J. Math. Kyoto Univ. (JMKYAZ)
27-3 (1987) 501-505

A note on local isometric imbeddings of complex projective spaces

Dedicated to the memory of Masaru Morinaka

By

Yoshio AGAOKA*)

Introduction.

In this note we consider the problem of local isometric imbeddings (or immersions) of complex projective spaces $P^n(\boldsymbol{C})$ endowed with the standard metric into the Euclidean spaces. On this subject, the following results are already known:

(1) $P^n(\boldsymbol{C})$ is globally isometrically imbedded into $\boldsymbol{R}^{n^2+2n}$ (Kobayashi [5]).

(2) $P^n(\boldsymbol{C})$ admits a solution of the Gauss equation in codimension n^2-1 (Agaoka [2]).

(3) $P^n(\boldsymbol{C})$ cannot be isometrically immersed into $\boldsymbol{R}^{3n-1}$ even locally (Agaoka-Kaneda [3]).

But there is a great difference between the dimension appeared in (1), (2) and (3). And even in the case $n=2$, the least dimensional Euclidean space into which $P^2(\boldsymbol{C})$ can be locally isometrically imbedded is not determined. (For details, see [2] p. 130.)

The purpose of this note is to improve the estimates of the type (3), namely we prove the following theorem.

Theorem. *Let $P^n(\boldsymbol{C})$ $(n\geqq 2)$ be the complex projective space endowed with the standard metric. If $P^n(\boldsymbol{C})$ can be locally isometrically immersed into $\boldsymbol{R}^{2n+k}$, then $k\geqq \frac{1}{5}(6n-4)$.*

This theorem gives a better result than that of [3] in the case $n\geqq 5$. To prove this theorem, we use several facts on the exterior algebra, which we show in § 1. Using these lemmas, we prove Theorem in § 2.

§ 1. Lemmas on the exterior algebra.

Let V be a finite dimensional real vector space with a positive definite inner product (,). Using the metric (,), an element of $\wedge^2 V$ can be considered as a

Received March 25, 1986

*) Postdoctoral Fellow of Japan Society for the Promotion of Science.

skew symmetric endomorphism of V in the following way. For $a_i, b_i \in V$, we define a linear map $\sum_{i=1}^{k} a_i \wedge b_i: V \to V$ by

$$(\sum_{i=1}^{k} a_i \wedge b_i)v = \sum_{i=1}^{k} \{(b_i, v)a_i - (a_i, v)b_i\} \qquad (v \in V).$$

We denote by $\operatorname{rank}(\sum_{i=1}^{k} a_i \wedge b_i)$, $\operatorname{Im}(\sum_{i=1}^{k} a_i \wedge b_i)$ the rank and the image of this linear map, respectively. Note that $\operatorname{rank}(\sum_{i=1}^{k} a_i \wedge b_i)$ is always even because $\sum_{i=1}^{k} a_i \wedge b_i$ is skew symmetric. For the vectors $a_1, \cdots, a_k$, we denote by $\langle a_1, \cdots, a_k \rangle$ the linear subspace of V spanned by $a_1, \cdots, a_k$. By definition it is clear that $\operatorname{Im}(\sum_{i=1}^{k} a_i \wedge b_i)$ is contained in the space $\langle a_1, \cdots, a_k, b_1, \cdots, b_k \rangle$ and hence $\operatorname{rank}(\sum_{i=1}^{k} a_i \wedge b_i) \leqq 2k$.

Lemma 1. *Let V be a real vector space and $a_1, \cdots, a_k, b_1, \cdots, b_k$ be elements of V. If $\operatorname{rank}(\sum_{i=1}^{k} a_i \wedge b_i) \geqq 2l$, then*

$$\dim \langle a_1, \cdots, a_k \rangle, \ \dim \langle b_1, \cdots, b_k \rangle \geqq l.$$

Proof. We assume that $\dim \langle a_1, \cdots, a_k \rangle \leqq l-1$. Then rearranging the indices of $\{a_i\}$ and $\{b_i\}$ if necessary, we may assume that $\{a_1, \cdots, a_p\}$ is linearly independent and $\langle a_1, \cdots, a_p \rangle = \langle a_1, \cdots, a_k \rangle$ $(p \leqq l-1)$. Then the vectors $a_{p+1}, \cdots, a_k$ are expressed in the form

$$\begin{cases} a_{p+1} = \alpha_{p+1,1} a_1 + \cdots + \alpha_{p+1,p} a_p \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_k = \alpha_{k,1} a_1 + \cdots + \alpha_{k,p} a_p . \end{cases}$$

($\alpha_{i,j}$ are real numbers.) Hence we have

$$\sum_{i=1}^{k} a_i \wedge b_i = a_1 \wedge (b_1 + \alpha_{p+1,1} b_{p+1} + \cdots + \alpha_{k,1} b_k) + \cdots + a_p \wedge (b_p + \alpha_{p+1,p} b_{p+1} + \cdots + \alpha_{k,p} b_k),$$

and this implies that $\operatorname{rank}(\sum_{i=1}^{k} a_i \wedge b_i) \leqq 2p \leqq 2(l-1)$, which contradicts the condition $\operatorname{rank}(\sum_{i=1}^{k} a_i \wedge b_i) \geqq 2l$. Hence we have $\dim \langle a_1, \cdots, a_k \rangle \geqq l$. In the same way we have $\dim \langle b_1, \cdots, b_k \rangle \geqq l$. q.e.d.

The next lemma is easy to prove and we omit the proof.

Lemma 2. *Let V_1, V_2, V_3 be subspaces of V. If $\dim(V_1 \cap V_2) \geqq k$ and $\dim(V_2 \cap V_3) \geqq l$, then $\dim(V_1 \cap V_3) \geqq k + l - \dim V_2$.*

Now we prove the following key lemma.

Lemma 3. *Let $a_1, \cdots, a_k, b_1, \cdots, b_k$ be elements of V and V_1, V_2 be subspaces of V spanned by $\{a_1, \cdots, a_k\}$, $\{b_1, \cdots, b_k\}$, respectively. If $\dim V_1 \geqq n_1$, $\dim V_2 \geqq n_2$ and $\dim V_1 \cap V_2 \leqq l$, then $\operatorname{rank}(\sum_{i=1}^{k} a_i \wedge b_i) \geqq 2(n_1 + n_2 - k - l)$. Or equivalently, if*

$\dim V_1 \geqq n_1$, $\dim V_2 \geqq n_2$, *and* rank $(\sum_{i=1}^{k} a_i \wedge b_i) \leqq 2(n_1+n_2-k-l)$, *then* $\dim V_1 \cap V_2 \geqq l$.

Proof. We put $p=\dim V_1$ $(\geqq n_1)$. Then rearranging the indices if necessary, we may assume that the vectors $a_1, \cdots, a_p$ are linearly independent and the rest of vectors $a_{p+1}, \cdots, a_k$ are expressed as a linear combination of $a_1 \cdots, a_p$, as in the proof of Lemma 1. Using the same notation, we put

$$\begin{cases} \bar{b}_1 = b_1+\alpha_{p+1,1}b_{p+1}+\cdots+\alpha_{k,1}b_k \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \bar{b}_p=b_p+\alpha_{p+1,p}b_{p+1}+\cdots+\alpha_{k,p}b_k . \end{cases}$$

Then we have

$$\sum_{i=1}^{k} a_i \wedge b_i = \sum_{i=1}^{p} a_i \wedge \bar{b}_i .$$

Moreover we have

$$\langle b_1, \cdots, b_k \rangle = \langle \bar{b}_1, \cdots, \bar{b}_p, b_{p+1}, \cdots, b_k \rangle$$

and hence

$$\dim \langle \bar{b}_1, \cdots, \bar{b}_p \rangle \geqq n_2-(k-p) \geqq n_1+n_2-k .$$

Now we put $q=\dim \langle \bar{b}_1, \cdots, \bar{b}_p \rangle$ $(\geqq n_1+n_2-k)$ and rearranging the indices, we assume that $\bar{b}_1, \cdots, \bar{b}_q$ are linearly independent. We express the rest of vectors $\bar{b}_{q+1}, \cdots, \bar{b}_p$ in the form:

$$\begin{cases} \bar{b}_{q+1} = \beta_{q+1,1}\bar{b}_1+\cdots+\beta_{q+1,q}\bar{b}_q \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \bar{b}_p = \beta_{p,1}\bar{b}_1+\cdots+\beta_{p,q}\bar{b}_q . \end{cases}$$

Then by putting

$$\begin{cases} \bar{a}_1 = a_1+\beta_{q+1,1}a_{q+1}+\cdots+\beta_{p,1}a_p \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \bar{a}_q = a_q+\beta_{q+1,q}a_{q+1}+\cdots+\beta_{p,q}a_p , \end{cases}$$

we have

$$\sum_{i=1}^{p} a_i \wedge \bar{b}_i = \sum_{i=1}^{q} \bar{a}_i \wedge \bar{b}_i .$$

In addition, we have

$$\langle a_1, \cdots, a_p \rangle = \langle \bar{a}_1, \cdots, \bar{a}_q, a_{q+1}, \cdots, a_p \rangle .$$

In particular the vectors $\bar{a}_1, \cdots, \bar{a}_q$ are linearly independent. We put $r=\dim \langle \bar{a}_1, \cdots, \bar{a}_q \rangle \cap \langle \bar{b}_1, \cdots, \bar{b}_q \rangle$. Then we have $r \leqq l$ because the subspace $\langle \bar{a}_1, \cdots, \bar{a}_q \rangle \cap \langle \bar{b}_1, \cdots, \bar{b}_q \rangle$ is contained in $V_1 \cap V_2$. By changing the indices, we assume that $\{\bar{a}_{r+1}, \cdots, \bar{a}_q, \bar{b}_1, \cdots, \bar{b}_q\}$ is the basis of the space $\langle \bar{a}_1, \cdots, \bar{a}_q \rangle + \langle \bar{b}_1, \cdots, \bar{b}_q \rangle$. Then in the similar way as above, we may assume that the vectors $\bar{a}_1, \cdots, \bar{a}_r$ span the subspace $\langle \bar{a}_1, \cdots, \bar{a}_q \rangle \cap \langle \bar{b}_1, \cdots, \bar{b}_q \rangle$ and they are expressed in the form:

$$\begin{cases} \bar{a}_1 = r_{1,1}\bar{b}_1 + \cdots + r_{1,q}\bar{b}_q \\ \cdots\cdots\cdots\cdots\cdots \\ \bar{a}_r = r_{r,1}\bar{b}_1 + \cdots + r_{r,q}\bar{b}_q \,. \end{cases}$$

Then we have

$$\begin{aligned} \sum_{i=1}^{q} \bar{a}_i \wedge \bar{b}_i = & (\bar{a}_{r+1} - r_{1,r+1}\bar{b}_1 - \cdots - r_{r,r+1}\bar{b}_r) \wedge \bar{b}_{r+1} \\ & + \cdots \\ & + (\bar{a}_q - r_{1,q}\bar{b}_1 - \cdots - r_{r,q}\bar{b}_r) \wedge \bar{b}_q \\ & + \sum_{1 \leqq i < j \leqq r} (r_{j,i} - r_{i,j}) \bar{b}_i \wedge \bar{b}_j \,. \end{aligned}$$

Because the vectors $\bar{a}_{r+1} - r_{1,r+1}\bar{b}_1 - \cdots - r_{r,r+1}\bar{b}_r, \cdots, \bar{a}_q - r_{1,q}\bar{b}_1 - \cdots - r_{r,q}\bar{b}_r, \bar{b}_1, \cdots, \bar{b}_q$ are linearly independent, it follows that rank $(\sum_{i=1}^{q} \bar{a}_i \wedge \bar{b}_i) \geqq 2(q-r) \geqq 2(n_1+n_2-k-r)$ $\geqq 2(n_1+n_2-k-l)$. q.e.d.

§ 2. Proof of Theorem.

We first express the curvature R of $P^n(\boldsymbol{C})$ as a linear endomorphism of $\wedge^2 V$ where V is the tangent space of $P^n(\boldsymbol{C})$ at one point. (Hence V is a $2n$ dimensional real vector space endowed with the positive definite inner product.) Using the suitable orthonormal basis $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ of V, the curvature $R: \wedge^2 V \to \wedge^2 V$ is given by

$$\begin{aligned} R(X_i \wedge X_j) &= X_i \wedge X_j + Y_i \wedge Y_j \,, \\ R(X_i \wedge Y_i) &= 2X_i \wedge Y_i + 2 \sum_{k=1}^{n} X_k \wedge Y_k \,, \\ R(X_i \wedge Y_j) &= X_i \wedge Y_j + X_j \wedge Y_i \,, \\ R(Y_i \wedge Y_j) &= X_i \wedge X_j + Y_i \wedge Y_j \,, \end{aligned}$$

for $1 \leqq i, j \leqq n$ $(i \neq j)$. (For details, see [2] p. 132.)

Now we assume that $P^n(\boldsymbol{C})$ is locally isometrically immersed into $\boldsymbol{R}^{2n+k}$. Then the curvature R is expressed in the following form (the Gauss equation):

$$R = \sum_{i=1}^{k} L_i \wedge L_i \,,$$

where $L_1, \cdots, L_k$ are symmetric linear endomorphisms of V defined by the second fundamental form of the isometric immersion. (See [4], [1].) For the element $X \in V$, we denote by $V(X)$ the subspace of V spanned by the elements $L_1(X), \cdots, L_k(X)$. Then, since rank $R(X_i \wedge Y_i) = 2n$, we have from Lemma 1

$$\dim V(X_i)\,, \quad \dim V(Y_i) \geqq n \,.$$

Next, since rank $R(X_1 \wedge X_2) = 4$, we have

$$\dim V(X_1) \cap V(X_2) \geqq 2n - k - 2 \,.$$

(In fact, we put $n_1=n_2=n$, $l=2n-k-2$, $V_1=V(X_1)$, $V_2=V(X_2)$ and apply Lemma 3.) In the same way, using the fact rank $R(Y_1\wedge X_2)=4$, we have

$$\dim V(Y_1)\cap V(X_2)\geqq 2n-k-2\,.$$

Therefore by Lemma 2, we have

$$\begin{aligned}\dim V(X_1)\cap V(Y_1) &\geqq 4n-2k-4-\dim V(X_2)\\ &\geqq 4n-3k-4\,.\end{aligned}$$

(Note that $\dim V(X_2)\leqq k$.) On the other hand, since rank $R(X_1\wedge Y_1)=2n$, and $\mathrm{Im}\, R(X_1\wedge Y_1)\subset V(X_1)+V(Y_1)\subset V$, we have $V=V(X_1)+V(Y_1)$. In particular

$$\begin{aligned}2n &= \dim V(X_1)+\dim V(Y_1)-\dim V(X_1)\cap V(Y_1)\\ &\leqq 2k-\dim V(X_1)\cap V(Y_1)\,,\end{aligned}$$

i.e.,

$$\dim V(X_1)\cap V(Y_1)\leqq 2k-2n\,.$$

Combining with the above inequality, we have $4n-3k-4\leqq 2k-2n$, namely, we have $k\geqq \frac{1}{5}(6n-4)$, which completes the proof of Theorem.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

Present Address
Faculty of Integrated Arts and Sciences
Hiroshima University

References

[1] Y. Agaoka, Isometric immersions of $SO(5)$, J. Math. Kyoto Univ., **24** (1984), 713–724.

[2] Y. Agaoka, On the curvature of Riemannian submanifolds of codimension 2, Hokkaido Math. J., **14** (1985), 107–135.

[3] Y. Agaoka and E. Kaneda, On local isometric immersions of Riemannian symmetric spaces, Tôhoku Math. J., **36** (1984), 107–140.

[4] H. Jacobowitz, Curvature operators on the exterior algebra, Linear and Multilinear Algebra, **7** (1979), 93–105.

[5] S. Kobayashi, Isometric imbeddings of compact symmetric spaces, Tôhoku Math. J., **20** (1968), 21–25.