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Projective structures on Riemann surfaces and Kleinian groups

By

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§1. Introduction and notations.

Let S be a compact Riemann surface of genus $p \geqq 2$, and let $\pi: U \to S$ be a holomorphic universal covering of S with the covering transformation group Γ, where U is the upper half plane $\{z \in \boldsymbol{C}: \operatorname{Im} z > 0\}$. Then, Γ is a finitely generated Fuchsian group of the first kind on U and consists of hyperbolic Möbius transformations. We denote by $B_2(L, \Gamma)$ the Banach space of all holomorphic quadratic differentials for Γ defined on the lower half plane L. Namely, $B_2(L, \Gamma)$ is the set of all holomorphic functions ϕ on L satisfying

$$\phi(\gamma(z))\gamma'(z)^2 = \phi(z)\,, \qquad \text{for all } z \in L,\ \gamma \in \Gamma\,, \tag{1.1}$$

with the norm

$$\|\phi\|_L = \sup_{z \in L} (2 \operatorname{Im} z)^2 |\phi(z)|\ .$$

More generally, for a Kleinian group G and for a G-invariant union Δ of components of G we denote by $B_2(\Delta, G)$ the Banach space consisting of all holomorphic functions ψ on Δ satisfying

$$\psi(g(z))g'(z)^2 = \psi(z)\,, \qquad \text{for all } z \in \Delta,\ g \in G\,,$$

$$\text{and} \qquad \psi(z) = O(|z|^{-4})\,, \quad z \to \infty\,, \qquad \text{if } \infty \in \Delta$$

with the norm

$$\|\psi\|_\Delta = \sup_{z \in \Delta} \rho_\Delta(z)^{-2} |\psi(z)|\ ,$$

where $\rho_\Delta(z)|dz|$ is the Poincaré metric on the component of Δ containing z.

For every ϕ in $B_2(L, \Gamma)$, there exists a locally schlicht meromorphic function f_ϕ on L with $\{f_\phi, z\} = \phi(z)$; here $\{f, \cdot\}$ means the *Schwarzian derivative* of f

$$\{f, \cdot\} = (f''/f')' - (f''/f')^2/2\,.$$

Throughout this paper, we shall denote by W_ϕ ($\phi \in B_2(L, \Gamma)$) a locally schlicht meromorphic function on L which is uniquely determined by ϕ such that

$$\{W_\phi, z\} = \phi(z)$$

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and

$$W_\phi(z)=(z+i)^{-1}+O(|z+i|) \quad \text{as } z\to -i\,.$$

From (1.1) we verify that the function W_ϕ induces a group homomorphism θ_ϕ: $\Gamma\to\mathrm{PSL}(2, \boldsymbol{C})$ defined by

$$\theta_\phi(\gamma)\circ W_\phi = W_\phi\circ\gamma\,, \quad \gamma\in\Gamma\,, \tag{1.2}$$

and we say that W_ϕ determines a *projective structure* on S, or that θ_ϕ (or the pair (W_ϕ, θ_ϕ)) is a *deformation* of Γ (cf. Gunning [4], Kra[6]).

Here, we consider the set $K(\Gamma)$ of ϕ in $B_2(L, \Gamma)$ such that $\Gamma^\phi=\theta_\phi(\Gamma)$ is a Kleinian group. As is well known, (Bers' embedding of) *Teichmüller space* $T(\Gamma)$ of Γ, which has been investigated by many authors (cf. [1], [7], [10], [11], [12]), is a connected open subset of $K(\Gamma)$, where $T(\Gamma)$ is the set of all ϕ in $B_2(L, \Gamma)$ such that W_ϕ admits a quasiconformal extension to $\hat{\boldsymbol{C}}$. And the case where W_ϕ is a (unbranched and unbounded) covering mapping on L is studied in Kra[6] and Kra-Maskit[8]. They showed that the set of all such ϕ is compact in $B_2(L, \Gamma)$.

The purpose of this paper is to investigate the structure of Int $K(\Gamma)$, the interior of $K(\Gamma)$ in $B_2(L, \Gamma)$. Our main results assert that the set of ϕ in $B_2(L, \Gamma)$ for which W_ϕ is a covering mapping on L is small in a certain sense (Theorem 2) and that all small deformations of a b-group are *not* Kleinian groups (Theorem 3).

§2. Preliminaries.

We shall state some known results for deformations of Γ.

Proposition 1 ([6]). *Let ϕ be in $B_2(L, \Gamma)$. Then, the followings are equivalent:*
(i) *$\Gamma^\phi(=\theta_\phi(\Gamma))$ acts discontinuously on $W_\phi(L)$,*
(ii) *W_ϕ is a covering mapping on L, and*
(iii) *$W_\phi(L)\neq\hat{\boldsymbol{C}}$.*
Furthermore, in the above cases $W_\phi(L)$ is an invariant component of Γ^ϕ.

To state the next proposition, we define three classes of Kleinian groups. A finitely generated non-elementary Kleinian group G is a *quasi-Fuchsian group* if G has two simply connected invariant components, a *b-group* if G has only one simply connected invariant component, and a *totally degenerate group* if the region of discontinuity of G is connected and simply connected. Of course, a totally degenerate group is a b-group.

Proposition 2 ([9]). *Let ϕ be in $K(\Gamma)$. Suppose that θ_ϕ is an isomorphism of Γ onto Γ^ϕ and Γ^ϕ is purely loxodromic. Then, Γ^ϕ is a quasi-Fuchsian group or a totally degenerate group.*

The following proposition implies that outside of $T(\Gamma)$ in $K(\Gamma)$ is generally ample.

Proposition 3 ([9] Theorem 5 and Remark 3). *There exists a Fuchsian group Γ satisfying the following conditions*:

(a) *U/Γ is a compact Riemann surface of genus $p \geqq 2$,*

(b) Int $(K(\Gamma)-T(\Gamma))$ *is not empty.*

As for Int $K(\Gamma)$, we know the following:

Proposition 4 ([6]). *For each ϕ in* Int $K(\Gamma)$, *θ_ϕ is an isomorphism, and Γ^ϕ is purely loxodromic.*

We denote by $S(\Gamma)$ the set of all ϕ in $B_2(L, \Gamma)$ such that W_ϕ is schlicht. Obviously, $T(\Gamma) \subset S(\Gamma) \subset K(\Gamma)$, and it is known that $S(\Gamma)$ is compact in $B_2(L, \Gamma)$. Furthermore,

Proposition 5 ([12]). Int $S(\Gamma) = T(\Gamma)$.

§3. Structure of Int $K(\Gamma)$.

It follows from Propositions 2 and 4 that for every ϕ in Int $K(\Gamma)$, Γ^ϕ is a quasi-Fuchsian group or a totally degenerate group. Let K be a component of Int $K(\Gamma)$, and let ϕ be in K. Then, there is a small $r>0$ such that

$$B(r; \phi) = \{\psi \in B_2(L, \Gamma) \colon \|\psi-\phi\|_L < r\} \subset K .$$

Taking a ψ in $B(r; \phi)$, we define a family $\{\chi_\lambda\}$ of isomorphisms of Γ^ϕ with a complex parameter λ in the unit disk $D=\{\lambda \in \boldsymbol{C} \colon |\lambda|<1\}$ by

$$\chi_\lambda = \theta_{\phi_\lambda} \circ \theta_\phi^{-1} , \tag{3.1}$$

where $\phi_\lambda = \phi + \lambda(\phi-\psi) \in B(r; \phi)$. Since χ_λ depends holomorphically on λ and $\chi_\lambda(\Gamma^\phi) = \Gamma^{\phi_\lambda}$ is a Kleinian group for every λ, the family $\{\chi_\lambda\}$ satisfies the condition of Theorem in Bers [2]. Hence, from this theorem, χ_λ is a quasiconformal deformation of Γ^ϕ for each $\lambda \in D$, that is, there exists a quasiconformal self-mapping w_λ of $\hat{\boldsymbol{C}}$ for each $\lambda \in D$ such that

$$\chi_\lambda(\gamma) = w_\lambda \circ \gamma \circ w_\lambda^{-1} , \qquad \text{for all } \gamma \in \Gamma^\phi . \tag{3.2}$$

Furthermore, from the proof of the theorem we verify that if $|\lambda|<1/3$, then there exists a function f_λ such that f_λ is holomorphic on $\Omega(\Gamma^\phi)$, the region of discontinuity of Γ^ϕ, and

$$\mu_\lambda(z) = \begin{cases} \rho_{\Omega(\Gamma^\phi)}(z)^{-2}\overline{f_\lambda(z)}, & \text{if } z \in \Omega(\Gamma^\phi) , \\ 0, & \text{if } z \in \Lambda(\Gamma^\phi) , \end{cases} \tag{3.3}$$

where $\Lambda(\Gamma^\phi)$ is the limit set of Γ^ϕ, μ_λ the complex dilatation of w_λ, and $\rho_{\Omega(\Gamma^\phi)}|dz|$ the Poincaré metric on the component of $\Omega(\Gamma^\phi)$ containing z.

From (3.2) and (3.3), we verify that

$$\mu_\lambda(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu_\lambda(z), \qquad \text{for all } \gamma \in \Gamma^\phi .$$

On the other hand, the Poincaré density $\rho_{\Omega(\Gamma^{\phi})}$ satisfies the condition:

$$\rho_{\Omega(\Gamma^{\phi})}(\gamma(z))\,|\gamma'(z)| = \rho_{\Omega(\Gamma^{\phi})}(z), \qquad \text{for all } \gamma\in\Gamma^{\phi}.$$

Hence we conclude that

$$f_\lambda\in B_2(\Omega(\Gamma^{\phi}), \Gamma^{\phi}) \text{ and } \|f_\lambda\|_{\Omega(\Gamma^{\phi})}<1, \tag{3.4}$$

for all λ in $\{|\lambda|<1/3\}$. Thus, we have:

Theorem 1. *Let K be an arbitrary component of* Int $K(\Gamma)$. *Then, for* ϕ_0, ϕ_1 *in* K, Γ^{ϕ_0} *and* Γ^{ϕ_1} *are quasiconformally equivalent, i.e., there exists a quasiconformal self-mapping* w *of* $\hat{C}$ *such that*

$$\theta_{\phi_1}\circ\theta_{\phi_0}^{-1}(\gamma) = w\circ\gamma\circ w^{-1}, \qquad \textit{for all } \gamma\in\Gamma^{\phi_0}. \tag{3.5}$$

Moreover, if the norm $\|\phi_0-\phi_1\|_L$ *is sufficiently small, then we can take a quasiconformal self-mapping* w *of* $\hat{C}$ *satisfying* (3.5) *as follows.*

There exists an f *in* $B_2(\Omega(\Gamma^{\phi_0}), \Gamma^{\phi_0})$ *such that*

$$\mu(z) = \begin{cases} \rho(z)^{-2}\overline{f(z)}, & z\in\Omega(\Gamma^{\phi_0}), \\ 0, & z\in\Lambda(\Gamma^{\phi_0}), \end{cases} \tag{3.6}$$

where μ *is the complex dilatation of* w, $\rho(z)|dz|$ *the Poincaré metric on the component of* $\Omega(\Gamma^{\phi_0})$ *containing* z.

Since a quasi-Fuchsian group and a b-group are not quasiconformally equivalent to each other, we have immediately from this theorem

Corollary. *The Teichmüller space* $T(\Gamma)$ *of* Γ *is equal to the component of* Int $K(\Gamma)$ *containing the origin.*

Remark. It is easily seen that Theorem 1 and Corollary are valid for *every* finitely generated Fuchsian group of the first kind.

Next, we shall investigate the function W_ϕ for ϕ in Int $K(\Gamma)-T(\Gamma)$.

Theorem 2. *For every* ϕ *in* Int $K(\Gamma)-T(\Gamma)$, *the function* W_ϕ *is not a covering mapping on L. Consequently,* $W_\phi(L)=\hat{C}$.

Proof. Suppose that there exists a ϕ_0 in Int $K(\Gamma)-T(\Gamma)$ for which W_{ϕ_0} is a covering mapping, and denote by K the component of Int $K(\Gamma)$ containing ϕ_0. Then, from Propositions 2 and 4, Γ^{ϕ_0} is a quasi-Fuchsian group or a totally degenerate group. Since $W_{\phi_0}(L)$ is a simply connected component of Γ^{ϕ_0} from Proposition 1, W_{ϕ_0} is schlicht by the monodromy theorem. If Γ^{ϕ_0} is a quasi-Fuchsian group, then ϕ_0 is in $T(\Gamma)$ by a theorem in Kra[7]. Thus, Γ^{ϕ_0} must be a totally degenerate group. We take a ϕ_1 sufficiently close to ϕ_0 so that the second statement of Theorem 1 holds. Then, both a locally schlicht meromorphic function $W_{\phi_1}\circ W_{\phi_0}^{-1}$ and a quasiconformal self-mapping w of $\hat{C}$ induce the same group isomorphism

$\theta_{\phi_1}\circ\theta_{\phi_0}^{-1}$ of Γ^{ϕ_0}, where w is a quasiconformal mapping obtained in Theorem 1. Since the Schwarzian derivative $\{W_{\phi_1}\circ W_{\phi_0}^{-1}, \cdot\}$ of $W_{\phi_1}\circ W_{\phi_0}^{-1}$ on $W_{\phi_0}(L)$ belongs to $B_2(W_{\phi_0}(L), \Gamma^{\phi_0})$ and the complex dilatation of w is given as (3.6), we can conclude that $W_{\phi_1}\circ W_{\phi_0}^{-1}$ is a Möbius transformation α from Gardiner-Kra[3] Theorem 11.2. Namely, ϕ_1 belongs to $S(\Gamma)$. This implies that ϕ_0 is in Int $S(\Gamma)$, and from Proposition 5, we have a contradiction. Thus, we proved the theorem.

§4. Small deformations of *b*-groups.

Bers[1] showed that a finitely generated (quasi-) Fuchsian group is *quasiconformally stable*, that is, roughly speaking, a small deformation of the Fuchsian group is always a deformation induced by a quasiconformal self-mapping of $\hat{\boldsymbol{C}}$. And he also showed that a totally degenerate group is *not* so. Here, concerning with his results, we shall show that all groups obtained by small deformations of a *b*-group are *not* Kleinian groups.

Let G be a *b*-group with the invariant component Δ. Then, for each ϕ in $B_2(\Delta, G)$, we can take a locally schlicht meromorphic function f_ϕ on Δ satisfying

$$\{f_\phi, z\} = \phi(z), \quad z\in\Delta\,,$$

and for a fixed point $z_0\in\Delta$

$$f_\phi(z) = z+O(|z-z_0|^3), \text{ as } z\to z_0\,.$$

Note that $f_0(z)=z$. We easily see that f_ϕ induces a group homomorphism χ_ϕ of G as (1.2).

Theorem 3. *Let G be a b-group with the invariant component Δ. Then for each $\varepsilon>0$ there exists a ϕ in $B_2(\Delta, G)$ such that*

(i) $||\phi||_\Delta<\varepsilon$, *and*

(ii) *$\chi_\phi(G)$ is not a Kleinian group.*

Remarks. 1) In [12], we have shown that all f_ϕ for $||\phi||_\Delta<\varepsilon$ are not schlicht on Δ. Obviously, $\chi_\phi(G)$ is a Kleinian group if f_ϕ is schlicht on Δ. Therefore, Theorem 3 is an extension of this result.

2) Jørgensen-Klein [5] show that an algebraic limit of finitely generated Kleinian groups is also Kleinian. Since χ_ϕ is an isomorphism for almost all ϕ in $B_2(\Delta, G)$ (cf. [6] p. 545), we can take ϕ in Theorem 3 such that $\chi_\phi(G)$ is isomorphic to G.

Proof. Take a conformal mapping h of the lower half plane L onto Δ and set $\Gamma=hGh^{-1}$. Then, Γ is a finitely generated Fuchsian group of the first kind isomorphic to G via h and L/Γ is a Riemann surface conformally equivalent to Δ/G.

Suppose that for every ϕ in $B_2(\Delta, G)$ satisfying (i), $\chi_\phi(G)$ is a Kleinian group. Then, a locally schlicht meromorphic function $f_\phi\circ h$ on L induces a group homomorphism of Γ onto a Kleinian group $\chi_\phi(G)$, and

$$|\{f_\phi \circ h, z\}|(2\operatorname{Im} z)^2 \leqq |\phi(h(z))h'(z)^2|(2\operatorname{Im} z)^2 + |\{h, z\}|(2\operatorname{Im} z)^2$$
$$= |\phi(h(z))|(\rho_\Delta(h(z)))^{-2} + |\{h, z\}|(2\operatorname{Im} z)^2 \leqq \|\phi\|_\Delta + \|h\|_L < +\infty ,$$

because $\{h, \cdot\}$ is in $B_2(L, \Gamma)$ by Nehari's theorem. Thus, the Schwarzian derivative $\{f_\phi \circ h, \cdot\}$ on L belongs to $K(\Gamma)$. In particular, $\{h, \cdot\} = \{f_0 \circ h, \cdot\}$ belongs to $K(\Gamma)$. Furthermore, by considering $\{f_\phi \circ h, \cdot\}$ for all ϕ satisfying (i), we verify that $\{h, \cdot\}$ is in Int $K(\Gamma)$. Since h is a covering (schlicht) mapping on L, we have a contradiction by the same way as in the proof of Theorem 2.

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