/d\lambda^k)\varphi_j(1,\lambda)=0\,,\\ &(-1)^N(d^N/d\lambda^N)\varphi_j^+(1,\lambda)>0\,,\quad (d^k/d\lambda^k)\varphi_j^+(1,\lambda)=0\,,\end{aligned}$$

for $j=1, 2$ and $k>N$. Hence $\Delta(\lambda)$ is a polynomial of degree N. (This fact has already shown by M.G. Krein [6] for the case that $s(x)=x$, $k(x)\equiv 0$ and $\rho=1$.) Therefore the assertion of the theorem follows from Lemmas 3.1 and 3.2 in this case, too. q.e.d.

Finally we observe how the points of S_1 and S_2 defined by (2.11) are distributed. By using (3.1) and (3.2) we see that if $|\Delta(\lambda)|<2\sqrt{\rho}$, then $\varphi_1^+(1,\lambda)\Delta'(\lambda)>0$. Since by (3.7) $\varphi_1^+(1,\cdot)$ is a polynomial of degree N in the case of (2.5), Lemma 3.1 coupled with $\lim_{\lambda\downarrow-\infty}\varphi_1^+(1,\lambda)=\infty$ tells us the following result, which is also well known in the case where $\mathfrak{G}$ is reduced to a Hill's operator (cf. [2; Theorem 3.1.1]).

Lemma 3.3. *$S_1\cup S_2\subset \boldsymbol{R}\backslash S_*$. $S_1\cap(-\infty,\lambda_0]$ consists of a single point and $S_2\cap(-\infty,\lambda_0]$ is empty. $\varphi_1^+(1,\lambda_0)\leq 0$ and $\varphi_2(1,\lambda_0)>0$. For each $j=1, 2$, $S_j\cap[\mu_n^{(1)},\mu_n^{(2)}]$ consists of a single point for $1\leq n\leq N-1$ if (2.5) holds, for $n\in\boldsymbol{N}$ otherwise. $S_1\cap[\lambda_{2l-1},\infty)=\phi$ if (2.5) holds. Further, when (2.5) is satisfied, $S_2\cap[\lambda_{2l-1},\infty)$ is empty or a one-point set according to $1\in\mathrm{Supp}(dm)$ or $1\notin\mathrm{Supp}(dm)$.*

4. Spectrum of periodic generalized diffusion operators on the real line.

The aim of this section is to prove Theorems 2 and 3.

By the same argument as the standard one for Hill's equations, the equation

$$r^2-\Delta(\lambda)r+\rho=0$$

has two distinct solutions $r_j(\lambda)$, $j=1, 2$ with $0<|r_1(\lambda)|<|r_2(\lambda)|$ for $\lambda\in\boldsymbol{C}\backslash S$ (cf. [5; § 2]). We note that the functions $r_j(\lambda)$ are both analytic in $\boldsymbol{C}\backslash S$. For $\lambda\in S$, we put $r_j(\lambda)=\lim_{\nu\downarrow 0}r_j(\lambda+\sqrt{-1}\,\nu)$, $j=1, 2$ conventionally. It is easy to see that the analytic continued $D^{1/2}(\lambda)$ satisfies

$$\lim_{\nu\downarrow 0}D^{1/2}(\lambda+\sqrt{-1}\,\nu)=\begin{cases}(-1)^n|D(\lambda)|^{1/2}, & \mu_n^{(1)}<\lambda<\mu_n^{(2)},\\ \sqrt{-1}(-1)^{n+1}|D(\lambda)|^{1/2}, & \mu_n^{(2)}<\lambda<\mu_{n+1}^{(1)}.\end{cases}$$

This implies

$$\lim_{v\downarrow 0} r_j(\lambda+\sqrt{-1}\,v) = (\Delta(\lambda)+\sqrt{-1}(-1)^{n+j+1}\sqrt{|D(\lambda)|})/2\,, \qquad \mu_n^{(2)}<\lambda<\mu_{n+1}^{(1)},\ j=1, 2. \tag{4.1}$$

Further we note that for $\lambda\in\boldsymbol{R}$ with $\Delta(\lambda)>2\sqrt{\rho}\,[\Delta(\lambda)<-2\sqrt{\rho}]$,

$$r_j(\lambda) = \{\varphi_1(1, \lambda)+\rho\varphi_2^+(1, \lambda)+(-1)^j\sqrt{D(\lambda)}\}/2>0 \tag{4.2}$$

$$[\text{resp. } r_j(\lambda) = \{\varphi_1(1, \lambda)+\rho\varphi_2^+(1, \lambda)+(-1)^{j+1}\sqrt{D(\lambda)}\}/2<0]\,.$$

If $\lambda\in\boldsymbol{R}$ and $\varphi_1^+(1, \lambda)\varphi_2(1, \lambda)=0$, then by (2.9)

$$D(\lambda) = (\varphi_1(1, \lambda)-\rho\varphi_2^+(1, \lambda))^2\geq 0\,, \tag{4.3}$$

and hence

$$\begin{aligned} &r_1(\lambda) = \varphi_1(1, \lambda)\,, \quad r_2(\lambda) = \rho\varphi_2^+(1, \lambda)\,, \qquad \text{if} \quad |\varphi_1(1, \lambda)|\leq|\rho\varphi_2^+(1, \lambda)|\,; \\ &r_2(\lambda) = \varphi_1(1, \lambda)\,, \quad r_1(\lambda) = \rho\varphi_2^+(1, \lambda)\,, \qquad \text{if} \quad |\varphi_1(1, \lambda)|\geq|\rho\varphi_2^+(1, \lambda)|\,. \end{aligned} \tag{4.4}$$

We can also easily show the equalities in [5; (2.10)] in our case, and so we get by (2.15)

$$f_j(\lambda) = \frac{r_j(\lambda)-\varphi_1(1, \lambda)}{\varphi_2(1, \lambda)} = \frac{\rho\varphi_1^+(1, \lambda)}{r_j(\lambda)-\rho\varphi_2^+(1, \lambda)}\,, \qquad j=1, 2\,. \tag{4.5}$$

Since r_j's are analytic in $\boldsymbol{C}\backslash S$, both f_j's can be analytically continued through $\boldsymbol{R}\backslash(S\cup S_2)$, which we denote by f_j again.

Proof of Theorem 2. It follows by the same method as [5; Lemma 2.3] that the spectrum of $\mathfrak{G}$ is continuous and coincides with S. Also, by [5; (2.14)], which is valid in our case,

$$\lim_{v\downarrow 0} \mathcal{I}m\, f_{11}(u+\sqrt{-1}\,v) = \varphi_2(1, u) \lim_{v\downarrow 0} 1/\mathcal{I}m\, \{r_2(u+\sqrt{-1}\,v)-r_1(u+\sqrt{-1}\,v)\}\,,$$

and the limit is uniform in u on each compact set in S_*. It follows from (2.16), (4.1) and the fact $(-1)^n\varphi_2(1, u)>0$, $\mu_n^{(2)}<u<\mu_{n+1}^{(1)}$ that $\rho_{11}(u)=|\varphi_2(1, u)|/\pi\sqrt{|D(u)|}$ for $u\in S_*$. Similarly the other formulas ρ_{jk} with $u\in S_*$ follow.

Let $\mu\in\mathring{S}\backslash S_*$, i.e. $\mu=\mu_n^{(1)}=\mu_n^{(2)}$ for some $n\in\boldsymbol{N}$. Then $\varphi_1^+(1, \mu)=\varphi_2(1, \mu)=\Delta'(\mu)=0$ and

$$r_j(\mu) = \varphi_1(1, \mu) = \rho\varphi_2^+(1, \mu) = (-1)^n\rho^{1/2}, \qquad j=1, 2\,. \tag{4.6}$$

Then it follows from (3.3) and (3.4) that

$$\Delta''(\mu) = -\Delta(\mu)\Psi(\mu)\,, \quad D''(\mu) = -8\rho\Psi(\mu)\,.$$

This implies

$$|D(u)|^{1/2} = 2(\rho\Psi(\mu))^{1/2}|u-\mu|+O((u-\mu)^2)\,, \qquad u\rightarrow\mu,\ u\in S_*\,. \tag{4.7}$$

Further, by virtue of (3.1), it holds as $u\rightarrow\mu$, $u\in S_*$ that

$$(4.8)\qquad \begin{aligned}\varphi_1(1, u) &= \varphi_1(1, \mu)+\varphi_1(1, \mu)\Psi_{12}(\mu)(u-\mu)+O((u-\mu)^2)\,,\\ \varphi_2(1, u) &= \varphi_1(1, \mu)\Psi_{11}(\mu)(u-\mu)+O((u-\mu)^2)\,.\end{aligned}$$

Also, by means of (3.2), as $u\to\mu$, $u\in S_*$,

$$(4.9)\qquad \begin{aligned}\varphi_1^+(1, u) &= -\varphi_2^+(1, \mu)\Psi_{22}(\mu)(u-\mu)+O((u-\mu)^2)\,,\\ \varphi_2^+(1, u) &= \varphi_2^+(1, \mu)-\varphi_2^+(1, \mu)\Psi_{12}(\mu)(u-\mu)+O((u-\mu)^2)\,,\end{aligned}$$

whence the formulas ρ_{jk} with $u\in\mathring{S}\backslash S_*$ follow. q.e.d.

Proof of Theorem 3. First we note that if $\mu=\mu_n^{(i)}\in S_1\cap S_2$, then (3.1), (3.2) and (4.6) imply $\Delta'(\mu)=0$ and $\mu_n^{(1)}=\mu_n^{(2)}$, i.e. $\mu\in\mathring{S}$. Hence $\mu_n^{(i)}\notin\mathring{S}$ implies $\mu_n^{(i)}\notin S_1\cap S_2$.

Let $\mu=\mu_n^{(i)}\notin S_1\cap S_2$ and $\mu_n^{(1)}<\mu_n^{(2)}<\infty$. Note that (2.9) implies in general

$$(\varphi_1(1, \lambda)-\rho\varphi_2^+(1, \lambda))^2 = D(\lambda)-4\rho\varphi_1^+(1, \lambda)\varphi_2(1, \lambda)\,.$$

Hence we have $\varphi_1(1, \mu)-\rho\varphi_2^+(1, \mu)\neq 0$ in this case. We also note that

$$(4.10)\qquad \begin{aligned}&\Delta'(\mu)\neq 0\,,\\ &|D(u)|^{1/2} = 2\rho^{1/4}|\Delta'(\mu)|^{1/2}|u-\mu|^{1/2}+O(|u-\mu|^{3/2})\,,\end{aligned}$$

as $u\to\mu$, $u\in\mathring{S}$. Then (2.18) is clear in this case.

Let $\mu=\mu_n^{(i)}\in S_1\backslash S_2$ and $\mu_n^{(1)}<\mu_n^{(2)}<\infty$. Then (4.6) and (4.10) follow. Hence by (2.17) and (4.9) we get (2.18) again.

Let $\mu=\mu_n^{(i)}\in S_2\backslash S_1$ and $\mu_n^{(1)}<\mu_n^{(2)}<\infty$. In this case, we have also (4.6) and (4.10). Hence (2.18) follows from (2.17) and (4.8). q.e.d.

5. Spectrum of periodic generalized diffusion operators with sticky elastic boundary conditions.

In this section we show Theorems 4, 5 and 6. In the following we fix a triplet $\gamma=(\gamma_1, \gamma_2, \gamma_3)\in\Gamma$. Let us put

$$\begin{aligned}\Lambda^\gamma &= \{\lambda\in\boldsymbol{R}\colon (\gamma_1\lambda-\gamma_3)\psi_1^\gamma(1, \lambda)+\gamma_2\rho\psi_1^{\gamma+}(1, \lambda) = 0\}\,,\\ \Lambda_j^\gamma &= \{\lambda\in\Lambda^\gamma\colon \gamma_2\psi_1^\gamma(1, \lambda)+(1-\gamma_2)\rho\psi_1^{\gamma+}(1, \lambda) = r_j(\lambda)\}\,, \qquad j = 1, 2\,.\end{aligned}$$

It is easy to see that $\Lambda^\gamma=\Lambda_1^\gamma\cup\Lambda_2^\gamma$. Further we note

Lemma 5.1. 1) $\Lambda^\gamma\subset\boldsymbol{R}\backslash S_*$. 2) $\Sigma_p^\gamma=\Lambda_1^\gamma\backslash S$.

Proof. 1) Let $\lambda\in\Lambda^\gamma$. If $\varphi_1^+(1,\lambda)\varphi_2(1, \lambda)=0$, then λ belongs to $\boldsymbol{R}\backslash S_*$ by (4.3). Assume $\varphi_1^+(1, \lambda)\varphi_2(1, \lambda)\neq 0$. Then the real pair $\{\gamma_1\lambda-\gamma_3, \gamma_2\}$ solves the equation

$$(5.1)\qquad \begin{aligned}&(\gamma_1\lambda-\gamma_3)\psi_1^\gamma(1, \lambda)+\gamma_2\rho\psi_1^{\gamma+}(1, \lambda)\\ &\quad = \gamma_2\rho\varphi_1^+(1, \lambda)+(\gamma_1\lambda-\gamma_3)\gamma_2(\varphi_1(1, \lambda)-\rho\varphi_2^+(1, \lambda))\\ &\qquad -(\gamma_1\lambda-\gamma_3)^2\varphi_2(1, \lambda) = 0\,.\end{aligned}$$

Therefore the discriminant is nonnegative:

$$(\varphi_1(1, \lambda)-\rho\varphi_2^+(1, \lambda))^2+4\rho\varphi_1^+(1, \lambda)\varphi_2(1, \lambda)\geq 0 .$$

The left hand side coincides with $D(\lambda)$ by means of (2.9), which yields $\lambda\in \boldsymbol{R}\backslash S_*$.
2) First of all we note that $\Sigma_p^\gamma\subset \boldsymbol{R}\backslash S$. Indeed, if $\lambda\in S$, then by means of [5; (2.9), (2.10)] a non-trivial linear combination of $\varphi_1(x, \lambda)$ and $\varphi_2(x, \lambda)$ is written as $\rho^{x/2}\{e^{\sqrt{-1}\alpha x}(1+\beta x)p_1(x)+e^{-\sqrt{-1}\alpha x}p_2(x)\}$, where α and β are real numbers and $p_1(x)$ and $p_2(x)$ are periodic with period 1. Hence there are no linear combinations of $\varphi_1(x, \lambda)$ and $\varphi_2(x, \lambda)$ belonging to $L^2(\boldsymbol{R}_+, m^\gamma)$. Consequently $\lambda\notin\Sigma_p^\gamma$.

We now also get by (2.20) that

$$\begin{aligned}(5.2)\qquad \mathcal{I}m\, h^\gamma(\lambda) &= \mathcal{I}m \frac{1}{-f_1(\lambda)-r_1\lambda+r_3} \\ &\quad -\frac{r_1(|r_1\lambda-r_3|^2-|f_1(\lambda)|^2)}{|f_1(\lambda)+r_1\lambda-r_3|^2(|r_1\lambda-r_3|^2+1)}\mathcal{I}m\,\lambda, \qquad \text{if } r_2=1,\\ \mathcal{I}m\, h^\gamma(\lambda) &= \mathcal{I}m f_1(\lambda), \qquad \text{if } r_2=0.\end{aligned}$$

Suppose that $r_2=1$. Then, by (2.21) and (5.2), Σ_p^γ coincides with the set of all poles λ of the function $(f_1(\lambda)+r_1\lambda-r_3)^{-1}$ in $\boldsymbol{R}\backslash S$. Incidentally, let $\lambda\in A_1^\gamma\backslash S$. Then by (4.5)

$$\psi_1^\gamma(1, \lambda)-r_1(\lambda) = -\varphi_2(1, \lambda)(f_1(\lambda)+r_1\lambda-r_3) = 0 ,$$

from which $\lambda\in\Sigma_p^\gamma$ provided $\varphi_2(1, \lambda)\neq 0$. But if $\varphi_2(1, \lambda)=0$ and $\varphi_1(1, \lambda)\neq\rho\varphi_2^+(1, \lambda)$, then $r_1(\lambda)=\varphi_1(1, \lambda)$ and

$$(f_1(\lambda)+r_1\lambda-r_3)(\varphi_1(1, \lambda)-\rho\varphi_2^+(1, \lambda)) = 0 .$$

This gives $\lambda\in\Sigma_p^\gamma$. (4.3) asserts that $\lambda\in S$ if $\varphi_1(1, \lambda)-\rho\varphi_2^+(1, \lambda)=\varphi_2(1, \lambda)=0$. Therefore we have $A_1^\gamma\backslash S\subset\Sigma_p^\gamma$. Tracing along the above argument reversely, we have the converse inclusion. Thus the conclusion follows in this case.

Suppose next that $r_2=0$. Then Σ_p^γ coincides with the set of all poles of the function $f_1(\lambda)$ in $\boldsymbol{R}\backslash S$. Let $\lambda\in A_1^\gamma\backslash S$. Then $\varphi_2(1, \lambda)=0$ by (5.1) and $r_1(\lambda)=\rho\varphi_2^+(1, \lambda)$. Hence $r_2(\lambda)=\varphi_1(1, \lambda)\neq r_1(\lambda)$ by (4.4) and $D(\lambda)>0$. This implies $\lambda\in\Sigma_p^\gamma$. The converse inclusion is seen similarly. q.e.d.

Now we give

Proof of Theorem 4. 1) Let $\lambda\in S_*$. Then $r_1(\lambda)\in \boldsymbol{C}\backslash\boldsymbol{R}$ and $\varphi_2(1, \lambda)\in\boldsymbol{R}\backslash\{0\}$ by (4.3). Hence, by (5.2) $\lim_{\nu\downarrow 0}\mathcal{I}m\, h^\gamma(\lambda+\sqrt{-1}\nu)\neq 0$, which ensures $S_*\subset\Sigma^\gamma$. The spectrum being closed, it then follows that $S\subset\Sigma^\gamma$. This combined with the inclusion $\Sigma_p^\gamma\subset\boldsymbol{R}\backslash S$ proves $S\subset\Sigma_c^\gamma$.

Suppose next that $\lambda\in\boldsymbol{R}\backslash S$. Then we can find a neighborhood $U(\lambda)$ of λ such that r_1 is analytic in $U(\lambda)$ and $r_1(\xi)\in\boldsymbol{R}$ for $\xi\in U(\lambda)\cap\boldsymbol{R}$. Hence either λ is a pole of h^γ or else $\lim_{\nu\downarrow 0}\mathcal{I}m\, h^\gamma(\xi+\sqrt{-1}\nu)=0$, $\xi\in(\lambda-\varepsilon, \lambda+\varepsilon)$, for some $\varepsilon>0$. This means $\lambda\notin\Sigma_c^\gamma$.

2) Note that substitution of (4.5) into (2.20) implies

$$h^{\gamma}(\lambda)=\frac{(\tau_1\lambda-\tau_3)(\varphi_1(1,\lambda)-r_1(\lambda))+\tau_2\varphi_2(1,\lambda)}{\{\tau_2(\varphi_1(1,\lambda)-r_1(\lambda))-(\tau_1\lambda-\tau_3)\varphi_2(1,\lambda)\}\{|\tau_1\lambda-\tau_3|^2+\tau_2\}}.$$

Hence, if $u\in\boldsymbol{R}\backslash(\Lambda^{\gamma}\cup S_2)$, then we have

$$\lim_{v\downarrow 0}\mathcal{Im}\, h^{\gamma}(u+\sqrt{-1}\,v)=\lim_{v\downarrow 0}\frac{\varphi_2(1,u)\,\mathcal{Im}\, r_1(u+\sqrt{-1}\,v)}{|\psi_1^{\gamma}(1,u)-\tau_2 r_1(u+\sqrt{-1}\,v)|^2}. \tag{5.3}$$

The limit in (5.3) is uniform in u on each compact interval in S_*.

Now the formula (2.22) with $u\in S_*$ follows from (2.21), (4.1) and the fact that $(-1)^n\varphi_2(1,u)>0$ for $\mu_n^{(2)}<u<\mu_{n+1}^{(1)}$.

We next note that if $\mu\in\mathring{S}\backslash S_*$, i.e. $\mu=\mu_n^{(1)}=\mu_n^{(2)}$ for some $n\in\boldsymbol{N}$, then $\varphi_1^{+}(1,\mu)=\varphi_2(1,\mu)=\Delta'(\mu)=0$. Further (5.1), (5.2) and (5.3) hold and $\Delta(u)=\Delta(\mu)+O((u-\mu)^2)$ as $u\rightarrow\mu$, $u\in S_*$. Hence

$$\begin{aligned}&|\varphi_1(1,u)|\sqrt{|D(u)|}\\&\quad=2(\rho\Psi(\mu))^{1/2}(-1)^n\varphi_1(1,\mu)\Psi_{11}(\mu)(u-\mu)^2+O(|u-\mu|^3),\\&(\psi_1^{\gamma}(1,u)-\tau_2\Delta(u)/2)^2+\tau_2|D(u)|/4\\&\quad=\{\varphi_1^2(1,\mu)\{(\tau_1\mu-\tau_2)\Psi_{11}(\mu)-\tau_2\Psi_{12}(\mu)\}^2+\tau_2\rho\Psi(\mu)\}(u-\mu)^2+O(|u-\mu|^3),\end{aligned}$$

as $u\rightarrow\mu$, $u\in S_*$. These imply (2.22) for $u\in\mathring{S}\backslash S_*$. q.e.d.

Proof of Theorem 5. Let $\mu=\lambda_j<\infty$ and $j\in[0,2l]\cap\boldsymbol{Z}$. Then (4.10) follows. We divide the proof into four cases.

1) Let $\lambda_j\in S_2$ and $\tau_2=0$. Then (4.6) holds with an appropriate $n\in\boldsymbol{N}$. Further by (2.25), (4.8) and (4.10)

$$\rho^{\gamma}(u)=\frac{|\varphi_1(1,\mu)|\rho^{1/4}|\Delta'(\mu)|^{1/2}\Psi_{11}(\mu)|u-\mu|^{3/2}+O(|u-\mu|^{5/2})}{\pi\varphi_1^2(1,\mu)\Psi_{11}^2(\mu)(u-\mu)^2+O(|u-\mu|^3)},$$

as $u\rightarrow\mu$, $u\in\mathring{S}$. This proves (2.23) in this case.

2) Let $\lambda_j\in S_2$ and $\tau_2=1$. Then $\psi_1^{\gamma}(1,\mu)-\Delta(\mu)/2=0$. Hence by (4.8) and (4.10) it follows that

$$\rho^{\gamma}(u)=\frac{|\varphi_1(1,\mu)|\rho^{1/4}|\Delta'(\mu)|^{1/2}\Psi_{11}(\mu)|u-\mu|^{3/2}+O(|u-\mu|^{5/2})}{\pi\rho^{1/2}|\Delta'(\mu)||u-\mu|+O(|u-\mu|^2)},$$

as $u\rightarrow\mu$, $u\in\mathring{S}$. This proves (2.23) in this case.

3) Assume that $\lambda_j\notin S_2$ and $\tau_j^{\gamma}=0$. In this case it holds that

$$\psi_1^{\gamma}(1,u)-\Delta(u)/2=O(|u-\mu|),\qquad u\rightarrow\mu,\ u\in\mathring{S}.$$

Hence by (2.22) and (4.10)

$$\rho^{\gamma}(u)=\frac{|\varphi_2(1,\mu)|\rho^{1/4}|\Delta'(\mu)|^{1/2}|u-\mu|^{1/2}+O(|u-\mu|^{3/2})}{\pi\rho^{1/2}|\Delta'(\mu)||u-\mu|+O(|u-\mu|^2)},$$

as $u \to \mu$, $u \in \mathring{S}$ and (2.23) follows.
4) Assume that $\lambda_j \notin S_2$ and $\tau_j^\gamma \neq 0$. Then $\Psi_1^\gamma(1, \mu) - \gamma_2 \Delta(\mu)/2 \neq 0$. Hence by (4.10) the numerator and the denominator in (2.22) are respectively equal to

$$|\varphi_2(1, \mu)|\rho^{1/4}|\Delta'(\mu)|^{1/2}|u-\mu|^{1/2} + O(|u-\mu|^{3/2}), \quad \text{and}$$
$$\pi(\psi_1^\gamma(1, \mu) - \gamma_2\Delta(\mu)/2)^2 + O(|u-\mu|)$$

as $u \to \mu$, $u \in \mathring{S}$. This implies (2.23). q.e.d.

In order to prove Theorem 6 we need to see that f_j's are tractable on $\boldsymbol{R}\backslash(S \cup S_2)$.

Lemma 5.2. *For each $\lambda \in \boldsymbol{R}\backslash(S \cup S_2)$ and $j=1, 2$*

$$(5.4) \qquad f_j'(\lambda) = \frac{(-1)^{j+1}|r_{3-j}(\lambda)|}{\sqrt{D(\lambda)}} \int_{0+}^{1+} \{\varphi_1(x, \lambda) + f_j(\lambda)\varphi_2(x, \lambda)\}^2 dm(x).$$

Proof. Let $\lambda \in \boldsymbol{R}\backslash S_2$ and $\Delta(\lambda) > 2\sqrt{\rho}$. Then it follows from (4.2) and (4.5) that

$$(5.5) \qquad f_j(\lambda) = \{-\varphi_1(1, \lambda) + \rho\varphi_2^+(1, \lambda) + (-1)^j\sqrt{D(\lambda)}\}/2\varphi_2(1, \lambda).$$

Hence

$$f_j'(\lambda) = [(-1)^j\{r_{3-j}(\lambda)\varphi_1'(1, \lambda) + r_j(\lambda)\rho\varphi_2^{+\prime}(1, \lambda)\}$$
$$- f_j(\lambda)\sqrt{D(\lambda)}\varphi_2'(1, \lambda)]/\varphi_2(1, \lambda)\sqrt{D(\lambda)}.$$

By (3.1) and (3.2) the right hand side of the last equality coincides with that of (5.4).

In the case of that $\lambda \in \boldsymbol{R}\backslash S_2$ and $\Delta(\lambda) < -2\sqrt{\rho}$, (5.4) also follows in the same way as above. q.e.d.

We now proceed to

Proof of Theorem 6. First of all we note that $\gamma_2=0$ yields $\tau_{2j-1}^\gamma = \tau_{2j}^\gamma$; $A_{2j-1} < A_{2j}$ and $\gamma_2=1$ imply $\tau_{2j-1}^\gamma < \tau_{2j}^\gamma$; if $A_{2j-1} > A_{2j}$ and $\gamma_2=1$, then $\tau_{2j-1}^\gamma \gtreqless \tau_{2j}^\gamma$ according to $\gamma_1 \gtreqless \varepsilon_j$.

We divide the proof into six cases. In the following we denote by ξ_j the unique element of $S_2 \cap [\lambda_{2j-1}, \lambda_{2j}]$ if it is not empty.
1) Let $\lambda_{2j-1} < \xi_j < \lambda_{2j} < \infty$ and $r_1(\xi_j) = \varphi_1(1, \xi_j)$ for some $j \in [1, l] \cap \boldsymbol{N}$. Then $r_2(\xi_j) = \rho\varphi_2^+(1, \xi_j) \neq \varphi_1(1, \xi_j)$. The second expression in (4.5) coupled with (4.4) yields

$$(5.6) \qquad \lim_{\lambda \to \xi_j, \lambda \in \boldsymbol{R}\backslash S} f_1(\lambda) = \rho\varphi_1^+(1, \xi_j)/\{\varphi_1(1, \xi_j) - \rho\varphi_2^+(1, \xi_j)\} \in \boldsymbol{R}.$$

In view of (4.2)

$$(5.7) \qquad \lim_{\lambda \downarrow \lambda_{2j-1}} f_1(\lambda) = A_{2j-1} \in \boldsymbol{R},$$

$$(5.8) \qquad \lim_{\lambda \uparrow \lambda_{2j}} f_1(\lambda) = A_{2j} \in \boldsymbol{R}.$$

It then follows from Lemma 5.2 that $f_1(\lambda)$ is continuous increasing on $G_j=(\lambda_{2j-1}, \lambda_{2j})$ and $-\infty<A_{2j-1}<A_{2j}<\infty$. Hence the equation $\gamma_2 f_1(\lambda)+\gamma_1\lambda-\gamma_3=0$, $\lambda\in G_j$ has the unique solution ν_j^γ if and only if $\tau_{2j-1}^\gamma<0<\tau_{2j}^\gamma$, from which

$$(5.9)\qquad \Lambda_1^\gamma\cap G_j=\begin{cases}\{\nu_j^\gamma\}, & \tau_{2j-1}^\gamma<0<\tau_{2j}^\gamma\,,\\ \phi\,, & \text{otherwise.}\end{cases}$$

2) Let $\lambda_{2j-1}<\xi_j<\lambda_{2j}<\infty$ and $r_2(\xi_j)=\varphi_1(1, \xi_j)$ for some $j\in[1, l]\cap \boldsymbol{N}$. Also suppose $\Delta(\lambda_{2j-1})=2\sqrt{\rho}$. Then by (3.1) and Lemma 3.3

$$\varphi_2(1, \lambda)<0 \qquad \text{for} \quad \lambda\in[\lambda_{2j-1}, \xi_j)\,,$$
$$\varphi_2(1, \lambda)>0 \qquad \text{for} \quad \lambda\in(\xi_j, \lambda_{2j}]\,.$$

Further $r_1(\xi_j)=\rho\varphi_2^+(1, \xi_j)<\varphi_1(1, \xi_j)$. Hence it follows from the first expression in (4.5) that

$$(5.10)\qquad \lim_{\lambda\uparrow\xi_j} f_1(\lambda)=+\infty,\quad \lim_{\lambda\downarrow\xi_j} f_1(\lambda)=-\infty.$$

(5.10) is valid for the case $\Delta(\lambda_{2j-1})=-2\sqrt{\rho}$, too. On the other hand, in view of (4.2),

$$(5.11)\qquad \lim_{\lambda\downarrow\lambda_{2j-1}} f_1(\lambda)=\lim_{\lambda\downarrow\lambda_{2j-1}} f_2(\lambda)=A_{2j-1}\in\boldsymbol{R}\,,$$
$$\lim_{\lambda\uparrow\lambda_{2j}} f_1(\lambda)=\lim_{\lambda\uparrow\lambda_{2j}} f_2(\lambda)=A_{2j}\in\boldsymbol{R}\,.$$

By virtue of (4.4) and the second expression in (4.5)

$$\lim_{\lambda\to\xi_j,\lambda\in\boldsymbol{R}\setminus S} f_2(\lambda)=\rho\varphi_1^+(1, \xi_j)/\{\varphi_1(1, \xi_j)-\rho\varphi_2^+(1, \xi_j)\}\in\boldsymbol{R}\,.$$

Thus Lemma 5.2 implies that $-\infty<A_{2j}<A_{2j-1}<\infty$ and $\varepsilon_j>0$. The equation $\gamma_2 f_1(\lambda)+\gamma_1\lambda-\gamma_3=0$, $\lambda\in G_j$ has two solutions ν_{j1}^γ, ν_{j2}^γ provided $\gamma_1>\varepsilon_j$ and $\tau_{2j-1}^\gamma<0<\tau_{2j}^\gamma$, a unique solution ν_j^γ provided $\gamma_1>\varepsilon_j$, $(-1)^k\tau_k^\gamma\le 0$, or $0\le\gamma_1\le\varepsilon_j$, $(-1)^k\tau_k^\gamma>0$, where $k=2j-1$ or $2j$, no solutions otherwise. Since $\Lambda_1^\gamma\cap G_j=\{\xi_j\}$ for $\gamma=(0, 0, 1)$ in this case, putting $\nu_j^{(0,0,1)}=\xi_j$, we get

$$(5.12)\qquad \Lambda_1^\gamma\cap G_j=\begin{cases}\{\nu_{j1}^\gamma, \nu_{j2}^\gamma\}, & \text{if}\quad \gamma_1>\varepsilon_j \text{ and } \tau_{2j-1}^\gamma<0<\tau_{2j}^\gamma\,,\\ \{\nu_j^\gamma\}, & \text{if}\quad \gamma_1>\varepsilon_j,\ (-1)^k\tau_k^\gamma\le 0, \text{ or } 0\le\gamma_1\le\varepsilon_j\,,\\ & \qquad (-1)^k\tau_\gamma^k>0, \text{ for } k=2j-1 \text{ or } 2j\,,\\ \phi\,, & \text{otherwise.}\end{cases}$$

3) Let $\xi_j=\lambda_{2j-1}<\lambda_{2j}<\infty$ for some $j\in[1, l]\cap\boldsymbol{N}$. We also assume $\Delta(\lambda_{2j-1})=2\sqrt{\rho}$. It then follows from the assumption that $\varphi_2(1, \lambda_{2j-1})=0$ and

$$r_i(\lambda_{2j-1})=\varphi_1(1, \lambda_{2j-1})=\rho\varphi_2^+(1, \lambda_{2j-1})=\sqrt{\rho}\,,\qquad i=1, 2\,.$$

Hence by using (5.5) and l'Hospital principle

$$\lim_{\lambda\downarrow\lambda_{2j-1}} f_1(\lambda) = \lim_{\lambda\downarrow\lambda_{2j-1}} \{-\varphi_1'(1, \lambda)+\rho\varphi_2^{+\prime}(1, \lambda)-\Delta(\lambda)\Delta'(\lambda)D(\lambda)^{-1/2}\}/2\varphi_2'(1, \lambda).$$

Note that $\varphi_2'(1, \lambda_{2j-1})>0$ by (3.1) and $\Delta(\lambda_{2j-1})\Delta'(\lambda_{2j-1})>0$. Further

$$D(\lambda) = 2\Delta(\lambda_{2j-1})\Delta'(\lambda_{2j-1})(\lambda-\lambda_{2j-1})+o(\lambda-\lambda_{2j-1}) \quad \text{as} \quad \lambda \downarrow \lambda_{2j-1}.$$

Hence

$$\lim_{\lambda\downarrow\lambda_{2j-1}} f_1(\lambda) = -\infty. \tag{5.13}$$

This is also true for the case $\Delta(\lambda_{2j-1})=-2\sqrt{\rho}$. Clealy (5.8) holds in this case, too. Therefore Lemma 5.2 again gives us that $-\infty<A_{2j-1}<A_{2j}<\infty$. There is the unique solution ν_j^γ of the equation $\gamma_2 f_1(\lambda)+\gamma_1\lambda-\gamma_3=0$, $\lambda\in G_j$ only for $\tau_{2j-1}^\gamma<0<\tau_{2j}^\gamma$. Hence (5.9) follows.

4) Let $\lambda_{2j-1}<\lambda_{2j}=\xi_j<\infty$ for some $j\in[1, l]\cap \boldsymbol{N}$. Then in the same way as in 3) we have $-\infty<A_{2j-1}<A_{2j}=\infty$ and (5.9).

5) We note by (4.2) and (4.5) that for $\lambda<\lambda_0$

$$f_1(\lambda) = -\varphi_1(1, \lambda)/\varphi_2(1, \lambda)+2\rho/\varphi_2(1, \lambda)(\Delta(\lambda)+\sqrt{D(\lambda)}).$$

By virtue of (2.9) and by the fact $\varphi_1(x, \lambda)=\varphi_1(x, 0)$ for $\lambda\in\boldsymbol{R}$ and $0\leq x\leq a=\inf\{x\colon x>0, m(x)>0\}$, it follows that

$$\varphi_2(1, \lambda)/\varphi_1(1, \lambda) = \int_0^1 \varphi_1^{-2}(x, \lambda)ds(x) \to \int_0^a \varphi_1^{-2}(x, 0)ds(x) \quad \text{as} \quad \lambda\downarrow -\infty.$$

Therefore

$$\lim_{\lambda\downarrow-\infty} f_1(\lambda) = A_{-1}\geq -\infty.$$

Since $\varphi_2(1, \lambda_0)$ is positive, it follows from the first expression of $f_1(\lambda)$ in (4.5) that

$$\lim_{\lambda\uparrow\lambda_0} f_1(\lambda) = A_0<\infty.$$

By means of Lemma 5.2, $-\infty\leq A_{-1}<A_0<\infty$ and the equation $\gamma_2 f_1(\lambda)+\gamma_1\lambda-\gamma_3=0$, $\lambda\in G_0=(-\infty, \lambda_0)$ has a unique solution ν_0^γ if and only if $\tau_{-1}^\gamma<0<\tau_0^\gamma$. Consequently (5.9) with $j=0$ follows.

6) Suppose the condition (2.5). Then we have by (4.2) and (4.5) that for $\lambda>\lambda_{2l-1}=\mu_1^{(N)}$

$$f_1(\lambda) = -\varphi_1(1, \lambda)/\varphi_2(1, \lambda)+2\rho/\varphi_2(1, \lambda)(\Delta(\lambda)+(-1)^N\sqrt{D(\lambda)}),$$

$$f_2(\lambda) = -\rho\varphi_2^{+}(1, \lambda)/\varphi_2(1, \lambda)+2\rho/\varphi_2(1, \lambda)(\Delta(\lambda)+(-1)^N\sqrt{D(\lambda)}).$$

In the same way as in 5),

$$\varphi_2(1, \lambda)/\varphi_1(1, \lambda) \to \int_0^a \varphi_1^{-2}(x, 0)ds(x) \quad \text{as} \quad \lambda\uparrow\infty,$$

from which $\lim_{\lambda\uparrow\infty} f_1(\lambda)=A_{2l}\in\boldsymbol{R}$. We should notice that $\tau_{2l}^\gamma=\infty$ in the case of

$\gamma_1>0$, and also that $\varepsilon_l=0$. On the other hand, since the function $g(x)\equiv\varphi_j(x+1, \lambda)$ solves the equation

$$g(x) = g(0)+g^+(0)s(x)+\int_{0+}^{x+}(s(x)-s(y))g(y)(-\lambda dm(y)+dk(y)),$$

we have

$$\varphi_j(x+1, \lambda) = \varphi_j(1, \lambda)\varphi_1(x, \lambda)+\rho\varphi_j^+(1, \lambda)\varphi_2(x, \lambda), \qquad j=1, 2.$$

Then the substitution of $x=-1$ gives us

$$\varphi_1(-1, \lambda) = \varphi_2^+(1, \lambda), \quad \varphi_2(-1, \lambda) = -\varphi_2(1, \lambda)/\rho.$$

Consequently, by (2.9)

$$\varphi_2(1, \lambda)/\rho\varphi_2^+(1, \lambda) = -\varphi_2(-1, \lambda)/\varphi_1(-1, \lambda) = \int_{-1}^{0}\varphi_1^{-2}(x, \lambda)ds(x)$$
$$\to \int_b^0\varphi_1^{-2}(x, 0)ds(x) \qquad \text{as} \quad \lambda\uparrow\infty,$$

where $b=\sup\{x: x<0, m(x)<0\}$. If $1\notin\mathrm{Supp}(dm)$, then $b<0$ and

$$\lim_{\lambda\uparrow\infty}f_1(\lambda) = \left(\int_b^0\varphi_1^{-2}(x, 0)\,ds(x)\right)^{-1}\equiv B_{2l}\in\boldsymbol{R}. \tag{5.14}$$

If $1\in\mathrm{Supp}(dm)$, then $\varphi_2(1, \lambda)\neq 0$ for $\lambda>\lambda_{2l-1}$ by Lemma 3.3. Since (5.7) with $j=l$ holds, we deduce from Lemma 5.2 that $-\infty<A_{2l-1}<A_{2l}<\infty$ and the equation $\gamma_2f_1(\lambda)+\gamma_1\lambda-\gamma_3=0$, $\lambda\in G_l=(\lambda_{2l-1}, \infty)$ has a unique solution ν_l^γ if and only if $\tau_{2l-1}^\gamma<0<\tau_{2l}^\gamma$. Thus (5.9) with $j=l$ follows.

If $1\notin\mathrm{Supp}(dm)$ and $\xi_l=\lambda_{2l-1}$, then (5.13) with $j=l$ follows. Therefore $-\infty=A_{2l-1}<A_{2l}<\infty$ and we get (5.9) with $j=l$.

In the case of that $1\notin\mathrm{Supp}(dm)$, $\lambda_{2l-1}<\xi_l$ and $r_1(\xi_l)=\varphi_1(1, \xi_l)$, both (5.6) and (5.7) are valid for $j=l$. So $-\infty<A_{2l-1}<A_{2l}<\infty$ and (5.9) follows with $j=l$.

Finally let $1\notin\mathrm{Supp}(dm)$, $\lambda_{2l-1}<\xi_l$ and $r_2(\xi_l)=\varphi_1(1, \xi_l)$. Then (5.10) and (5.11) holds with $j=l$. Noting that (5.14) and $-\infty<A_{2l}<0<B_{2l}<\infty$, we see by Lemma 5.2 that $-\infty<A_{2l}<A_{2l-1}<\infty$ and the equation $\gamma_2f_1(\lambda)+\gamma_1\lambda-\gamma_3=0$, $\lambda\in G_l$ has two solutions ν_{1l}^γ, ν_{2l}^γ in the case of $\gamma_1>0$ and $\tau_{2l-1}^\gamma<0$, a unique solution ν_l^γ in the case of $\gamma_1>0$ and $\tau_{2l-1}^\gamma\geq 0$, or $\gamma_1=0$ and $(-1)^k\tau_k^\gamma>0$ with $k=2l-1$ or $2l$, no solutions in the other cases. Also note that $\Lambda_1^\gamma\cap G_l=\{\xi_l\}\equiv\{\nu_l^\gamma\}$ for $\gamma=(0, 0, 1)$. Therefore (5.12) with $j=l$ is obtained.

Since $\Sigma_p^\gamma\cap G_j=\Lambda_1^\gamma\cap G_j$ by virtue of Lemma 5.1, we complete the proof. q.e.d.

6. Examples of periodic generalized diffusion operator.

In this section we give two examples. The first one is a second order differential operator with constant coefficients and the second one is a periodic difference operator.

Example 1. Let b and k be real numbers, and set

$$\mathfrak{G} = d^2/dx^2 - bd/dx + k .$$

Then $\rho = e^b$ and

$$ds(x) = e^{bx}dx , \quad dm(x) = e^{-bx}dx , \quad dk(x) = -ke^{-bx}dx .$$

The solutions $\varphi_j(x, \lambda)$, $j=1, 2$ of (2.8) are given by

$$\varphi_1(x, \lambda) = \{(b+\delta(\lambda)) \exp(-\delta(\lambda)x/2) - (b-\delta(\lambda)) \exp(\delta(\lambda)x/2)\} e^{bx/2}/2\delta(\lambda) ,$$
$$\varphi_2(x, \lambda) = \{\exp(\delta(\lambda)x/2) - \exp(-\delta(\lambda)x/2)\} e^{bx/2}/\delta(\lambda) ,$$

for $\lambda \neq \lambda^\circ \equiv b^2/4 - k$, where $\delta(\lambda)$ is the square root of the discriminant of the equation

$$\xi^2 - b\xi + \lambda + k = 0 .$$

For $\lambda = \lambda^\circ$ we have

$$\varphi_1(x, \lambda^\circ) = e^{bx/2}(1 - bx/2) , \quad \varphi_2(x, \lambda^\circ) = xe^{bx/2} .$$

Now if we take analytically continued version of $\delta(\lambda)$ such that $\delta(\lambda)/2 = (\lambda^\circ - \lambda)^{1/2}$ for $\lambda < \lambda^\circ$, then

$$\Delta(\lambda) = e^{b/2}(e^{\delta(\lambda)/2} + e^{-\delta(\lambda)/2}) ,$$
$$D(\lambda) = e^b(e^{\delta(\lambda)/2} - e^{-\delta(\lambda)/2})^2 ,$$
$$r_j(\lambda) = \exp\{(b + (-1)^j\delta(\lambda))/2\} , \qquad j = 1, 2 ,$$
$$f_1(\lambda) = (b - \delta(\lambda))/2 .$$

Thus

$$S = [\lambda_0, \infty) , \quad \lambda_0 = \lambda^\circ = b^2/4 - k .$$

When $\mathfrak{G}$ is considered on $\boldsymbol{R}$, the spectral measure density functions are as follows.

$$\rho_{11}(u) = 1/2\pi\sqrt{u-\lambda_0} , \quad \rho_{22}(u) = (u+k)/2\pi\sqrt{u-\lambda_0} ,$$
$$\rho_{12}(u) = \rho_{21}(u) = b/4\pi\sqrt{u-\lambda_0} , \qquad u > \lambda_0 .$$

For $\mathfrak{G}^\gamma$, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \Gamma$ we get

$$\rho^\gamma(u) = \sqrt{u-\lambda_0}/\pi\{\gamma_2(u-\lambda_0) + (\gamma_1 u + \gamma_2 b/2 - \gamma_3)^2\} , \qquad u > \lambda_0 ,$$
$$\tau_0^\gamma = \gamma_1\lambda_0 + \gamma_2 b/2 - \gamma_3 ,$$
$$\Sigma_p^\gamma = \begin{cases} \{\nu_0^\gamma\} , & \text{if } \tau_0^\gamma > 0 , \\ \phi , & \text{if } \tau_0^\gamma \leq 0 . \end{cases}$$

Example 2. Given $0 < \rho, \xi < \infty$, we put for $x \in \boldsymbol{R}$

$$s(x)=\begin{cases}(\rho^x-1)/(\rho-1)\,, & \text{if}\quad \rho\neq 1\,,\\ x\,, & \text{if}\quad \rho=1\,,\end{cases}$$

$$m(x)=\begin{cases}\sum_{k\in Z}\{(\rho+1)(\rho^k-1)\xi/(\rho-1)\rho^k\}\,\chi_{[k,k+1)}(x)\,, & \text{if}\quad \rho\neq 1\,,\\ \sum_{k\in Z}2\xi k\chi_{[k,k+1)}(x)\,, & \text{if}\quad \rho=1\,,\end{cases}$$

Then the operator $\mathfrak{G}u(x)=du^+(x)/dm(x)$ is nothing more than the periodic difference operator

$$\begin{aligned}\mathfrak{G}u(k) &= \{u^+(k)-u^-(k)\}/\{m(k)-m(k-1)\},\\ u^\pm(k) &= \{u(k\pm 1)-u(k)\}/\{s(k\pm 1)-s(k)\},\end{aligned}\qquad k\in\boldsymbol{Z}\,.$$

Now we get easily

$$\begin{aligned}&\varphi_1(x,\lambda)=1\,,\quad \varphi_2(x,\lambda)=s(x)\,,\qquad 0\leq x\leq 1\,,\\ &\varDelta(\lambda)=\rho+1-\rho m^\circ\lambda\,,\quad m^\circ\equiv(\rho+1)\xi/\rho\,.\end{aligned}$$

Set $\cos\theta(u)=\varDelta(u)/2\sqrt{\rho}$ and $\sin\theta(u)=\sqrt{|D(u)|}/2\sqrt{\rho}$. Then in the same way as in [5; (2.10)] we have for $0\leq x<1$, $k\in\boldsymbol{Z}$ and $u\in\boldsymbol{R}$,

$$\begin{aligned}\varphi_1(x+k,u) &= \rho^{k/2}\{\sin(k+1)\theta(u)+\sqrt{\rho}(m^\circ u-1)\sin k\theta(u)\\ &\qquad -\sqrt{\rho}\,m^\circ us(x)\sin k\theta(u)\}/\sin\theta(u)\,,\\ \varphi_2(x+k,u) &= \rho^{(k-1)/2}[\sin k\theta(u)-\{\rho(m^\circ u-1)\sin k\theta(u)\\ &\qquad +\sqrt{\rho}\sin(k-1)\theta(u)\}s(x)]/\sin\theta(u)\,.\end{aligned}$$

Moreover,

$$S=[\lambda_0,\lambda_1]\,,\quad \lambda_0=(\sqrt{\rho}-1)^2/\rho m^\circ,\quad \lambda_1=(\sqrt{\rho}+1)^2/\rho m^\circ.$$

If $\mathfrak{G}$ is considered on $\boldsymbol{R}$, then for $\lambda_0<u<\lambda_1$

$$\begin{aligned}&\rho_{11}(u)=1/\pi\sqrt{|D(u)|},\quad \rho_{22}(u)=\rho m^\circ u/\pi\sqrt{|D(u)|},\\ &\rho_{12}(u)=\rho_{21}(u)=\{\rho(1-m^\circ u)-1\}/2\pi\sqrt{|D(u)|}.\end{aligned}$$

Next we consider $\mathfrak{G}^\gamma$, $\gamma=(\gamma_1,\gamma_2,\gamma_3)\in\Gamma$ on $\boldsymbol{R}_+$. Then

$$\begin{aligned}&A_{-1}=A_2=-1\,,\quad A_0=\sqrt{\rho}-1\,,\quad A_1=-\sqrt{\rho}-1\,;\\ &\rho^\gamma(u)=\sqrt{|D(u)|}/2\pi\{(\psi_1^\gamma(1,u)-\gamma_2\varDelta(u)/2)^2+\gamma_2|D(u)|/4\},\ \lambda_0<u<\lambda_1;\\ &\Sigma_p^\gamma=\{\nu_0^\gamma,\nu_1^\gamma\}\quad \text{if}\quad 0<\gamma_1<\rho m^\circ/2,\ \tau_1^\gamma<0<\tau_0^\gamma,\\ &\Sigma_p^\gamma=\{\nu_0^\gamma\}\quad \text{if}\quad 0<\gamma_1<\rho m^\circ/2,\ \tau_1^\gamma\geq 0,\text{ or }\gamma_1=0\,,\\ &\qquad\qquad \tau_{-1}^\gamma<0<\tau_0^\gamma,\text{ or }\gamma_1\geq\rho m^\circ/2,\ \tau_0^\gamma>0\,,\\ &\Sigma_p^\gamma=\{\nu_1^\gamma\}\quad \text{if}\quad 0<\gamma_1<\rho m^\circ/2,\ \tau_0^\gamma\leq 0,\text{ or }\gamma_1=0\,,\\ &\qquad\qquad \tau_1^\gamma<0<\tau_2^\gamma,\text{ or }\gamma_1\geq\rho m^\circ/2,\ \tau_1^\gamma<0\,,\\ &\Sigma_p^\gamma=\phi\qquad \text{otherwise.}\end{aligned}$$

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