J. Math. Kyoto Univ. (JMKYAZ)
28-2 (1988) 373-381

Around Yor's theorem on the Brownian sheet and local time

By

Endre CSÁKI, Antónia FÖLDES and Yuji KASAHARA*)

1. Introduction.

Let $\{X_t; t\in \boldsymbol{R}_+\}$ ($\boldsymbol{R}_+=[0, \infty)$) be a standard Brownian motion starting at 0 and let $\{L(t, x); (t, x)\in \boldsymbol{R}_+\times \boldsymbol{R}\}$ be its local time; i.e., $L(t, x)$ is a jointly continuous process such that

$$\int_A L(t, x)dx=\int_0^t 1_A(X_s)ds\,, \qquad \text{a.s.},$$

for all $t\geqq 0$ and every measurable set A. We also let $\ell(t)=(1/2)L(t, 0)$. The following characterizations of $\ell(t)$ are well-known (see pages 43 and 48 of Itô-McKean [6] and page 130 of Ikeda-Watanabe [5]): Let

$d_\varepsilon(t)$=the number of times that $|X_s|$ crosses down from ε to 0 up to time t,

$\eta_\varepsilon(t)$=the number of excursion intervals in $(0, t]$ of length $\geqq\varepsilon$

and

$\xi_\varepsilon(t)$=the total length of excursion intervals in $(0, t]$ of length $<\varepsilon$.

Then, it holds that

$$\lim_{\varepsilon\downarrow 0} \varepsilon d_\varepsilon(t)=2\ell(t)\,, \qquad \text{a.s.}, \tag{1.1}$$

$$\lim_{\varepsilon\downarrow 0} \sqrt{\pi\varepsilon/2}\,\eta_\varepsilon(t)=2\ell(t)\,, \qquad \text{a.s.}, \tag{1.2}$$

and

$$\lim_{\varepsilon\downarrow 0} \sqrt{\pi/(2\varepsilon)}\,\xi_\varepsilon(t)=2\ell(t)\,, \qquad \text{a.s.} \tag{1.3}$$

Some limit theorems for the fluctuations are obtained by one of the authors ([8], see also [2, 7]):

Theorem A. *As* $\varepsilon\downarrow 0$, *we have that*

$$\varepsilon^{-1/2}\{\varepsilon d_\varepsilon(t)-2\ell(t)\} \xrightarrow{\mathscr{D}} \sqrt{2}\,B(\ell(t))\,,$$

$$\varepsilon^{-1/4}\{\sqrt{\pi\varepsilon/2}\,\eta_\varepsilon(t)-2\ell(t)\} \xrightarrow{\mathscr{D}} (2\pi)^{1/4}B(\ell(t))$$

*) Research of the first two authors is supported by Hungarian National Foundation for Scientific Research Grant No. 1808.
Communicated by Prof. S. Watanabe, May 6, 1987

and

$$\varepsilon^{-1/4}\{\sqrt{\pi/(2\varepsilon)}\,\xi_{\varepsilon}(t)-2\ell(t)\} \xrightarrow{\mathscr{D}} (2\pi/9)^{1/4}B(\ell(t))$$

in the function space $D(\boldsymbol{R}_{+}; \boldsymbol{R})$, *where* $\{B(t)\}_{t\geq 0}$ *is a standard Brownian motion independent of* $\{\ell(t)\}_{t\geq 0}$.

(Throughout the paper $\xrightarrow{\mathscr{D}}$ denotes the convergence in law.)

In the present paper we introduce another parameter $x(>0)$ and extend Theorem A for two-parametered processes $(t, x)\mapsto d_{\varepsilon x}(t)$, $(t, x)\mapsto\eta_{\varepsilon x}(t)$ and $(t, x)\mapsto\xi_{\varepsilon x}(t)$, and show that, under suitable normalizations, *Brownian sheets* appear in the limiting processes. By a Brownian sheet we mean a jointly continuous Gaussian process $\{W(t, x); (t, x)\in\boldsymbol{R}_{+}^{2}\}$ with mean zero and covariance $E[W(t, x)W(s, y)]=\min(t, s)\cdot\min(x, y)$. This problem is motivated by the following result due to M. Yor ([10]), who obtained a Brownian sheet from the local time of a one-dimensional Brownian motion.

Theorem B. (M. Yor) *As* $\lambda\to\infty$,

$$\left(X_{t}, \ell(t), \frac{\sqrt{\lambda}}{2}\left\{L\left(t, \frac{a}{\lambda}\right)-2\ell(t)\right\}\right) \xrightarrow{\mathscr{D}} (X_{t}, \ell(t), \sqrt{2}W(\ell(t), a)),$$

over the function space $C(\boldsymbol{R}_{+}^{2}; \boldsymbol{R}^{3})$, *where* $\{W(t, x); (t, x)\in\boldsymbol{R}_{+}^{2}\}$ *is a Brownian sheet independent of* $\{X_{t}\}_{t\geq 0}$.

Here, $C(\boldsymbol{R}_{+}^{2}; \boldsymbol{R}^{3})$ is the space of all continuous functions from $\boldsymbol{R}_{+}^{2}$ to $\boldsymbol{R}^{3}$ endowed with the topology of compact uniform convergence.

In the present paper, we shall prove similar results for downcrossings and other characteristics. Main results will be stated in Section 2. In Section 3 we shall study the case of random walks, and in Section 4 we shall give other limit theorems for d_{ε} and η_{ε}. The proof of the results in Section 2 is based on point process method and is quite different from Yor's while the results in Sections 3 and 4 will be proved by reducing to Yor's theorem (Theorem B) using strong invariance principles.

2. Main results.

Theorem B is considered in the function space $C(\boldsymbol{R}_{+}^{2}; \boldsymbol{R}^{3})$ but this space is not suited for downcrossings because $d_{\varepsilon}(t)$, $\eta_{\varepsilon}(t)$ and $\xi_{\varepsilon}(t)$ are discontinuous. Therefore, we shall consider the function space $D(\boldsymbol{R}_{+}^{n}; \boldsymbol{R}^{m})$, the space of all functions from $\boldsymbol{R}^{n}$ to $\boldsymbol{R}^{m}$ which are continuous from above and have limits from below. We endow this space with Bickel-Wichura's S-topology: f_{k} converges to f in $D(\boldsymbol{R}_{+}^{n}; \boldsymbol{R}^{m})$ if and only if there exist $\lambda_{k}(t)=(\lambda_{k}^{(1)}(t_{1}), \cdots, \lambda_{k}^{(n)}(t_{n}))$, $k\geq 1$, satisfying the following three conditions.

(i) $\lambda_{k}^{(j)}(\cdot)$ is a strictly increasing, continuous function from $\boldsymbol{R}_{+}$ to itself such that $\lambda_{k}^{(j)}(0)=0$ and $\lambda_{k}^{(j)}(\infty)=\infty$, $(1\leq j\leq n,\ k\geq 1)$.

(ii) $\lambda_{k}(t)$ converges to $\lambda_{\infty}(t)\equiv t$ uniformly on every compact subset of $\boldsymbol{R}_{+}^{n}$

as $k\to\infty$, $(j=1, \cdots, n)$.

(iii) $f_k(\lambda_k(t))$ converges to $f(t)$ uniformly on every compact subset of $\boldsymbol{R}_+^n$ as $k\to\infty$.

For the details we refer to Bickel-Wichura [1]. Notice that $d_{1/a}(t)$, $\eta_{1/a}(t)$ and $\xi_a(t)$ are nondecreasing in a and t, and hence choosing suitable versions we may regard them as random elements of $D(\boldsymbol{R}_+^2;\boldsymbol{R})$.

Our main results are the following.

Theorem 1. *As* $\varepsilon\to+0$,

$$(\varepsilon^{-1/2}\{\varepsilon d_{\varepsilon/a}(t)-2a\ell(t)\},\ \ell(t)) \xrightarrow{\mathcal{D}} (\sqrt{2}W(\ell(t), a), \ell(t)) \quad in\ D(\boldsymbol{R}_+^2;\boldsymbol{R}^2).$$

where $\{W(t, a); t, a\geqq 0\}$ *is a Brownian sheet independent of* $\{\ell(t); t\geqq 0\}$.

Theorem 2. *As* $\varepsilon\to+0$,

$$(\varepsilon^{-1/4}\{\sqrt{\varepsilon}\,\eta_{\varepsilon/a}(t)-\sqrt{8a/\pi}\,\ell(t)\},\ \ell(t)) \xrightarrow{\mathcal{D}} \left(\left(\frac{8}{\pi}\right)^{1/4}W(\ell(t), \sqrt{a}), \ell(t)\right) \quad in\ D(\boldsymbol{R}_+^2;\boldsymbol{R}^2),$$

where $W(t, a)$ *is the same as in Theorem* 1.

Theorem 3. *As* $\varepsilon\to+0$,

$$\left(\varepsilon^{-1/4}\left\{\frac{1}{\sqrt{\varepsilon}}\xi_{\varepsilon a}(t)-\sqrt{8a/\pi}\,\ell(t)\right\},\ \ell(t)\right) \xrightarrow{\mathcal{D}} \left(\left(\frac{8}{9\pi}\right)^{1/4}W(\ell(t), a^{3/2}), \ell(t)\right) \quad in\ D(\boldsymbol{R}_+^2;\boldsymbol{R}^2),$$

where $W(t, a)$ *is the same as in Theorem* 1.

Proof of Theorem 1. Throughout we shall use the convention that $d_\varepsilon(t)\equiv 0$ when $\varepsilon=\infty$, and $f^{-1}(t)$ denotes the right-continuous inverse function of the non-decreasing function $f(t)$. Also $X\overset{d}{=}Y$ means that X and Y are equally distributed. Notice that $d_\varepsilon(\ell^{-1}(t))$ may be expressed as follows:

$$d_\varepsilon(\ell^{-1}(t))=N_1((0, t]\times(\varepsilon, \infty))$$

where $N_1(dt, da)$ is a Poisson random measure on $(0, \infty)^2$ such that $E[N_1(dt, dx)]=2x^{-2}dtdx$, (see page 130 of Ikeda-Watanabe [5]). We also remark that for every fixed $t>0$, $\ell^{-1}(\ell(t))$ is the first hitting time of 0 after time t. Therefore, see that

$$0\leqq d_\varepsilon(\ell^{-1}(\ell(t)))-d_\varepsilon(t)\leqq 1, \quad \text{a.e.} \tag{2.1}$$

Lemma 2.1. *Let*

$$W_\varepsilon(t, a)=\frac{1}{\sqrt{2\varepsilon}}\{\varepsilon d_{\varepsilon/a}(\ell^{-1}(t))-2at\}, \qquad t, a\geqq 0.$$

Then, as $\varepsilon\to+0$,

$$(W_\varepsilon(t, a), \ell^{-1}(t)) \xrightarrow{\mathcal{D}} (W(t, a), \ell^{-1}(t)), \qquad in\ D(\boldsymbol{R}_+^2; \boldsymbol{R}^2),$$

where $W(t, a)$ *is the same as in Theorem* 1.

Proof. Let $N(t, a)=N_1\Big((0, t]\times\Big(\frac{1}{a}, \infty\Big)\Big)$, $t, a\geqq 0$. Since N is a two-parametered Poisson process with $E[N(t, a)]=2at$, it is easy to see that, as $\varepsilon\downarrow 0$,

$$W_\varepsilon(t, a)=\frac{1}{\sqrt{2\varepsilon}}\Big\{\varepsilon N\Big(t, \frac{a}{\varepsilon}\Big)-2at\Big\} \xrightarrow{\mathcal{D}} W(t, a), \qquad in\ D(\boldsymbol{R}_+^2; \boldsymbol{R}).$$

Indeed, for every fixed $t\geqq 0$, the convergence of $W_\varepsilon(t, \cdot)$ to $W(t, \cdot)$ is the usual invariance principle and furthermore, for $t>s\geqq 0$, $W_\varepsilon(t, \cdot)-W_\varepsilon(s, \cdot)$ is independent of $\{W_\varepsilon(u, \cdot); u\leqq s\}$. Therefore, we have that all finite-dimensional distributions of W_ε converge weakly to those of W. In order to prove the tightness in S-topology it is sufficient to show the following two conditions:

(i) $W_\varepsilon(t, 0)=W_\varepsilon(0, a)=0$, a.s., for every $t\geqq 0$ and $a\geqq 0$.

(ii) There exists a constant $K>0$ such that $E[W_\varepsilon(B)^2W_\varepsilon(C)^2]\leqq K\mu(B\cup C)^2$ for every neighboring blocks B and C of $\boldsymbol{R}_+^2$, where $W_\varepsilon(A)$ is the increment of W_ε around A and μ denotes the Lebesgue measure.

For details we refer to Bickel-Wichura [1]. Let us check the above conditions. (i) is clear by definition, and (ii) is also easy because $W_\varepsilon(B)$ and $W_\varepsilon(C)$ are independent and $E[W_\varepsilon(A)^2]=\mu(A)$. Therefore, (ii) is satisfied with $K=1$. Thus we have seen that $W_\varepsilon(t, a)\xrightarrow{\mathcal{D}} W(t, a)$ in $D(\boldsymbol{R}_+^2; \boldsymbol{R})$. It remains to show the independence of $W(\cdot, \cdot)$ and $\ell(\cdot)$. Let $0\leqq a_1<\cdots<a_n$. Then for every $\varepsilon>0$, $Z_\varepsilon(t)=(W_\varepsilon(t, a_1), \cdots, W_\varepsilon(t, a_n), \ell^{-1}(t))$ is an $\boldsymbol{R}^{n+1}$-valued Lévy process. Let $Z^*(t)=(W^*(t, a_1), \cdots, W^*(t, a_n), \psi^*(t))$ be any limit process of $\{Z_\varepsilon\}_\varepsilon$. (As we have seen above, $\{Z_\varepsilon\}_\varepsilon$ is tight in $D(\boldsymbol{R}_+; \boldsymbol{R}^{n+1})$.) Of course $W^*\stackrel{d}{=}W$ and $\psi^*\stackrel{d}{=}\psi$, and furthermore, Z^* has independent increments since so has Z_ε. Since $(W^*, 0)$ and $(0, \psi^*)$ are respectively the Gaussian and the Poisson part of Lévy process Z^*, we conclude that W^* and ℓ^* are independent by Lévy-Itô's theorem.

(Q.E.D.)

We continue the proof of Theorem 1. It is well known as Skorohod's theorem that convergence in law can be realized by an almost everywhere convergence without changing the laws. Therefore, by Lemma 2.1, we can construct stochastic processes W_ε^*, ℓ_ε^* $(\varepsilon>0)$, W^* and ℓ^* (on a suitable probability space) such that

$$(W_\varepsilon^*(t, a), \ell_\varepsilon^{*-1}(t)) \stackrel{d}{=} (W_\varepsilon(t, a), \ell^{-1}(t)), \tag{2.2}$$

$$(W^*(t, a), \ell^{*-1}(t)) \stackrel{d}{=} (W(t, a), \ell^{-1}(t)), \tag{2.3}$$

and

(2.4) $$\lim_{\varepsilon\downarrow 0}(W_{\varepsilon}^{*}, \ell_{\varepsilon}^{*-1})=(W, \ell^{*-1}), \quad \text{in } D(\boldsymbol{R}_{+}^{2}; \boldsymbol{R}^{2}), \text{ a.s.}$$

Since $\ell^{*}(t)$ is continuous, it follows from (2.4) that $\ell_{\varepsilon}^{*}(t)$ converges to $\ell^{*}(t)$ uniformly for t on every finite interval with probability one. Therefore, we see that $W_{\varepsilon}^{*}(\ell_{\varepsilon}^{*}(t), a)$ converges to $W^{*}(\ell^{*}(t, a))$ in $D(\boldsymbol{R}_{+}^{2}; \boldsymbol{R})$ almost surely, which combined with (2.2) and (2.3) implies

(2.5) $$(W_{\varepsilon}(\ell(t), a), \ell(t)) \xrightarrow{\mathcal{D}} (W(\ell(t)), a), \ell(t)).$$

Keeping (2.1) and definition of W_{ε} in mind, we see that (2.5) proves the assertion of Theorem 1. (Q. E. D.)

Proof of Theorem 2. It is known that $\eta_{\varepsilon}(\ell^{-1}(t))=N_{2}((0, t]\times(\varepsilon, \infty))$, where $N_{2}(dt, dx)$ is a Poisson random measure on $(0, \infty)^{2}$ with $E[N_{2}(dt, dx)]=\sqrt{2/\pi}\, x^{-3/2}dtdx$ (see page 130 of Ikeda-Watanabe [5]). Therefore, the assertion of Lemma 2.1 holds for

$$W_{\varepsilon}(t, a)=\left(\frac{\pi}{8}\right)^{1/4}\varepsilon^{-1/4}\{\sqrt{\varepsilon}\eta_{\varepsilon/a^{2}}(\ell^{-1}(t))-\sqrt{8/\pi}\,at\}, \qquad t, a\geqq 0.$$

Now the rest of the proof of Theorem 2 is the same as that of Theorem 1. (Q. E. D.)

Proof of Theorem 3. Notice that $\xi_{\varepsilon}(\ell^{-1}(t))$ may be expressed as follows:

$$\xi_{\varepsilon}(\ell^{-1}(t))=\iint_{(0, t]\times(0, \varepsilon)} xN_{2}(ds, dx),$$

where N_{2} is the same as in the proof of Theorem 2. Let

$$X_{\varepsilon}(t, a)=\varepsilon^{-1/4}\left\{\frac{1}{\sqrt{\varepsilon}}\xi_{\varepsilon a}(\ell^{-1}(t))-\sqrt{8a/\pi}\,t\right\}, \qquad t, a\geqq 0.$$

Then $X_{\varepsilon}(t, a)$ may be expressed as follows:

$$X_{\varepsilon}(t, a)=\varepsilon^{-3/4}\iint_{(0, t]\times(0, a\varepsilon]} x\tilde{N}_{2}(ds, dx),$$

where, $\tilde{N}_{2}(dt, dx)=N_{2}(dt, dx)-E[N_{2}(dt, dx)]$. Therefore, for every $0\leqq a<b$,

$$E[X_{\varepsilon}(t, a)X_{\varepsilon}(t, b)]=\varepsilon^{-3/2}t\int_{0}^{\varepsilon a} x^{2}\sqrt{2/\pi}\,x^{-3/2}dx$$
$$=\sqrt{8/(9\pi)}\,a^{3/2}t.$$

Thus we easily see that the finite-dimensional marginal distributions of $X_{\varepsilon}(t, a)$ converge to those of $(8/(9\pi))^{1/4}W(t, a^{3/2})$ (see, for example, Theorem 4.1 of Kasahara-Watanabe [9]). Tightness in $D(\boldsymbol{R}_{+}^{2}; \boldsymbol{R})$ as well as the rest of the proof of Theorem 3 can be established in a similar way as in the proof of Theorem 1, and we omit the details. (Q. E. D.)

To conclude this section we remark that, by the scaling property of Brownian motion, Theorems 1-3 may be rewritten as follows:

Theorem 1′. *As* $\lambda\to\infty$,

$$(\lambda^{-1/4}\{d_{1/a}(\lambda t)-2a\ell(\lambda t)\},\ \lambda^{-1/2}\ell(\lambda t))$$
$$\xrightarrow{\mathcal{D}} (\sqrt{2}W(\ell(t),\ a),\ \ell(t)) \qquad in\ D(\boldsymbol{R}_+^2\,;\ \boldsymbol{R}^2).$$

where $W(t,\ a)$ *is the same as in Theorem* 1.

Theorem 2′. *As* $\lambda\to\infty$,

$$(\lambda^{-1/4}\{\eta_{1/a}(t)-\sqrt{8a/\pi}\,\ell(\lambda t)\},\ \lambda^{-1/2}\ell(\lambda t))$$
$$\xrightarrow{\mathcal{D}} \left(\left(\frac{8}{\pi}\right)^{1/4}W(\ell(t),\ \sqrt{a}),\ \ell(t)\right) \qquad in\ D(\boldsymbol{R}_+^2\,;\ \boldsymbol{R}^2),$$

where $W(t,\ a)$ *is the same as in Theorem* 1.

Theorem 3′. *As* $\lambda\to\infty$,

$$(\lambda^{-1/4}\{\xi_a(t)-\sqrt{8a/\pi}\,\ell(\lambda t)\},\ \lambda^{-1/2}\ell(\lambda t))$$
$$\xrightarrow{\mathcal{D}} \left(\left(\frac{8}{9\pi}\right)^{1/4}W(\ell(t),\ a^{3/2}),\ \ell(t)\right) \qquad in\ D(\boldsymbol{R}_+^2\,;\ \boldsymbol{R}^2),$$

where $W(t,\ a)$ *is the same as in Theorem* 1.

3. Local times of random walk.

Let $X_1,\ X_2,\ \cdots$, be a sequence of independent, identically distributed (i. i. d.) random variables taking values on integers $\boldsymbol{Z}$. Throughout we assume that

$$E(X_1)=0\,, \qquad \mathrm{Var}\,(X_1)=\sigma^2<\infty\,,$$

and

$$\text{g. c. d.}\{k\,;\ P(X_1=k)>0\}=1\,.$$

We further put a technical assumption that

$$E\{|X_1|^{9+\varepsilon}\}<\infty \qquad \text{for some } \varepsilon>0\,.$$

The random walk $S_0=0,\ S_n=X_1+X_2+\cdots+X_n,\ n\geqq 1$, is recurrent and the local time is defined by

$$N(t,\ j)=\#\{k\,;\ 0\leqq k\leqq t,\ S_k=j\}, \qquad t\in\boldsymbol{R}_+,\ j\in\boldsymbol{Z}\,.$$

For real x, we define $N(t,\ x)=N(t,\ [x])$.

Theorem 4. *For* $\lambda\to\infty$, *and* $\varepsilon=\varepsilon(\lambda)\downarrow 0$ *such that* $\sqrt{\varepsilon}\log\lambda\to 0$ *and* $\lambda\varepsilon^2\to\infty$, *we have*

$$\left(\varepsilon^{1/2}\lambda^{-1/4}\left\{N\left(\lambda t,\ \frac{a}{\varepsilon}\right)-N(\lambda t,\ 0)\right\},\ \lambda^{-1/2}N(\lambda t,\ 0)\right)$$
$$\xrightarrow{\mathcal{D}} \left(\left(\frac{2}{\sigma}\right)^{3/2}W(\ell(t),\ a),\ \frac{2}{\sigma}\ell(t)\right),$$

in $D(\boldsymbol{R}_+^2; \boldsymbol{R}^2)$, *where* $\ell(\cdot)$ *and* $W(\cdot, \cdot)$ *are the same as in Theorem* 1.

Proof. From Theorem 1.2 of Borodin ([3]) we have that, under the above conditions,

$$\sup_x |L(n\sigma^2, x)-\sigma^2 N(n, x)| = O(n^{1/4}\log n), \quad \text{a.s.,} \qquad \text{as } n\to\infty\,,$$

for an appropriate construction of $L(\cdot, \cdot)$ and $N(\cdot, \cdot)$. Consequently,

$$\varepsilon^{1/2}\lambda^{-1/4}\left\{N\left(\lambda t, \frac{a}{\varepsilon}\right)-N(\lambda t, 0)\right\}$$

$$=\sigma^{-2}\varepsilon^{1/2}\lambda^{-1/4}\left\{L\left(\lambda\sigma^2 t, \left[\frac{a}{\varepsilon}\right]\right)-L(\lambda\sigma^2 t, 0)\right\}+O(\varepsilon^{1/2}\log\lambda)\,.$$

By the scaling property of the Brownian motion, letting $a_\varepsilon=\varepsilon[a/\varepsilon]$, we see that

$$\sigma^{-2}\varepsilon^{1/2}\lambda^{-1/4}\left\{L\left(\lambda\sigma^2 t, \left[\frac{a}{\varepsilon}\right]\right)-L(\lambda\sigma^2 t, 0)\right\}$$

is distributed like

$$\sigma^{-3/2}(\lambda')^{1/2}\{L(t, a_\varepsilon/\lambda')-L(t, 0)\}, \qquad \lambda'=\varepsilon\sigma\sqrt{\lambda}\,,$$

which converges to $2\sqrt{2}\sigma^{-3/2}W(\ell(t), a)$ as $\lambda\varepsilon^2\to\infty$, by Theorem B. Similarly one gets that $\lambda^{-1/2}N(\lambda t, 0)$ converges in law to $(2/\sigma)\ell(t)$. (Q.E.D.)

Remark. Obviously $N(t, a)$ can be replaced by any other characteristic of the local time for which Theorem 1.2 or 1.4 of Borodin ([3]) holds.

4. Downcrossings and excursions around various points.

In the present section we consider downcrossings and excursions around various points. Let $\{X_t\}_{t\geq 0}$ be a standard Brownian motion as before, and for $a\in\boldsymbol{R}$ define

$d_\varepsilon(t; a)$=the number of times that $|X_s-a|$ crosses down from $x=\varepsilon$ to $x=0$ by time t

and

$\eta_\varepsilon(t; a)$=the number of excursion intervals of X_t-a in $(0, t]$ of length $\geqq\varepsilon$.

Theorem 5. *For* $\lambda\to\infty$, *and* $\varepsilon=\varepsilon(\lambda)\downarrow 0$ *such that* $\lambda\varepsilon(\log 1/\varepsilon)^2\to 0$, *we have that*

$$\left(X_t, \ell(t), \sqrt{\lambda}\varepsilon\left\{d_\varepsilon\left(t; \frac{a}{\lambda}\right)-d_\varepsilon(t; 0)\right\}\right)$$

$$\xrightarrow{\mathcal{D}} (X_t, \ell(t), 2\sqrt{2}W(\ell(t), a)),$$

in $D(\boldsymbol{R}_+^2; \boldsymbol{R}^3)$, *where* $\ell(\cdot)$ *and* $W(\cdot, \cdot)$ *are the same as in Theorem B.*

Proof. It is proved by Borodin (Theorem 1.5 of [2]) that

$$\varepsilon d_\varepsilon(t\,;a)-L(t,a)=O\Big(\sqrt{\varepsilon}\log\frac{1}{\varepsilon}\Big),\quad \text{a.s.,}\qquad \text{as } \varepsilon\downarrow 0\,, \tag{4.1}$$

uniformly for $(t, a)\in[0, T]\times\boldsymbol{R}$ for every fixed $T>0$. Therefore, we get

$$\sqrt{\lambda}\varepsilon\Big\{d_\varepsilon\Big(t\,;\frac{a}{\lambda}\Big)-d_\varepsilon(t\,;0)\Big\}$$

$$=\sqrt{\lambda}\Big\{L\Big(t,\frac{a}{\lambda}\Big)\Big)-L(t,0)\Big\}+\sqrt{\lambda}O\Big(\sqrt{\varepsilon}\log\frac{1}{\varepsilon}\Big),\ \text{a.s.,}\quad \text{as } \varepsilon\downarrow 0\,,$$

where $O(\sqrt{\varepsilon}\log 1/\varepsilon)$ is uniform in $(t, a, \lambda)\in[0, T]\times\boldsymbol{R}\times\boldsymbol{R}$ for every fixed $T>0$. This combined with Theorem B proves the theorem. (Q. E. D.)

Corollary. *For $\varepsilon=\varepsilon(\lambda)$ such that $\varepsilon\downarrow 0$ and $\sqrt{\varepsilon}\log\lambda\to 0$, as $\lambda\to\infty$, we have,*

$$\Big(\frac{1}{\sqrt{\lambda}}X_{\lambda t},\ \frac{1}{\sqrt{\lambda}}\ell(\lambda t),\ \varepsilon\lambda^{-1/4}\{d_\varepsilon(\lambda t\,;a)-d_\varepsilon(\lambda t\,;0)\}\Big)$$

$$\xrightarrow{\mathscr{D}}(X_t,\ \ell(t),\ 2\sqrt{2}W(\ell(t),a))$$

in $D(\boldsymbol{R}_+^2\,;\boldsymbol{R}^3)$.

Proof. By the scale change property of the Brownian motion, the assertion of Theorem 5 may be rewritten as

$$\Big(\frac{1}{\lambda}X_{\lambda^2 t},\ \frac{1}{\lambda}\ell(\lambda^2 t),\ \sqrt{\lambda}\varepsilon\{d_{\lambda\varepsilon}(\lambda^2 t\,;a)-d_{\lambda\varepsilon}(\lambda^2 t\,;0)\}\Big)$$

$$\xrightarrow{\mathscr{D}}(X_t,\ \ell(t),\ 2\sqrt{2}W(\ell(t),a))\,.$$

Replacing λ^2 by λ' and ε by $\varepsilon'/\sqrt{\lambda'}$ we have the assertion. (Q. E. D.)

Theorem 6. *For $\varepsilon=\varepsilon(\lambda)$ such that $\varepsilon\downarrow 0$ and $\sqrt{\lambda}\varepsilon^{1/4}\log 1/\varepsilon=O(1)$ as $\lambda\to\infty$, we have*

$$\Big(X_t,\ \ell(t),\ \sqrt{\varepsilon}\lambda\Big\{\eta_\varepsilon\Big(t\,;\frac{a}{\lambda}\Big)-\eta_\varepsilon(t\,;0)\Big\}\Big)$$

$$\xrightarrow{\mathscr{D}}\Big(X_t,\ \ell(t),\ \frac{4}{\sqrt{\pi}}W(\ell(t),a)\Big),$$

in $D(\boldsymbol{R}_+^2\,;\boldsymbol{R}^3)$, where $\ell(\cdot)$ and $W(\cdot,\cdot)$ are the same as in Theorem B.

Proof. The proof is essentially the same as that of Theorem 5. The only difference is that we use the following result due to Csörgő and Révész ([4]) in place of (4.1).

$$\sup_{(a,t)\in\boldsymbol{R}\times[0,T]}|\sqrt{\varepsilon}\eta_\varepsilon(t\,;a)-\sqrt{2/\pi}L(t,a)|=o\Big(\varepsilon^{1/4}\log\frac{1}{\varepsilon}\Big),\quad \text{a.s.}$$

(Q. E. D.)

Corollary. *For $\varepsilon=\varepsilon(\lambda)$ such that $\varepsilon\downarrow 0$ and $\varepsilon^{1/4}\log\lambda=O(1)$ as $\lambda\to\infty$, we have*

$$(4.2)\qquad \left(\frac{1}{\sqrt{\lambda}}X_{\lambda t},\ \frac{1}{\sqrt{\lambda}}\ell(\lambda t),\ \sqrt{\varepsilon}\lambda^{-1/4}\{\eta_\varepsilon(\lambda t\,;\,a)-\eta_\varepsilon(\lambda t\,;\,0)\}\right)$$

$$\xrightarrow{\mathcal{D}}\left(X_t,\ \ell(t),\ \frac{4}{\sqrt{\pi}}W(\ell(t),\ a)\right),$$

in $D(\boldsymbol{R}_+^2\,;\,\boldsymbol{R}^3)$, *where* $W(\cdot,\ \cdot)$ *is the same as in Theorem* 1.

Proof. Apply first the scale change property as in the proof of Corollary of Theorem 5 and next replace λ^2 by λ' and ε by ε'/λ'. Then we have the assertion. (Q. E. D.)

MATHEMATICAL INSTITUTE OF
HUNGARIAN ACADEMY OF SCIENCES
MATHEMATICAL INSTITUTE OF
HUNGARIAN ACADEMY OF SCIENCES
INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA

References

[1] P. J. Bickel and M. J. Wichura, Convergence criteria for multiparameter stochastic processes and some applications, Ann. Math. Stat., **42** (1971), 1656-1670.
[2] A. N. Borodin, On the character of convergence to Brownian local time I, Probab. Th. Rel. Fields, **72** (1986), 231-250.
[3] A. N. Borodin, On the character of convergence to Brownian local time II, Ibid., **72** (1986), 251-277.
[4] M. Csörgő and P. Révész, Measure du voisinage and occupation density, Ibid., **73** (1986), 211-226.
[5] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland, Amsterdam/Kodansha, Tokyo, 1981.
[6] K. Itô and H. P. McKean, Jr., Diffusion Processes and their Sample Paths, Springer, Berlin-Heidelberg-New York, 1965.
[7] Y. Kasahara, On Lévy's downcrossing theorem, Proc. Japan Acad., **56** (1980), Ser. A, 455-458.
[8] Y. Kasahara, Limit theorems for Lévy processes and Poisson point processes and their applications to Brownian excursions, J. Math. Kyoto Univ., **24** (1984), 521-538.
[9] Y. Kasahara and S. Watanabe, Limit theorem for point processes and their functionals, J. Math. Soc. Japan **38** (1986), 543-574.
[10] M. Yor, Le drap brownien comme limite en loi de temp locaux lineaires, Séminaire de Probabilités XVII, Lect. Notes in Math. 986, Springer-Verlag, Berlin-Heidelberg-New York, 1983.