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On the positivity of the fundamental solutions for parabolic systems

By

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§1. Introduction and results.

It is well known that the bounded solution $u(t, x)$ of the heat equation

$$\begin{cases} \left(\dfrac{\partial}{\partial t}-\triangle\right)u(t, x)=0, \qquad 0<t\leqq T(<\infty),\ x\in\mathbf{R}^n, \\ u(0, x)=u_0(x) \end{cases}$$

is nonnegative if so is the initial data $u_0(x)$ in $C_0^\infty(\mathbf{R}^n)$. This follows from the fact that $u(t, x)$ is given by

$$u(t, x)=\int E(t, x-y)u_0(y)dy,$$

where $E(t, x)=(4\pi t)^{-n/2}\exp(-|x|^2/4t)$ is the fundamental solution of the heat equation.

It is shown in Kimura-Otsuka [3] that this property never holds for the single equation of higher order in x with constant coefficients.

In this paper we will deal with parabolic systems of the first order in t, and give a necessary and sufficient condition for the positivity in this sense (see Theorem 1.2 and remarks below).

Let us consider the Cauchy problem

$$\begin{cases} Lu\equiv\dfrac{\partial}{\partial t}u(t, x)-\sum_{|\alpha|\leqq 2m}A_\alpha(t, x)\left(\dfrac{\partial}{\partial x}\right)^\alpha u(t, x)=0, \qquad t_0<t\leqq T,\ x\in\mathbf{R}^n, & (1.1) \\ u(t_0, x)=u_0(x), & (1.2) \end{cases}$$

where $u(t, x)={}^t(u_1(t, x), u_2(t, x),\dots, u_N(t, x))$ and $u_0(x)={}^t(u_{0,1}(x),\dots, u_{0,N}(x))$ are N-dimensional column vectors and

$$A_\alpha(t, x)=(a_{\alpha,j,k}(t, x))_{1\leqq j,k\leqq N} \tag{1.3}$$

are $N\times N$ matrices. We assume that:

(i) *$a_{\alpha,j,k}(t, x)$ are real-valued, bounded functions which are uniformly continuous in*

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(t, x) and uniformly Hölder continuous in x with exponent σ.

(ii) *L is parabolic, that is, there exists a positive constant δ such that the real part of each solution $\lambda_j(t, x; i\xi)$ of*

$$\det[\lambda I - A_{2m}(t, x; i\xi)] = 0 \tag{1.4}$$

is smaller than $-\delta|\xi|^{2m}$ for any $(t, x, \xi) \in [0, T] \times \mathbf{R}^n \times \mathbf{R}^n$, where

$$A_p(t, x; i\xi) = \sum_{|\alpha|=p} A_\alpha(t, x)(i\xi)^\alpha. \tag{1.5}$$

Under these conditions, Eidel'man [2] constructed the fundamental solution $E(t, s, x, y) = (e_{j,k}(t, s, x, y))_{1 \leqq j,k \leqq N}$ of the Cauchy problem:

$$\begin{cases} L_{t,x}E(t, s, x, y) = O, & (0 \leqq) s < t \leqq T(<\infty),\ x, y \in \mathbf{R}^n \\ E(s+0, s, x, y) = \delta(x-y)I, \end{cases}$$

where I is the $N \times N$ unit matrix. Then he showed the following theorem on the wellposedness of the Cauchy problem.

Theorem 1.1 (see Eidel'man [2] Chap. 3 Theorem 5.3). *For any $u_0(x)$ in $C^{(2m,\sigma)}(\mathbf{R}^n)$ and $0 \leqq t_0 < T$ there exists a unique solution $u(t, x)$ of (1.1)–(1.2) in $C^{(2m,\sigma,0)}([t_0, T] \times \mathbf{R}^n)$, and this solution is given by*

$$u(t, x) = \int E(t, t_0, x, y) u_0(y) dy. \tag{1.6}$$

Here $C^{(2m,\sigma)}(\mathbf{R}^n)$ is the space of all functions defined on $\mathbf{R}^n$ which have bounded and uniformly Hölder continuous derivatives with exponent σ up to order $2m$. And $C^{(2m,\sigma,0)}([t_0, T] \times \mathbf{R}^n)$ is the space of all functions defined on $[t_0, T] \times \mathbf{R}^n$ whose x-derivatives up to order $2m$ are bounded, uniformly continuous in (t, x) and uniformly Hölder continuous in x with exponent σ.

Now let us define the positivity of the operator L.

Definition 1.1. *We say that the operator L has the positivity if it has the following property*:

$$\begin{cases} \text{If } u_{0,j}(x) \geqq 0 \text{ for any } 1 \leqq j \leqq N, \text{ then } u_j(t, x) \geqq 0 \text{ for any} \\ 1 \leqq j \leqq N,\ t_0 \leqq t \leqq T \text{ and } x \in \mathbf{R}^n, \text{ where } u(t, x) \text{ is the unique} \\ \text{solution of (1.1)–(1.2) and } 0 \leqq t_0 < T. \end{cases} \tag{1.7}$$

This is equivalent to the positivity of each element of the fundamental solution:

Definition 1.2. *We say that the fundamental solution $E(t, s, x, y)$ has the positivity if*

$$e_{j,k}(t, s, x, y) \geqq 0 \text{ for } 0 \leqq s < t \leqq T,\ x, y \in \mathbf{R}^n,\ 1 \leqq j, k \leqq N. \tag{1.8}$$

Then our result is the following.

Theorem 1.2. *The fundamental solution has the positivity if and only if*

(i) $m=1$,

(ii) $A_\alpha(t, x)$ *is diagonal if* $|\alpha|\geqq 1$,

and (iii) $a_{0,j,k}(t, x)\geqq 0$ *if* $j\neq k$.

Remarks. (i) The maximum principle holds for parabolic systems satisfying these conditions (see Protter-Weinberger [6]).

(ii) In Miyajima-Okazawa [5] and Miyajima [4], similar results are obtained for single equations with coefficients independent of t.

(iii) For single equations of the second order, we have the following estimate from below:

Theorem 1.3 (see Aronson [1] for example). *There exist some positive constants* ε_0 *and* δ_0 *such that*

$$e(t, s, x, y)\geqq \delta_0(t-s)^{-n/2}\exp\left(-\varepsilon_0\frac{|x-y|^2}{t-s}\right)$$

for any $0\leqq s<t\leqq T(<\infty)$, $x, y\in\mathbf{R}^n$.

(iv) It is easily seen that in the proofs below we can replace (1.8) by the weaker condition as follows:

$$(1.9)\qquad \begin{cases} \textit{For any } (s, y) \textit{ in } [0, T)\times\mathbf{R}^n, \textit{ there exists a} \\ \textit{positive constant } r_0 \textit{ such that} \\ e_{j,k}(t, s, x, y)\geqq 0 \text{ if } s<t<s+r_0 \text{ and } |x-y|<r_0. \end{cases}$$

Thus, if (1.9) holds, then conditions (i)–(iii) of Theorem 1.2 hold. So (1.8) also holds by this theorem. That is, condition (1.8) is equivalent to (1.9).

We will prove Theorem 1.2 in several steps as following.

Proposition 1.1. *If* (1.8) *holds, then* $m=1$ *and the principal part* $A_2(t, x; i\xi)$ *is diagonal.*

Proposition 1.2. *Assume that* $m=1$ *and* $A_2(t, x; i\xi)$ *is diagonal. If* (1.8) *holds, then the first order term* $A_1(t, x; i\xi)$ *is also diagonal.*

Proposition 1.3. *Assume that* $m=1$ *and* $A_p(t, x; i\xi)$ $(p=1, 2)$ *are diagonal. If* (1.8) *holds, then* $a_{0,j,k}(t, x)\geqq 0$ *unless* $j=k$.

Proposition 1.4. *If conditions* (i), (ii) *and* (iii) *of Theorem* 1.2 *are fulfilled, then* (1.8) *holds.*

We will prove Proposition 1.1 in §2, Proposition 1.2 in §3, and Propositions 1.3 and 1.4 in §4.

Notations. In the following, we will often denote various constants by the same letter such as C, while the constant c will remain unchanged throughout this paper.

The integration with no indication of the domain will be extended over the whole space $\mathbf{R}^n$, and $(2\pi)^{-n}d\xi$ will be abbreviated as $đ\xi$.

If a capital letter such as A or $A_\alpha(x)$ denotes a matrix, then its matrix element will be often denoted by the corresponding small letter with suffix like $a_{j,k}$ or $a_{\alpha,j,k}(x)$. And $|A|$ will denote $\max\limits_{j,k} |a_{j,k}|$.

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§2. Proof of Proposition 1.1.

Let $E(t,s,x,y)$ be the fundamental solution of Eidel'man for the system (1.1):

$$E(t, s, x, y)=E_0(t, s, x, y)+E_1(t, s, x, y).$$

Here

$$E_0(t, s, x, y)=G(t, s, x-y, y),$$

$$G(t, s, x, y)=\int \exp(ix\xi)Q(t, s, \xi, y)đ\xi, \qquad đ\xi=(2\pi)^{-n}d\xi$$

and $Q(t, s, \xi, y)$ is the solution of the ordinary differential equation

$$\text{(2.1)} \qquad \begin{cases} \dfrac{d}{dt}Q(t, s, \xi, y)=A_{2m}(t, y;\ i\xi)Q(t, s, \xi, y), & s<t<T, \\ Q(s, s, \xi, y)=I. \end{cases}$$

The second term $E_1(t, s, x, y)$ is obtained as

$$E_1(t, s, x, y)=\int_s^t d\tau\int E_0(t, \tau, x, z)\Phi(\tau, s, z, y)dz,$$

where $\Phi(t, s, x, y)$ is the solution of the integral equation

$$\Phi(t, s, x, y)-\int_s^t d\tau\int H(t, \tau, x, z)\Phi(\tau, s, z, y)dz=H(t, s, x, y)$$

with

$$\begin{aligned} H(t, s, x, y)&=-LE_0(t, s, x, y) \\ &=\left\{A_{2m}\left(t, x;\ \frac{\partial}{\partial x}\right)-A_{2m}\left(t, y;\ \frac{\partial}{\partial x}\right)\right\}E_0(t, s, x, y) \\ &\quad+\sum_{p=0}^{2m-1}A_p\left(t, x;\ \frac{\partial}{\partial x}\right)E_0(t, s, x, y). \end{aligned}$$

This integral equation is solved by successive approximation (cf. $\tilde{\Phi}(t, s, x, y)$ in the next section). According to Eidel'man [2], we have the following estimates.

Lemma 2.1 (see [2] Chap. 1 §3).

$$\text{(i)} \qquad |Q(t, s, \xi, y)|\leqq C\exp(-c_0(t-s)|\xi|^{2m}),$$

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} E_0(t, s, x, y)\right| \leqq C(t-s)^{-(n+|\alpha|)/2m} \exp(-c\rho(t-s, x-y))$$

for $|\alpha| \leqq 2m$, *where* C, c, c_0 *are positive constants and* $\rho(\tau, z) = (|z|^{2m}/\tau)^{1/(2m-1)}$.

(ii) $$|\Phi(t, s, x, y)| \leqq C(t-s)^{-(n+2m-\sigma)/2m} \exp(-c\rho(t-s, x-y)),$$

$$|E_1(t, s, x, y)| \leqq C(t-s)^{-(n-\sigma)/2m} \exp(-c\rho(t-s, x-y)).$$

Now let us fix any point (t_0, x_0) in $[0, T) \times \mathbf{R}^n$ and put

$$F(t, x) = t^{n/2m} E(t_0 + t, t_0, x_0 + t^{1/2m} x, x_0).$$

Then we have the following proposition.

Proposition 2.1. *As t tends to 0, $F(t, x)$ converges uniformly to the function*

$$F_0(x) = \int e^{ix\xi} \exp(A_{2m}(t_0, x_0; i\xi))\, d\xi.$$

Proof. Without loss of generality, we may assume that $t_0 = 0$ and $x_0 = 0$. Let us put

$$F_j(t, x) = t^{n/2m} E_j(t, 0, t^{1/2m} x, 0), \qquad j = 0, 1.$$

First, from Lemma 2.1 (ii), we have

$$|F_1(t, x)| \leqq Ct^{\sigma/2m} \exp(-c\rho(1, x)) \leqq Ct^{\sigma/2m}.$$

So $F_1(t, x)$ converges uniformly to 0 as $t \longrightarrow +0$.

As for $F_0(t, x)$, we can write as

$$F_0(t, x) = \int e^{ix\xi} Q(t, 0, t^{-1/2m}\xi, 0)\, d\xi.$$

So we have

(2.2) $$|F_0(t, x) - F_0(x)| \leqq \int |Q(t, 0, t^{-1/2m}\xi, 0) - \exp(A_{2m}(0, 0; i\xi))|\, d\xi.$$

Then the proposition follows from the next Lemma 2.2. In fact, from Lemma 2.1 (i) and (2.3), the integrand in the right-hand side of (2.2) is smaller than $C \exp(-c_0|\xi|^{2m})$. So we can apply Lebesgue's dominated convergence theorem to (2.2).

Lemma 2.2.

(2.3) $$\lim_{t \to +0} Q(t, 0, t^{-1/2m}\xi, 0) = \exp(A_{2m}(0, 0; i\xi)).$$

Proof. $Q(t, \xi) \equiv Q(t, 0, \xi, 0)$ is the solution of the integral equation

(2.4) $$Q(t, \xi) = I + \int_0^t A_{2m}(\tau, 0; i\xi) Q(\tau, \xi)\, d\tau.$$

So $Q(t, \xi)$ is constructed in the form

(2.5) $$Q(t, \xi) = \sum_{l=0}^{\infty} Q_l(t, \xi),$$

$$Q_0(t, \xi)=I$$

and

$$Q_l(t, \xi)=\int_0^t A_{2m}(\tau, 0; i\xi)Q_{l-1}(\tau, \xi)\, d\tau$$

$$=t\int_0^1 A_{2m}(tr, 0; i\xi)Q_{l-1}(tr, \xi)\, dr, \qquad l\geqq 1.$$

By induction, we can show that $Q_l(t, \xi)$ is homogeneous in ξ of degree $2ml$, and has the estimate

$$|Q_l(t, \xi)|\leqq(C_0 t|\xi|^{2m})^l/l!,$$

where C_0 is independent of l, t and ξ. So the right-hand side of (2.5) converges uniformly in t and satisfies the equations (2.4) and (2.1). Since

$$Q_l(t, t^{-1/2m}\xi)=\int_0^1 A_{2m}(tr, 0; i\xi)Q_{l-1}(tr, (tr)^{-1/2m}\xi)r^{l-1}dr,$$

we can easily show by induction that

$$|Q_l(t, t^{-1/2m}\xi)|\leqq(C_0|\xi|^{2m})^l/l!$$

and

$$\lim_{t\to+0} Q_l(t, t^{-1/2m}\xi)=(A_{2m}(0, 0; i\xi))^l/l!.$$

Thus $\sum_{l=0}^{\infty} Q_l(t, t^{-1/2m}\xi)$ converges absolutely and uniformly in t, and again by Lebesgue's theorem, (2.3) holds.

Proposition 2.2.

$$\int x^\alpha F_0(x)\, dx=\begin{cases} I & \text{if } \alpha=0, \\ O & \text{if } 0<|\alpha|<2m. \end{cases} \tag{2.6}$$

Proof. In order to simplify the notation, let us assume that $t_0=0$ and $x_0=0$. It follows from Proposition 2.1 that

$$\int x^\alpha F_0(x)dx=\left(i\frac{\partial}{\partial\xi}\right)^\alpha [\exp(A^{2m}(0, 0; i\xi))]|_{\xi=0}. \tag{2.7}$$

By definition, we can write as

$$\exp(A_{2m}(0, 0; i\xi))=I+A_{2m}(0, 0; i\xi)A'(\xi),$$

where $A'(\xi)$ is analytic in ξ. So it is easily shown that the right-hand side of (2.7) is equal to I if $\alpha=0$, and equal to O if $0<|\alpha|<2m$.

Proof of Proposition 1.1. We may assume $t_0=0$ and $x_0=0$. Let us assum that $m\geqq 2$. From Proposition 2.2, we have

$$\int f_{0,j,j}(x)dx=1 \tag{2.7}$$

and

$$\int |x|^2 f_{0,j,j}(x)dx=0$$

for the diagonal elements $f_{0,j,j}(x)$, $1\leqq j\leqq N$. This implies that $f_{0,j,j}(x)$ is somewhere negative. So if t is sufficiently small, $e_{j,j}(t, 0, x, 0)$ is somewhere negative, too. Thus, the condition (1.8) never holds unless $m=1$.

Now let $j\neq k$. Then we have from (2.6) that

$$\int f_{0,j,k}(x)dx=0.$$

On the other hand, $f_{0,j,k}(x)$ is nonnegative if (1.8) holds. So $f_{0,j,k}(x)$ should be identically zero. Thus, from the condition (1.8) it follows that $F_0(x)$ is diagonal, and so is exp $(A_{2m}(0, 0; i\xi))$ for any $\xi\in\mathbf{R}^n$. Then, it is easy to show that $A_{2m}(0, 0; i\xi)$ is also diagonal. In fact, from the formula $\frac{d}{dt}\exp(tA)=A\exp(tA)$, we can write $A_{2m}(0, 0; i\xi)$ as

$$A_{2m}(0, 0; i\xi)=\exp(-tA_{2m}(0, 0; i\xi))\frac{d}{dt}\exp(tA_{2m}(0, 0; i\xi)).$$

But the right-hand side is diagonal because of the homogeneity of $A_{2m}(t, x: i\xi)$ with respect to ξ. Thus $A_{2m}(0, 0; i\xi)$ itself is diagonal and the proposition is proved.

§3. Proof of Proposition 1.2.

In this section we will assume that $m=1$ and the principal part $A_2(t, x; i\xi)$ is diagonal. Let us rewrite the operator L into the following form:

$$L=\frac{\partial}{\partial t}-A\left(t, x; \frac{\partial}{\partial x}\right)-B\left(t, x; \frac{\partial}{\partial x}\right),$$

where

$$A(t, x; i\xi)=\operatorname{diag}[a_1(t, x, i\xi),\dots, a_N(t, x; i\xi)],$$

$$a_j(t, x; i\xi)=\sum_{p=0}^{2}a_{p,j,j}(t, x; i\xi),$$

$$B(t, x; i\xi)=(b_{j,k}(t, x; i\xi))_{1\leqq j,k\leqq N},$$

$$b_{j,k}(t, x; i\xi)=a_{1,j,k}(t, x; i\xi)+a_{0,j,k}(t, x)\qquad \text{if } j\neq k \tag{3.1}$$

and

$$b_{j,j}(t, x; i\xi)\equiv 0,\qquad 1\leqq j\leqq N$$

(see (1.5) for definition of $a_{p,j,k}$). We will construct a fundamental solution $\tilde{E}(t, s, x, y)$ in a slightly different form from that of Eidel'man [2], and then we will show that $\tilde{E}(t, s, x, y)$ coincides with $E(t, s, x, y)$:

$$\tilde{E}(t, s, x, y)=\tilde{E}_0(t, s, x, y)+\tilde{E}_1(t, s, x, y),$$

where

$$\tilde{E}_0(t, s, x, y)=\mathrm{diag}\,[e_1(t, s, x, y),\dots, e_N(t, s, x, y)]$$

is the fundamental solution for the operator $\frac{\partial}{\partial t}-A\left(t, x; \frac{\partial}{\partial x}\right)$. That is,

$$\left(\frac{\partial}{\partial t}-a_j\left(t, x; \frac{\partial}{\partial x}\right)\right)e_j(t, s, x, y)=0, \qquad 0\leqq s<t\leqq T,\ x, y\in\mathbf{R}^n$$

and

$$\lim_{t\to s+0} e_j(t, s, x, y)=\delta(x-y)$$

for each $j=1,\dots, N$. $e_j(t,s,x,y)$ is constructed in Eidel'man [2] (see Introductory section of Chap. 1):

$$e_j(t, s, x, y)=g_j(t, s, x-y, y)+\varphi_j(t, s, x, y), \tag{3.2}$$

where

$$g_j(t, s, x, y)=\int\exp\,[ix\xi+\int_s^t a_{2,j,j}(\tau, y; i\xi)\ d\tau]d\xi, \tag{3.3}$$

and $\varphi_j(t, s, x, y)$ is obtained by an analogous way to $E_1(t, s, x, y)$ discussed in the previous section. Here we have the following estimates.

Lemma 3.1. (i) $$\left|\left(\frac{\partial}{\partial x}\right)^\alpha g_j(t, s, x, y)\right|\leqq C(t-s)^{-(n+|\alpha|)/2}\exp\left(-c\frac{|x|^2}{t-s}\right)$$

if $|\alpha|\leqq 2$.

(ii) $$\left|\left(\frac{\partial}{\partial x}\right)^\alpha \varphi_j(t, s, x, y)\right|\leqq C(t-s)^{-(n+|\alpha|-\sigma)/2}\exp\left(-c\frac{|x-y|^2}{t-s}\right)$$

if $|\alpha|\leqq 1$.

In order that $\tilde{E}(t, s, x, y)$ is a fundamental solution, we will construct $\tilde{E}_1(t, s, x, y)$ in the form:

$$\tilde{E}_1(t, s, x, y)=\int_s^t d\tau\int\tilde{E}_0(t, \tau, x, z)\tilde{\Phi}(\tau, s, z, y)\ dz,$$

where $\tilde{\Phi}(t, s, x, y)$ is the solution of the integral equation

$$\tilde{\Phi}(t, s, x, y)-\int_s^t d\tau\int\tilde{H}(t, \tau, x, z)\tilde{\Phi}(\tau, s, z, y)dz=\tilde{H}(t, s, x, y)$$

with

$$\tilde{H}(t, s, x, y)=-L\tilde{E}_0(t, s, x, y)=B\left(t, x; \frac{\partial}{\partial x}\right)\tilde{E}_0(t, s, x, y).$$

So $\tilde{\Phi}(t, s, x, y)$ is obtained by iteration of $\tilde{H}$:

$$\tilde{\Phi}(t, s, x, y)=\sum_{l=1}^{\infty}\tilde{\Phi}_l(t, s, x, y),$$

where

$$\tilde{\Phi}_1(t, s, x, y)=\tilde{H}(t, s, x, y)$$

and

$$\tilde{\Phi}_l(t, s, x, y)=\int_s^t d\tau\int\tilde{H}(t, \tau, x, z)\tilde{\Phi}_{l-1}(\tau, s, z, y)\ dz$$

for $l\geqq 2$. Then we have the following lemma.

Lemma 3.2. (i) *There exist some constants C and C_0 such that*

$$|\tilde{\Phi}_l(t, s, x, y)|\leqq C\Gamma\left(\frac{l}{2}\right)^{-1}C_0{}^l(t-s)^{(l-n-2)/2}\exp\left(-c\frac{|x-y|^2}{t-s}\right). \tag{3.4}$$

(ii)
$$\tilde{E}(t, s, x, y)\equiv E(t, s, x, y).$$

Proof. (i) It is not difficult to show

$$((t-\tau)(\tau-s))^{-n/2}\int\exp\left(-c\frac{|x-z|^2}{t-\tau}-c\frac{|z-y|^2}{\tau-s}\right)\ dz \tag{3.5}$$
$$=\left(\frac{\pi}{c}\right)^{n/2}(t-s)^{-n/2}\exp\left(-c\frac{|x-y|^2}{t-s}\right)$$

if $s<\tau<t$. So (3.4) follows from the formula

$$\int_s^t(t-\tau)^{-1/2}(\tau-s)^{(l-2)/2}d\tau=\sqrt{\pi}\,\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{l+1}{2}\right)^{-1}(t-s)^{(l-1)/2}.$$

(ii) In the same way as Eidel'man, we can show that this $\tilde{E}(t, s, x, y)$ is a fundamental solution which gives the unique bounded solution for the Cauchy problem (1.1)–(1.2) (see (1.6)). Thus $\tilde{E}(t, s, x, y)$ has to coincide with $E(t, s, x, y)$ discussed in previous sections. We will leave the details to the reader.

Now let us fix (t_0, x_0) in $[0, T)\times\mathbf{R}^n$ and put

$$\tilde{F}(t, x)=t^{(n-1)/2}\tilde{E}_1(t_0+t, t_0, x_0+t^{1/2}x, x_0).$$

Proposition 3.1. *If $j\neq k$, the (j, k)-element $\tilde{f}_{j,k}(t, x)$ of $\tilde{F}(t, x)$ converges to the function*

$$\tilde{f}_{0,j,k}(x)=\int e^{ix\xi}\left[\int_0^1\exp(a(r, \xi))\ dr\right]a_{1,j,k}(t_0, x_0; i\xi)d\xi \tag{3.6}$$

as $t\longrightarrow +0$, where

$$a(r, \xi)=ra_{2,j,j}(t_0, x_0; i\xi)+(1-r)a_{2,k,k}(t_0, x_0; i\xi). \tag{3.7}$$

Proof. We may assume that $t_0=0$ and $x_0=0$. Let us put

$$\tilde{F}(t, x)=\tilde{F}_1(t, x)+\tilde{F}_2(t, x),$$

where

$$\tilde{F}_1(t, x)=t^{(n-1)/2}\int_0^t d\tau\int\tilde{E}_0(t, \tau, t^{1/2}x, z)\tilde{\Phi}_1(\tau, 0, z, 0)dz.$$

Then we have

$$|\tilde{F}_2(t, x)|\leqq Ct^{(n-1)/2}\int_0^t d\tau\int|\tilde{E}_0(t, \tau, t^{1/2}x, z)|\sum_{l=2}^{\infty}|\tilde{\Phi}_l(\tau, 0, z, 0)|dz$$

$$\leqq C't^{(n-1)/2}\int_0^t((t-\tau)\tau)^{-n/2}d\tau\int\exp\left(-c\frac{|t^{1/2}x-z|^2}{t-\tau}-c\frac{|z|^2}{\tau}\right)dz$$

$$=C''t^{1/2}\exp(-c|x|^2).$$

So $\tilde{F}_2(t, x)$ converges to O as t tends to 0.

As for $\tilde{F}_1(t, x)=(\tilde{f}_{1,j,k}(t, x))_{1\leqq j,k\leqq N}$, we have

$$\tilde{f}_{1,j,k}(t, x)=t^{(n-1)/2}\int_0^t d\tau\int e_j(t, \tau, t^{1/2}x, z)b_{j,k}\left(\tau, z; \frac{\partial}{\partial z}\right)e_k(\tau, 0, z, 0)dz, \tag{3.8}$$

where $b_{j,k}$ are defined by (3.1). Then the proposition is proved by applying to (3.8) the Lemma 3.3 below, which we will prove in the latter half of this section.

Lemma 3.3. *Let*

$$q_0(t, x)=t^{(n-1)/2}\int_0^t d\tau\int e_j(t, \tau, t^{1/2}x, z)b(\tau, z)\frac{\partial}{\partial z_\nu}e_k(\tau, 0, z, 0)dz$$

and

$$q_1(t, x)=t^{(n-2)/2}\int_0^t d\tau\int e_j(t, \tau, t^{1/2}x, z)b(\tau, z)e_k(\tau, 0, z, 0)dz,$$

where $b(\tau, z)$ is a bounded and uniformly continuous function on $[0, T]\times\mathbf{R}^n$ and $1\leqq\nu\leqq n$. Then we have

$$\lim_{t\to+0}q_0(t, x)=b(0, 0)\int_0^1 dr\int\exp(ix\xi+a(r, \xi))i\xi_\nu d\xi$$

and

$$\lim_{t\to+0}q_1(t, x)=b(0, 0)\int_0^1 dr\int\exp(ix\xi+a(r, \xi))d\xi, \tag{3.9}$$

where $a(r, \xi)$ is defined by (3.7).

Proof of Proposition 1.2. Let us assume that (1.8) holds. Since $\tilde{E}_0(t, s, x, y)$ is diagonal, we have $\tilde{f}_{0,j,k}(x)\geqq 0$ if $j\neq k$. On the other hand, (3.6) implies that $\tilde{f}_{0,j,k}(x)$ is an odd function. So $\tilde{f}_{0,j,k}(x)$ must be identically equal to zero. That is, $a_{1,j,k}(t_0, x_0; i\xi)\equiv 0$. Thus the proposition is proved.

Proof of Lemma 3.3. Introducing new variables $r=t^{-1}\tau$, $\zeta=t^{-1/2}z$ and $\omega=t^{1/2}$, we have

$$q_0(t, x)=\int_0^1 dr\int[\omega^n e_j(t, tr, \omega x, \omega\zeta)]b(tr, \omega\zeta)\times[\omega^{n+1}\left(\frac{\partial}{\partial x_\nu}e_k\right)(tr, 0, \omega\zeta, 0)]\,d\zeta. \tag{3.10}$$

From Lemma 3.1, we have the estimates

$$|\omega^n e_j(t,\ tr,\ \omega x,\ \omega\zeta)| \leqq C\omega^n(t-tr)^{-n/2}\exp\left(-c\frac{|\omega x-\omega\zeta|^2}{t-tr}\right)$$

$$=C(1-r)^{-n/2}\exp\left(-c\frac{|x-\zeta|^2}{1-r}\right)$$

and

$$\left|\omega^{n+1}\left(\frac{\partial}{\partial x_\nu}e_k\right)(tr,\ 0,\ \omega\zeta,\ 0)\right| \leqq Cr^{-(n+1)/2}\exp\left(-c\frac{|\zeta|^2}{r}\right).$$

So the integrand of (3.10) is bounded by an integrable function

$$C(1-r)^{-n/2}r^{-(n+1)/2}\exp\left(-c\frac{|x-\zeta|^2}{1-r}-c\frac{|\zeta|^2}{r}\right).$$

Similarly, we have

$$q_1(t,\ x)=\int_0^1 dr\int[\omega^n e_j(t,\ tr,\ \omega x,\ \omega\zeta)]b(tr,\ \omega\zeta)[\omega^n e_k(tr,\ 0,\ \omega\zeta,\ 0)]d\zeta.$$

and this integrand is bounded by

$$C((1-r)r)^{-n/2}\exp\left(-c\frac{|x-\zeta|^2}{1-r}-c\frac{|\zeta|^2}{r}\right).$$

Using Lebesgue's Theorem and the following Lemma 3.4, we have

$$\lim_{t\to+0} q_0(t,\ x)=\int_0^1 dr\int\left[\int\exp\{i(x-\zeta)\xi+(1-r)a_{2,j,j}(0,\ 0;\ i\xi)\}d\xi\right]$$

$$\times b(0,\ 0)\left[\int\exp\{i\zeta\xi+ra_{2,k,k}(0,\ 0;\ i\xi)\}i\xi_\nu d\xi\right]d\zeta,$$

and

$$\lim_{t\to+0} q_1(t,\ x)=\int_0^1 dr\int\left[\int\exp\{i(x-\zeta)\xi+(1-r)a_{2,j,j}(0,\ 0;\ i\xi)\}d\xi\right]$$

$$\times b(0,\ 0)\left[\int\exp\{i\zeta\xi+ra_{2,k,k}(0,\ 0;\ i\xi)\}d\xi\right]d\zeta.$$

Then the lemma follows from the formula

$$\int\left[\int e^{ix\xi}u(\xi)d\xi\right]\left[\int e^{ix\xi}v(\xi)d\xi\right]dx=\int u(\xi)v(-\xi)d\xi.$$

Lemma 3.4.

(3.11) $$\lim_{t\to+0}\omega^n e_j(tr,\ ts,\ \omega x,\ \omega y)=\int\exp\{i(x-y)\xi+(r-s)a_{2,j,j}(0,\ 0;\ i\xi)\}d\xi$$

and

(3.12) $$\lim_{t\to+0}\omega^{n+1}\left(\frac{\partial}{\partial x_\nu}e_j\right)(tr,\ ts,\ \omega x,\ \omega y)=\int\exp\{i(x-y)\xi+(r-s)a_{2,j,j}(0,\ 0;\ i\xi)\}\ i\xi_\nu d\xi$$

provided $0 \leqq s < r \leqq 1$.

Proof. From (3.2), we have

$$\omega^n e_j(tr, ts, \omega x, \omega y) = \omega^n g_j(tr, ts, \omega(x-y), \omega y) + \omega^n \int_{ts}^{tr} d\tau \int g_j(tr, \tau, \omega x - z, z) \phi_j(\tau, ts, z, \omega y) dz \equiv I_1 + I_2.$$

First, using Lemma 3.1 and (3.5), we have

$$|I_2| \leqq C\omega^n \int_{ts}^{tr} (tr-\tau)^{-n/2} (\tau - ts)^{-(n+2-\sigma)/2} d\tau \int \exp\left(-c\frac{|\omega x - z|^2}{tr-\tau} - c\frac{|z-\omega y|^2}{\tau - ts}\right) dz$$

$$= C' t^{\sigma/2} (r-s)^{(\sigma-n)/2} \exp\left(-c\frac{|x-y|^2}{r-s}\right) \longrightarrow 0$$

as $t \longrightarrow +0$.

As for I_1, it follows from (3.3) that

$$I_1 = \omega^n \int \exp\{i\omega(x-y)\xi + \int_{ts}^{tr} a_{2,j,j}(\tau, \omega y; i\xi) d\tau\} d\xi$$

$$= \int \exp\{i(x-y)\eta + \int_s^r a_{2,j,j}(t\rho, \omega y; i\eta) d\rho\} d\eta,$$

where we put $\eta = \omega\xi$ and $\rho = t^{-1}\tau$. Then, by Lebesgue's Theorem, we have

$$\lim_{t \to +0} I_1 = \int \exp\{i(x-y)\eta + (r-s) a_{2,j,j}(0, 0; i\eta)\} d\eta.$$

Thus (3.11) is proved.

We can show (3.12) in the same way. In fact, we have

$$\omega^{n+1}\left(\frac{\partial}{\partial x_\nu} g_j\right)(tr, ts, \omega(x-y), \omega y) = \omega^{n+1} \int \exp\{i\omega(x-y)\xi + \int_{ts}^{tr} a_{2,j,j}(\tau, \omega y; i\xi) d\tau\} i\xi_\nu d\xi$$

$$= \int \exp\{i(x-y)\eta + \int_s^r a_{2,j,j}(t\rho, \omega y; i\eta) d\rho\} i\eta_\nu d\eta$$

$$\longrightarrow \int \exp\{i(x-y)\eta + (r-s) a_{2,j,j}(0, 0; i\eta)\} i\eta_\nu d\eta$$

as $t \longrightarrow +0$.

§4. Proof of Propositions 1.3 and 1.4.

In this section we will assume that $m=1$ and $A_p(t, x: i\xi)$ $(p=1, 2)$ are diagonal. And as in the preceding section, let us write

$$L=\frac{\partial}{\partial t}-A\Big(t, x; \frac{\partial}{\partial x}\Big)-B(t, x),$$

where $A(t, x; i\xi)$ is diagonal, $B(t, x)=(b_{j,k}(t, x))_{1\leqq j,k\leqq N}$ is a matrix of functions acting as an operator of multiplication and

$$b_{j,j}(t,x)\equiv 0, \qquad 1\leqq j\leqq N.$$

The proof of Proposition 1.3 is parallel to that of Proposition 1.2. Let us construct a fundamental solution

$$\bar{E}(t, s, x, y)=\tilde{E}_0(t, s, x, y)+\bar{E}_1(t, s, x, y), \tag{4.1}$$

where $\tilde{E}_0(t, s, x, y)$ was defined in §3, and

$$\bar{E}_1(t, s, x, y)=\int_s^t d\tau\int \tilde{E}_0(t, \tau, x, z)\bar{\Phi}(\tau, s, z, y)\ dz,$$

$$\bar{\Phi}(t, s, x, y)=\sum_{l=1}^{\infty}\bar{\Phi}_l(t, s, x, y),$$

$$\bar{\Phi}_1(t, s, x, y)=B(t, x)\tilde{E}_0(t, s, x, y)$$

and

$$\bar{\Phi}_l(t, s, x, y)=\int_s^t d\tau\int \bar{\Phi}_1(t, \tau, x, z)\bar{\Phi}_{l-1}(\tau, s, z, y)\ dz$$

for $l\geqq 2$.

Then we can easily prove the following lemma similar to the Lemma 3.2.

Lemma 4.1. (i) *There exist some constants C and C_0 such that*

$$|\bar{\Phi}_l(t, s, x, y)|\leqq C\Gamma(l)^{-1}C_0^l(t-s)^{l-1-(n/2)}\exp\Big(-c\frac{|x-y|^2}{t-s}\Big).$$

(ii)
$$\bar{E}(t, s, x, y)\equiv E(t, s, x, y).$$

Now let us fix (t_0, x_0) in $[0, T)\times\mathbf{R}^n$ and put

$$\bar{F}(t, x)=t^{(n-2)/2}\bar{E}_1(t_0+t, t_0, x_0+t^{1/2}x, x_0).$$

Then we have the following proposition.

Proposition 4.1. *If $j\neq k$, the (j, k)-element $\bar{f}_{j,k}(t, x)$ of $\bar{F}(t, x)$ converges to the function*

$$\bar{f}_{0,j,k}(x)=b_{j,k}(t_0, x_0)\int_0^1 dr\int \exp\,[ix\xi+a(r, \xi)]d\xi, \tag{4.2}$$

as $t\longrightarrow +0$, where $a(r,\xi)$ is defined by (3.7).

Proof. We may assume that $t_0=0$ and $x_0=0$. Let us put

$$\bar{F}(t, x)=\bar{F}_1(t, x)+\bar{F}_2(t, x),$$

where

$$\bar{F}_1(t, x)=t^{(n-2)/2}\int_0^t d\tau\int \bar{E}_0(t, \tau, t^{1/2}x, z)\bar{\Phi}_1(\tau, 0, z, 0)dz.$$

Then we have

$$|\bar{F}_2(t, x)|\leqq Ct^{(n-2)/2}\int_0^t d\tau\int|\bar{E}_0(t, \tau, t^{1/2}x, z)|\sum_{l=2}^{\infty}|\bar{\Phi}_l(\tau, 0, z, 0)|dz$$

$$\leqq C' t^{(n-2)/2}\int_0^t((t-\tau)\tau)^{-n/2}\tau d\tau\int\exp\left(-c\frac{|t^{1/2}x-z|^2}{t-\tau}-c\frac{|z|^2}{\tau}\right)dz$$

$$=C''t\exp(-c|x|^2).$$

So $\bar{F}_2(t, x)$ converges to O as $t \longrightarrow +0$.

As for $\bar{F}_1(t, x)$, we have

$$\bar{f}_{1,j,k}(t, x)=t^{(n-2)/2}\int_0^t d\tau\int e_j(t, \tau, t^{1/2}x, z)b_{j,k}(\tau, z)e_k(\tau, 0, z, 0)dz.$$

Then this proposition follows from (3.9) of Lemma 3.3.

Proof of Proposition 1.3. Let us assume that (1.8) holds. Then we have $\bar{f}_{0,j,k}(x)\geqq 0$ if $j\neq k$. On the other hand, $\int\exp(ix\xi+a(r, \xi))d\xi$ is positive at $x=0$ for any $0\leqq r\leqq 1$. So it follows from (4.2) that $b_{j,k}(t_0, x_0)\geqq 0$. Since (t_0, x_0) is an arbitrary point in $[0, T)\times\mathbf{R}^n$, the proposition is proved.

Proof of Proposition 1.4. Under conditions (i), (ii) of the Theorem 1.2, the fundamental solution $E(t, s, x, y)$ is constructed in the form (4.1). Then, under the condition (iii), the elements $\bar{\Phi}_{l,j,k}(t, s, x, y)$ of each $\bar{\Phi}_l(t, s, x, y)$ are nonnegative, because $e_j(t, s, x, y)$ $(j=1,\dots, N)$ are nonnegative (see Theorem 1.3). So $e_{j,k}(t, s, x, y)$ are also nonnegative. Thus the proof is completed.

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