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On well-posedness of the Cauchy problem for p-parabolic systems, II

By

Ahmed EL-FIKY*)

§1. Introduction.

Let $A(x,D)$ be a matrix of pseudo-differential operator of order p in the form

$$A(x, D)=H(x, D)\Lambda^{p}+B(x, D), \qquad x\in R^{l}, \tag{1.1}$$

where $H(x, \xi)$ is $m\times m$ homogeneous matrix of degree 0 in $\xi(|\xi|\geqq 1)$ and smooth in x and ξ. $B(x, \xi)$ belongs to the class $S_{1,0}^{p_0}$, $0\leqq p_0<p$, modulo smoothing operators. Here, the symbol of Λ belongs to $S_{1,0}^{1}$ (see for example, H. Kumano-go [2]) and coincides with $|\xi|$ for $|\xi|\geqq 1$ and p is a positive number.

The purpose of this paper is to show that the condition

$$\sup_{x\in R^{l},\ \xi\in S_{\xi}^{l-1}} \mathrm{Re}\ \lambda_i(x, \xi)<0, \qquad 1\leqq {}^{\forall}i\leqq m \tag{1.2}$$

is necessary and sufficient in order that there exist positive constants a, b and β such that the estimate

$$\|(\lambda I-A(x, D))U(x)\|\geqq a(|\lambda|-\beta_0)\|U\|+b\|U\|_p, \qquad \text{for } {}^{\forall}U\in H^{p},\ {}^{\forall}\lambda,\ \mathrm{Re}\ \lambda\geqq\beta_0 \tag{1.3}$$

holds.

Here $U(x)$ is m-vector, $\|\cdot\|$, $\|\cdot\|_p$ denote L^2 and H^p-norm respectively. $\lambda_i(x, \xi)$, $(i=1, 2,\dots m)$ are the roots of the characteristic equation

$$\det\ (\lambda I-H(x, \xi))=0.$$

Note that the sufficiency was proved in [1] by using a partition of unity of the unit sphere S_{ξ}^{l-1} and a partition of unity in R_x^l as in Mizohata [3]. Therefore, we need only to show the necessity of the condition (1.2).

In this article we shall use the method of micro-localization of pseudo-differential operators which was developed by Mizohata [4] and [5]. In §2. we give the definition of micro-localizer and state our result. In §3. we give the proofs of the proposition 2.1 and lemma 2.1.

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§2. Statement of the result.

In this section we give the definitions of the micro-localizer $\alpha_n(D)\ \beta(x)$ and state our propositions and lemmas.

The following definitions are due to Mizohata [4] and [5].

Definition 2.1.

Let $(x_0,\ \xi^0)\in R^l\times R^l/0$ and $|\xi^0|=1$. Let $\alpha(\xi)\in C_0^\infty$, $0\leqq\alpha(\xi)\leqq 1$, $=1$ on $\{\xi,\ |\xi-\xi^0|\leqq r_0 2\}$ and $=0$ on $\{\xi,\ |\xi-\xi^0|\geqq r_0\}$, $r_0<1$. Put

$$\alpha_n(\xi)=\alpha\left(\frac{\xi}{n}\right). \tag{2.1}$$

We note that

$$\left\{\begin{array}{ll}\text{i)} & \alpha_n(\xi) \text{ has its support in } \{\xi,\ |\xi-n\xi_0\}|\leqq nr_0\}, \text{ and}=1\\ & \text{on}\{\xi,\ |\xi-n\xi^0|\leqq nr_0/2\}.\\ \text{ii)} & |\alpha_n^{(\mu)}(\xi)|\leqq c_{(\mu)}/n^{|\mu|}, \quad \text{for } \mu\geqq 0.\end{array}\right. \tag{2.2}$$

Next, $\beta(x)\in C_0^\infty$, $=1$ on $\{x,\ |x-x_0|\leqq r_0/2\}$,

and $=0$ on $\{x,\ |x-x_0|\geqq r_0\}$.

Notice that r_0 is usually chosen small and we call it the size of micro-localizer.

Assume that the condition (1.2) is violated, namely for any given $\varepsilon(>0)$ small, there exist $(x_0,\ \xi^0)$, $\xi^0(\in R^l,\ |\xi^0|=1)$ and one of the characteristic roots, say $\lambda_1(x^0,\ \xi_0)$, such that

$$\text{Re}\ \lambda_1(x_0,\ \xi^0)\geqq-\varepsilon. \tag{2.3}$$

Let $c={}^t(c_1,\ c_2,\dots,\ c_m)$ be an eigen-vector corresponding to $\lambda_1\ (x_0,\ \xi^0)$, then

$$H(x_0,\ \xi^0)\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}=\lambda_1(x_0,\ \xi^0)\begin{bmatrix}c_1\\ \vdots\\ c_m\end{bmatrix},\ \sum_{j=1}^m|c_j|^2=1. \tag{2.4}$$

Now, consider the sequence

$$U_n(x)=\alpha_n(D)\beta(x)\tilde{\varphi}(x)\begin{bmatrix}c_1\\ \vdots\\ c_m\end{bmatrix}, \tag{2.5}$$

where $\alpha_n(D)\beta(x)$ is the micro-localizer which was defined above and $\tilde{\varphi}_n(x)$ is defined as follows;

let $\varphi(\xi)$ be a function with support in $|\xi|\leqq 1$, and

$\int|\varphi(\xi)|^2 d\xi=1$. Then putting $\quad \varphi_n(\xi)=\varphi(\xi-n\xi^0)$, we define

$$\tilde{\varphi}_n(x)=F[\varphi_n(\xi)]=(2\pi)^{-l}\int e^{ix\xi}\varphi_n(\xi)\mathrm{d}\xi$$

$$=(2\pi)^{-l}\int e^{ix\xi}\varphi(\xi-n\xi^0)\mathrm{d}\xi=e^{inx\xi_0}\tilde{\varphi}(x),$$

where

$$\tilde{\varphi}(x)=(2\pi)^{-l}\int e^{ix\xi}\varphi(\xi)\mathrm{d}\xi. \tag{2.6}$$

Hereafter according to U_n defined by (1.5), we take λ in (1.3) defined by

$$\lambda_n=\beta_0+\varepsilon n^p+\lambda_1(x_0,\ \xi^0)n^p. \tag{2.7}$$

Let us notice that it holds

$$\mathrm{Re}\ \lambda_n\geqq\beta_0>0.$$

(1.3), (2.5) and (2.7) imply

$$\|(\lambda_n I-A(x,\ D)U_n(x)\|\geqq b\|U_n(x)\|_p,\qquad n=1,\ 2,\ldots \tag{1.3}'$$

On the other hand, we can show that the estimate (1.3)′ fails to hold, by taking $\varepsilon=b/4$.

Now we consider

$$(\lambda_n I-H(x_0,\ \xi^0)\Lambda^p)U_n(x) \tag{2.8}$$

$$=(\lambda_n-\lambda_1(x_0,\ \xi^0)\Lambda^p)\alpha_n(D)\beta(x)\tilde{\varphi}_n(x)\begin{bmatrix}c_1\\ \vdots\\ c_m\end{bmatrix}.$$

Then, we state

Lemma 2.1. Put $\lambda_n=\beta_0+\frac{b}{4}n^p+\lambda_1(x_0,\ \xi^0)n^p$, *then we have*

$$\|(\lambda_n I-H(x_0,\ \xi^0)\Lambda^p)U_n(x)\|$$

$$\leqq\left(2\beta_0+\frac{b}{2}n^p+cr_0n^p\right)\|\beta(x)\tilde{\varphi}(x)\|,$$

where c is a positive constant independent of n and r_0.

(see §3. for the proof). Next we consider

$$(H(x,\ D)-H(x_0,\ \xi_0))\Lambda^p\alpha_n(D)\beta(x)\tilde{\varphi}_n(x)\begin{bmatrix}c_1\\ \vdots\\ c_m\end{bmatrix} \tag{2.9}$$

Now we micro-localize the symbol $H(x,\ \xi)$. In order to make this article self-contained, we explain it with proofs (see [5]). First, we define a C^∞—function $\tilde{\chi}(x)$,

$x \in R^l$ as follows;

(2.10) $$\tilde{x}(x)=\begin{cases} x & \text{for} |x-x_0| \leqq r_0 \\ x_0 & \text{for} |x-x_0| \geqq 2r_0, \text{ (constant map)}. \end{cases}$$

If $r_0 \leqq |x-x_0| \leqq 2r_0$, then $|\tilde{x}(x)-x_0| \leqq 2r_0$.
Similarly, let $\xi \longmapsto \tilde{\xi}(\xi)$ be a C^∞-mapping satisfying

(2.11) $$\tilde{\xi}(\xi)=\begin{cases} \xi_0 & \text{for} |\xi-\xi_0| \leqq r_0 \\ \xi_0 & \text{for} |\xi-\xi^0| \geqq 2r_0, \text{ (constant map)}. \end{cases}$$

If $r_0 \leqq |\xi-\xi_0| \leqq 2r_0$, then $|\tilde{\xi}(\xi)-\xi^0| \leqq 2r_0$.
Putting

$$\tilde{\xi}_n(\xi)=n\tilde{\xi}(\xi/n),$$

we localize $H(x, \xi)$ in the following way

(2.12) $$H_{n,loc}(x, \xi)=H(\tilde{x}(x), \tilde{\xi}_n(\xi)).$$

By using (2.10) and (2.11), we see easily that

1) $H_{n,loc}(x, \xi)=H(x, \xi)$ for $|x-x_0| \leqq r_0$ and $|\xi-n\xi_0| \leqq nr_0$

2) $H_{n,loc}(x_0, \xi)=H(x^0, n\xi)$ for $|x-x_0| \geqq 2r_0$ and $|\xi-n\xi^0| \geqq 2nr_0$.

3) $|\text{entry of } (H_{n,loc}(x, \xi)-H(x_0, n\xi^0))| \leqq \text{const } r_0$

where const. is independent of r_0 and n.

With these preparations (2.9) becomes

(2.9)′ $$(H_{n,loc}(x, D)-H(x_0, \xi^0))\Lambda^p\alpha_n(D)\beta(x)\tilde{\varphi}_n(x)\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

$$+H(x, D)-H_{n,loc}(x, D))\Lambda^p\alpha_n(D)\beta(x)\tilde{\varphi}(x)\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}.$$

Before we state our propositions, we introduce a convenient terminology.

Definition. We say a sequence of operators $a_n(x, D)$, is negligible if for any large L, $\|a(x, D)\|_{L(L^2, L^2)}$ is estimated by $C_L n^{-L}$ when $n \to \infty$.

Proposition 2.1

$$(H(x, D)-H_{n,loc}(x, D))\Lambda^p\alpha_n(D)\beta(x)\tilde{\varphi}_n(x)\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \textit{ is negligible.}$$

(see §3 for the proof).
Next, by virtue of sharp Gårding inequality, we have

Proposition 2.2 *Let $p>0$, then we obtain*

$$\|(H_{n,loc}(x, D)-H(x_0, \xi^0))\Lambda^p\alpha_n(D)\beta(x)\tilde{\varphi}(x)\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}\|$$

$$\leqq c'\cdot r_0 n^p\|\beta(x)\tilde{\varphi}(x)\|+\tilde{c}n^{p-1/2}\|B(x)\tilde{\varphi}(x)\|,$$

where c' and $\tilde{c}$ are positive constants independent of r_0 and n.

For $B(x, D)\alpha_n(D)\beta(x)\tilde{\varphi}_n(x)\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$, by virtue of Calderón-Vaillancourt theorem,

we get

Lemma 2.2 *Let $B(x, \xi)\in S^{p_0}_{1,0}$, $0\leqq p_0<p$, then we obtain*

$$\|B(x, D)\alpha_n(D)\beta(x)\tilde{\varphi}(x)\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}\|\leqq const. \quad n^{p_0}\|\beta(x)\tilde{\varphi}(x)\|,$$

where const. is independent of n and r_0.

From these Lemmas and Propositions, we obtain

$$\|(\lambda_n I-A(x, D))U_n(x)\|\leqq(b/2+\text{const. } r_0)n^p\|\beta(x)\tilde{\varphi}(x)\|, \tag{2.13}$$

if n is large, where const. is independent of n and r_0.
On the other hand, we consider

$$\|U_n(x)\|_p=(\sum_{j=1}^m\|\langle\Lambda\rangle^p\alpha_n(D)\beta(x)\tilde{\varphi}(x)c_j\|^2)^{1/2}$$

$$=\langle\Lambda\rangle^p\alpha_n(D)\beta(x)\tilde{\varphi}_n(x)\|,$$

where $\widehat{\langle\Lambda\rangle u}(\xi)=(1+|\xi|^2)^{1/2}\hat{u}(\xi)$.

Since $(1+|\xi|^2)^{p/2}\geq|\xi|^p\geqq(1-r_0)^p n^p$, for $\xi\in\text{supp}(\alpha_n(\xi))$,

we obtain

$$\|\langle\Lambda\rangle^p\alpha_n(D)\beta(x)\tilde{\varphi}_n(x)\|\geqq(1-r_0)^p n^p\|\alpha_n(D)\beta(x)\tilde{\varphi}(x)\|.$$

Now, by commuting $\alpha_n(D)$ with $\beta(x)$, we get

$$\alpha_n(D)\beta(x)\tilde{\varphi}(x)=\beta(x)\alpha_n(D)\tilde{\varphi}(x) \tag{2.14}$$

$$+\sum_{1\leq|\nu|N}\nu!^{-1}\beta_{(\nu)}(x)\alpha_n^{(\nu)}(D)\tilde{\varphi}_n(x)+r_N(x, D; n)\tilde{\varphi}(x).$$

Here $\alpha_n(D)\tilde{\varphi}_n(x)=\tilde{\varphi}_n(x)$, since $\alpha_n(\xi)=1$ for $\xi\in\text{supp}(\psi_n(\xi))$.

Hence,

$$\beta(x)\alpha_n(D)\tilde{\varphi}_n(x)=\beta(x)\tilde{\varphi}_n(x)=e^{inx\xi^0}\beta(x)\tilde{\varphi}(x),$$

and its L^2-norm is $\|\beta(x)\psi(x)\|$.

Taking into account that $\alpha_n^{(\nu)}(\xi)\psi_n(\xi)=0$, for $|\nu|\geqq 1$, we see that all terms of the second part of the right-hand side of (2.14) are all zero. Therefore, it suffices to consider the remainder term.

From (2.14), we have

$$r_N(x,\ \xi,\ n)=(N+1)\int_0^1(1-\theta)^N r_{N,\theta}(x,\ \xi,\ n)d\theta,$$

(2.15)
$$r_{N,\theta}(x,\ \xi,\ n)$$
$$=\sum_{|\nu|=N+1}\nu!^{-l}(2\pi)^{-l}\iint e^{-iy\eta}\alpha_n^{(\nu)}(\xi+\eta)\beta_{(\nu)}(x+\theta y)dyd\eta.$$

Put

(2.16)
$$I(x,\ \xi,\ \theta,\ \eta)=\iint e^{-iy\eta}\alpha_n^{(\nu)}(\xi+\eta)\beta_{(\nu)}(x+\theta y)dyd\eta,$$

then by integration by parts, we obtain

$$I(x,\ \xi,\ \theta,\ \eta)=\iint e^{-iy\eta}\frac{(1-\Delta_n)^l((\alpha_n^{(\nu)}(\xi+\eta))}{(1+|\eta|^2)^l}(1-\Delta_y)^y\Big(\frac{(\beta_{(\nu)}(x+\theta y)}{(1+|y|^2)^l}\Big)dyd\eta$$

Since $|\alpha_n^{(\nu)}(\xi)|\leqq c_{(\nu)}/n^{|\nu|}$, for $\nu\geq 0$

and $\text{supp}(\alpha_n(\xi))\subset\{\xi;\ |\xi-n\xi^0|\leqq nr_0.\}$

we obtain

$$|(1-\Delta_n)^l\alpha_n^{(\nu)}(\xi+\eta)|\leqq c^1(\nu)/n^{|\nu|},$$

where c^1 is a constant independent of n. So that,

$$|I(x,\ \xi,\ \theta,\ n)|\leqq\text{const.}\cdot n^{-N-1},$$

where const. is independent of θ and n. We have the same type inequality for $\partial_\xi^s\partial_x^q I(x,\ \xi,\ \theta,\ n)$:

:
$$|\partial_\xi^s\partial_x^q I(x,\ \xi,\ \theta\ ,n)|\leqq\text{const.}\cdot n.\ ^{-N-1}$$

Thus we have

$$|r_N(x,\ \xi,\ n)|\leqq\text{const.}\cdot n^{-N-1},$$

and

$$|\partial_\xi^s\partial_x^q r_N(x,\ \xi,\ n)|\leqq\text{const.}\cdot n^{-N-1},$$

By applying Calderón-vaillancourt theorem to $r_N(x,\ D,\ n)$, we obtain

(2.17) $$\|r_N(x, D, n)\|_{\mathscr{L}(L^2,L^2)} \leqq \text{const.}\cdot n^{-N-1},$$

where const. is independent of n.
Summing up the above results, we obtain

(2.18) $$\|U_n(x)\| \geqq (1\text{-}r_0)^p n^p \|\beta(x)\tilde{\varphi}(x)\| - (\text{negligible terms}).$$

By taking r_0 small, (2.13) and (2.18) shows that the estimate (1.2) fails to hold. Thus the proof is complete.

§3. Proofs of Lemma 2.1 and Proposition 2.1.

Here we give the proofs of lemma 2.1 and proposition 2.1 which are used in §2.

Proof of Lemma 2.1. First, denote $\beta(x)\tilde{\varphi}(x)$ by $v_n(x)$, and take into account of (2.4) and (2.5), then we have

$$\begin{aligned}&\|(\lambda_n I - H(x_0, \xi^0)\Lambda^p) U_n(x)\| \\ &\quad = \|(\lambda_n - \lambda_1(x_0, \xi^0)\Lambda^p)\alpha_n(D) v_n(x)| \\ &\quad = \|(\lambda_n - \lambda_1(x_0, \xi^0)|\xi|^p)\alpha_n(\xi)\hat{v}_n(\xi)\|.\end{aligned}$$

Next, since $\lambda_n = \beta_0 + \frac{b}{4}n^p + \lambda_1(x_0, \xi^0)n^p$, we obtain

$$\begin{aligned}&\|(\lambda_n - \lambda_1(x_0, \xi^0)|\xi|^p)\alpha_n(\xi)\ddot{v}_n(\xi)\| \\ &\quad \leqq \beta_0 + \frac{b}{4}n^p + |\lambda_1(x^0, \xi^0)| \sup_{|\xi - n\xi^0| \leq nr_0} (|\ |\xi|^p - n\xi^0|^p|) \|\alpha_n v_n\|\end{aligned}$$

By using Mean-value theorem, we get

$$\sup_{|\xi - n\xi^0| \leq nu_0} (|\ |\xi|^p - |n\xi^0|^p\ |) \leqq \text{const.}\cdot r_0 \cdot n^p,$$

where const. depends only on p.
Hence

$$\begin{aligned}&\|(\lambda_n - \lambda_1(x_0, \xi^0)\Lambda^p)\alpha_n(D)\beta(x)\tilde{\varphi}_n(x)\| \\ &\quad \leqq \left(\beta_0 + \frac{b}{4}n + c\cdot r_0 n^p\right)\|\alpha_n(D)\beta(x)\tilde{\varphi}_n(x)\|.\end{aligned}$$

Since

$$\|\alpha_n(D)\beta(x)\tilde{\varphi}_n(x)\| \leqq \|\beta(x)\tilde{\varphi}(x)\| + (\text{negligible}),$$

we obtain

$$\begin{aligned}&\|(\lambda_n - \lambda_1(x_0, \xi^0)\Lambda^p)\alpha_n(D)\beta(x)\tilde{\varphi}_n(x)\| \\ &\quad \leqq \left(2\beta_0 + \frac{b}{2}n^p + 2cn^p\right)\|\beta(x)\tilde{\varphi}(x)\|.\end{aligned}$$

Thus the proof is complete.

Proof of Proposition 2.1 Put

$$H(x, \xi)-H_{n,loc}(x, \xi)=H'(x, \xi)$$

Then $H'(x, \xi)=0$ for $\{x; |x-x_0|\leqq r_0\}$ and $\{\xi; |\xi-n\xi^0|\leqq nr_0\}$.
Hence, for $x\in\mathrm{supp}(\beta(x))$, $H'(x, \xi)$ vanishes. Now, considering the asympototic expression of the commutator $[H'\Lambda^p\alpha_n, \beta(x)]$,

$$(3.1)\qquad (H'(x, D)\Lambda^p\alpha_n(D))\beta(x)$$
$$=\sum_{|\nu|\leq N}\nu!^{-1}\beta_{(\nu)}(x)(H'(x, D)\Lambda^p\alpha_n(D))^{(\nu)}+r'_N(x, D, n),$$

we see that all terms of the first part of the right-hand side of (3.1) are zero operator. So it suffice to consider the remainder term.
By using the same argument as we used in §2, together with the properties

$$|(H'(x, \xi)|\xi|^p)^{(\nu)}|\leqq c|\xi|^{p-|\nu|}$$

and

$$|\alpha_n^{(\nu)}(\xi)|\leqq c'_\nu n^{-|\nu|}\qquad \nu\geqq 0,$$

We see that

$$\|r'_N(x, D, n)\|_{\mathscr{L}(L^2, L^2)}\leqq c_N n^{p-N-1}.$$

Thus the proof is complete.

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DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ALEXANDRIA UNIVERSITY, EGYPT.

References

[1] A. El-Fiky, On well-posedness of the Cauchy problem for p-parabolic systems I, to appear in J. Math. Kyoto Univ.

[2] H. Kumano-go, Pseudo-differential operators, M.I.T. Press, 1981.

[3] S. Mizohata, Le Problème de Cauchy pour les systèmes hyperboliques et paraboliques, Mem. Coll. Sci. Univ. Kyoto, ser A. **32-2** (1959), 181–212.

[4] S. Mizohata, On the Cauchy problem, Academic Press (Notes and Reports in Mathematics in Science and Engineering, vol. 3), and Science Press (Peking, China), 1986.

[5] S. Mizohata, Some comments on micro-localizer, (unpublished note).