J. Math. Kyoto Univ. (JMKYAZ)
27-3 (1987) 417–431

On a short time expansion of the fundamental solution of heat equations by the method of Wiener functionals

By

Hideaki UEMURA

0. Introduction.

Let (M, g) be a d-dimensional compact smooth Riemannian manifold and $p(t, x, y)$ be the fundamental solution of the heat equation $\frac{\partial}{\partial t}u=Q_x u$ with respect to the Riemannian volume $\sqrt{\det g(x)}dx$ where $Q=\frac{1}{2}\Delta_M+h\partial$, Δ_M being the Laplace-Beltrami operator and $h\partial$ a smooth vector field. H.P. McKean and I.M. Singer [6] studied the following asymptotic expansion of $p(t, x, x)$:

$$(0.1)\qquad (2\pi t)^{d/2}p(t, x, x) = 1+k_1(x)t+k_2(x)t^2+\cdots+k_n(x)t^n+o(t^n) \qquad \text{as} \quad t\downarrow 0$$

and obtained that

$$(0.2)\qquad k_1(x) = R(x)/12-\operatorname{div} h(x)/2-|h(x)|^2/2$$

where $R(x)$ is the scalar curvature, $\operatorname{div} h(x)$ and $|h(x)|$ the divergence and the Riemannian norm of $h(x)$, respectively.

In the case of $h=0$, they also showed that

$$(0.3)\qquad k_2(x)=\left(\frac{5}{2}R^2(x)-||R_{ij}(x)||^2+||R_{ijkl}(x)||^2\right)/720+\text{const. }\Delta_M R(x)$$

where $||R_{ij}(x)||$ and $||R_{ijkl}(x)||$ are the Riemannian norms of the Ricci tensor $R_{ij}(x)$ and of the curvature tensor $R_{ijkl}(x)$ respectively. The universal constant of $\Delta_M R(x)$ was found by S.A. Molchanov [7] to be $\frac{1}{120}$.

(0.2) can be obtained by a direct calculation, but it is too complicated to obtain (0.3). In fact, McKean and Singer avoided such a direct calculation and, instead, they first determined the possible types of monomials in components of the curvature tensor and its derivatives which will appear in k_2, and showed that coefficients of these monomials are universal, i.e. independent of a manifold and its dimension.

Received, Feb. 7, 1986.

Finally they determined them by computing on concrete manifolds.

In the present paper we give a probabilistic approach to computing above $k_1, k_2, \cdots$. Our method consists of, first, representing $p(\varepsilon^2, x, x)$, $\varepsilon>0$, as a Wiener functional expectation $E[\Phi(\varepsilon, x, w)]$ and then expanding the functional as

$$(0.4)\qquad \Phi(\varepsilon, x, w)=\varepsilon^{-d}(\Phi_0(x, w)+\varepsilon\Phi_1(x, w)+\varepsilon^2\Phi_2(x, w)+\cdots +\varepsilon^n\Phi_n(x, w)+o(\varepsilon^n)) \quad \text{as} \quad \varepsilon\downarrow 0.$$

Then (0.1) can be obtained by taking the expectation of (0.4).

Since we are dealing with the fundamental solution, the above expectation cannot be understood as the usual sense but as a certain sense of disintegration of Wiener measure expectation. Recently, S. Watanabe [10] discussed such a generalization of Wiener measure expectation in the framework of Malliavin calculus: By introducing a family of Sobolev spaces formed of both smooth and generalized Wiener functionals, the above expectation $E[\Phi(\varepsilon, x, w)]$ and the expansion of $\Phi(\varepsilon, x, w)$ can be given a correct mathematical sense and the coefficients $\Phi_n(x, w)$ can be computed explicitly. Evaluating $E[\Phi_n(x, w)]$ is reduced to computing conditional expectations of certain multiple Wiener integrals and we obtain a general rule of such computations in Theorem 3.1. This corresponds to the principle of "pairwise contractions" in McKean-Singer [6] (cf. Also [2]), usually proved by appealing to H. Weyl's invariant theory, which enables us to determine the possible types of monomials in components of curvature and its derivatives. Moreover, Theorem 3.1 asserts the universality of coefficients of these monomials.

Here, the author wishes to express his thanks to Professors S. Watanabe, S. Kotani and T. Fujita for their valuable suggestions.

1. Asymptotic expansion of Wiener functionals.

In this section we summarize notions and results in S. Watanabe [9] as are necessary for the further discussions. We use the notation of $\boldsymbol{D}_p^s(E)$, $s\in\boldsymbol{R}$, $1\leq p<\infty$, to denote the *Sobolev space* of E-valued Wiener functionals defined on the r-dimensional Wiener space (W_0^r, P), where E is a separable Hilbert space. Roughly speaking, $\boldsymbol{D}_p^s(E)$ consists of E-valued Wiener functionals $F(w)$ which satisfy $\|F\|_{p,s}=\|(I-L)^{s/2}F\|_p<\infty$, where L is *the Ornstein-Uhlenbeck operator* (*the number operator*) and $\|\cdot\|_p$ is the norm of $L^p(E)$, the usual L^p-space of E-valued Wiener functionals. We omit E and write simply $\boldsymbol{D}_p^s$ if $E=\boldsymbol{R}$. Let $H\subset W_0^r$ be the usual *Cameron-Martin Hilbert subspace* of W_0^r. The *H-derivative* D; $DF(w)[h]=\lim_{\varepsilon\downarrow 0}\frac{F(w+\varepsilon h)-F(w)}{\varepsilon}$, $h\in H$, and its dual D^* are well defined as continuous operators D: $\boldsymbol{D}_p^{s+1}(E)\rightarrow\boldsymbol{D}_p^s(H\otimes E)$ and D^*: $\boldsymbol{D}_p^{s+1}(H\otimes E)\rightarrow\boldsymbol{D}_p^s(E)$, and $L=-D^*D$. Here $H\otimes E$ is the Hilbert space formed of all linear operators $H\rightarrow E$ of Hilbert-Schmidt type endowed with the Hilbert-Schmidt norm. (cf. N. Ikeda and S. Watanabe [5], S. Watanabe [9], H. Sugita [8])

Set $\boldsymbol{D}^\infty(E)=\bigcap_{s>0}\bigcap_{1<p<\infty}\boldsymbol{D}_p^s(E)$, $\tilde{\boldsymbol{D}}^\infty(E)=\bigcap_{s>0}\bigcup_{1<p<\infty}\boldsymbol{D}_p^s(E)$, $\tilde{\boldsymbol{D}}^{-\infty}(E)=\bigcup_{s>0}\bigcap_{1<p<\infty}\boldsymbol{D}_p^{-s}(E)$

and $\boldsymbol{D}^{-\infty}(E)=\bigcup_{s>0}\bigcup_{1<p<\infty}\boldsymbol{D}_p^{-s}(E)$. For $G\in\boldsymbol{D}^\infty$ and $\Phi\in\boldsymbol{D}^{-\infty}$ (or for $G\in\tilde{\boldsymbol{D}}^\infty$ and $\Phi\in\tilde{\boldsymbol{D}}^{-\infty}$), $G\cdot\Phi$ $(=\Phi\cdot G)\in\boldsymbol{D}^{-\infty}$ is defined by ${}_{\boldsymbol{D}^{-\infty}}\langle G\cdot\Phi, F\rangle_{\boldsymbol{D}^\infty}={}_{\boldsymbol{D}^{-\infty}}\langle\Phi, G\cdot F\rangle_{\boldsymbol{D}^\infty}$ [resp.$={}_{\tilde{\boldsymbol{D}}^{-\infty}}\langle\Phi, G\cdot F\rangle_{\tilde{\boldsymbol{D}}^\infty}$] where $F\in\boldsymbol{D}^\infty$.

Let $F(w)=(F_1(w),\cdots,F_d(w))\in\boldsymbol{D}^\infty(\boldsymbol{R}^d)$ be given, equivalently $F_i(w)\in\boldsymbol{D}^\infty$, $i=1,\cdots,d$. The *Malliavin covariance* $\sigma(w)=(\sigma_{ij}(w))$ of F is defined by $\sigma_{ij}=\langle DF_i, DF_j\rangle_H$, $i, j=1,\cdots,d$. If F is *non-degenerate* in the sense that $\sigma(w)$ is positive definite *a.s.* and $\det\sigma(w)^{-1}\in\bigcap_{1<p<\infty}L^p$, then for any $T\in\mathcal{S}'(\boldsymbol{R}^d)$, a tempered Schwartz distribution on $\boldsymbol{R}^d$, its *pull-back* $T(F)$ under the Wiener map $w\to F(w)$ can be defined as an element of $\tilde{\boldsymbol{D}}^{-\infty}$ and, for every $G\in\tilde{\boldsymbol{D}}^\infty$, the natural coupling ${}_{\tilde{\boldsymbol{D}}^{-\infty}}\langle T(F), G\rangle_{\tilde{\boldsymbol{D}}^\infty}={}_{\boldsymbol{D}^{-\infty}}\langle G\cdot T(F), 1\rangle_{\boldsymbol{D}^\infty}$, which we denote also by $E[T(F)\cdot G]$ or $E[G\cdot T(F)]$, coincide with $T(\phi)$ *where* $\phi\in\mathcal{S}(\boldsymbol{R}^d)$ *is given by* $\phi(x)=E[G\,|\,F=x]\cdot p(x)$, $p(x)$ being the C^∞-density of law of F. That $x\to\phi(x)$ is C^∞ can be deduced from the expression $\phi(x)=E[G\cdot\delta_x(F)]$ and continuity of pull-back.

Let $F(\varepsilon, w)$ be a family of Wiener functionals indexed by ε, $0<\varepsilon\leq 1$. If $F(\varepsilon, w)\in\boldsymbol{D}^\infty(E)$ for all ε and, for every $s>0$ and every $1<p<\infty$, $\|F(\varepsilon, w)\|_{p,s}=o(\varepsilon^n)$ as $\varepsilon\downarrow 0$, n being a fixed integer, we say $F(\varepsilon, w)=o(\varepsilon^n)$ *as* $\varepsilon\downarrow 0$ *in* $\boldsymbol{D}^\infty(E)$. In a similar way, we can speak of $F(\varepsilon, w)=o(\varepsilon^n)$ in $\tilde{\boldsymbol{D}}^\infty(E)$, in $\tilde{\boldsymbol{D}}^{-\infty}(E)$ and in $\boldsymbol{D}^{-\infty}(E)$. For instance, $F(\varepsilon, w)=o(\varepsilon^n)$ in $\tilde{\boldsymbol{D}}^\infty(E)$ if for every $s>0$ there exists $p=p_s\in(1,\infty)$ such that $F(\varepsilon, w)\in\boldsymbol{D}_p^s(E)$ for all ε and $\|F(\varepsilon, w)\|_{p,s}=o(\varepsilon^n)$.

Let $F(\varepsilon, w)\in\boldsymbol{D}^\infty(E)$, $0<\varepsilon\leq 1$. We say that $F(\varepsilon, w)$ *has the asymptotic expansion*

$$(1.1)\qquad F(\varepsilon, w)\sim f_0(w)+\varepsilon f_1(w)+\cdots+\varepsilon^n f_n(w)+\cdots\qquad \textit{in } \boldsymbol{D}^\infty(E) \textit{ as } \varepsilon\downarrow 0$$

if $f_i\in\boldsymbol{D}^\infty(E)$, $i=0, 1,\cdots$ *exist such that, for every* n,

$$F(\varepsilon, w)=f_0(w)+\varepsilon f_1(w)+\cdots+\varepsilon^n f_n(w)+o(\varepsilon^n)\qquad \textit{in } \boldsymbol{D}^\infty(E) \textit{ as } \varepsilon\downarrow 0\,.$$

Asymptotic expansion in the space $\tilde{\boldsymbol{D}}^\infty(E)$, $\tilde{\boldsymbol{D}}^{-\infty}(E)$ and $\boldsymbol{D}^{-\infty}(E)$ can be defined in a similar way.

Let $F(\varepsilon, w)\in\boldsymbol{D}^\infty(\boldsymbol{R}^d)$, $0<\varepsilon\leq 1$. We say that $F(\varepsilon, w)$ is *uniformly non-degenerate* if $F(\varepsilon, w)$ is non-degenerate for every ε and furthermore

$$(1.2)\qquad \overline{\lim_{\varepsilon\downarrow 0}}\,\|\det\sigma(\varepsilon, w)^{-1}\|_p<\infty\qquad \text{for every } p\in(1,\infty)\,,$$

where $\sigma(\varepsilon, w)$ is the Malliavin covariance of $F(\varepsilon, w)$. The following theorem is due to S. Watanabe [10].

Theorem 1.1. *Let a family* $F(\varepsilon, w)\in\boldsymbol{D}^\infty(\boldsymbol{R}^d)$, $0<\varepsilon\leq 1$, *be uniformly non-degenerate and has the asymptotic expansion* (1.1). *Then, for every* $T\in\mathcal{S}'(\boldsymbol{R}^d)$, $T(F(\varepsilon, w))\in\tilde{\boldsymbol{D}}^{-\infty}$ *has the asymptotic expansion in* $\tilde{\boldsymbol{D}}^{-\infty}$ *as* $\varepsilon\downarrow 0$:

$$(1.3)\qquad T(F(\varepsilon, w))\sim\Phi_0(w)+\varepsilon\Phi_1(w)+\cdots\qquad \textit{as } \varepsilon\downarrow 0 \textit{ in } \tilde{\boldsymbol{D}}^{-\infty}.$$

The coefficients $\Phi_i(w)$, $i=0, 1,\cdots$ *are computed from the formal Taylor expansion* *of T*:

$$T(F(\varepsilon, w)) = T(f_0)+\partial T(f_0)(\varepsilon f_1+\varepsilon^2 f_2+\cdots) + \frac{1}{2}\partial^2 T(f_0)(\varepsilon f_1+\varepsilon^2 f_2+\cdots)\otimes(\varepsilon f_1+\varepsilon^2 f_2+\cdots) + \cdots \tag{1.4}$$

where ∂T *is a distribution derivative of* T, *i.e.* $\partial T=\left(\frac{\partial}{\partial x^1}T, \cdots, \frac{\partial}{\partial x^d}T\right)$, $\partial T(f_0)\cdot(\varepsilon f_1+\varepsilon^2 f_2+\cdots)=\sum_{i=1}^{d}\left(\frac{\partial}{\partial x^i}T\right)(f_0)\cdot f^i$ *if we write* $\varepsilon f_1+\varepsilon^2 f_2+\cdots=(f^1, \cdots, f^d)$, *and* $\partial^2 T(f_0)(\varepsilon f_1+\varepsilon^2 f_2+\cdots)\otimes(\varepsilon f_1+\varepsilon^2 f_2+\cdots)=\sum_{i,j=1}^{d}\left(\frac{\partial^2}{\partial x^i \partial x^j}T\right)(f_0)f^i\cdot f^j$. $\Phi_k(w)$ *is a sum of the coefficient of* ε^k *in* (1.4). *For example,* $\Phi_0=T(f_0)$, $\Phi_1=\partial T(f_0)f_1$ *and* $\Phi_2=\frac{1}{2}(\partial^2 T(f_0)f_1\otimes f_1)+\partial T(f_0)f_2$.

Corollary. *With the same assumptions of the above theorem,*

$$E[T(F(\varepsilon, w))]\sim E[\Phi_0(w)]+\varepsilon E[\Phi_1(w)]+\cdots \quad \text{as} \quad \varepsilon \downarrow 0. \tag{1.5}$$

2. An application to solutions of S.D.E..

In this section we apply Theorem 1.1 to the case that $F(\varepsilon, w)$ is obtained as a solution of a stochastic differential equation (S.D.E.). Consider the following S.D.E. on $\boldsymbol{R}^d$ over the r-dimensional Wiener space (W_0^r, P).

$$\begin{cases} dX_t(w) = \varepsilon \sum_{\alpha=1}^{r} L_\alpha(X_t(w))\circ dw^\alpha(t)+\varepsilon^2 L_0(X_t(w))\,dt \\ X_0(w) = x_0 \end{cases} \tag{2.1}$$

where $x_0\in\boldsymbol{R}^d$, ε, $0<\varepsilon\leq 1$, is a fixed constant and $L_\alpha=(L_\alpha^1, \cdots, L_\alpha^d)$ with $L_\alpha^i\in C_b^\infty(\boldsymbol{R}^d)$, $\alpha=0, 1, \cdots, r$, and $\circ dw^\alpha(t)$ is a stochastic integral of the Stratonovich type. Here $C_b^\infty(\boldsymbol{R}^d)$ is the totality of bounded C^∞-functions whose derivatives are all bounded. We denote by $X^\varepsilon(t, x_0, w)$ the solution of S.D.E. (2.1). Note that $X^1(\varepsilon^2 t, x_0, w)\overset{\mathcal{L}}{\sim} X^\varepsilon(t, x_0, w)$, and we will treat $X^\varepsilon(1, x_0, w)$ instead of $X^1(\varepsilon^2, x_0, w)$. Then it is easy to see that, for each fixed x_0 and ε, $X^\varepsilon(1, x_0, w)\in\boldsymbol{D}^\infty(\boldsymbol{R}^d)$ and it has an asymptotic expansion as $\varepsilon\downarrow 0$ in $\boldsymbol{D}^\infty(\boldsymbol{R}^d)$:

$$X^\varepsilon(1, x_0, w)\sim f_0(w)+\varepsilon f_1(w)+\cdots \quad \text{as } \varepsilon\downarrow 0 \text{ in } \boldsymbol{D}^\infty(\boldsymbol{R}^d) \tag{2.2}$$

and moreover $f_p(w)=(f_p^1(w), \cdots, f_p^d(w))\in\boldsymbol{D}^\infty(\boldsymbol{R}^d)$, $p=0, 1, \cdots$, are obtained explicitly. These facts would be proved in the following Proposition 2.1. To describe them, however, we have to introduce the following notations: For m, $m^*\in\boldsymbol{Z}$ such that $m\geq 1$, $m^*\geq 0$ and $m^*\leq m$, and $\boldsymbol{k}=\{k(1), k(2), \cdots, k(m^*)\}\subset\boldsymbol{N}$ satisfying $1\leq k(1)<k(2)<\cdots<k(m^*)\leq m$ if $m^*\geq 1$ and otherwise being empty, we set

$$(2.3) \qquad F(m, m^*, \boldsymbol{k}) = \{\boldsymbol{i} = (i_1, \cdots, i_m) \in \{0, 1, \cdots, r\}^m;$$
$$i_j \neq 0 \quad \text{iff} \quad j \in \boldsymbol{k}\}.$$

For $\boldsymbol{i}=(i_1, i_2, \cdots, i_m) \in F(m, m^*, \boldsymbol{k})$, denote the multiple Wiener integral $S^{\boldsymbol{i}}(t, w)$ on (W_0^r, P) by

$$(2.4) \qquad S^{\boldsymbol{i}}(t, w) = \int_0^t \circ dw^{i_1}(t_1) \int_0^{t_1} \circ dw^{i_2}(t_2) \cdots \int_0^{t_{m-1}} \circ dw^{i_m}(t_m)$$

where $w^0(t)=t$, $w(t)=(w^1(t), w^2(t), \cdots, w^r(t))$.

Proposition 2.1. *For each fixed $x_0 \in \boldsymbol{R}^d$ and ε, $0<\varepsilon\leq 1$, $X^\varepsilon(1, x_0, w)$, the solution of S.D.E.* (2.1), *belongs to $\boldsymbol{D}^\infty(\boldsymbol{R}^d)$ and it has an asymptotic expansion as $\varepsilon \downarrow 0$ in $\boldsymbol{D}^\infty(\boldsymbol{R}^d)$ as* (2.2). *Moreover $f_p(w)$ is given by*

$$(2.5) \qquad \begin{cases} f_0(w) = x_0 \\ f_p(w) = \displaystyle\sum_{\substack{m\geq 1, m^*\geq 0 \\ m^*+2(m-m^*)=p}} \sum_{\boldsymbol{i}\in F(m, m^*, k)} (V_{i_m}\circ \cdots \circ V_{i_2})(L_{i_1})(x_0) \cdot S^{\boldsymbol{i}}(1, w) \\ \qquad\qquad p = 1, 2, \cdots . \end{cases}$$

where V_α is a differential operator corresponding to L_α: $V_\alpha(x) = L_\alpha^i(x)\dfrac{\partial}{\partial x^i}$.

Proof. Applying Itô's formula to (2.1) repeatedly, for any l we have

$$(2.6) \quad X^\varepsilon(1, x_0, w) = x_0 + \sum_{\substack{1\leq m\leq l \\ m^*\geq 0}} \sum_{\boldsymbol{i}\in F(m, m^*, k)} \varepsilon^{m^*+2(m-m^*)} (V_{i_m}\circ \cdots \circ V_{i_2})(L_{i_1})(x_0) \cdot S^{\boldsymbol{i}}(1, w)$$
$$+ \sum_{\substack{m=l+1 \\ m^*\geq 0}} \sum_{\boldsymbol{i}\in F(m, m^*, k)} \varepsilon^{m^*+2(m-m^*)} \int_0^1 \circ dw^{i_1}(t_1) \int_0^{t_1} \circ dw^{i_2}(t_2) \cdots$$
$$\cdots \int_0^{t_{m-1}} (V_{i_m}\circ \cdots \circ V_{i_2})(L_{i_1})(X_{t_m}^\varepsilon) \circ dw^{i_m}(t_m).$$

It is easy to see that $S^{\boldsymbol{i}}(1, w)$ and $\int_0^1 \circ dw^{i_1}(t_1) \int_0^{t_1} \circ dw^{i_2}(t_2) \cdots \int_0^{t_{m-1}} (V_{i_m}\circ \cdots \circ V_{i_2})(L_{i_1})$ $(X_{t_m}^\varepsilon) \circ dw^{i_m}(t_m)$ belong to $\boldsymbol{D}^\infty(\boldsymbol{R}^d)$ and that $m^*+2(m-m^*) \geq l+1$ if $m \geq l+1$ and $m^* \leqq m$. Therefore $X^\varepsilon(1, x_0, w)$ has an asymptotic expansion as $\varepsilon \downarrow 0$ and $f_p(w)$ can be obtained as a coefficient of ε^p in (2.6).

From now we assume that $a^{ij}(x) = \sum_{\alpha=1}^r L_\alpha^i(x) L_\alpha^j(x)$ is positive definite at x_0. Then $X^\varepsilon(1, x_0, w)$ is non-degenerate for each ε, so for all $x \in \boldsymbol{R}^d$ we can define $\delta_x(X^\varepsilon(1, x_0, w))$ as an element of $\tilde{\boldsymbol{D}}^{-\infty}$, but it is never uniformly non-degenerate. However, setting $F(\varepsilon, w) = (X^\varepsilon(1, x_0, w) - f_0(w))/\varepsilon$, it is easy to show that $F(\varepsilon, w)$ is uniformly non-degenerate, so in this case the smooth density $p(\varepsilon^2, x_0, x)$ of the law of $X^\varepsilon(1, x_0, w)$ exists and has the asymptotic expansion when $x = x_0$ because, first, $p(\varepsilon^2, x_0, x_0) = E[\delta_{x_0}(X^\varepsilon(1, x_0, w))] = \varepsilon^{-d} E[\delta_0(F(\varepsilon, w))]$, secondly by Theorem 1.1, $\delta_0(F(\varepsilon, w))$ has an asymptotic expansion, and finally by Corollary of Theorem 1.1, $E[\delta_0(F(\varepsilon, w))]$ can be expanded as $\varepsilon \downarrow 0$: We set

(2.7) $$(2\pi)^{d/2}\varepsilon^d p(\varepsilon^2, x_0, x_0) \sim c_0(x_0)+\varepsilon c_1(x_0)+\varepsilon^2 c_2(x_0)+\cdots \quad \text{as } \varepsilon\downarrow 0.$$

We describe $c_n(x_0)$ explicitly in Proposition 2.2 with the following notations: When m_ν, m_ν^*, $\boldsymbol{k}_\nu$ are given for all $\nu=1,\cdots,j$, $\hat{\boldsymbol{i}}\in\prod_{\nu=1}^{j} F(m_\nu, m_\nu^*, \boldsymbol{k}_\nu)$ is defined by

(2.8) $$\hat{\boldsymbol{i}} = (\boldsymbol{i}^{(1)}, \cdots, \boldsymbol{i}^{(j)}) = (i_{1,1}, \cdots, i_{1,m_1}, i_{2,1}, \cdots, i_{j,m_j})$$

and we denote $\hat{\boldsymbol{i}}=(i(1), i(2), \cdots, i(m_1+\cdots+m_j))$, and by $\{k(1), k(2), \cdots, k(m_1^*+\cdots\cdots+m_j^*)\}$, we mean the empty set if $m_1^*+\cdots+m_j^*=0$ and otherwise the totality of

(2.9) $$k(l) = k_n(l-\sum_{\nu=1}^{n-1} m_\nu^*)+\sum_{\nu=1}^{n-1} m_\nu \quad \text{when} \quad \sum_{\nu=1}^{n-1} m_\nu^* < l \leqq \sum_{\nu=1}^{n} m_\nu^*.$$

For $\hat{\boldsymbol{i}}\in\prod_{\nu=1}^{j} F(m_\nu, m_\nu^*, \boldsymbol{k}_\nu)$, define the multiple Wiener integral $S^{\hat{\boldsymbol{i}}}(t, w)$ on (W_0^r, P) by

(2.10) $$S^{\hat{\boldsymbol{i}}}(t, w) = S^{\boldsymbol{i}^{(1)}}(t, w)\cdot S^{\boldsymbol{i}^{(2)}}(t, w) \cdots S^{\boldsymbol{i}^{(j)}}(t, w).$$

For $\boldsymbol{i}'=(i(1'), \cdots, i(n'))\in F_n=\{1, \cdots, d\}^n$, define

(2.11) $$\partial^{\boldsymbol{i}'} = \frac{\partial^n}{\partial x^{i(1')}\cdots\partial x^{i(n')}}.$$

Then by Theorem 1.1 we have

(2.12) $$\delta_0(F(\varepsilon, w)) \sim \sum_{n=0}^{\infty} \varepsilon^n \sum_{j=1}^{n}\sum_{A_n}\sum_{\boldsymbol{i}'\in F_j} \frac{1}{j!}\partial^{\boldsymbol{i}'}\delta_0(f_1(w)) f_{p_1}^{i(1')}(w)\cdots f_{p_j}^{i(j')}(w)$$

$$\text{as } \varepsilon\downarrow 0 \text{ in } \tilde{\boldsymbol{D}}^{-\infty}$$

and by Corollary of Theorem 1.1

(2.13) $$c_n(x_0) = (2\pi)^{d/2}\sum_{j=1}^{n}\sum_{A_n}\sum_{\boldsymbol{i}'\in F_j}\frac{1}{j!}E[\partial^{\boldsymbol{i}'}\delta_0(f_1(w)) f_{p_1}^{i(1')}(w)\cdots f_{p_j}^{i(j')}(w)]$$

where $f_p(w)$ is as in (2.2) and

(2.14) $$A_n = \{\boldsymbol{p} = (p_1, \cdots, p_j)\in\boldsymbol{N}^j;\ p_1+\cdots+p_j-j = n,\ p_1\geq 2, \cdots, p_j\geq 2\}.$$

In (2.5) we have the explicit description for $f_p(w)$, $p=0, 1, \cdots$, so it is easy to see that $f_p(-w)=f_p(w)$ when p is even and otherwise $f_p(-w)=-f_p(w)$, and that $\partial^{\boldsymbol{i}'}\delta_0(f_1(-w))=(-1)^j\partial^{\boldsymbol{i}'}\delta_0(f_1(w))$ when $\boldsymbol{i}'\in F_j$. Thus if we change w to $-w$ in (2.13), we can easily see $c_n(x_0)=0$ *if n is odd.* Now putting (2.5) to (2.13) we obtain the following Proposition 2.2.

Proposition 2.2. *$c_{2n}(x_0)$ is given by*

(2.15) $$c_{2n}(x_0) = (2\pi)^{d/2}\sum_{j=1}^{2n}\sum_{A_{2n}}\sum_{B}\sum_{C}\sum_{D}\prod_{\nu=1}^{j}(V_{i_{\nu,m_\nu}}\circ\cdots\circ V_{i_{\nu,2}})(L_{i_{\nu,1}}^{i(\nu')})(x_0)\cdot$$
$$\cdot\frac{1}{j!}E[\partial^{\boldsymbol{i}'}\delta_0(\sum_{\alpha=1}^{r} L_\alpha(x_0)w^\alpha(1))S^{\hat{\boldsymbol{i}}}(1, w)]$$

where A_{2n} *is as in* (2.14) *and*

$$B=\{(m_\nu, m_\nu^*)\in \boldsymbol{Z}^2,\ \nu=1,\cdots,j;\ m_\nu^*+2(m_\nu-m_\nu^*)=p_\nu,\ 1\le m_\nu\le p_\nu,\ 0\le m_\nu^*\le m_\nu\} \tag{2.16}$$

$$C=\{\boldsymbol{k}_\nu=\{k_\nu(1),\cdots,k_\nu(m_\nu^*)\}\subset \boldsymbol{N},\ \nu=1,\cdots,j;\ 1\le k_\nu(1)<\cdots<k_\nu(m_\nu^*)\le m_\nu\} \tag{2.17}$$

$$D=\{(\hat{\boldsymbol{i}},\boldsymbol{i}')\in\prod_{\nu=1}^{j}F(m_\nu,m_\nu^*,\boldsymbol{k}_\nu)\times F_j\} \tag{2.18}$$

Especially in the case of $r=d$, denoting L as a matrix $((L^i_\alpha(x_0))$, then $\det L\neq 0$ by the assumption for $L^i_\alpha(x_0)$. Thus, for $\boldsymbol{i}'\in F_j$, $\partial^{\boldsymbol{i}'}\delta_0(\sum_{\alpha=1}^{d}L_\alpha(x_0)w^\alpha(1))=\sum_{\tilde{\boldsymbol{i}}'\in F_j}(L^{-1})^{\boldsymbol{i}'\tilde{\boldsymbol{i}}'}\partial^{\tilde{\boldsymbol{i}}'}\delta_0(w(1))$ where $(L^{-1})^{\boldsymbol{i}'\tilde{\boldsymbol{i}}'}=\prod_{\nu=1}^{d}(L^{-1})^{i(\nu')\tilde{i}(\nu')}$ if we express $\tilde{\boldsymbol{i}}'=(\tilde{i}(1'),\cdots,\tilde{i}(j'))$ and by $(L^{-1})^{ij}$ (i,j)-component of L^{-1}. Therefore we have the following proposition.

Proposition 2.3. *In the case of* $r=d$, $c_{2n}(x_0)$ *is represented as follows;*

$$c_{2n}(x_0)=(2\pi)^{d/2}\sum_{j=1}^{2n}\sum_{A_{2n}}\sum_{B}\sum_{C}\sum_{D}\sum_{\tilde{\boldsymbol{i}}'\in F_j}\prod_{\nu=1}^{j}(V_{i_{\nu,m_\nu}}\circ\cdots\circ V_{i_{\nu,2}})(L^{i(\nu')}_{i_{\nu,1}})(x_0)(L^{-1})^{\boldsymbol{i}'\tilde{\boldsymbol{i}}'}\cdot\frac{1}{j!}E[\partial^{\tilde{\boldsymbol{i}}'}\delta_0(w(1))S^{\hat{\boldsymbol{i}}}(1,w)] \tag{2.19}$$

where A_{2n}, B, C, *and* D *are as in Proposition* 2.2.

3. Computation of $E[\partial^{\boldsymbol{i}'}\delta_0(\boldsymbol{w}(1))S^{\hat{\boldsymbol{i}}}(1,\boldsymbol{w})]$.

We consider the d-dimensional Wiener space (W^d_0, P) and compute $E[\partial^{\boldsymbol{i}'}\delta_0(w(1))S^{\hat{\boldsymbol{i}}}(1,w)]$ by the following methods: First we note that

$$E[\partial^{\boldsymbol{i}'}\delta_0(w(1))S^{\hat{\boldsymbol{i}}}(1,w)]=(-1)^j\partial^{\boldsymbol{i}'}E[\delta_x(w(1))S^{\hat{\boldsymbol{i}}}(1,w)]|_{x=0} \tag{3.1}$$

when $\boldsymbol{i}'\in F_j$, and that

$$E[\delta_x(w(1))S^{\hat{\boldsymbol{i}}}(1,w)]=E[S^{\hat{\boldsymbol{i}}}(1,w)\,|\,w(1)=x]\cdot(2\pi)^{-d/2}\exp\left(\frac{-|x|^2}{2}\right). \tag{3.2}$$

Set $\tilde{w}(t)=w(t)-tw(1)+tx$, then

$$E[S^{\hat{\boldsymbol{i}}}(1,w)\,|\,w(1)=x]=E[S^{\hat{\boldsymbol{i}}}(1,\tilde{w})] \tag{3.3}$$

and finally $E[S^{\hat{\boldsymbol{i}}}(1,\tilde{w})]$ can be computed by the definition of the stochastic integral, i.e. $S^{\hat{\boldsymbol{i}}}(1,\tilde{w})$ is approximated by the step functions, that is the multiplication of the form $\tilde{w}^\alpha(t)-\tilde{w}^\alpha(t')$, and the expectation of this type can be computed easily.

It is more convenient to use Itô's stochastic integral instead of Stratonovich's one. So in the following, $S^{\hat{\boldsymbol{i}}}(t,w)$, first introduced in (2.4) and (2.10), is defined by Itô's integrals. It is clear that the same conclusions of Theorem 3.1 below remain valid in the Stratonovich case with different universal constants.

First, we consider $E[S^{\hat{i}}(1, w) | w(1)=x]$. For this we introduce some notions.

Definition 3.1. Suppose all m_ν, m_ν^*, $\boldsymbol{k}_\nu$, $\nu=1, \cdots, j$ are given, so is $\{k(1), \cdots, k(m_1^*+\cdots+m_j^*)\}$. The set $\tau \subset \{k(1), \cdots, k(m_1^*+\cdots+m_j^*)\}$ is called *a collection of singles* iff $m_1^*+\cdots+m_j^*-{}^\sharp\tau$ is even, ${}^\sharp\tau$ denoting the power of τ.

Definition 3.2. Suppose A be any set whose power is even $2l$. Then σ_A is called *a pairing decomposition* of A iff $\sigma_A=\{\alpha_1, \beta_1\} \cup \cdots \cup \{\alpha_l, \beta_l\}$ where $\{\alpha_i, \beta_i, i=1, \cdots, l\}=A$. Here the order of the sets $\{\alpha_1, \beta_1\}, \cdots, \{\alpha_l, \beta_l\}$ is arbitrary. Especially when $A=\{1, \cdots, 2l\}$, we denote σ_{2l} instead of σ_A, and when $A=\{k(1), \cdots, k(m_1^*+\cdots+m_j^*)\} \setminus \tau$, σ_τ instead of σ_A.

Now ϕ denotes $\tau \cup \sigma_\tau$, i.e. ϕ: $\{k(1), \cdots, k(m_1^*+\cdots+m_j^*)\} = \{\gamma_1, \cdots, \gamma_h\} \cup \{\alpha_1, \beta_1\} \cup \cdots \cup \{\alpha_l, \beta_l\}$ where $2l+h=m_1^*+\cdots+m_j^*$ and $\{\gamma_1, \cdots, \gamma_h, \alpha_1, \cdots, \alpha_l, \beta_l, \cdots, \beta_l\}=\{k(1), \cdots, k(m_1^*+\cdots+m_j^*)\}$.

Lemma 3.1. *Suppose all m_ν, m_ν^*, $\boldsymbol{k}_\nu$, $\nu=1, \cdots, j$, are given. Then for all $\hat{\boldsymbol{i}} \in \prod_{\nu=1}^{j} F(m_\nu, m_\nu^*, \boldsymbol{k}_\nu)$ and $x \in \boldsymbol{R}^d$,*

$$E[S^{\hat{i}}(1, w) | w(1)=x] = \sum_{\phi} c(\phi)\delta_{i(\alpha_1)i(\beta_1)} \cdots \delta_{i(\alpha_l)i(\beta_l)} x^{i(\gamma_1)} \cdots x^{i(\gamma_h)} \tag{3.4}$$

where $c(\phi)$ is a universal constant depending only on m_ν, m_ν^*, $\boldsymbol{k}_\nu$, $\nu=1, \cdots, j$, and ϕ. Especially $c(\phi)$ is independent of d.

For the proof of this lemma, we need the following two propositions.

Proposition 3.1. *Let $w(t)$ be the 1-dimensional Brownian motion, and $\hat{w}(t)$ be the 1-dimensional Brownian bridge on $[0, 1]$, i.e. $\hat{w}(t)=w(t)-tw(1)$. Then for all $t_i \in [0, 1]$, $i=1, \cdots, 2l$,*

$$E[\hat{w}(t_1)\hat{w}(t_2) \cdots \hat{w}(t_{2l})] = \sum_{\sigma_{2l}} \prod_{i=1}^{l} E[\hat{w}(t_{\alpha_i})\hat{w}(t_{\beta_i})]$$

where $\sigma_{2l}=\bigcup_{i=1}^{l} \{\alpha_i, \beta_i\}$.

Proof. The following fact is well known;

$$(\hat{w}(t_1), \cdots, \hat{w}(t_{2l})) \sim N(0, v_{ij}) \quad \text{where} \quad v_{ij} = (t_i \wedge t_j)(1-t_i \vee t_j).$$

So, this proposition can be easily proved by derivations of a characteristic function.

Proposition 3.2. *On (W_0^d, P), set $w=(w^1, \cdots, w^d) \in W_0^d$. Then for all $t_i \in [0, 1]$, $i=1, \cdots, 2l$,*

$$E[\hat{w}^{i(1)}(t_1) \cdots \hat{w}^{i(2l)}(t_{2l})] = \sum_{\sigma_{2l}} \prod_{j=1}^{l} E[\hat{w}^{i(\alpha_j)}(t_{\alpha_j})\hat{w}^{i(\beta_j)}(t_{\beta_j})] \cdot \delta_{i(\alpha_j)i(\beta_j)}.$$

Proof. Let θ be a decomposition of the set $\{1, \cdots, 2l\}$ into a union of d disjoint subsets, i.e. θ: $\{1, \cdots, 2l\}=\bigcup_{j=1}^{d} A_j$ where $\bigcup_{j=1}^{d} A_j=\{1, \cdots, 2l\}$ and $A_i \cap A_j=\phi$ if

$i \neq j$. Then

$$E[\hat{w}^{i(1)}(t_1) \cdots \hat{w}^{i(2l)}(t_{2l})] = \sum_{\theta} \prod_{j=1}^{d} E[\prod_{\alpha \in A_j} \hat{w}^{i(\alpha)}(t_\alpha)]\delta_{A_j} \prod_{m \neq n}(1-\delta_{A_m A_n}).$$

Here, $\delta_{A_j}=1$ if $i(\alpha)=i(\beta)$ for all α, $\beta \in A_j$, and otherwise $\delta_{A_j}=0$, and $1-\delta_{A_m A_n}=0$ if $i(\alpha)=i(\beta)$ for some $(\alpha, \beta) \in (A_m, A_n)$, and otherwise $1-\delta_{A_m A_n}=1$. Because $E[\hat{w}^1(t_1) \cdots \hat{w}^1(t_{2k+1})]=0$, it suffices to assume that each $A_j, j=1, \cdots, d$, has even powers. Now, for all $\sigma_{2l}=\{\alpha_1, \beta_1\} \cup \cdots \cup \{\alpha_l, \beta_l\}$, we set Θ_σ the totality of θ such that for each i, $1 \leq i \leq l$, there exists j satisfying $\{\alpha_i, \beta_i\} \subset A_j$, and we denote $\sigma_{A_j}=\bigcup_{\nu=1}^{a_j} \{\alpha_{j,\nu}, \beta_{j,\nu}\}$. Then by Proposition 3.1,

$$\begin{aligned} &E[\hat{w}^{i(1)}(t_1) \cdots \hat{w}^{i(2l)}(t_{2l})] \\ &= \sum_{\theta} \sum_{\sigma_{A_1} \cdots \sigma_{A_d}} \prod_{j=1}^{d} \prod_{\nu=1}^{a_j} E[\hat{w}^{i(\alpha_{j,\nu})}(t_{\alpha_{j,\nu}})\hat{w}^{i(\beta_{j,\nu})}(t_{\beta_{j,\nu}})] \cdot \delta_{A_j} \prod_{m \neq n}(1-\delta_{A_m A_n}) \\ &= \sum_{\sigma_{2l}} \sum_{\Theta_\sigma} \prod_{j=1}^{l} E[\hat{w}^{i(\alpha_j)}(t_{\alpha_j})\hat{w}^{i(\beta_j)}(t_{\beta_j})] \cdot \delta_{A_1} \cdots \delta_{A_d} \prod_{m \neq n}(1-\delta_{A_m A_n}). \end{aligned}$$

Noting that, on $\bigcup_{j=1}^{n} B_j$ where $B_j, j=1, \cdots, n$, are any sets,

$$\begin{aligned} \mathbf{1} &= (1-\mathbf{1}_{B_1}) \cdots (1-\mathbf{1}_{B_n}) + \sum_i (1-\mathbf{1}_{B_1}) \cdots \overset{i}{\vee} \cdots (1-\mathbf{1}_{B_n})\mathbf{1}_{B_i} \\ &\quad + \sum_{i \neq j} (1-\mathbf{1}_{B_1}) \cdots \overset{i}{\vee} \cdots \overset{j}{\vee} \cdots (1-\mathbf{1}_{B_n})\mathbf{1}_{B_i}\mathbf{1}_{B_j} + \cdots + \mathbf{1}_{B_1} \cdots \mathbf{1}_{B_n} \end{aligned}$$

where $\mathbf{1}_{B_j}$ denotes the indicator function of B_j, and $\overset{i}{\vee}$ the exclusion of the i-th component, we can easily show that

$$\sum_{\Theta_\sigma} \delta_{A_1} \cdots \delta_{A_d} \prod_{m \neq n}(1-\delta_{A_m A_n}) = \delta_{i(\alpha_1)i(\beta_1)} \cdots \delta_{i(\alpha_l)i(\beta_l)}.$$

Therefore we obtain the assertion of Proposition 3.2.

Proof of Lemma 3.1. Let $\tilde{w}(t)$ be $w(t)-tw(1)+tx=\hat{w}(t)+tx$, and let $\tilde{w}^0(t)=t$, then by (3.3)

$$\begin{aligned} &E[S^{\hat{i}}(1, w) \mid w(1)=x] = E[S^{\hat{i}}(1, \tilde{w})] \\ &= E[\int_0^1 d\tilde{w}^{i(1)}(t_1) \int_0^{t_1} d\tilde{w}^{i(2)}(t_2) \cdots \int_0^{t_{m_1+\cdots+m_j-1}} d\tilde{w}^{i(m_1+\cdots+m_j)}(t_{m_1+\cdots+m_j})] \end{aligned}$$

(where as for each $d\tilde{w}^{i(m_1+\cdots+m_\nu+1)}$, $\nu=1, \cdots, j-1$, the integral is taken on [0, 1])

$$\begin{aligned} &= \lim_{|\Delta_1| \to 0} \sum_{\Delta_1} \lim_{|\Delta_2| \to 0} \sum_{\Delta_2 | t_{p(1)}} \cdots \lim_{|\Delta_{m_1+\cdots+m_j}| \to 0} \sum_{\Delta_{m_1+\cdots+m_j} | t_{p(m_1+\cdots+m_j-1)}} \\ &\quad E[\Delta\tilde{w}^{i(1)}(t_{p(1)}) \cdots \Delta\tilde{w}^{i(m_1+\cdots+m_j)}(t_{p(m_1+\cdots+m_j)})] \end{aligned}$$

where $\Delta_i=\{0, 1/2^n, \cdots, (2^n-1)/2^n, 1\}$ is a division on [0, 1] such that $\Delta_i \subset \Delta_j$ if $i<j$ (i.e. Δ_j is a refinement of Δ_i), $t_{p(1)} \in \Delta_1$, $t_{p(i)} \in \Delta_i | t_{p(i-1)}$ if we denote by $\Delta_j | t_{p(i)}$ a

division on $[0, t_{p(i)}]$ by Δ_j, but especially $\Delta_j | t_{p(m_1+\cdots+m_\nu)}$, $\nu=1, \cdots, j-1$, divisions on $[0, 1]$, and $\Delta\tilde{w}^{i(\nu)}(t_{p(\nu)})=\tilde{w}^{i(\nu)}(t'_{p(\nu)})-\tilde{w}^{i(\nu)}(t_{p(\nu)})$ in which $t'_{p(\nu)}$ is the next point of $t_{p(\nu)}$ in $\Delta_\nu | t_{p(\nu-1)}$. Then

$$\begin{aligned}
&E[\Delta\tilde{w}^{i(1)}(t_{p(1)}) \cdots \Delta\tilde{w}^{i(m_1+\cdots+m_j)}(t_{p(m_1+\cdots+m_j)})] \\
&= E[\Delta\tilde{w}^{i(k(1))}(t_{p(k(1))}) \cdots \Delta\tilde{w}^{i(k(m_1^*+\cdots+m_j^*))}(t_{p(k(m_1^*+\cdots+m_j^*))})]\cdot P_1(t) \\
&= \sum_{\tau} E[\Delta\hat{w}^{i(\alpha_1)}(t_{p(\alpha_1)}) \cdots \Delta\hat{w}^{i(\alpha_l)}(t_{p(\alpha_l)})\Delta\hat{w}^{i(\beta_1)}(t_{p(\beta_1)}) \cdots \Delta\hat{w}^{i(\beta_l)}(t_{p(\beta_l)})] \\
&\qquad \cdot x^{i(\gamma_1)} \cdots x^{i(\gamma_h)}\cdot P_2(t) \\
&= \sum_{\phi} E[\Delta\hat{w}^{i(\alpha_1)}(t_{p(\alpha_1)})\Delta\hat{w}^{i(\beta_1)}(t_{p(\beta_1)})] \cdots E[\Delta\hat{w}^{i(\alpha_l)}(t_{p(\alpha_l)})\Delta\hat{w}^{i(\beta_l)}(t_{p(\beta_l)})] \\
&\qquad \cdot\delta_{i(\alpha_1)i(\beta_1)}\cdots\delta_{i(\alpha_l)i(\beta_l)}\cdot x^{i(\gamma_1)}\cdots x^{i(\gamma_h)}\cdot P_2(t) \\
&= \sum_{\phi} P_3(t)\cdot\delta_{i(\alpha_1)i(\beta_1)} \cdots \delta_{i(\alpha_l)i(\beta_l)}\cdot x^{i(\gamma_1)} \cdots x^{i(\gamma_h)}
\end{aligned}$$

where $P_1(t)=\prod_{\{\nu;i(\nu)=0\}}(t'_{p(\nu)}-t_{p(\nu)})$ which depends only on m_ν, m_ν^*, and $\boldsymbol{k}_\nu$, $\nu=1, \cdots, j$, $P_2(t)=P_1(t)\prod_{\{\nu\,;\,k(\nu)\in\tau\}}(t'_{p(k(\nu))}-t_{p(k(\nu))})$ which depends only on $P_1(t)$ and τ, and $P_3(t)$ depends only on $P_2(t)$ and ϕ. So $P_3(t)$ depends only on ϕ, m_ν, m_ν^*, and $\boldsymbol{k}_\nu$, $\nu=1, \cdots, j$, and is independent of d. Thus

$$E[\hat{S^i}(1, w) | w(1)=x] = \sum_{\phi} c(\phi)\cdot\delta_{i(\alpha_1)i(\beta_1)} \cdots \delta_{i(\alpha_l)i(\beta_l)}\cdot x^{i(\gamma_1)} \cdots x^{i(\gamma_h)}$$

where $c(\phi)$ is a universal constant which depends only on ϕ, m_ν, m_ν^*, and $\boldsymbol{k}_\nu$, $\nu=1, \cdots, j$.

Example 3.1. In the case that $j=2$, $m_1=m_2=2$, $m_1^*=m_2^*=1$, and $k_1(1)=k_2(1)=2$,

$$\begin{aligned}
E[\hat{S^i}(1, w) | w(1)=x] &= E[(\int_0^1 dt_1 \int_0^{t_1} dw^{i(2)}(t_2))(\int_0^1 dt_3 \int_0^{t_3} dw^{i(4)}(t_4)) | w(1)=x] \\
&= \frac{1}{12}\delta_{i(2)i(4)}+\frac{1}{4}x^{i(2)}x^{i(4)}\,.
\end{aligned}$$

Example 3.2 (*Levy's stochastic area*). In the case $j=2$, $m_1=m_2=2$, $m_1^*=m_2^*=2$, and $\boldsymbol{k}_1=\boldsymbol{k}_2=\{1, 2\}$,

$$\begin{aligned}
&E[\hat{S^i}(1, w) | w(1)=x] \\
&= E[(\int_0^1 dw^{i(1)}(t_1) \int_0^{t_1} dw^{i(2)}(t_2))(\int_0^1 dw^{i(3)}(t_3) \int_0^{t_3} dw^{i(4)}(t_4)) | w(1)=x] \\
&= \frac{1}{4}\delta_{i(1)i(2)}\delta_{i(3)i(4)}+\frac{1}{12}\delta_{i(1)i(3)}\delta_{i(2)i(4)}-\frac{1}{12}\delta_{i(1)i(4)}\delta_{i(2)i(3)} \\
&\quad -\frac{1}{4}\delta_{i(3)i(4)}x^{i(1)}x^{i(2)}+\frac{1}{12}\delta_{i(2)i(4)}x^{i(1)}x^{i(3)}-\frac{1}{12}\delta_{i(2)i(3)}x^{i(1)}x^{i(4)} \\
&\quad -\frac{1}{12}\delta_{i(1)i(4)}x^{i(2)}x^{i(3)}+\frac{1}{12}\delta_{i(1)i(3)}x^{i(2)}x^{i(4)}-\frac{1}{4}\delta_{i(1)i(2)}x^{i(3)}x^{i(4)}
\end{aligned}$$

$$+\frac{1}{4}x^{i(1)}x^{i(2)}x^{i(3)}x^{i(4)}.$$

Now consider *Lévy's stochastic area* $A(t, w)$, i.e. for $w \in W_0^2$

$$A(t, w) = \frac{1}{2}\left(\int_0^t w^1(s)dw^2(s) - \int_0^t w^2(s)dw^1(s)\right) \tag{3.5}$$

and let's compute $E[A(1, w)^2 | w(1)=x]$. By (3.5)

$$A(1, w)^2 = \frac{1}{4}\left(\int_0^1 w^1(t)dw^2(t)\right)^2 + \frac{1}{4}\left(\int_0^1 w^2(t)dw^1(t)\right)^2$$
$$-\frac{1}{2}\left(\int_0^1 w^1(t)dw^2(t)\right)\left(\int_0^1 w^2(t)dw^1(t)\right),$$

so, by the above equation, we obtain

$$E[A(1, w)^2 | w(1)=x] = \frac{1}{12} + \frac{|x|^2}{12}$$

Remark 3.1. This can also be obtained by the following way: Let $K(x, \lambda) = E[\exp\{\sqrt{-1}\lambda A(1, w)\} | w(1)=x]$, then by M. Yor [11]

$$K(x, \lambda) = \left(\frac{\lambda}{2}/\sinh\left(\frac{\lambda}{2}\right)\right)\cdot \exp\left\{\left(1-\frac{\lambda}{2}\coth\frac{\lambda}{2}\right)\frac{|x|^2}{2}\right\}.$$

Especially $K(x, \lambda) = K(x, -\lambda)$, so $E[A(1, w)^{2k+1} | w(1)=x] = 0$, and $E[A(1, w)^2 | w(1)=x] = \frac{1}{12} + \frac{|x|^2}{12}$.

Finally we treat $E[(\partial^{i'}\delta_0(w(1))S^{\hat{i}}(1, w)]$. Before stating the theorem, we introduce the following notation: Suppose m_ν, m_ν^*, and $\boldsymbol{k}_\nu$, $\nu=1, \cdots, j$, are given and $m_1^* + \cdots + m_j^* + n = 2l$. Let $A = \{k(1), \cdots, k(m_1^* + \cdots + m_j^*)\} \cup \{1', \cdots, n'\}$, then we denote ψ instead of σ_A, i.e.

$$\psi: \{k(1), \cdots, k(m_1^* + \cdots + m_j^*)\} \cup \{1', \cdots, n'\} = \{\alpha_1, \beta_1\} \cup \cdots \cup \{\alpha_l, \beta_l\}. \tag{3.6}$$

Remark 3.2. All ψ can be decomposed as follows: τ be a collection of singles on $\{k(1), \cdots, k(m_1^* + \cdots + m_j^*)\}$ such that its power is less than n i.e. τ: $\{k(1), \cdots, k(m_1^* + \cdots + m_j^m)\} = \{\gamma_1, \cdots, \gamma_h\}$ and $h \leq n$. σ_τ be $\{\alpha_1, \beta_1\} \cup \cdots \cup \{\alpha_p, \beta_p\}$. Define σ'_τ as a pairing decomposition on $\{\gamma_1, \cdots, \gamma_h, 1', \cdots, n'\}$ such that each partner of γ_i, $i=1, \cdots, h$, is some ν' $(1 \leq \nu \leq n)$, i.e. σ'_τ: $\{\gamma_1, \cdots, \gamma_h, 1', \cdots, n'\} = \{\alpha_{p+1}, \beta_{p+1}\} \cup \cdots \cup \{\alpha_l, \beta_l\}$ and $\{\alpha_{p+1}, \cdots, \alpha_{p+h}\} = \{\gamma_1, \cdots. \gamma_h\}$ Then for all ψ there exist τ, σ_τ, and σ'_τ as above such that $\psi = \sigma_\tau \cup \sigma'_\tau$.

Theorem 3.1. *For all* $\hat{\boldsymbol{i}} \in \prod_{\nu=1}^{j} F(m_\nu, m_\nu^*, \boldsymbol{k}_\nu)$ *and* $\boldsymbol{i}' \in F_n$,

$$E[\partial^{i'}\delta_0(w(1))S^{\hat{i}}(1, w)] = (2\pi)^{-d/2}\sum_{\psi} c(\psi)\delta_{i(\alpha_1)i(\beta_1)}\cdots\delta_{i(\alpha_l)i(\beta_l)}$$

if $m_1^*+\cdots+m_j^*+n=2l$, *and otherwise* 0, *where* $c(\psi)$ *is a universal constant which depends only on* ψ, m_ν, m_ν^*, $\boldsymbol{k}_\nu$, $\nu=1,\cdots,j$, *and* n.

Proof. As stated at the beginning of this section

$$E[\partial^{i'}\delta_0(w(1))S^{\hat{i}}(1,w)]=(-1)^n\partial^{i'}E[S^{\hat{i}}(1,w)\,|\,w(1)=x](2\pi)^{-d/2}\exp\left(\frac{-|x|^2}{2}\right)\Bigg|_{x=0}$$

So, by Lemma 3.1

$$E[S^{\hat{i}}(1,w)\,|\,w(1)=x]=\sum_{\phi}c(\phi)\delta_{i(\alpha_1)i(\beta_1)}\cdots\delta_{i(\alpha_p)i(\beta_p)}x^{i(\gamma_1)}\cdots x^{i(\gamma_h)} \tag{3.7}$$

and $\phi=\sigma_\tau\cup\tau$ for some collection of singles τ. Therefore if $n<h$, or $n\geq h$ and $n-h$ is odd, $\partial^{i'}x^{i(\gamma_1)}\cdots x^{i(\gamma_h)}\exp\left(\frac{-|x|^2}{2}\right)\Big|_{x=0}=0$, and if $n\geq h$ and $n-h$ is even

$$\partial^{i'}x^{i(\gamma_1)}\cdots x^{i(\gamma_h)}\exp\left(\frac{-|x|^2}{2}\right)\Bigg|_{x=0}=\sum_{\sigma'_\tau}(-1)^{(n-h)/2}\delta_{i(\alpha_{p+1})i(\beta_{p+1})}\cdots\delta_{i(\alpha_l)i(\beta_l)} \tag{3.8}$$

where σ'_τ is as in Remark 3.2. Thus, combined (3.8) with (3.7), we can conclude the proof of theorem.

4. Main theorem.

In this section, we consider the asymptotic expansion of the pole $p(t,x,x)$ as $t\downarrow 0$ for the minimal fundamental solution $p(t,x,y)$ of the heat equation $\frac{\partial u}{\partial t}=\frac{1}{2}\Delta_M u$ on a Riemannian manifold, Δ_M being the Laplace-Beltrami operator. Note that $p(t,x,y)$ is the density of the law of X_t with respect to the Riemannian volume, where X_t is the minimal Brownian motion on M starting at x at $t=0$. In the previous section we studied the solutions of S.D.E. on $\boldsymbol{R}^d$ as Wiener functionals. We first state a localization result which reduces our problem to that of S.D.E. on $\boldsymbol{R}^d$.

Lemma 4.1. *On a Riemannian manifold* (M,g), *for any fixed point* $x_0\in M$ *and positive numbers* ν, R, $0<2\nu<R$, *such that* $W(x_0,\nu+R)=\{y\in\boldsymbol{R}^d;\ |x_0-y|\leq\nu+R\}$ *is in a local chart of* x_0, *let* X_t *be the minimal solution of the following S.D.E. on* M;

$$\begin{cases} dX_t=L_\alpha(X_t)\circ dw^\alpha(t)+L_0(X_t)dt \\ X_0=x_0 \end{cases}$$

where $(a^{ij})=LL^*$ *is elliptic,* $\tilde{X}_t$ *be its minimal diffusion on* $W(x_0,R)$, $p(t,x_0,y)$ *be the density of the law of* X_t *with respect to the Riemannian volume, and* $\tilde{p}(t,x_0,y)$ *that of* $\tilde{X}_t$. *Then there exist positive constants* c_1, c_2 *such that for all* $y\in W(x_0,R-2\nu)$

$$p(t,x_0,y)-\tilde{p}(t,x_0,y)\leq\frac{c_1}{\nu^d}\exp\left(-c_2\frac{\nu^2}{t}\right). \tag{4.1}$$

Proof. Let $\tau(w)=\inf\{t\geq 0, X_t \notin W(x_0, R)\}$. By R. Azencott [1], there exist positive constants c, c' such that

$$P(\tau(w)\leq t,\ |X_t-y|\leq r)\leq c\left(\frac{r}{\nu}\right)^d \exp\left(-c'\frac{\nu^2}{t}\right)$$

where $r<\nu/2$ and $|x_0-y|+2\nu<R$. Noting that

$$p(t, x_0, y) = \lim_{r\downarrow 0} P(X_t\in W(y, r))/\mathrm{vol}(W(y, r)),$$

$$\tilde{p}(t, x_0, y) = \lim_{r\downarrow 0} P(X_t\in W(y, r),\ \tau(w)>t)/\mathrm{vol}(W(y, r))$$

and

$$P(X_t\in W(y, r))-P(X_t\in W(y, r),\ \tau(w)>t) = P(\tau(w)\leq t,\ X_t\in W(y, r)),$$

we can easily conclude (4.1).

Remark 4.1. By the above lemma, it suffices to treat $\tilde{X}_t$ instead of X_t in our problem, and $\tilde{X}_t$ can be identified with a minimal diffusion on a compact subset of $\boldsymbol{R}^d$. So, again by the above lemma with $M=\boldsymbol{R}^d$, we can replace X_t by a diffusion $\tilde{\tilde{X}}_t$ on $\boldsymbol{R}^d$ obtained as the global solution of S.D.E. on $\boldsymbol{R}^d$ whose coefficients coincide with $L_\alpha(x)$ in a coodinate neighborhood of x_0.

By Remark 4.1, it is enough to analyze $\frac{1}{2}\Delta_M$-diffusion *on a local chart of* x_0. So taking a normal coodinate with center x_0 and extending this coodinate to the global Euclidean coodinate of $\boldsymbol{R}^d$, it is enough to study the solution of the following S.D.E. on $\boldsymbol{R}^d$ over the d-dimensional Wiener space (W_0^d, P);

$$\text{(4.2)}\qquad \begin{cases} dX_t^{\varepsilon,k} = \varepsilon\sigma^{\alpha k}(X_t^\varepsilon)\circ dw^\alpha(t)+\varepsilon^2\sigma_0^k(X_t^\varepsilon)dt \\ X_0^\varepsilon = 0 \end{cases}$$

where $X_t^\varepsilon=(X_t^{\varepsilon,1}, X_t^{\varepsilon,2}, \cdots, X_t^{\varepsilon,d})$, $\sigma(x)=g^{-1/2}(x)$, $\sigma_0^k(x)=-\frac{1}{2}g^{ij}(x)\Gamma_{ij}^k(x)-\frac{1}{2}\left(\frac{\partial}{\partial x^j}\sigma^{\alpha k}(x)\right)\cdot\sigma^{\alpha j}(x)$ in some neighborhood of 0 and both $\sigma^{\alpha k}(x)$ and $\sigma_0^k(x)$ belong to $C_b^\infty(\boldsymbol{R}^d)$. Here $\{\Gamma_{ij}^k\}$ are the Christoffel symbols and (g^{ij}) the inverse of (g_{ij}).

By Cartan's formula, we have the following expansions of $\sigma(x)$ and $\sigma_0(x)$ in the normal coodinate. (cf. B.Y. Chen and L. Vanhecke [3])

$$\text{(4.3)}\qquad \sigma^{pq}(x) = \delta_{pq}-\frac{1}{6}\sum_{i,j} R_{piqj}x^ix^j-\frac{1}{12}\sum_{i,j,k}\nabla_i R_{pjqk}x^ix^jx^k$$
$$-\frac{1}{120}\sum_{i,j,k,l}\left(3\nabla_{ij}^2 R_{pkql}-\frac{7}{3}\sum_s R_{sipj}R_{skql}\right)x^ix^jx^kx^l+o(|x|^5)$$

$$\text{(4.4)}\qquad \sigma_0^m(x) = -\frac{1}{3}\sum_i R_{mi}x^i+\frac{1}{24}\sum_{i,j}(\nabla_m R_{ij}-6\nabla_i R_{mj})x^ix^j$$
$$+\frac{1}{360}\sum_{i,j,k}(9\nabla_{im}^2 R_{jk}-36\nabla_{ij}^2 R_{mk}+8\sum_p R_{ip}R_{pjmk}$$
$$-16\sum_{p,q} R_{ipjq}R_{mpkq})x^ix^jx^k+o(|x|^4)$$

where ∇ means the covariant derivative and values of monomials in components of the curvature and its derivatives are taken at the origin. Now we can apply above results to obtain the expansion (2.7). Then,

$$\begin{aligned} c_2(x_0) &= (2\pi)^{d/2}\Big\{\Big(-\frac{1}{3}R_{\alpha k}(x_0)\Big)E\Big[\frac{\partial}{\partial x^k}\delta_0(w(1))\int_0^1 w^\alpha(t)dt\Big] \\ &\quad +\Big(-\frac{1}{3}R_{\alpha\beta k\gamma}(x_0)\Big)E\Big[\frac{\partial}{\partial x^k}\delta_0(w(1))\int_0^1\int_0^t w^\gamma(s)\circ dw^\beta(s)\circ dw^\alpha(t)\Big]\Big\} \\ &= R(x_0)/12\,. \end{aligned}$$

Similarly we can compute $c_4(x_0)$ to be $\frac{1}{720}\Big(\frac{5}{2}R^2(x_0)-||R_{ij}(x_0)||^2+||R_{ijkl}(x_0)||^2\Big)$ $+\Delta R(x_0)/120$. This computation is elementary but quite complicated, and for higher $c_{2n}(x_0)$ it is, of course, too much complicated. So we would give some informations for $c_{2n}(x_0)$ in the following Theorem 4.1. Before stating the theorem, we introduce the notion of *order*.

Definition 4.1 (cf. P. Gilkey [4]). $\mathcal{R}$ be a totality of monomials in components of the curvature tensor and its covariant derivatives. A function ord: $\mathcal{R}\to\boldsymbol{N}$ is defined as follows:

$$\operatorname{ord}(\nabla^m_{p_1\cdots p_m}R_{ijkl}) = 2+m, \quad \text{and} \quad \operatorname{ord}(R_1R_2) = \operatorname{ord}(R_1)+\operatorname{ord}(R_2),\ R_1,\ R_2\in\mathcal{R}\,.$$

If $\operatorname{ord}(R)=m$, $R\in\mathcal{R}$, then we say *R is of order m.*

Theorem 4.1. *$c_{2n}(x_0)$ is a linear combination of the contractions of the elements in $\mathcal{R}$ of order $2n$. Moreover the constants of this combination are universal, i.e. independent of a manifold and its dimension. Here, a contraction means a contraction of all indices with respect to a pairing of them.*

Remark 4.2. Theorem 4.1 is not new, in fact, M. Beals, C. Fefferman and R. Grossman [2] and P. Gilkey [4] have showed the same theorem by first determining the order of elements of $\mathcal{R}$ which appear in $c_{2n}(x_0)$ and then appealing to Weyl's invariant theory to show that these elements are given by the contraction. Our assertion is that this theorem can also be proved by the probabilistic methods.

Proof of Theorem 4.1. Fist we determine the order of elements of $\mathcal{R}$ which appear in $c_{2n}(x_0)$. By Cartan's formula we have the following expansions of $\sigma^{pq}(x)$ and $\sigma_0^k(x)$ in (4.2):

$$(4.5)\qquad \begin{cases} \sigma^{pq}(x) = \delta_{pq}+\sum_{j=2}^{n} R_{\alpha_1\cdots\alpha_j pq}x^{\alpha_1}\cdots x^{\alpha_j}+o(|x|^n) \\ \sigma_0^k(x) = \sum_{j=1}^{n} R'_{\alpha_1\cdots\alpha_j k}x^{\alpha_1}\cdots x^{\alpha_j}+o(|x|^n) \end{cases}$$

where $R_{\alpha_1\cdots\alpha_j pq}$ [resp. $R'_{\alpha_1\cdots\alpha_j k}$] is a symbolic representation for a linear combination of elements in $\mathcal{R}$ whose order is j [resp. $j+1$]. So if we set $(V_{i_{\nu,m_\nu}}\circ\cdots\circ V_{i_{\nu,2}})\cdot$ $\cdot(L^{i(\nu')}_{i_{\nu,1}})(x_0)=R_{i(k_\nu(1))\cdots i(k_\nu(m^*_\nu))i(\nu')}(x_0)$ in Propositions 2.2 or 2.3 applied to (4.2), this is a linear combination of elements of order $p_\nu - 1$ in $\mathscr{R}$. Consequently, by (2.15) or (2.19),

$$c_{2n}(x_0) = (2\pi)^{d/2} \sum_{j=1}^{2n} \sum_{A_{2n}} \sum_{B} \sum_{C} \sum_{D} R_{i(k(1))\cdots i(k(m_1^*+\cdots+m_j^*))i(1')\cdots i(j')}(x_0) \cdot \frac{1}{j!} E[\partial^{i'}\delta_0(w(1))S^{\hat{i}}(1, w)] \tag{4.6}$$

where the sets A_{2n}, B, C, and D are as in (2.14), (2.16), (2.17) and (2.18), respectively, and $R_{i(k(1))\cdots i(k(m_1^*+\cdots+m_j^*))i(1')\cdots i(j')}(x_0) = \prod_{\nu=1}^{j} R_{i(k_\nu(1))\cdots i(k_\nu(m_\nu^*))i(\nu')}(x_0)$, so it is a combination of elements of order $p_1 + \cdots + p_j - j = 2n$. Now we apply Theorem 3.1 to evaluate $E[\partial^{i'}\delta_0(w(1))S^{\hat{i}}(1, w)]$ which completes the proof.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] R. Azencott, Densité des diffusions en temps petit, Développements asymptotiques, Séminaire de Prob. XVIII (1982/83), Lect. Notes in Math. 1059, Springer-Verlag (1984), 402–498.

[2] M. Beals, C. Fefferman and R. Grossman, Strictly pseudoconvex domains in $\boldsymbol{C}^n$, Bull. Amer. Math. Soc. (N.S.), **8**-2 (1983), 125–322.

[3] B.Y. Chen and L. Vanhecke, Differential geometry of geodesic spheres, J. für reine und angew. Math., **325** (1981), 28–67.

[4] P. Gilkey, The spectral geometry of a Riemannian manifold, J. Diff. Geo., **10** (1975), 601–618.

[5] N. Ikeda and S. Watanabe, An introduction to Malliavin's calculus, Stochastic analysis, Proc. of the Taniguchi international symp. Katata and Kyoto 1982, ed. by K. Itô, Kinokuniya, Tokyo, 1984, 1–52.

[6] H. P. McKean and I.M. Singer, Curvature and the eigenvalues of the Laplacian, J. Diff. Geo., **1** (1967), 43–69.

[7] S. A. Molchanov, Diffusion processes and Riemannian geometry, Russian Math. Surveys, **30**-1 (1975), 1–63.

[8] H. Sugita, Sobolev spaces of Wiener functionals and Malliavin's calculus, J. Math. Kyoto Univ., **25**-1 (1985), 31–48.

[9] S. Watanabe, Lectures on stochastic differential equations and Malliavin calculus, Tata Inst. Fund. Res. (1984).

[10] S. Watanabe, Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels, Ann. Prob., **15**-1 (1987), 1–39.

[11] M. Yor, Remarques sur une formule de Paul Lévy; Séminaire de Prob. XIV (1978/79), Lect. Notes in Math. 784, Springer-Verlag (1980), 343–346.