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A property of operators characterized by iteration and a necessary condition for hypoellipticity

By

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§0. Introduction.

Let $\mathcal{A}(x, D_x)$ be a linear partial differential operator of order $m(\geqq 1)$ with coefficients of class C^∞ in a bounded open set Ω of $\boldsymbol{R}^n$. For $\sigma\geqq 1$, let us denote by $G^\sigma(\bar{\Omega}, \mathcal{A})$ the vector space consisting of all $u\in L^2(\Omega)$ such that $\mathcal{A}^k u\in L^2(\Omega)$ for $k=1, 2, \cdots$, and furthermore

$$\|\mathcal{A}^k u\|\leqq L^{k+1}(k!)^{\sigma m} \qquad \text{for } k=0, 1, 2, \cdots$$

with some positive constant L (where $\|\ \|$ denotes the norm in $L^2(\Omega)$). In particular, $G^\sigma(\bar{\Omega}, \Delta)$ (where Δ denotes Laplacian) coincides with the Gevrey class of index σ in $\bar{\Omega}$.

G. Métivier ([6] and [7]) proved that, if $\mathcal{A}$ is formally selfadjoint and $G^1(\bar{\Omega}, \mathcal{A})\subset \boldsymbol{A}(\Omega)$ (where $\boldsymbol{A}(\Omega)$ denotes the space of real analytic functions in Ω), then $\mathcal{A}$ is elliptic in Ω (see also T. Kotake—M.S. Narasimhan [4]).

On the other hand, Y. Morimoto [8] showed that, for the operator

$$\mathcal{A}_\gamma = D_{x_1}^2+\exp(-1/|x_1|^\gamma)D_{x_2}^2 \quad \left(D_{x_j}=\frac{1}{i}\frac{\partial}{\partial x_j},\ j=1, 2\right)$$

there exists a function u_0 belonging to $G^1(\bar{\Omega}, \mathcal{A}_\gamma)$ but not of class C^∞ in Ω if $\gamma\geqq 1$ (where Ω is a neighborhood of the origin in $\boldsymbol{R}^2$). Furthermore, he proved that the operator $P_\gamma=D_t^2+\mathcal{A}_\gamma$ is not hypoelliptic in $\boldsymbol{R}\times\Omega$ if $\gamma\geqq 1$, in the following way: Let us define

$$u(t, x_1, x_2) = \sum_{k\geqq 0} t^{2k}\mathcal{A}_\gamma^k u_0(x_1, x_2)/(2k)!\,.$$

Then, since $u_0\in G^1(\bar{\Omega}, \mathcal{A}_\gamma)$, the series on the right hand side converges in $L^2(\Omega)$ if $|t|$ is small enough. But $u(t, x_1, x_2)$ is a non-smooth solution of the equation

$$P_\gamma u\equiv 0 \qquad \text{satisfying} \qquad u(0, x_1, x_2) = u_0(x_1, x_2)\,.$$

Hence, P_γ is not hypoelliptic. (P_γ is hypoelliptic if $0<\gamma<1$. See S. Kusuoka—

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D. Strook [5] and Y. Morimoto [9].)

By the above argument, we see that $G^1(\bar{\Omega}, \mathcal{A}_\gamma)$ should be contained in $C^\infty(\Omega)$ for P_γ to be hypoelliptic. (Note that $\mathcal{A}_\gamma$ itself is hypoelliptic whenever $\gamma>0$. See V.S. Fediĭ [2].) Now, we can expect that the same is true for the operators of higher order, that is, $G^1(\bar{\Omega}, \mathcal{A})\subset C^\infty(\Omega)$ if $\mathcal{A}(x, D_x)$ is of order $m(\geqq 1)$ and if $P=D_t^m+\mathcal{A}(x, D_x)$ is hypoelliptic. In the present paper, noting this property and based on the method of G. Métivier [7], we shall show a necessary condition for the operators of the form $D_t^m+\mathcal{A}(x, D_x)$ to be hypoelliptic.

The plan of this paper is as follows. We state our main results in §1. In §2, we prove some auxiliary lemmas, and in §3 we give the proofs of the theorems stated in §1. In §4 we shall apply our results to some concrete examples of operators analogeous to $\mathcal{A}_\gamma$.

In the course of the preparation of this paper, the author was communicated that Prof. Y. Morimoto independently obtained almost the same results as Theorem 1 and Theorem 2 below. The present paper is also based on the works of Y. Morimoto [8] and [9].

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§1. Main results.

Notations. In this paper, we use $\|\ \|$ to denote the norm in $L^2(\Omega)$, or that in $L^2(\boldsymbol{R}^n)$. The norm in the Sobolev space $H^s(\boldsymbol{R}^n)$ for $s>0$ is denote by $\|\ \|_s$. Furthermore, for an open set $\omega\subset\Omega$, we denote $H^s(\omega)$ by the Sobolev space for $s>0$ in ω. $H^\infty(\omega)$ is defined by $\bigcap_{s>0} H^s(\omega)$.

First, let us introduce the space of functions, which is connected with the method of Y. Morimoto [9].

Definition. For $r>0$, $H_r^{\log}(\boldsymbol{R}^n)$ is the vector space which consicts of all $u\in L^2(\boldsymbol{R}^n)$ such that

$$\|u\|_{\log,r}^2=\int\{\log(2+|\xi|^2)\}^{2r}|\hat{u}(\xi)|^2d\xi<+\infty\ ,$$

where

$$\hat{u}(\xi)=(2\pi)^{-n/2}\int u(x)e^{-ix\cdot\xi}dx\ .$$

Now, we state the results for formally selfadjoint operators.

Theorem 1. *Let $\mathcal{A}(x, D_x)$ be formally selfadjoint partial differential operator of order $m(\geqq 1)$ with coefficients of class $C^\infty(\bar{\Omega})$. Assume that, for an open set $\omega\subset\Omega$ and for some real numbers $\sigma\geqq 1$ and $s>0$, the restriction $u|_\omega$ of any function $u\in G^\sigma(\bar{\Omega}, \mathcal{A})$ belongs to $H^s(\omega)$. Then, for any open set $\omega_1\subset\subset\omega$, there exists a positive constant C such that*

$$(1.1)\qquad ||u||^2_{\log,\sigma m} \leqq C(||\mathcal{A}u||^2+||u||^2) \qquad \text{for all } u \in C_0^\infty(\omega_1)\,.$$

Theorem 2. *Let $\mathcal{A}(x, D_x)$ be a formally selfadjoint partial differential operator of order $m(\geqq 1)$ with coefficients of class $C^\infty(\bar{\Omega})$. Assume that, for an open set $\omega \subset \Omega$ and for a real number $\sigma \geqq 1$, the restriction $u|_\omega$ of any function $u \in G^\sigma(\bar{\Omega}, \mathcal{A})$ belongs to $H^\infty(\omega)$. Then, for any open set $\omega_1 \subset\subset \omega$ and for any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$(1.2)\qquad ||u||^2_{\log,\sigma m} \leqq \varepsilon||\mathcal{A}u||^2+C_\varepsilon||u||^2 \qquad \text{for all } u \in C_0^\infty(\omega_1)\,.$$

Now, we can present a necessary condition for hypoellipticity.

Corollary. *Let $\mathcal{A}(x, D_x)$ be a formally selfadjoint partial differential operator of order $m(\geqq 1)$ with coefficients of class $C^\infty(\bar{\Omega})$. If the operator $D_t^m+\mathcal{A}(x, D_x)$ is hypoelliptic in $\boldsymbol{R}\times\Omega\subset\boldsymbol{R}^{n+1}$, then* (1.2) *with $\sigma=1$ holds for any open set $\omega_1 \subset\subset \Omega$.*

Proof of Corollary. If (1.2) with $\sigma=1$ does not hold for $\omega_1 \subset\subset \Omega$, then it follows from theorem 2 that there exists a $u_0 \in G^1(\bar{\Omega}, \mathcal{A})$ not belonging to $H^\infty(\omega)$ $(\omega_1 \subset\subset \omega \subset\subset \Omega)$. Since $u_0 \in G^1(\bar{\Omega}, \mathcal{A})$, the series

$$u(t, x) = \sum_{k\geqq 0} (it)^{mk}(-\mathcal{A})^k u_0(x)/(mk)!$$

converges in $L^2((-\delta, \delta)\times\Omega)$ for small $\delta>0$, and it satisfies the equation $(D_t^m+\mathcal{A}(x, D_x))u(t, x)\equiv 0$.

On the other hand, $u(t, x)$ is not of class C^∞ in $(-\delta, \delta)\times\Omega$, because $u(0, x)=u_0(x)\notin H^\infty(\omega)$. Hence, the operator $D_t^m+\mathcal{A}(x, D_x)$ is not hypoelliptic. Q.E.D.

Let us now generalize the above Theorem 1 to the operators which are not necessarily formally selfadjoint but have maximally accretive extensions (see T. Kato [9] page 279 for the definition of maximally accretive operator).

Theorem 3. *Let $\mathcal{A}(x, D_x)$ be a partial differential operator of order $m(\geqq 1)$ with coefficients of class $C^\infty(\bar{\Omega})$ which has a maximally accretive realization* $(A, \mathrm{D}(A))$ *in $L^2(\Omega)$. Assume that, for an open set $\omega \subset \Omega$ and for some real numbers $\sigma \geqq 1$ and $s>0$, the restriction $u|_\omega$ of any function $u \in G^\sigma(\bar{\Omega}, \mathcal{A})$ belongs to $H^s(\omega)$. Then, for any open set $\omega_1 \subset\subset \omega$ and for any $\delta(1<\delta<\sigma m)$, there exists a positive constant C such that*

$$(1.3)\qquad ||u||^2_{\log,\sigma m-\delta} \leqq C(||\mathcal{A}u||^2+||u||^2) \qquad \text{for all } u \in C_0^\infty(\omega_1)\,.$$

If A is a maximally accretive operator, then $A+\lambda$ has bounded inverse for any $\lambda \in \boldsymbol{C}$ with $\operatorname{Re}\lambda>0$ and $||(A+\lambda)^{-1}|| \leqq (\operatorname{Re}\lambda)^{-1}$. Moreover, $\mathrm{D}(A)$ (the domain of definition of A) is dense in $L^2(\Omega)$. So, $-A$ generates a contractive semi-group $\{G(t)\}_{t\geqq 0}$ by Hille-Yosida's theorem. We will make use of $G(t)$ in the proof of theorem 3.

Remark. We can apply Theorem 3 to the operators of the form

$$\mathcal{A} = -\sum_{j=1}^{p} X_j^2+X_0+c\,,$$

where $X_0, X_1, \cdots, X_p$ are real vector fields of class $C^\infty(\bar{\Omega})$ and $c \in C^\infty(\bar{\Omega})$ (see G. Métivier [7] §5).

§2. Auxiliary lemmas.

In this section, we treat two spectral resolution of positively selfadjoint operators. One is related to formally selfadjoint differential operator $\mathcal{A}(x, D_x)$, and the other is related to $\Lambda=(1+|D_x|^2)^{1/2}$. Using them, we introduce some spaces of functions connected with $G^\sigma(\bar{\Omega}, \mathcal{A})$, and characterize the space $H_r^{\log}(\boldsymbol{R}^n)$. Furthermore, in Lemma 2.4 and Lemma 2.5 below, we modify the interpolative method of G. Métivier [7] §3 to make use of our theorems.

I) Spectral resolutions.

Let $(A, \mathrm{D}(A))$ be a realization in $L^2(\Omega)$ of a formally selfadjoint differential operator $\mathcal{A}(x, D_x)$, of order $m(\geqq 1)$, with coefficients of class $C^\infty(\bar{\Omega})$. In this section, we assume that A is positively selfadjoint with domain $\mathrm{D}(A)$. (In general case, we make use of the square root of a Friedrichs extension of $\mathcal{A}^2$. The detail will be stated in the next section.) We denote by $E(t)$ $(-\infty<t<+\infty)$ the spectral resolution of $(A, \mathrm{D}(A))$:

$$A=\int_{-\infty}^\infty t dE(t)\,, \quad E((-\infty, 0))=0\,.$$

First, let us put

$$F(t)=E(t-1)-E(\frac{t}{e}-1)\,.$$

Then, it is easy to see that

$$u=\int_1^\infty F(t)u\frac{dt}{t} \qquad \text{for all } u\in L^2(\Omega)\,, \tag{2.1}$$

where the integral on the right hand side converges in $L^2(\Omega)$. Moreover,

$$\|u\|^2=\int_1^\infty \|F(t)u\|^2\frac{dt}{t} \qquad \text{for all } u\in L^2(\Omega)\,, \tag{2.2}$$

where $\| \quad \|$ denotes the norm in $L^2(\Omega)$.

Next, we introduce the following integrals $J_k(u)$ and $N_\alpha^\sigma(u)$ connected with $\mathrm{D}(A^k)$ and $G^\sigma(\bar{\Omega}, \mathcal{A})$.

$$J_k(u)=\int_1^\infty t^{2k}\|F(t)u\|^2\frac{dt}{t} \qquad \text{for integer } k\geqq 0\,.$$

$$N_\alpha^\sigma(u)=\int_1^\infty \exp(\alpha t^{1/\sigma m})\|F(t)u\|^2\frac{dt}{t} \qquad \text{for } \sigma\geqq 1 \text{ and } \alpha>0\,.$$

Furthermore, concerning the integral $N_\alpha^\sigma(u)$, we define the followings:

Definition (1). For $\sigma\geqq 1$, we denote by $\mathrm{D}^\sigma(A)$ the vector space consisting of all functions $u\in \mathrm{D}(A^{\infty})$ $(=\bigcap_{k=0}^{\infty}\mathrm{D}(A^{k}))$ such that

$$||A^{k}u||\leqq L^{k+1}(k!)^{\sigma m} \qquad \text{for } k=0, 1, 2, \cdots$$

with some positive constant L.
(2). For $\sigma\geqq 1$ and $\alpha>0$, we denote by $\mathrm{D}_{\alpha}^{\sigma}(A)$ the vector space consisting of all functions $u\in \mathrm{D}(A^{\infty})$ satisfying

$$N_{\alpha}^{\sigma}(u)<\infty .$$

Now, we see the followings:

Lemma 2.1 (i). *An element $u\in L^{2}(\Omega)$ belongs to $\mathrm{D}(A^{k})$ if and only if $J_{k}(u)<\infty$. Moreover, for all $u\in \mathrm{D}(A^{k})$,*

$$||(A+1)^{k}u||^{2}\leqq J_{k}(u)\leqq e^{2k}||(A+1)^{k}u||^{2} \tag{2.3}$$

(ii). *For $\sigma\geqq 1$, $\bigcup_{\alpha>0}\mathrm{D}_{\alpha}^{\sigma}(A)=\mathrm{D}^{\sigma}(A)$.*

(iii). *For any $u\in L^{2}(\Omega)$, $\tau\mapsto F(\tau)u$ is a measurable mapping from $[1, \infty)$ to $\mathrm{D}_{\alpha}^{\sigma}(A)$ satisfying*

$$N_{\alpha}^{\sigma}(F(\tau)u)\leqq \exp(\alpha(e\tau)^{1/\sigma m})||u||^{2} . \tag{2.4}$$

Proof of (i). To verify (2.3), it suffices to remark that

$$\left(\frac{t}{e}\right)^{k}||F(t)u||\leqq||(A+1)^{k}F(t)u||\leqq t^{k}||F(t)u|| . \tag{2.5}$$

Integration of (2.5) with respect to t yields (2.3).

Proof of (ii). If $u\in \mathrm{D}_{\alpha}^{\sigma}(A)$, then by (2.3)

$$\begin{aligned}||(A+1)^{k}u||^{2}&\leqq\int_{1}^{\infty}t^{2k}||F(t)u||^{2}\frac{dt}{t}\\&\leqq\sup_{t\geqq 0}t^{2k}\exp(-\alpha t^{1/\sigma m})N_{\alpha}^{\sigma}(u)\\&\leqq\left(\frac{2\sigma m}{\alpha}\right)^{2\sigma mk}(k!)^{2\sigma m}N_{\alpha}^{\sigma}(u) .\end{aligned}$$

Hence, we see that $u\in \mathrm{D}^{\sigma}(A)$.

Conversely, if $u\in L^{2}(\Omega)$ satisfies

$$||(A+1)^{k}u||\leqq L^{k+1}(k!)^{\sigma m} \qquad \text{for } k=0, 1, 2, \cdots$$

with some positive constant L, then by (2.3) we see that

$$\int_{1}^{\infty}t^{2k}||F(t)u||^{2}\frac{dt}{t}\leqq L^{2}(eL)^{2k}(k!)^{2\sigma m} \qquad \text{for } k=0, 1, 2, \cdots .$$

Hence, it follows from Hölder's inequality that

$$\int_1^\infty t^{k/\sigma m}||F(t)u||^2\frac{dt}{t}$$
$$\leqq\left(\int_1^\infty t^{2k}||F(t)u||^2\frac{dt}{t}\right)^{1/2\sigma m}\left(\int_1^\infty ||F(t)u||^2\frac{dt}{t}\right)^{1-1/2\sigma m}$$
$$\leqq L^2(eL)^{k/\sigma m}\,k!\qquad \text{for } k=0,1,2,\cdots.$$

Let us divide the both sides of the last inequality by $2^k(eL)^{k/\sigma m}k!$, and sum up them with respect to k. Then, we see that $N^\sigma_\alpha(u)<\infty$ with $\alpha=2^{-1}(eL)^{-1/\sigma m}$.

Proof of (iii). By (2.5), we have $F(\tau)u\in D(A^\infty)$.

On the other hand, by definition of $N^\sigma_\alpha(u)$,

$$N^\sigma_\alpha(F(\tau)u)\leqq\int_1^\infty \exp(\alpha t^{1/\sigma m})||F(t)F(\tau)u||^2\frac{dt}{t}.$$

Noticing that $F(t)F(\tau)u\equiv 0$ for $t>e\tau$, we have

$$\begin{aligned}N^\sigma_\alpha(F(\tau)u) &\leqq \exp(\alpha(e\tau)^{1/\sigma m})\int_1^\infty ||F(t)F(\tau)u||^2\frac{dt}{t}\\ &=\exp(\alpha(e\tau)^{1/\sigma m})||F(\tau)u||^2\\ &\leqq \exp(\alpha(e\tau)^{1/\sigma m})||u||^2.\end{aligned}$$

Q.E.D.

Next, we introduce the operator $\Lambda=(1+|D_x|^2)^{1/2}$ in $L^2(\boldsymbol{R}^n)$ with bomain $H^1(\boldsymbol{R}^n)$, that is,

$$\Lambda u(x)=(2\pi)^{-n/2}\int e^{ix\cdot\xi}(1+|\xi|^2)^{1/2}\hat{u}(\xi)d\xi.$$

The spectral resolution of Λ is denoted by $\tilde{E}(\lambda)$ $(-\infty<\lambda<+\infty)$:

$$\Lambda u=\int_{-\infty}^\infty \lambda d\tilde{E}(\lambda),\quad \tilde{E}((-\infty,1))=0.$$

Furthermore, let us put

$$\tilde{F}(\lambda)=\tilde{E}(\lambda-1)-\tilde{E}(\frac{\lambda}{e}-1).$$

Then, by the same way as in (2.1) and (2.2), we have

(2.6) $$u=\int_1^\infty \tilde{F}(\lambda)u\frac{d\lambda}{\lambda},\ \text{and}$$

(2.7) $$||u||^2=\int_1^\infty ||\tilde{F}(\lambda)u||^2\frac{d\lambda}{\lambda}\qquad \text{for all } u\in L^2(\boldsymbol{R}^n),$$

where $||\ \ ||$ denotes the norm in $L^2(\boldsymbol{R}^n)$.

Next, let us introduce the following integrals $\tilde{J}_s(u)$ and $\tilde{I}_r(u)$ connected with $H^s(\boldsymbol{R}^n)$ and $H^{\log}_r(\boldsymbol{R}^n)$.

(2.8) $$\tilde{J}_s(u)=\int_1^\infty \lambda^{2s}||\tilde{F}(\lambda)u||^2\frac{d\lambda}{\lambda}\qquad \text{for } s>0.$$

$$\tilde{I}_r(u) = \int_1^\infty (\log \lambda)^{2r} \|\tilde{F}(\lambda)u\|^2 \frac{d\lambda}{\lambda} \qquad \text{for } r>0\,. \tag{2.9}$$

Lemma 2.2 (i). *An element $u \in L^2(\boldsymbol{R}^n)$ belongs to $H^s(\boldsymbol{R}^n)$ if and only if $\tilde{J}_s(u) < \infty$. Moreover, there exists a positive constant C_s such that*

$$C_s^{-1}\|u\|_s^2 \leqq \tilde{J}_s(u) \leqq C_s\|u\|_s^2 \qquad \text{for all } u \in H^s(\boldsymbol{R}^n)\,. \tag{2.10}$$

(ii). *An element $u \in L^2(\boldsymbol{R}^n)$ belongs to $H_r^{\log}(\boldsymbol{R}^n)$ if and only if $\tilde{I}_r(u) < \infty$. Moreover, there exists a positive constant C_r such that*

$$C_r^{-1}\|u\|_{\log,r}^2 \leqq \tilde{I}_r(u) \leqq C_r\|u\|_{\log,r}^2 \qquad \text{for all } u \in H_r^{\log}(\boldsymbol{R}^n)\,. \tag{2.11}$$

Proof of (i). Noticing that the norms $\|u\|_s$ and $\|(\Lambda+1)^s u\|$ are equivalent, we obtain (i) by the same argument as in the proof of (ii) of lemma 2.1.

Proof of (ii). Let us consider the operator

$$\{\log(\Lambda+1)\}^r = \int_0^\infty \{\log(\lambda+1)\}^r \, d\tilde{E}(\lambda) \qquad \text{for } r>0\,.$$

Then, for $\lambda \geqq 1$,

$$\begin{aligned}&(\log \lambda)^{2r}\|\tilde{F}(\lambda)u\|^2 - \|\{\log(\Lambda+1)\}^r \tilde{F}(\lambda)u\|^2 \\ &\quad = \int_1^\infty \{(\log \lambda)^{2r} - (\log \mu)^{2r}\} \, d_\mu \|\tilde{E}(\mu-1)\tilde{F}(\lambda)u\|^2\,.\end{aligned}$$

The integral on the right hand side is non-negative, because

$$d_\mu \tilde{E}(\mu-1)\tilde{F}(\lambda)u = 0 \qquad \text{if } \mu>\lambda\,.$$

Hence, it follows from the above argument that

$$\|\{\log(\Lambda+1)\}^r \tilde{F}(\lambda)u\|^2 \leqq (\log \lambda)^{2r}\|\tilde{F}(\lambda)u\|^2 \leqq \|\{\log(e\Lambda+e)\}^r \tilde{F}(\lambda)u\|^2\,,$$

where the definition of $\{\log(e\Lambda+e)\}^r$ is analogous to that of $\{\log(\Lambda+1)\}^r$. Integrations of the inequalities with respect to λ yield (ii), noticing that the norms $\|u\|_{\log,r}$, $\|\{\log(\Lambda+1)\}^r u\|$ and $\|\{\log(e\Lambda+e)\}^r u\|$ are equivalent. Q.E.D.

II) Interpolation.

Now, we shall prove the following lemmas which play important roles in the proofs of the theorems.

Lemma 2.3. *Let $t \mapsto f(t)$ be a measurable mapping from $[1, \infty)$ to $H^s(\boldsymbol{R}^n)$ $(s>0)$ satisfying*

$$M(f;\theta) = \int_1^\infty \|f(t)\|^2 t^{2\theta} \frac{dt}{t} < \infty\,, \quad \textit{and}$$

$$N(f; s, \alpha, \beta, \theta) = \int_1^\infty \exp(-\alpha t^{1/\beta}) \tilde{J}_s(f(t)) t^{2\theta} \frac{dt}{t} < \infty$$

for some positive numbers α, β, θ and s (for brevity, let us denote $M(f) = M(f;\theta)$

and $N(f)=N(f;\, s,\, \alpha,\, \beta,\, \theta)$).

Then, the integral

$$v=\int_1^\infty f(t)\frac{dt}{t} \tag{2.12}$$

converges with respect to the norm in $L^2(\boldsymbol{R}^n)$. *Moreover,* $v \in H_{\theta\beta}^{\log}(\boldsymbol{R}^n)$ *and*

$$\|v\|_{\log,\theta\beta}^2 \leqq C(M(f)+N(f))\,, \tag{2.13}$$

where C is a constant independent of f.

Remark. In the integral defining $N(f)$, the factor $t^{2\theta}$ is negligible if we take $\alpha>0$ larger. Also, $\tilde{J}_s(f(t))$ can be replaced by $\|f(t)\|_s^2$. So, the condition $N(f)<\infty$ says that $\|f(t)\|_s$ increases at most in exponential order as $t\to\infty$.

Lemma 2.4. *Let* $t\mapsto f(t)$ *be a measurable mapping from* $[1,\infty)$ *to* $H^\infty(\boldsymbol{R}^n)$ *satisfying*

$$M(f;\,\theta)<\infty \quad \textit{and} \quad N(f;\, s,\, \alpha,\, \beta,\, \theta)<\infty$$

for some fixed positive numbers α, β *and* θ, *and for any* $s>0$ (*for brevity, let us denote* $N(f;\, s)=N(f;\, s,\, \alpha,\, \beta,\, \theta)$).

Then, for any $\varepsilon>0$, *there exists a* $s>0$ *independent of f such that*

$$\int_1^\infty (\log \lambda)^{2\theta\beta}\|\tilde{F}(\lambda)v\|^2\frac{d\lambda}{\lambda} \leqq \varepsilon(M(f)+N(f;\, s))\,,$$

where v is the same as in (2.12).

Proof of lemma 2.3. The integral on the right hand side of (2.12) is convergent, because

$$\left(\int_1^\infty \|f(t)\|\frac{dt}{t}\right)^2 \leqq \int_1^\infty t^{2\theta}\|f(t)\|^2\frac{dt}{t}\cdot\int_1^\infty t^{-2\theta}\frac{dt}{t}=(2\theta)^{-1}M(f)<\infty\,.$$

Let us put $\exp(t^{1/\beta})=\tau$ and $g(\tau)=f(t)$. Then, we see that

$$\begin{aligned} M(f)+N(f) &= \beta\int_e^\infty \{\|g(\tau)\|^2+\tau^{-\alpha}\cdot\tilde{J}_s(g(\tau))\}(\log\tau)^{2\theta\beta-1}\frac{d\tau}{\tau} \\ &= \beta\int_e^\infty\int_1^\infty (1+\lambda^{2s}\cdot\tau^{-\alpha})(\log\tau)^{2\theta\beta-1}\|\tilde{F}(\lambda)g(\tau)\|^2\frac{d\lambda}{\lambda}\frac{d\tau}{\tau}\,, \end{aligned} \tag{2.14}$$

In the last inequality, we have used (2.7) and (2.8).

On the other hand, since $\tilde{F}(\lambda)$ is a bounded linear operator in $L^2(\boldsymbol{R}^n)$,

$$\begin{aligned} \tilde{F}(\lambda)v &= \tilde{F}(\lambda)\beta\int_e^\infty g(\tau)\,(\log\tau)^{-1}\frac{d\tau}{\tau} \\ &= \beta\int_e^\infty \tilde{F}(\lambda)g(\tau)\,(\log\tau)^{-1}\frac{d\tau}{\tau}\,. \end{aligned}$$

Hence, it follows from Schwarz's inequality that

$$(2.15)\qquad \|\tilde{F}(\lambda)v\|^2 \leqq \beta^2 \int_e^\infty (1+\lambda^{2s}\cdot\tau^{-\alpha})(\log\tau)^{2\theta\beta-1}\|\tilde{F}(\lambda)g(\tau)\|^2\frac{d\tau}{\tau}$$
$$\times\int_e^\infty (1+\lambda^{2s}\cdot\tau^{-\alpha})^{-1}(\log\tau)^{-2\theta\beta-1}\frac{d\tau}{\tau}$$

Let us estimate the right hand side of (2.15). Changing variable τ by $\mu=\log\tau$, and putting a $(\lambda)=\frac{s}{\alpha}\log\lambda$, we obtain

$$\int_e^\infty(1+\lambda^{2s}\cdot\tau^{-\alpha})^{-1}(\log\tau)^{-2\theta\beta-1}\frac{d\tau}{\tau}$$
$$=\int_1^\infty(1+\lambda^{2s}\cdot e^{-\alpha\mu})^{-1}\mu^{-2\theta\beta-1}d\mu$$
$$\leqq\int_{a(\lambda)}^\infty\mu^{-2\theta\beta-1}d\mu+\lambda^{-2s}\int_1^{a(\lambda)}e^{\alpha\mu}d\mu\,.$$

Furthermore, since $e^{\alpha\mu}\leqq\lambda^s$ for $\mu\leqq a(\lambda)$, we see that

$$(2.16)\qquad \int_e^\infty(1+\lambda^{2s}\cdot\tau^{-\alpha})^{-1}(\log\tau)^{-2\theta\beta-1}\frac{d\tau}{\tau}$$
$$\leqq(2\theta\beta)^{-1}\left[\frac{s}{\alpha}\log\lambda\right]^{-2\theta\beta}+\lambda^{-s}\left[\frac{s}{\alpha}\log\lambda\right]$$
$$\leqq\left\{(2\theta\beta)^{-1}\left[\frac{s}{\alpha}\right]^{-2\theta\beta}+\frac{s}{\alpha}\left[\frac{2\theta\beta+1}{s}\right]^{2\theta\beta+1}\right\}(\log\lambda)^{-2\theta\beta}$$
$$=s^{-2\theta\beta}\{(2\theta\beta)^{-1}\alpha^{2\theta\beta}+\alpha^{-1}(2\theta\beta+1)^{2\theta\beta+1}\}(\log\lambda)^{-2\theta\beta}\,.$$

The third line of (2.16) holds because

$$\lambda^{-s}(\log\lambda)^{2\theta\beta+1}\leqq\left[\frac{2\theta\beta+1}{s}\right]^{2\theta\beta+1}\qquad\text{for }\lambda\geqq1\,.$$

Therefore, it follows from (2.15) and (2.16) that

$$(\log\lambda)^{2\theta\beta}\|\tilde{F}(\lambda)v\|^2$$
$$\leqq C\int_e^\infty(1+\lambda^{2s}\cdot\tau^{-\alpha})(\log\tau)^{2\theta\beta-1}\|\tilde{F}(\lambda)g(\tau)\|^2\frac{d\tau}{\tau}$$

where C depends only on s, α, β and θ.

Hence, it follows from (2.14) that

$$\int_1^\infty(\log\lambda)^{2\theta\beta}\|\tilde{F}(\lambda)v\|^2\frac{d\lambda}{\lambda}\leqq\beta^{-1}C(M(f)+N(f))\,.$$

Noticing (ii) of Lemma 2.2, we have Lemma 2.3. Q.E.D.

Proof of Lemma 2.4. In (2.16), let us take $s>0$ such that

$$s^{-2\theta\beta}\{(2\theta\beta)^{-1}\alpha^{2\theta\beta}+\alpha^{-1}(2\theta\beta+1)^{2\theta\beta+1}\}\leqq\varepsilon\cdot\beta^{-1}\,.$$

Then, (2.16) becomes

$$\int_e^\infty (1+\lambda^{2s}\cdot\tau^{-\alpha})^{-1}(\log\tau)^{-2\theta\beta-1}\frac{d\tau}{\tau}\leqq\varepsilon\cdot\beta^{-1}(\log\lambda)^{-2\theta\beta}.$$

Hence, it follows from (2.15) that

$$\begin{aligned}&(\log\lambda)^{2\theta\beta}\|\tilde{F}(\lambda)v\|^2\\ &\leqq\varepsilon\beta\int_e^\infty(1+\lambda^{2s}\cdot\tau^{-\alpha})\,(\log\tau)^{2\theta\beta-1}\|\tilde{F}(\lambda)g(\tau)\|^2\frac{d\tau}{\tau}.\end{aligned}$$

Now, if we integrate the last inequality with respect to λ, we obtain

$$\begin{aligned}&\int_1^\infty(\log\lambda)^{2\theta\beta}\|\tilde{F}(\lambda)v\|^2\frac{d\lambda}{\lambda}\\ &\leqq\varepsilon\beta\int_1^\infty\int_e^\infty(1+\lambda^{2s}\cdot\tau^{-\alpha})\,(\log\tau)^{2\theta\beta-1}\|\tilde{F}(\lambda)g(\tau)\|^2\frac{d\tau}{\tau}\frac{d\lambda}{\lambda}.\end{aligned}$$

The conclusion of lemma 2.4 follows from (2.14). Q.E.D.

§3. Proofs of Theorems.

First, let us prove Theorem 2, because the proof of Theorem 1 is parallel to that of Theorem 2.

Proof of Theorem 2. The differential operator $\mathcal{B}=\mathcal{A}^2$ is formally positive and selfadjoint. Let us denote by $(B, \mathrm{D}(B))$ the Friedrichs extension of $\mathcal{B}$ in $L^2(\Omega)$, defined in the following way:

Let $\mathcal{V}$ be the vector space consisting of all functions $u\in L^2(\Omega)$ satisfying $\mathcal{A}u\in L^2(\Omega)$ (where $\mathcal{A}$ is operated to u in distribution sense). The domain of definition of B is

$\mathrm{D}(B)=\{u\in\mathcal{V}$; *there exists a function* $f\in L^2(\Omega)$ *such that*

$$(\mathcal{A}u, \mathcal{A}v)+(u, v)=(f, v)\qquad \textit{for all } v\in\mathcal{V}\}$$

where (,) denotes the scalar product in $L^2(\Omega)$.

Now, we define Bu by $f-u$ in the above definition of $\mathrm{D}(B)$. It is evident that $(Bu, v)=(\mathcal{A}u, \mathcal{A}v)$ for all $u\in\mathrm{D}(B)$ and for all $v\in\mathcal{V}$. In particular, $(Bu, u)=\|\mathcal{A}\|^2$ for all $u\in\mathrm{D}(B)$.

Now, it is clear that, for $u\in\mathrm{D}(B^\infty)$, $B^pu=\mathcal{A}^{2p}u$ and

$$\|\mathcal{A}^{2p}u\|=\|B^pu\|\qquad\text{for } p=0, 1, 2, \cdots.$$

Next, let us denote by $B^{1/2}$ the positive square root of B. Then, since $\|\mathcal{A}v\|^2=(Bv, v)=\|B^{1/2}v\|^2$ for all $v\in\mathrm{D}(B)$, we see that, for $u\in\mathrm{D}(B^\infty)$,

$$\|\mathcal{A}^{2p+1}u\|^2=\|\mathcal{A}B^pu\|^2=(B\cdot B^pu, B^pu)=\|B^{p+1/2}u\|^2\quad\text{for } p=0, 1, 2, \cdots.$$

Hence, we can show the inclusion $\mathrm{D}^\sigma(B^{1/2})\subset G^\sigma(\bar{\Omega}, \mathcal{A})$.

Now, we fix $\alpha_1>0$ and define $V=\mathrm{D}^{\sigma}_{\alpha_1}(B^{1/2})$. The above argument and (ii) of lemma 2.1 imply that V is a Banach space and contained in $G^{\sigma}(\bar{\Omega}, \mathcal{A})$. Furthermore, let us take a function $\phi\in C_0^{\infty}(\omega)$ satisfying $0\leqq\phi\leqq 1$ in ω and $\phi\equiv 1$ in ω_1. Then, the hypothesis of theorem 2, i.e., $G^{\sigma}(\bar{\Omega}, \mathcal{A})\subset H^{\infty}(\omega)$, yields that ϕu belongs to $H^{\infty}(\boldsymbol{R}^n)$ for any $u\in V$. Hence, the following holds by the closed graph theorem.

For any $s>0$, there exists a positive constant $C_s=C_s(\phi)$ such that

$$\tilde{J}_s(\phi u)\leqq C_s N^{\sigma}_{\alpha_1}(u) \qquad \text{for all } u\in V. \tag{3.1}$$

(Recall that $\sqrt{\tilde{J}_s(u)}$ is equivalent to the norm in $H^s(\boldsymbol{R}^n)$. See (i) of lemma 2.2.)

Now, we can apply lemma 2.4 to $f(t)=\phi F(t)u$ for $u\in\mathrm{D}(B^{1/2})$ $(A=B^{1/2})$, with $\theta=1$, $\beta=\sigma m$ and $\alpha\gg\alpha_1$. Then, we have

$$\begin{aligned} M(f) &= \int_1^{\infty}||\phi F(t)u||^2t^2\frac{dt}{t}\leqq\int_1^{\infty}||F(t)u||^2t^2\frac{dt}{t} \\ &\leqq e^2||(B^{1/2}+1)u||^2 \end{aligned}$$

and for any $s>0$,

$$\begin{aligned} N(f;\, s) &= \int_1^{\infty}\exp(-\alpha t^{1/\sigma m})\tilde{J}_s(\phi F(t)u)\,t^2\frac{dt}{t} \\ &\leqq C_s\int_1^{\infty}\exp(-\alpha t^{1/\sigma m})N^{\sigma}_{\alpha_1}(F(t)u)t^2\frac{dt}{t} \\ &\leqq C'_s||u||^2\,. \end{aligned}$$

In the above inequalities, we have used lemma 2.1 and (3.1).

Therefore, since

$$v=\int_1^{\infty}f(t)\frac{dt}{t}=\int_1^{\infty}\phi F(t)u\frac{dt}{t}=\phi u\,,$$

ϕu belongs to $H^{\log}_{\sigma m}(\boldsymbol{R}^n)$. Furthermore, for arbitrary small $\varepsilon>0$, there exists a positive constant C'_s such that

$$\begin{aligned} &\int_1^{\infty}(\log\lambda)^{2\sigma m}||\tilde{F}(\lambda)\phi u||^2\frac{d\lambda}{\lambda} \\ &\qquad\leqq\varepsilon(e^2||(B^{1/2}+1)u||^2+C'_s||u||^2) \qquad \text{for all } u\in\mathrm{D}(B^{1/2})\,. \end{aligned} \tag{3.2}$$

Hence, by (ii) of lemma 2.2, we obtain the following inequality.

For any $\varepsilon>0$, there exists a positive constant C_ε such that

$$||\phi u||^2_{\log,\sigma m}\leqq\varepsilon||B^{1/2}u||^2+C_\varepsilon||u||^2 \qquad \text{for all } u\in\mathrm{D}(B^{1/2})\,.$$

Since, $\phi u=u$ and $||B^{1/2}u||^2=||\mathcal{A}u||^2$ for $u\in C_0^{\infty}(\omega_1)$ (note that $C_0^{\infty}(\omega_1)\subset\mathrm{D}(B)$), the last inequality gives Theorem 2. Q.E.D.

Proof of Theorem 3. The following proof is quite analogous to the proof of G. Métivier [7] §5. But, we give the proof for the sake of self-containedness. Since the proof is long, we devide it into two parts.

Part 1 (*preparation*). Let $(A, \mathrm{D}(A))$ denote a maximally accretive realization of $\mathcal{A}(x, D_x)$ in $L^2(\Omega)$. The contractive semi-group generated by $-A$ is denoted by $G(t)$ $(t>0)$. First, we prove the following lemmas.

Lemma 3.1. *Let us put*

$$H(t) = e^{-\lambda t}G(t) - e^{-e\lambda t}G(et)$$

for positive constant λ. *Then, we have*

$$u = \int_0^\infty H(t)u\frac{dt}{t} \qquad \text{for all } u \in \mathrm{D}(A)\,, \tag{3.3}$$

the integral on the right hand side being conver gent with respect to the norm in $L^2(\Omega)$.

Proof of Lemma 3.1. For $u \in \mathrm{D}(A)$, $H(t)u$ can be written as

$$H(t)u = \int_t^{et} e^{-\lambda\tau}G(\tau)\,(A+\lambda)u\,d\tau\,.$$

Furthermore, since $G(t)$ is contractive, i.e., $||G(t)|| \leqq 1$, we obtain

$$||H(t)u|| \leqq 2t\,e^{-\lambda t}||(A+\lambda)u||\,. \tag{3.4}$$

Hence, the right hand side of (3.3) converges with respect to the norm in $L^2(\Omega)$. Moreover, exchanging the order of integrations, we have

$$\begin{aligned}\int_0^\infty H(t)u\frac{dt}{t} &= \int_0^\infty e^{-\lambda\tau}G(\tau)\,(A+\lambda)u\,d\tau\int_{\tau/e}^{\tau}\frac{dt}{t} \\ &= \int_0^\infty e^{-\lambda\tau}G(\tau)\,(A+\lambda)u\,d\tau \\ &= (A+\lambda)^{-1}(A+\lambda)u = u\,.\end{aligned}$$

This proves (3.3). Q.E.D.

Lemma 3.2. *Taking a function* $\psi \in C_0^\infty(\boldsymbol{R})$ *satisfying*

$$\text{(i)}\quad \psi \geqq 0, \quad \text{(ii)}\quad \operatorname{supp}\psi \subset (1, 2) \quad \textit{and} \quad \text{(iii)}\quad \int\psi(\tau)d\tau = 1\,,$$

we put:

$$T(t) = \int_0^\infty \psi(\tau)H\left(\frac{\tau}{t}\right)d\tau\,.$$

Then ,we have

$$u = \int_0^\infty T(t)u\frac{dt}{t} \qquad \text{for all } u \in \mathrm{D}(A)\,, \tag{3.5}$$

the integral on the right hand side being convergent with respect to the norm in $L^2(\Omega)$.

Proof of Lemma 3.2. By (3.4), we obtain immediately the following.

(3.6) $$\|T(t)u\| \leqq \frac{4}{t} e^{-\lambda' t} \|(A+\lambda)u\| \qquad \text{for all } u \in \mathrm{D}(A).$$

So, the right hand side of (3.5) converges. Moreover,

$$\int_0^\infty T(t)u\frac{dt}{t} = \int_0^\infty \psi(\tau)\,d\tau \int_0^\infty H\Big(\frac{\tau}{t}\Big)u\frac{dt}{t} = \int_0^\infty H(s)u\frac{ds}{s}.$$

This proves (3.5). Q.E.D.

In the proof of theorem 3, we make use of $T(t)$ with a function of the Gevrey class of index $\rho (1<\rho<\sigma m)$, that is,

(3.7) $$\sup_{\tau>0} |\psi^{(k)}(\tau)| \leqq L^{k+1}(k!)^\rho \qquad \text{for } k=0, 1, 2, \cdots$$

with some positive constant L.

It is easy to see that, if we put $\tilde{G}(t)=e^{-\lambda' t}G(t)$ (semi-group generated by $-(A+\lambda)$) then

$$(A+\lambda)^k T(t) = t^k \int_0^\infty \psi^{(k)}(\tau)\tilde{G}\Big(\frac{\tau}{t}\Big)d\tau - \Big(\frac{t}{e}\Big)^k \int_0^\infty \psi^{(k)}(\tau)\tilde{G}\Big(\frac{e\tau}{t}\Big)d\tau$$
$$\text{for } k=0, 1, 2, \cdots.$$

So, noting that $\|\tilde{G}(t)\| \leqq e^{-\lambda' t}$, we obtain by (3.7) that

(3.8) $$\|(A+\lambda)^k T(t)u\| \leqq 2t^k L^{k+1}(k!)^\rho e^{-\lambda'/t}\|u\|$$
$$\text{for } k=0, 1, 2, \cdots \text{ and for all } u \in L^2(\Omega).$$

Now, we denote by W the Banach space which consists of all $u \in \mathrm{D}(A^\infty)$ satisfying

$$\|u\|_W = \sup_{k\geqq 0} \frac{\|(A+\lambda)^k u\|}{(k!)^{\sigma m} L^k} < \infty.$$

Then, (3.8) yields

(3.9) $$\|T(t)u\|_W \leqq 2L\, e^{-\lambda'/t} \exp((\sigma m-\rho)t^{1/\sigma m-\rho})\|u\| \qquad \text{for all } u \in L^2(\Omega).$$

On the other hand, since W is contained in $G^\sigma(\bar{\Omega}, \mathcal{A})$, the hypothesis of theorem 3 implies that, for any $u \in w$, ϕu belongs to $H^s(\boldsymbol{R}^n)$ (where $\phi \in C_0^\infty(\omega)$ is the same as in the proof of theorem 2). So, the following inequality holds by the closed graph theorem.

There exists a positive constant $C_1=C_1(\phi)$ such that

(3.10) $$\tilde{I}_s(\phi u) \leqq C_1\|u\|_W^2 \qquad \text{for all } u \in W.$$

(Recall that $\sqrt{\tilde{J}_s(u)}$ is equivalent to the norm in $H^s(\boldsymbol{R}^n)$. See (i) of lemma 2.2.)

Part 2. We take up the proof of theorem 3. According to lemma 3.2, we can write $u \in \mathrm{D}(A)$ in such a way that

$$u = u_0 + u_1$$

where

$$u_0=\int_0^1 T(t)u\frac{dt}{t} \quad\text{and}\quad u_1=\int_1^\infty T(t)u\frac{dt}{t}.$$

Now, let us show that both of ϕu_0 and ϕu_1 belong to $H^{\log}_{\sigma m-\delta}(\boldsymbol{R}^n)$, for any $u\in$ D(A). At first, (3.89) yields

$$\|u_0\|_W \leqq 2L\, e^{\sigma m-\rho}\int_0^1 e^{-\lambda/t}\frac{dt}{t}\cdot\|u\|<\infty .$$

Therefore, we see by (3.10) that ϕu_0 is an element of $H^s(\boldsymbol{R}^n)$. To show that $\phi u_1\in H^{\log}_{\sigma m-\delta}(\boldsymbol{R}^n)$, we apply lemma 2.3 to $f(t)=\phi T(t)u$ with $\beta=\sigma m-\rho$ and $\alpha>2(\sigma m-\rho)$. We have, for $0\leqq\theta<1$,

$$\begin{aligned} M(f) &= \int_1^\infty \|f(t)\|^2 t^{2\theta}\frac{dt}{t} \leqq \int_1^\infty \|T(t)u\|^2 t^{2\theta}\frac{dt}{t} \\ &\leqq 16\int_1^\infty t^{2\theta-3}e^{-2\lambda/t}\,dt\times\|(A+\lambda)u\|^2 \\ &\leqq 8(1-\theta)^{-1}\|(A+\lambda)u\|^2<\infty , \end{aligned}$$

and

$$\begin{aligned} N(f) &= \int_1^\infty \exp(-\alpha t^{1/\sigma m-\rho})\tilde{J}_s(f(t))t^{2\theta}\frac{dt}{t} \\ &\leqq C_1\int_1^\infty \exp(-\alpha t^{1/\sigma m-\rho})\|T(t)u\|_W^2\, t^{2\theta}\frac{dt}{t} \\ &\leqq C_2\|u\|^2<\infty . \end{aligned}$$

In the above inequalities, we have used (3.6), (3.10) and (3.9). Therefore, since

$$\phi u_1=\int_1^\infty f(t)\frac{dt}{t},$$

ϕu_1 belongs to $H^{\log}_{(\sigma m-\sigma)\theta}(\boldsymbol{R}^n)$. Now, for any given $\delta(1<\delta<\sigma m)$, we can choose ρ and θ such that $1<\rho<\sigma m$, $0<\theta<1$ and $(\sigma m-\rho)\theta=\sigma m-\delta$. This means that ϕu_1 belongs to $H^{\log}_{\sigma m-\delta}(\boldsymbol{R}^n)$ for any $u\in$ D(A). Hence, the following inequality holds by the closed graph theorem.

For any $\delta(1<\delta<\sigma m)$, there exists a positive constant C_3 such that

$$\|\phi u\|^2_{\log,\sigma m-\delta}\leqq C_3(\|Au\|^2+\|u\|^2) \qquad \text{for all } u\in \mathrm{D}(A).$$

Since $\phi u=u$ and $Au=\mathcal{A}u$ for $u\in C_0^\infty(\omega_1)$ in the last inequality, Theorem 3 is now completely proved. Q.E.D.

§4. Examples.

In this section, we shall verify that some operators similar to $\mathcal{A}_\gamma$ in Introduction of the present paper violate at least one of (1.2) or (1.3).

Let Ω be a bounded open set in $\boldsymbol{R}^2$ which contains the origin O. Let us consider the following differential operators defined in Ω:

$$\mathcal{A}_1 = D_{x_1}^2 + D_{x_2}(\phi(x_1, x_2)D_{x_2}), \quad \mathcal{A}_2 = D_{x_1}^2 + i\phi(x_1, x_2)D_{x_2}$$

and

$$\mathcal{A}_3 = iD_{x_1} + D_{x_2}(\phi(x_1, x_1)D_{x_2}),$$

where we denote $\phi(x_1, x_2) = \exp(-1/|x|^\gamma)$ with $\gamma > 0$.

They are of elliptic or parabolic type degenerated at the origin in infinite order, and hypoelliptic in Ω (see Fediĭ [2]).

Proposition 4.1 (i). *$\mathcal{A}_1$ violates the condition* (1.2) *with $\omega = \Omega$, $\sigma = 1$ and $m = 2$, if $\gamma \geqq 1$.*
(ii). *$\mathcal{A}_2$ ($\mathcal{A}_3$) violates the condition* (1.3) *with $\omega = \Omega$, $\sigma = 1$ and $m = 2$, if $\gamma > 2$ ($\gamma > 1$ respectively).*

Therefore, none of $D_{x_3}^2 + \mathcal{A}_j$ ($j = 1$, 2, or 3) is hypoelliptic in any neiborhood of the origin in $\boldsymbol{R}^3$.

Remark. $\mathcal{A}_1$ is formally selfadjoint. $\mathcal{A}_2$ and $\mathcal{A}_3$ have maximally accretive realizations in $L^2(\Omega)$.

Proof. Let us verify the assertion (i). The proof of (ii) is parallel to that of (i).

First, let h_t ($t \geqq 1$) be the family of linear transform in $\boldsymbol{R}^2$ defined by

$$h_t x = (tx_1, \exp(t^\gamma)x_2) \qquad \text{for } x = (x_1, x_2) \in \boldsymbol{R}^2.$$

Furthermore, we put $\Delta_t = \det h_t = t\exp(t^\gamma)$ and denote

$$\omega_R = \{(x_1, x_2) \in \boldsymbol{R}^2;\ |x| < R\} \qquad \text{for given } R > 0.$$

Notice that $u \circ h_t(x) = u(h_t x) \in C_0^\infty(\omega_R)$ for any $u \in C_0^\infty(\omega_R)$. So, we shall prove the assertion (i), by showing that the growth order of $||u \circ h_t||_{\log,2}^2$ as $t \to \infty$ is not smaller than that of $||\mathcal{A}_1(u \circ h_t)||^2 + ||u \circ h_t||^2$.

It is easy to see that

$$\Delta_t ||u \circ h_t||^2 = ||u||^2. \tag{4.1}$$

Now, let us denote $h_t D = (tD_{x_1}, \exp(t^\gamma)D_{x_2})$.

Then,

$$\mathcal{A}_1(u \circ h_t) \circ h_t^{-1}(x) = \mathcal{A}_1(h_t^{-1}x, h_t D)u(x)$$

and

$$\Delta_t ||\mathcal{A}_1(u \circ h_t)||^2 = ||\mathcal{A}_1(h_t^{-1}x, h_t D)u||^2. \tag{4.2}$$

On the other hand, since the total symbol $\mathcal{A}_1(x, \xi)$ of $\mathcal{A}_1$ is equal to

$$\xi_1^2 + \phi(x)\xi_2^2 - i\phi_{x_2}(x)\xi_2 \quad (\phi(x) = \phi(x_1, x_2)),$$

it is clear that, for $x\in\omega_R$,

$$|\mathcal{A}_1(x, D)u(x)|\leqq|\partial_{x_1}^2u(x)|+\phi(x)|\partial_{x_2}^2u(x)|+C\sqrt{\phi(x)}|\partial_{x_2}u(x)|\ ,$$

where $C=\sqrt{2}\sup_{x\in\omega_R}|\partial_{x_2}^2\phi(x)|^{1/2}$.

So, noting that $h_t^{-1}\omega_R$ is contained in ω_R if $t\geqq 1$, we obtain the following inequality: For $x\in\omega_R$,

$$|\mathcal{A}_1(h_t^{-1}x, h_tD)u(x)|\leqq t^2|\partial_{x_1}^2u(x)|+\phi(t, x)|\partial_{x_2}^2u(x)|+C\sqrt{\phi(t, x)}|\partial_{x_2}u(x)|\ ,$$

where we denote $\phi(t, x)=\exp(2t^{\gamma})\phi(h_t^{-1}x)=\exp(2t^{\gamma}-1/|h_t^{-2}x|^{\gamma})$.

In the last inequality, it is easy to show that $\phi(t, x)$ is uniformly bounded in ω_R as $t\to\infty$, if we take $R>0$ small enough. Thus, returning to (4.2), we obtain

$$\Delta_t\|\mathcal{A}_1(u\circ h_t)\|^2 = O(t^4) \quad \text{as } t\to\infty\ . \tag{4.3}$$

Concerning the left hand side of (1.2), we can write

$$\|u\circ h_t\|_{\log,2}^2 = \int\{\log(2+|h_t\xi|^2)\}^4|\hat{u}(\xi)|^2d\xi\times(\Delta_t)^{-1}\ .$$

Hence, Fatou's lemma yields that

$$\begin{aligned}(4.4)\quad &\liminf_{t\to\infty} t^{-4\gamma}\Delta_t\|u\circ h_t\|_{\log,2}^2\\ &=\liminf_{t\to\infty}\int\{t^{-\gamma}\log(2+|h_t\xi|^2)\}^4|\hat{u}(\xi)|^2d\xi\geqq\int a(\xi)|\hat{u}(\xi)|^2d\xi>0\ ,\end{aligned}$$

where $a(\xi)=0$ if $\xi_2=0$, and $a(\xi)=16$ if $\xi_2\neq 0$.

Now, we can see that the condition (1.2) with $\sigma=1$ and $m=2$ contradicts (4.1), (4.3) and (4.4), if $\gamma\geqq 1$. This completes the proof of (i). Q.E.D.

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