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Some remarks on the C^∞-Goursat problem

By

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§1. Introduction.

Let us consider the following partial differential operator with constant coefficients.

$$L=\sum_{i+j+|\alpha|\leq m} a_{ij\alpha}D_t^iD_x^jD_y^\alpha,\ t\geqq 0,\ x\in \boldsymbol{R}^1,\ y\in \boldsymbol{R}^n, \tag{1.1}$$

$$D_t=-i\frac{\partial}{\partial t},\ D_x=-i\frac{\partial}{\partial x},\ D_y=\left(-i\frac{\partial}{\partial y_1},\ -i\frac{\partial}{\partial y_2},\dots,\ -i\frac{\partial}{\partial y_n}\right)$$

$a_{ij\alpha}$: constant.

In this paper we assume that the hypersurface $t=0$ is s-tuple characteristics, namely

$$\text{(A)}\qquad \begin{cases} \text{i)} & a_{ij\alpha}=0 \text{ for } i+j+|\alpha|=m,\ i>m-s, \text{ and} \\ \text{ii)} & \sum_{j+|\alpha|=0} a_{m-s,j\alpha}\xi^j\eta^\alpha\neq 0. \end{cases}$$

Under the assumption (A), we consider the following problem. (We say Goursat problem for $t\geqq 0$)

$$\text{(P)}\qquad \begin{cases} Lu=0\ t\geqq 0,\ x\in R^1,\ y\in R^n \\ D_t^iu(0,\ x,\ y)=\phi_i(x,\ y)\in\mathscr{E}_{(x,y)},\ 0\leqq i\leqq m-s-1 \\ D_x^ju(t,\ 0,\ y)=\psi_j(t,\ y)\in\mathscr{E}_{(t,y)},\ 0\leqq j\leqq s-1,\ t\geqq 0 \end{cases}$$

where we impose among $\{\phi_i\}$ and $\{\psi_j\}$ the following compatibility condition;

$$\text{(C)}\qquad D_x^j\phi_i(0,\ y)=D_t^i\psi_j(0,\ y),\ 0\leqq i\leqq m-s-1,\ 0\leqq j\leqq s-1,\ y\in \boldsymbol{R}^n.$$

We say that the Goursat problem (P) is $\mathscr{E}$-wellposed if for any data $\{\phi_i\}$, $\{\psi_j\}$ with compatibility condition (C), there exists a unique solution $u(t,\ x,\ y)\in\mathscr{E}_{(t,x,y)}$ $t\geqq 0$.

T. Nishitani [4] had considered the following operator:

$$\text{(N)}\qquad P=\sum_{\substack{i+j+|\alpha|\leq m \\ i\leqq m-s}} a_{ij\alpha}D_t^iD_x^jD_y^\alpha,\ a_{m-s,s,0}\neq 0.$$

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And he had obtained a necessary and sufficient condition for the $\mathscr{E}$-wellposedness. For this operator (N) we obtained a Levi condition [2].

Let us call the operator (N) which was treated by Nishitani N-type. We have the following conjecture:

Conjecture 1. If the Goursat problem for (P) is $\mathscr{E}$-wellposed then operator L is N-type.

In this paper we are going to show that under some assumptions this conjecture 1 is true.

Remark 1.1. "Operator L is N-type" means that the coefficient of $D_t^{m-s}D_x^s$ doesn't vanish and the order of D_t is at most $m-s$, namely $a_{m-s,s,0}\neq 0$ and $a_{ij\alpha}=0$ for $i>m-s$.

Remark 1.2. If the Goursat problem is $\mathscr{E}$-wellposed then the linear mapping $\{\{\phi_i\},\ \{\psi_j\}\}\to u(t,\ x,\ y)$ is continuous from $\Pi\mathscr{E}_{(x,y)}\times\Pi\mathscr{E}_{(t,y)}$ into $\mathscr{E}(t,\ x,\ y)$.

§2. Result.

Firstly we show the following theorem:

Theorem 1. *If the Goursat problem* (P) *is $\mathscr{E}$-wellposed then $a_{m-s,s,0}\neq 0$. Where $a_{m-s,s,0}$ is the coefficient of $D_t^{m-s}D_t^s$ in* (1.1).

Proof. Let us show that assuming $a_{m-s,s,0}=0$ there exists Goursat data $\{\phi_i\}$, $\{\psi_j\}$ such that (P) has no solution in $\mathscr{E}$.

Consider the Goursat data:

$$\text{(2.1)}\qquad \begin{cases} D_t^iu(0,\ x,\ y)=0 & 0\leqq i\leqq m-s-1 \\ D_x^ju(t,\ 0,\ y)=t^{m-s}g_j(y) & 0\leqq j\leqq s-1. \end{cases}$$

For any $g_j(y)\in\mathscr{E}_y$, this Goursat data satisfy compatibility condition (C). Let u be the solution of $Lu=0$ with (2.1). Because of $a_{m-s,s,0}=0$, we have

$$\text{(2.2)}\qquad Lu|_{t=x=0}=\sum_{\substack{j+|\alpha|\leqq s\\ j<s}} a_{m-s,j,\alpha}\ D_y^\alpha g_j(y).$$

Therefore

$$\text{(2.3)}\qquad \sum_{\substack{j+|\alpha|\leqq m\\ j<s}} a_{m-s,j,\alpha}D_y^\alpha g_j(y)=0.$$

By the assumption (A), (2.3) is some restriction $\{g_j(y)\}$. So if we take $\{g_j(y)\}$ which does not satisfy (2.3) then (P) has no solution. q.e.d.

According to Theorem 1, if (P) is $\mathscr{E}$-well posed then L is the following:

$$\text{(2.4)}\qquad L=\sum_{i+j+|\alpha|\leqq m} a_{ij\alpha}D_t^iD_x^jD_y^\alpha,$$

$$a_{m-s,s,0}\neq 0,\ a_{ij\alpha}=0 \text{ for } i+j+|\alpha|=m \text{ and } i>m-s$$

Let L_m be the principal part of L,

(2.5) $$L_m(\tau, \xi, \eta)=\sum_{i+j+|\alpha|=m} a_{ij\alpha}\tau^i\xi^j\eta^\alpha.$$

Because of assumption (A),

(2.6) $$L_m(\tau, \xi, 0)=\xi^s\sum_{i=0}^{m-s} a_{i,m-i,0}\tau^i\xi^{m-s-i}.$$

By Theorem 1 $L_m(\tau, \xi, 0)$ is the polynomial of τ of degree $m-s$. Let the roots of $L_m(\tau, \xi, 0)=0$ be $\{\alpha_i\xi;\ i=1, 2,\dots, m-s\}$. Where $\{\alpha_i\}$ are the roots of $L_m(\tau, 1, 0)=0$. We have the following;

Therem 2. *If the Goursat problem* (P) *is $\mathscr{E}$-wellposed then the roots of $L_m(\tau, \xi, 0)=0$ are real for $\xi\in R^1$, i.e. $\{\alpha;\ i=1,\dots, m-s\}$ are real.*

Theorem 3. *If the Goursat problem* (P) *is $\mathscr{E}$-wellposed and the roots $\{\alpha_i\xi,\ \alpha_i\neq 0\}$ of $L_m(\tau, \xi, 0)=0$ are real and have same sign then L is N-type.*

Remark 2.1. In the case where the roots $\{\alpha_i;\ \alpha\neq 0\}$ of $L_m(\tau, 1, 0)=0$ are real and have different sign we can not show that the conjecture 1 is true. But under some strong assumptions the conjecture 1 is true. About this case we study in §6.

Let us assume

(2.6) $$\alpha_i\neq 0\ \ i=1, 2,\dots, m-s_0,\ \ \alpha_i=0\ \ i=m-s_0+1,\dots, m-s,\ \ s_0\geqq s.$$

§3. The properties of the roots of $L(\tau, \xi, 0)=0$.

Here we give a rouch sketch of the proof of Theorem 2 and Theorem 3. Assuming that the conclusion of Theorem does not hold we construct a sequence of the solutions of (P) which shows the continuity from Goursat data to solutions does not hold.

Firstly we consider the differential operator $L(D_t, D_x, 0)$. Let us write

(3.1) $$L(D_t, D_x, 0)=\Gamma(D_t, D_x).$$

(3.2) $$\Gamma(D_t, D_x)=\sum_{i+j\leqq m} a_{ij}D_t^iD_x^j,$$

where $a_{ij}=a_{ij0}$in (2.4), $a_{ij}=0$ for $i+j=m$ and $i>m-s$, $a_{m-s,s}\neq 0$.

Notice that if $\Gamma(\tau, \xi)=0$ for some (τ, ξ) then $\exp(i\tau t+i\xi x)$ is the solution of $\Gamma u=0$. In this section we investigate the properties of the roots $\tau(\xi)$ (or$\xi(\tau)$) of $\Gamma(\tau, \xi)=0$ considering that $\Gamma(\tau, \xi)$ is the polynomial of τ (or ξ).

By (3.2) we can write

(3.3) $$\Gamma(\tau, \xi)=\sum_{i=m-s+1}^{m-1}\tau^i\{\sum_{j=0}^{m-i-1} a_{ij}\xi^j\}+\sum_{i=0}^{m-s}\tau^i\{\sum_{j=0}^{m-i} a_{ij}\xi^j\},\ a_{m-s,s}\neq 0$$

Let us consider the roots of $\tau(\xi)$ of $\Gamma(\tau, \xi)=0$ and it's Puiseux expansion in the neighborhood of $\xi=+\infty$. Let

(3.4) $$\tau=c_1\xi^{\rho_1}+c_2\xi^{\rho_2}+\cdots,\quad \rho_1>\rho_2>\cdots,\quad c\neq 0.$$

By the "Newton's polygon construction" we have the following (refer to A. Lax [3]).

Lemma 3.1. *The roots of $\Gamma(\tau, \xi)=0$ have the following properties:*
i) *the number of roots with $\rho_1=1$ is $m-s_0$ and they have the Puiseux expansion of* (3.5)

(3.5) $$\tau_j(\xi)=\alpha_j\xi+c_{2,j}\xi^{\rho_{2,j}}+c_{3,j}\xi^{\rho_{3,j}}+\cdots,$$
$$1>\rho_{2,j}>\rho_{3,j}>\cdots,\quad j=1, 2,\dots, m-s_0$$

ii) *the number of roots with $\rho_1<1$ is s_0-s, let us write them*

(3.6) $$\tau_k(\xi)=c_{1,k}\xi^{\rho_{1,k}}+c_{2,k}\xi^{\rho_{2,k}}+\cdots,$$
$$1>\rho_{1,k}>\rho_{2,k}>\cdots,\quad k=m-s_0+1,\dots, m-s.$$

Remark 3.1. When $\rho_1=1$, the coefficient c_1 (in (3.4)) is determined by $\sum_{i=0}^{m-s} c_1^i a_{i,m-i}=0$. So we have (3.5).

Next, we consider the roots $\xi(\tau)$ of $\Gamma(\tau,\xi)=0$ and it's Puiseux expansion in the neighborhood of $\tau=\infty$. Let

(3.7) $$\xi=b_1\tau^{\sigma_1}+b_2\tau^{\sigma_2}+\cdots,\quad \sigma_1>\sigma_2>\cdots,\quad b_1\neq 0.$$

By the "Newton's polygon construction" we have the following:

Lemma 3.2. *The number of roots with $\sigma_1<1$ or $\xi\equiv 0$ is s. Let them be*

(3.8) $$\xi_j(\tau)=b_{1,j}\tau^{\sigma_{1,j}}+b_{2,j}\tau^{\sigma_{2,j}}+\cdots,\quad \sigma_{1,j}<1, j=1, 2,\dots, s_1$$
$$\sigma_{1,j}>\sigma_{2,j}>\cdots,$$

(3.9) $$\xi_j(\tau)\equiv 0,\quad j=s_1+1,\dots, s,\quad s_1\leqq s.$$

Here we consider the case where $\Gamma(D_t, D_x)$ is not N-type. In this case there exists $a_{h,k}$ such that

(3.10) $$\begin{cases} a_{hk}\neq 0 \text{ for } h>m-s,\ k\geqq 0,\ k+h<m \\ a_{ij}=0 \text{ for } i>h \\ a_{hj}=0 \text{ for } j>k \end{cases}$$

Then $\Gamma(\tau, \xi)$ becomes (3.11)

(3.11) $$\Gamma(\tau, \xi)=\tau^h(a_{h,k}\xi^k+a_{h,k-1}\xi^{k-1}+\cdots+a_{h,0})$$
$$+\sum_{i=m-s+1}^{h-1}\tau^i\{\sum_{j=0}^{m-i-1}a_{i,j}\xi^j\}+\sum_{i=0}^{m-s}\tau^i\{\sum_{j=0}^{m-i}a_{i,j}\xi^j\}.$$

Lemma 3.3. *If* (3.10) *holds then there exists a root $\xi(n)$ of $\Gamma(\varepsilon n, \xi)=0$ such that*

(3.12) $$\xi(n)=b_1n^{\theta_1}+b_2n^{\theta_2}+b_3n^{\theta_3}+\cdots,$$

$$0<\theta_1<1,\ \theta_1>\theta_2>\theta_3>\cdots,\ \mathrm{Im}\ b_1<0 \text{ for } \varepsilon=1 \text{ or } \varepsilon=-1.$$

Proof of Lemma 3.3 Let

$$A=\{(i, j);\ a_{i,j}\neq 0\}.$$

By (3.2) and (3.10) it holds

$$(m-s,\ s),\ (h,\ k)\in A,\ h>m-s,\ h+k<m.$$

Consider Newton's polygon. Namely consider the convex hull of A. There exists $(p, q)\in A$ and θ $(0<\theta<1)$ such that

$$\begin{cases} p+q<m,\ p>m-s,\ p+\theta q=m-s+\theta s, \\ i+\theta j\leqq m-s+\theta s \text{ for } \forall\ (i,\ j)\ \in\ A. \end{cases} \tag{3.13}$$

We put

$$A_0=\{(i, j);\ i+\theta j=m-s+\theta s,\ (i, j)\in A,\ (i, j)\neq(m-s,\ s)\}.$$

Let the formal solution of $\Gamma(\varepsilon n,\ \xi)=0$ be (3.14).

$$\xi=c_1 n^{\theta}+c_2 n^{\theta'}+c_3 n^{\theta''}+\cdots,\ \theta>\theta'>\theta''>\cdots. \tag{3.14}$$

Substitute (3.14) in $\Gamma(\varepsilon n,\ \xi)=0$ and notice the coefficient of $n^{m-s+\theta s}$.

$$a_{m-s,s}\varepsilon^{m-s}c_1^s+\sum_{(i,j)\in A_0} a_{i,j}\varepsilon^i c_1^j=0. \tag{3.15}$$

Let us write

$$q=\max_{(i,j)\in A_0} j,\ p+\theta q=m-s+\theta s. \tag{3.16}$$

Then (3.15) becomes the following;

$$a_{m-s,s}\varepsilon^{m-s}c_1^s+a_{p,q}\varepsilon^p c_1^q+a_{p'q'}\varepsilon^{p'}c_1^{q'}+\cdots=0 \tag{3.15'}$$

$$s>q>q'>\cdots.$$

By (3.13) we have (3.17).

$$\theta(s-q)=p-(m-s)\geqq 1. \tag{3.17}$$

Because of the fact that $p-(m-s)$ and $s-q$ are integer and $0<\theta<1$, we have

$$s-q\geqq 2. \tag{3.18}$$

Differentiating (3.15′) q times by c_1 we have

$$c_1^{s-q}+a_1'\varepsilon^{p-(m-s)}=0,\ a_1'\neq 0. \tag{3.19}$$

Firstly we show that (3.19) has a root c_1 with $\mathrm{Im}\ c_1<0$. When $s-q\geqq 3$, it is obvious. Let us consider the case where $s-q=2$. In this case $p-(m-s)=1$. In fact because of $p+q\leqq m-1$ it holds $p-(m-s)\leqq m-1-q-(m-s)=s-q-1=1$. Then (3.19) becomes (3.20).

(3.20) $$c_1^2+a_1'\varepsilon=0,\ a_1'\neq 0.$$

(3.20) has a root c_1 with Im $c_1<0$ if we take ε with $a'\varepsilon\neq -1$.

Because of Lemma 3.4, (3.15) has a root c_1 with Im $c_1<0$. q.e.d.

Lemma 3.4. *Let $P(z)$ be the polynomial of degree m. Let the roots of $P(z)=0$ be $z_1, z_2,\dots, z_m$ and M be the convex hull of $\{z_i; i=1, 2,\dots, m\}$. Then the roots of $\frac{d}{dz}P(z)=0$ are contained in M.*

§4. Proof of Teorem 2.

Suppose that α_1 is a root of $L_m(\tau, 1, 0)=0$ with Im $\alpha_1\neq 0$. In (3.5), put

(4.1) $$\xi=n\varepsilon',\ \varepsilon'=1 \text{ or } -1,$$

where we determine ε' with Im $\alpha_1\varepsilon'<0$.

We put

(4.2) $$\tau(n)=\tau_1(n\varepsilon')=\alpha_1\varepsilon' n+o(n).$$

And substitute this $\tau(n)$ for τ in (3.8) and (3.9).

(4.3) $$\xi_j(\tau(n)),\ j=1, 2,\dots, s.$$

By Lemma 3.2

(4.4) $$\xi_j(\tau(n))\sim c_j n^{\sigma_{1'j}},\ \sigma_{1'j}<1, \text{ for } 1\leqq j\leqq s_1$$
$$\xi_j(\tau(n))\equiv 0 \text{ for } s_1+1\leqq j\leqq s.$$

Firstly we assume that $\xi_j(\tau(n))$ $(j=1, 2,\dots,s)$ are distinct for large n. Let

(4.5) $$\begin{cases} u_n^0=\exp(in\varepsilon' x+i\tau(n)t) \\ u_n^1=\exp(i\xi(\tau(n))x+i\tau(n)t) \\ \quad\cdots\cdots\cdots\cdots \\ u_n^s=\exp(i\xi_s(\tau(n))x+i\tau(n)t) \end{cases}$$

And let

(4.6) $$u_n=u_n^0+A_1u_n^1+A_2u_n^2+\cdots+A_su_n^s,$$

where $A_i(i=1, 2,\dots, s)$ are constant which depend on n. $u_n^i(i=0, 1,\dots, s)$ are solutions of $L(D_t, D_x, D_y)u=\Gamma(D_t, D_x)u=0$, therefore u_n is the solution, too. We define $\{A_i\}$ as follows;

(4.7) $$D_x^k u_n(t, 0)=\{\exp\ (i\tau(n)t)\}\ \{(n\varepsilon')^k+(\xi_1(\tau(n)))^k A_1+\cdots\cdots$$
$$+(\xi_s(\tau(n)))^k A_s\}=0,\ k=0, 1,\dots, s-1.$$

$A_i(n)$ has at most polynomial order with respect to n. We have

$$(4.8)\qquad \begin{cases} D_t^k u_n(0,\ x)=(\tau(n))^k\{\exp(in\varepsilon' x)+A_1\exp(i\xi_1(\tau(n))x)+\cdots\cdots \\ \quad +A_s\exp(i\xi_s(\tau(n))x),\ k=0,\ 1,\dots,\ m-s-1. \\ D_x^j u_n(t,\ 0)=0,\ j=0,\ 1,\dots,\ s-1. \end{cases}$$

So the order of data with respect to n is polynomial(of n)$\times\exp(cn^\sigma)$ $(0<\sigma<1,$ $c>0)$. On the otherhand the order of u_n is exp $(c'nt)$ $(c'=|\mathrm{Im}\ (\alpha_1\varepsilon')|)$. Then the continuity of data to solution does not hold.
In the case where $\Gamma(\tau,\ \xi)=0$ has multiple roots, for instance $\xi_1(\tau)$ is p-tuple roots, we put

$$u_n^k=x^{k-1}\ \exp\ (i\xi_1 x\times i\tau(n)t),\ k=1,\ 2,\dots,\ p-1.$$

And the nearly same way as the first case we can show that the continuity of data to solution does not hold.

§5. Proof of Theorem 3.

At first we remark that.

Remark 5.1. *When $L(D_t,\ D_x,\ D_y)$ is not N-type without loss of generality we can consider that $L(D_t,\ D_x,\ 0)$ is not N-type.*

In fact putting

$$(5.1)\qquad u(t,\ x,\ y)=v(t,\ x,\ y)\exp(i\rho y)$$

where ρ is a parameter, then

$$L(D_t,\ D_x,\ D_y)\ u(t,\ x,\ y)\equiv\exp(i\rho y)\ \hat{L}(D_t,\ D_x,\ D_y)v$$
$$=\exp(i\rho y)\ L(D_t,\ D_x,\ \rho)v+\sum_{\substack{i+j+\alpha\leqq m \\ \alpha\geqq 1}}\tilde{a}_{ij\alpha}D_t^iD_x^jD_y^\alpha v.$$

So

$$(5.2)\qquad \hat{L}(D_t,\ D_x,\ 0)=L(D_t,\ D_x,\ \rho).$$

When $L(D_t,\ D_x,\ D_y)$ is not N-type, for suitable ρ, $L(D_t,\ D_x,\ \rho)$ is not N-type. We consider $\hat{L}(D_t,\ D_x,\ D_y)v=0$ instead of $L(D_t,\ D_x,\ D_y)u=0$. By (5.2), $\hat{L}(D_t, D_x, 0)$ is not N-type for suitable ρ.

Suppose that the roots of $L_m(\tau,\ 1,\ 0)=0$ are real and negative or 0. Moreover we assume $L(D_t,\ D_x,\ D_y)$ is not N-type. Because of Remark 5.1 we can assume that (3.10) holds. Let us recall (3.12).

$$(3.12)\qquad \xi(n)=b_1n^{\theta_1}+b_2n^{\theta_2}+b_3n^{\theta_3}+\cdots,$$
$$0<\theta_1<1,\ \theta_1>\theta_2>\theta_3>\cdots,\ \mathrm{Im}\ b_1<0.$$

Substitute this $\xi(n)$ for ξ in (3.5) and (3.6)

$$(3.5')\qquad \tau_j(\xi(n))=\alpha_j\xi(n)+c_{2,j}(\xi(n))^{\rho_{2,j}}+\cdots$$

$$=b_1\alpha_j n^{\theta_1}+c_{2,j}n^{\theta_{2},j}+\cdots$$

$$\text{Im } b_1\alpha_j>0,\ \theta_1>\theta_{2,j}>\cdots,\ j=1,2\cdots,\ m-s_0$$

(3.6′)
$$\tau_k(\xi(n))=c_{1,k}(\xi(n))^{\rho_{1},k}+c_{2,k}(\xi(n))^{\rho_{2},k}+\cdots$$
$$=\tilde{c}_{1,k}n^{\omega_{1},k}+\tilde{c}_{2,k}n^{\omega_{2},k}+\cdots.$$

$$\theta_1>\omega_{1,k}>\omega_{2,k}>\cdots,\ k=m-s_0+1,\dots,\ m-s.$$

At first we assume that $\tau_k(\xi)$ $(k=1,2,\dots,m-s)$ are distinct for large n. Let

(5.3)
$$\begin{cases} u_n^0=\exp\{i\varepsilon nt+i\xi(n)x\} \\ u_n^1=\exp\{i\tau_1(\xi(n))t+i\xi(n)x\} \\ \cdots\cdots\cdots\cdots \\ u_n^{m-s}=\exp\{i\tau_{m-s}(\xi(n))t+i\xi(n)x\} \end{cases}$$

(5.4)
$$u_n=u_n^0+B_1u_n^1+\cdots+B_{m-s}u_n^{m-s}.$$

We define the coefficient $\{B_k\}$ as follows.

(5.5)
$$D_t^k u_n(0,\ x)=\{\exp i\xi(n)x\}\{(\varepsilon n)^k+B_1(\tau_1(\xi(n)))^k+\cdots\cdots$$
$$+B_{m-s}(\tau_{m-s}(\xi(n)))^k\}=0.$$
$$k=0,\ 1,\ 2,\dots,\ m-s-1.$$

$B_k(n)$ has at most polynomial order of n. We have

(5.6)
$$\begin{cases} D_n^k u_n(0,\ x)=0,\ k=0,\ 1,\dots,\ m-s-1 \\ D_x^j u_n(t,\ 0)=(\xi(n))^j\{\exp(i\varepsilon nt)+B_1\exp(i\tau_1(\xi(n))t)+\cdots\cdots \\ \qquad +B_{m-s}\exp(i\tau_{m-s}(\xi(n))t)\},\ j=0,\ 1,\dots,\ s-1 \end{cases}$$

Because of $t\geqq 0$, the order of data with respect to n is polynomial (of n)$\times$exp $(c'n^{\omega})$, $\omega<\theta_1$.
On the otherhand the order of u_n is $\exp(cn^{\theta_1}x)$ $(c>0)$. Then the continuity of data to solution does not hold.
When $\Gamma(\tau,\ \xi)=0$ has multiple roots we treate in the same way as §4.

In the case where the roots of $L_m(\tau,\ 1,\ 0)=0$ are real and positive or 0 we take $\xi(n)$ with Im $b_1>0$ in Lemma 3.3. There exists such $\xi(n)$ is proved in the same way as Lemma 3.3.

§6. Remaining Case.

Finally we consider the remaining case. Suppose $L_m(\tau,\ 1,\ 0)=0$ has real roots with different sign. In this case we don't know that the conjecture is true or false.

Here we consider the simple example.

(6.1)
$$P=\partial_t^2\partial_x^2-\partial_x^4+\partial_t^3,\ \text{ where } \partial_t=\frac{\partial}{\partial t},\ \partial_x=\frac{\partial}{\partial x}.$$

Let the principal part of P be P_4;

(6.2) $$P_4(\tau, \xi)=\tau^2\xi^2-\xi^4.$$

The roots of $P(\tau, 1)=0$ are $\tau=1$ and $\tau=-1$. And obviously this P is not N-type. Concerning this example, the conjecture 1 is true. Namely

Proposition 6.1. *The Goursat problem for P is nct $\mathscr{E}$-wellposed.*

Proof. We prove this proposition by making the sequence of solutions of $Pu=0$ which does not hold the continuity of data to solution.

Let us consider the following Goursat problem.

(6.3) $$Pu=0$$

(6.4) $$u(t,0)=\exp(-tn),\ \partial_x u(t, 0)=0,\ u(0, x)=1,\ \partial_t u(0, x)=-n.$$

We remark that this Goursat data satisfy compatibility conditions. Let the formal solution of Problem (6.3)–(6.4) be the following:

(6.5) $$u_n=\sum_{j,k}\{u_{j,k}^{(n)}/j!k!\}t^j x^k.$$

Substituting (6.5) in (6.3) we have

(6.6) $$u_{j+2,k+2}^{(n)}=-u_{j+3,k}^{(n)}+u_{j,k+4}^{(n)} \qquad j,\ k\geqq 0.$$

By (6.4) it holds

(6.7) $$\begin{cases} u_{j,0}^{(n)}=(-n)^j,\ u_{j,1}^{(n)}=0 \text{ for } j\geqq 0, \\ u_{0,k}^{(n)}=0,\ u_{1,k}^{(n)}=0 \text{ for } k\geqq 1. \end{cases}$$

Concerning $u_{j,k}^{(n)}$, we have the following lemma.

Lemma 6.1. *It holds* i), ii) *and* iii).

i) *By* (6.6) *and* (6.7), $\{u_{j,k}^{(n)}\}$ *are determined unique, and formal solution* (6.5) *converge in* $(t,x)\in \boldsymbol{R}^2$.

ii) $u_{j,k}^{(n)}=0$ *when k is odd.*

iii) $u_{j,2k}^{(n)}=(-1)^j\{n^{j+k}+\sum_{s=1}^{j+k}p_s^{(j,k)}n^{j+k-s}\}$ *for* $j\geqq 2$, *where* $p_s^{(j,k)}\geqq 0$.

Let us notice $\partial_t^2 u_n(0, x)$. By (6.5) and Lemma 6.1,

(6.8) $$\partial_t^2 u_n(0, x)=\sum_k u_{2,k}^{(n)}/k!=\sum_k\{u_{2,2k}^{(n)}/(2k)!\}x^{2k}.$$

Using Lemma 6.1 again, we have

(6.9) $$u_{2,2k}^{(n)}\geqq n^{2+k}.$$

Then for $x>0$ it holds

(6.10) $$\partial_t^2 u^{(n)}(0, x)\geqq\sum_k\{n^{2+k}/(2k)!\}x^{2k}$$

$$=n^2\sum_k(\sqrt{n}\,x)^{2k}/(2k)!>(n^2/2)\exp(\sqrt{n}\,x).$$

Consider the sequence of solutions $\{u_n\}$. By (6.4), when $n\to\infty$ the order of n of Goursat data is at most polynomial. But by (6.10), the order of solution is exponential. This show that the continuity of data to solution does not hold. q.e.d.

Proof of Lemma 6.1. 1) Suppose $\{u_{j,k};\ j+k<p+q$ or $j+k=p+q,\ j<p\}$ are determined then by (6.6) $u_{p,q}$ is determined unique. The convergence of the formal solution is obvious (refer to [1]).

2) Goursat data (6.7) satisfy ii). Notice (6.6). If $k+2$ is odd then k and $k+4$ are odd. So by induction we prove ii).

3) By (6.6), we have

$$u^{(n)}_{j+2,2k+2}=-u^{(n)}_{j+3,2k}+u^{(n)}_{j,2k+4} \tag{6.11}$$

By (6.7)

$$u^{(n)}_{j,2k+4}=0 \text{ for } j=0,\ 1, \tag{6.12}$$

$$u^{(n)}_{j+3,0}=(-n)^{j+3}=(-1)^{j+3}n^{j+3}\quad j\geqq 0, \tag{6.13}$$

then $u^{(n)}_{j+3,0}$ has the form of iii) in Lemma 6.1. Suppose $u^{(n)}_{j+3,2k}$ has the form of iii) and $u^{(n)}_{j,2k+4}$ has the form of iii) or zero, then $u^{(n)}_{j+2,2k+2}$ becomes the following;

$$\begin{aligned}u^{(n)}_{j+2,2k+2}&=-(-1)^{j+3}\{n^{j+3+k}+\sum_{s=1}^{j+3+k}p_s^{(j+3,k)}n^{j+3+k-s}\}\\&\quad+(-1)^j\{\rho n^{j+k+2}+\sum_{s=1}^{j+k+2}p_s^{(j,k+2)}n^{j+k+2-s}\}\\&=(-1)^{j+2}\{n^{(j+2)+(k+1)}+(p_1^{(j+3,k)}+\rho)n^{j+2+k}\\&\quad+\sum_{s=2}^{j+3+k}(p_s^{(j+3,k)}+p_{s-1}^{(j,k+2)})n^{j+k+3-s}\}\end{aligned}$$

where $\rho=0$ for $j=0,\ 1$, $\rho=1$ for $j\geqq 2$.

Putting

$$\begin{cases}p_1^{(j+3,k)}+\rho=p_1^{(j+2,k+1)}\\ p_s^{(j+3,k)}+p_{s-1}^{(j,k+2)}=p_s^{(j+2,k+1)}\end{cases} \tag{6.12}$$

Then

$$u^{(n)}_{j+2,2k+2}=(-1)^{j+2}\{n^{(j+2)+(k+1)}+\sum_{s=1}^{j+2+k+1}p_s^{(j+3,k)}n^{j+k+3-s}\}. \tag{6.13}$$

So $u^{(n)}_{j+2,2k+2}$ has the form of iii). q.e.d.

Next, let us consider the following example.

$$\hat{P}=\partial_t^2\partial_x^2-\partial_x^4-\partial_t^3 \tag{6.14}$$

About this operator $\hat{P}$, we don't know that the conjecture 1 is true of false. But we have

Proposition 6.2. *The Goursat problem for $\hat{P}$ is not $\mathscr{E}$-wellposed for $t\leqq 0$. Namely*

$$(6.15)\qquad \begin{cases} \hat{P}u=0 \ x\in R^1, \ t\leqq 0 \\ \partial_t^i u(0, x)=\phi_i(x), \ i=0, 1 \\ \partial_x^j u(t, 0)=\psi_j(t), \ j=0, 1 \\ \partial_x^j\phi_i(0)=\partial_t^i\psi_j(0), \ i=0, 1, j=0, 1 \end{cases}$$

the problem (6.15) *is not $\mathscr{E}$-wellposed.*

Proof. Let $t=-t'$, Proposition 6.2 is reduced to Proposition 6.1.

Hereafter assuming $\mathscr{E}$-wellposedness for $t\geqq 0$ and $t\leqq 0$ we consider the conjecture:

Conjecture 2. If the Goursat problem (P) is $\mathscr{E}$-wellposed for $t\geqq 0$ and $t\leqq 0$, then the operator L is N-type.

Remark 6.1. When $t=0$ is simple characteristic, the operator is always N-type.

Remark 6.2. In the case where the order of differential operator is 3, the conjecture 1 is true (because of Theorem 2, Theorem 3 and Remark 6.1).

Let us consider the operator of order 4 with double characteristic.

$$(6.16)\qquad M=\partial_t^2\partial_x^2-\{a\partial_t^3+b\partial_t\partial_x^3+c\partial_x^4+\sum_{\substack{i+j\leqq 3\\ i\leqq 2}} a_{ij}\partial_t^i\partial_x^j\} \quad a_{ij};\ \text{real constant.}$$

We are going to show that the cinjecture 2 is true for M with $b\neq 0$ and a_{ij} small. The characteristic equation of principal part of M is

$$(6.17)\qquad \tau^2\xi^2=b\tau\xi^3+c\xi^4.$$

Suppose the roots of $\tau^2-b\tau-c=0$ are real and have different sygn.
Then

$$(6.18)\qquad c>0.$$

Here we assume

$$(6.19)\qquad a\neq 0 \text{ and } b\neq 0.$$

Without loss of generality, under the assumption (6.19), we can consider $a>0$, $b>0$ in (6.16) if necessary replacing $t\to -t$ and $x\to -x$. Let

$$(6.20)\quad \hat{M}=\partial_t^2\partial_x^2-\{a\partial_t^3+b\partial_t\partial_x^3+c\partial_x^4+\sum_{\substack{i+j\leqq 3\\ i\leqq 2}} a_{ij}^+\partial_t^i\partial_x^j-\sum_{\substack{i+j\leqq 3\\ i\leqq 2}} a_{ij}^-\partial_t^i\partial_x^j\},\ a,b,c>0,\ a_{ij}^+\geqq 0,\ a_{ij}^-\geqq 0.$$

where a_{ij}^+ and a_{ij}^- are the following;

when $a_{ij} \geqq 0$ we put $a_{ij}^{+}=a_{ij},\ a_{ij}^{-}=0$

when $a_{ij}<0$ we put $a_{ij}^{+}=0,\ a_{ij}^{-}=-a_{ij}$.

Concerning the coefficient a_{ij} we impose the following assumption;

$$(6.21)\qquad \begin{cases} \sum_{r=0}^{2} a_{r,0}^{-}\{8/(ab^2)\}^{3-r}+a_{2,1}^{-}(2/b) \leqq a/2, \\ \sum_{s=1}^{3} a_{0,s}^{-}\{4/(ab)\}^{4-s} \leqq c/2 \text{ and} \\ a_{1,1}^{-}\{4/(ab)\}^2+a_{1,2}^{-}\{4/(ab)\}+a_{2,1}^{-}(2/a) \leqq b/2. \end{cases}$$

Theorem 4. *If $a,b,c>0$ and* (6.21) *hold then the Goursat problem for $\hat{M}$ for $t \leqq 0$ is not $\mathscr{E}$-wellposed.*

§7. Proof of Theorem 4.

Suppose that the Goursat problem for $\hat{M}$ is $\mathscr{E}$-wellposed. Let us consider the following Goursat problem;

$$(7.1)\qquad \begin{cases} \hat{M}u=0, \\ u(t,\ 0)=\exp(n^2t)-\{1+n^2t+(n^2t)^2/2!\}, \\ \partial_x u(t,\ 0)=n\{\exp\ (n^2t)-(1+n^2t+(n^4t^2)/2!)\} \\ u(0,x)=0, \\ \partial_t u(0,\ x)=0. \end{cases}$$

Let u_n be the solution of (7.1), and (7.2) be the formal solution of (7.1).

$$(7.2)\qquad u_n(t,\ x)=\sum_{j,k}\{u_{jk}/(j!k!)\}t^j x^k$$

By (7.1) we have

$$(7.3)\qquad \begin{cases} u_{0,k}=0,\ u_{1,k}=0 \text{ for } k \geqq 0, \\ u_{j,0}=n^{2j},\ u_{j,1}=n^{2j+1} \text{ for } j \geqq 3. \end{cases}$$

Substituting (7.2) into $\hat{M}$u=0 it holds

$$(7.4)\qquad u_{j+2,k+2}=au_{j+3,k}+bu_{j+1,k+3}+cu_{j,k+4}+\sum_{\substack{r+s\leqq 2\\ r\leqq 2}}(a_{r,s}^{+}-a_{r,s}^{-})u_{j+r,k+s} \text{ for } j,\ k \geqq 0.$$

Here we remark that by (7.3) and (7.4) the formal solution (7.2) is determined unique.

Lemma 7.1. *If $a,\ b,\ c>0$ and* (6.21) *hold then the following four estimates hold for large n and for $j,\ k \geqq 0$.*

$$(7.5)\qquad u_{j+2,k+2} \geqq (a/2)^{[(k+2)/2]} n^{2(j+2)+k+2}$$

$$(7.6)\qquad u_{j+2,k+2} \geqq (a/2) u_{j+3,k}$$

$$u_{j+2,k+2} \geqq (b/2) u_{j+1,k+3} \tag{7.7}$$

$$u_{j+2,k+2} \geqq (c/2) u_{j,k+4}. \tag{7.8}$$

We prove this lemma later. By (7.2) we have

$$\partial_t^2 u_n|_{t=0} = \sum_{k\geqq 0} u_{2,k} x^k/k!. \tag{7.9}$$

By Lemma 7.1 we have the following estimate;

$$\partial_t^2 u_n(0, x) > \sum_{k=2}^{\infty} (a/2)^{[(k+2)/2]} n^{4+k+2} x^k/k! \text{ for } x>0. \tag{7.10}$$

This shows that $\partial_t^2 u(0, x)$ grows with exponential order of n for $x>0$. On the otherhand the Goursat data of (7.1) have polynomial order for $t\leqq 0$. Therefore the Goursat problem for $\hat{M}$ is not $\mathcal{E}$-wellposed for $t\leqq 0$.

Proof of Lemma 7.1. At first we remark that $u_{j,k}$ is the polynomial of n of degree at most $2j+k$. We rewrite (7.4).

$$\begin{aligned} u_{j+2,k+2} = (a/2)u_{j+3,k} + \{&(b/2)u_{j+1,k+3} + (c/2)u_{j,k+4} \\ &+ \sum_{\substack{r+s\leqq 3\\ r\leqq 2}} a_{r,s}^{+} u_{j+r,k+4} + (a/2)u_{j+3,k} + (b/2)u_{j+1,k+3} \\ &+ (c/2)u_{j,k+4} - \sum_{\substack{r+s\leqq 3\\ r\leqq 2}} a_{r,s}^{-} u_{j+r,k+s}\}. \end{aligned} \tag{7.4'}$$

Let us write $S_{j,k}^{f}$ the term $\{\cdots\cdots\}$ in (7.4′).

$$S_{j,k} = (a/2)u_{j+3,k} + (b/2)u_{j+1,k+3} + (c/2)u_{j,k+4} - \sum_{\substack{r+s\leqq 3\\ r\leqq 2}} a_{r,s}^{-} u_{j+r,k+s}. \tag{7.11}$$

Suppose $u_{j+2,k+2}$ with $j+k<p+q$ or $j+k=p+q$, $j<p$ satisfy Lemma 7.1. We shall show that $u_{p+2,q+2}$ satisfy Lemma 7.1.

If $S_{p,q}\geqq 0$ then $u_{p+2,+2}$ satisfy Lemma 7.1. So we want to show $S_{p,q}\geqq 0$.

Case 1. $p+q\leqq N$ where N is some finite number.

By the assumption of induction

$$(a/2)u_{p+3,q} > (a/2)^{[(q+2)/2]} n^{2p+q+6} \tag{7.12}$$

And $\sum a_{r,s}^{-} u_{p+r,q+s}$ is the polynomial of degree at most $2p+q+5$. Then for sufficient large n we have $S_{p,q}>0$.

Case 2–1. where $p+q>N$ and $p=0$.

By (7.3) it holds

$$S_{0,q} = (a/2)u_{3,q} - (a_{2,0}^{-} u_{2,q} + a_{2,1}^{-} u_{2,q+1}) \tag{7.13}$$

By the assumption of induction

$$u_{2,q+1} \leqq (2/b) u_{3,q} \tag{7.14}$$

$$u_{2,q} \leqq (2/b) u_{3,q-1} \leqq (2/b)(2/a) u_{2,q+1} \leqq (2/b)(2/b)(2/a) u_{3,q}. \tag{7.15}$$

By (6.2) and (7.13), (7.14), (7.15) it holds

$$S_{0,q} \geqq \{(a/2)-(2/b)a_{2,1}^{-}-(8/ab^2)a_{2,0}^{-}\}u_{3,q}>0. \tag{7.16}$$

Case 2–2. where $p+q>N$ and $q=0$ or $q=1$.

$$\begin{aligned} S_{p,q}= &\{(a/2)u_{p+3,q}-\sum_{r=0}^{2}a_{r,0}^{-}u_{p+r,q}\} \\ &+\{(b/2)u_{p+1,q+3}-(a_{1,1}^{-}u_{p+1,q+1}+a_{1,2}^{-}u_{p+1,q+2}+a_{2,1}^{-}u_{p+2,q+1})\} \\ &+\{(c/2)u_{p,q+4}-\sum_{s=1}^{3}a_{0,s}^{-}u_{p,q+s}\}\equiv S_{p,q}^{(1)}+S_{p,q}^{(2)}+S_{p,q}^{(3)}. \end{aligned} \tag{7.17}$$

Here $S_{p,q}^{(1)}$ stands for the first $\{\cdots\cdots\}$ in the right handside of (7.17), and $S_{p,q}^{(2)}$ stands for the second $\{\cdots\cdots\}$, $S_{p,q}^{(3)}$ stands for the last $\{\cdots\cdots\}$. First, we consider $S_{p,q}^{(1)}$. By (7.3) we hauve

$$u_{p+r,q}=n^{2(p+r)+q} \text{ for } q=0 \text{ or } q=1. \tag{7.18}$$

Then

$$\begin{aligned} S_{p,q}^{(1)} &= (a/2)u_{p+3,q}-\sum_{r=0}^{2}a_{r,0}^{-}u_{p+r,q} \\ &=(a/2)n^{2(p+3)+q}-\sum_{r=0}^{2}a_{r,0}^{-}n^{2(p+r)+q} \\ &=n^{2(p+3)+q}\{(a/2)-\sum_{r=0}^{2}a_{r,0}^{-}n^{2(r-3)}\}. \end{aligned} \tag{7.19}$$

So for large n we have

$$S_{p,q}^{(1)}>0. \tag{7.20}$$

Next, we consider $S_{p,q}^{(2)}$. By the assumption of induction we have

$$\begin{aligned} u_{p+1,q+1} &\leqq (2/a)u_{p,q+3} \leqq (2/a)(2/b)u_{p+1,q+2} \\ &\leqq (2/a)(2/b)(2/b)u_{p+2,q+1} \leqq (2/a)(2/b)(2/b)(2/a)u_{p+1,q+3} \end{aligned} \tag{7.21}$$

In the same way we have

$$u_{p+1,q+2} \leqq (4/ab)u_{p+1,q+3}, \quad u_{p+2,q+1} \leqq (2/a)u_{p+1,q+3}.$$

Therefore

$$S_{p,q}^{(2)} \geqq [(2/b)-\{a_{11}^{-}(4/ab)^2+a_{12}^{-}(4/ab)+a_{21}^{-}(2/a)\}]u_{p+1,q+3}. \tag{7.22}$$

Then by (6.21) it holds

$$S_{p,q}^{(2)}>0. \tag{7.23}$$

Lastly we consider $S_{p,q}^{(3)}$. By the assumption of induction we have

$$u_{p,q+3} \leqq (4/ab)u_{p,q+4}, \quad u_{p,q+2}(4/ab)^2u_{p,q+4}, \quad u_{p,q+1} \leqq (4/ab)^3u_{p,q+4}.$$

Then by (6.21) we have

$$S^{(3)}_{p,q} \geqq \{(c/2) - a^-_{0,3}(4/ab) - a^-_{0,2}(4/ab)^2 - a^-_{0,1}(4/ab)^3\} u_{p,q+4} \geqq 0.$$

Case 2–3 where $p+q>N$ and $p \geqq 1$, $q \geqq 2$.
In this case we separate $S_{p,q}$ into three parts in the same way as Case 2–2. We can estimate $S^{(2)}_{p,q}$ and $S^{(3)}_{p,q}$ in the very same way as Case 2–2. By the assumption of induction we have

$$S^{(1)}_{p,q} \geqq \{(a/2) - \sum_{r=0}^{2} a_{r,0}(8/ab^2)^{3-r}\} u_{p+3,q}.$$

Because of (6.21), it holds

$$S^{(1)}_{p,q} > 0.$$

Thus we complete the proof of Lemma 7.1.

Remark 7.1. In (6.21) we can replace $4/b^2$ by $2/c$.

Remark 7.2. When $b=0$ we don't know that the conjecture 2 is true or false except special case (c.f (6.1), (6.14)).

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