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A class of pseudo-differential operators of logarithmic type and infinitely degenerate hypoelliptic operators

By

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1. Introduction

Recently, the hypoellipticity for the infinitely degenerate operators has been intensively studied. ([7], [11], [6]) Here, we call the partial differential operator P be hypoelliptic in a open subset Ω of $\boldsymbol{R}^n$ iff for any $u\in\mathcal{E}'(\Omega)$ and any $\omega\subset\Omega$, $Pu\in C^\infty(\omega)$ implies $u\in C^\infty(\omega)$.

In [7], by the probabilistic method, they have shown that for a positive real number n and $\phi(t)=\exp(-|t|^{-n})$ if $t\neq 0$, $=0$ if $t=0$, the operator

$$L=D_1^2+D_2^2+\phi(x_2)D_3^2$$

is hypoelliptic in $\boldsymbol{R}^3$ if and only if $n<1$. Here $D_j=-i\partial/\partial x_j$.

Inspired by this result, in [11] Y. Morimoto has studied the hypoellipticity for a class of operators containing L by the method based on some a priori estimate which has its own interest. Also T. Hoshiro [6] has proved the same result by a different method. It seems that their works are influenced by the micro-local methods in the analytic or Gevrey class: The back ground of the method in [11] is Morrey-Nirenberg method (cf. [2] etc.) and Hörmander's micro-local method. On the other hand, the back ground of the method in [6] is Mizohata's $\alpha_n\beta_n$ method.

Thus, the development of the theory of the regularity of solution in the analytic or Gevrey class animates the study of the hypoellipticity in C^∞ class.

In this paper, inspired by this observation, we shall introduce a new class of pseudo-differential operators which is viewed as a version in a C^∞ class of Gevrey pseudo-differential operators of infinite order. Moreover, we shall apply it to the study of the hypoellipticity of the infinitely degenerate operators by the method of parametrix. This class enables us to obtain more sharp results than that in a framework of the class $S^m_{\rho\delta}$ introduced by L. Hörmander.

In section 3, we shall study the hypoellipticity of the parabolic operators. Our method is similar to that in [4] and [9] but requires us more precise argument. In section 4, we shall show that the operator

$$(D_1+i\phi(x_1)D_2)^2+\phi(x_1)D_2$$

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is hypoelliptic in $\boldsymbol{R}^2$ for any $n>0$. This will be done by the perturbation methed considering $(D_1+i\phi D_2)^2$ as principal term. We remark that in contrast with the case that the coefficients finitely degenerate, this perturbation method does not work well in the framework $S^m_{\rho\delta}$.

We do not know whether their methods in [11] or [6] are applicable to our case. This is an interesting problem.

We note that F. Treves has obtained the result on the hypoellipticity for the operators of principal type with infinitely degenerate coefficients. ([13]). We also mention V. S. Fedii [3] as a pioneering work for the infinitely degenerate operators.

Finally, we would like to thank Professor S. Mizohata for his advice.

2. Definitions, Calculus and Formal symbols

Let Ω be a subset of $\boldsymbol{R}^d$ and $m, \rho, \delta, \tau, \theta$ be a real number such that $0\leqq\rho\leqq 1$, $0\leqq\delta\leqq 1$, $\tau\geqq 0$, $\theta\geqq 0$, $1-\delta+\theta>0$ and $\rho+\tau>0$.

Definition 1. $(\theta>0, \tau>0)$ We denote by $\mathcal{L}^{m,\rho\delta}_{\tau\theta}(\Omega)$ the space of all functions $p(x, \xi)\in C^\infty(\Omega\times\boldsymbol{R}^d)$ satisfying the following condition: for every compact subset $K\subset\Omega$ there exists constant C and for every $\varepsilon>0$ there exist constants C_ε and R_ε such that

$$(1.1)\qquad |D^\beta_x D^\alpha_\xi p(x, \xi)|\leqq C_\varepsilon C^{|\alpha+\beta|}\langle\xi\rangle^{m-\rho|\alpha|}(\varepsilon|\alpha|/\log\langle\xi\rangle)^{\tau|\alpha|}\times\{|\beta|+|\xi|^\delta(\varepsilon|\beta|/\log\langle\xi\rangle)^\theta\}^{|\beta|}$$

for every α, β and $x\in K$, $\xi\in\boldsymbol{R}^d$ with $\varepsilon|\alpha|+R_\varepsilon\leqq\log\langle\xi\rangle$.

Here, $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$.

Definition 2. $(\tau>0, \delta<1)$ $\mathcal{L}^{m,\rho\delta}_{\tau 0}(\Omega)=\{p(x, \xi)\in C^\infty(\Omega\times\boldsymbol{R}^d)$: for every compact subset $K\subset\Omega$ and every β there exists constant C and for every $\varepsilon>0$ there exist constants C_ε and R_ε such that

$$(1.2)\qquad |D^\beta_x D^\alpha_\xi p(x, \xi)|\leqq C_\varepsilon C^{|\alpha|}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}(\varepsilon|\alpha|/\log\langle\xi\rangle)^{|\alpha|}$$

for every α, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $\varepsilon|\alpha|+R_\varepsilon\leqq\log\langle\xi\rangle$.$\}$

Definition 3. $(\theta>0, \rho>0)$ $\mathcal{L}^{m,\rho\delta}_{0\theta}(\Omega)=\{p(x, \xi)\in C^\infty(\Omega\times\boldsymbol{R}^d)$: for every compact subset $K\subset\Omega$ and every α, there exist constants C and R and for every $\varepsilon>0$, there exists a constant C_ε such that

$$(1.3)\qquad |D^\beta_x D^\alpha_\xi p(x, \xi)|\leqq C_\varepsilon C^{|\beta|}\langle\xi\rangle^{m-\rho|\alpha|}(|\beta|+|\xi|^\delta(\varepsilon|\beta|/\log\langle\xi\rangle)^\theta)^{|\beta|}$$

for every β, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $R\leqq|\xi|$.$\}$

Definition 4. $(\rho>0, \delta<1)$ $\mathcal{L}^{m,\rho\delta}_{00}(\Omega)=S^m_{\rho\delta}(\Omega)=\{p(x, \xi)\in C^\infty(\Omega\times\boldsymbol{R}^d)$: for every compact subset $K\subset\Omega$ every α, β, there exists constants C and R such that

$$(1.4)\qquad |D^\beta_x D^\alpha_\xi p(x, \xi)|\leqq C\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for every $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $R\leqq|\xi|$.}

For $p\in\mathcal{L}^{m,\rho\delta}_{\tau\theta}(\Omega)$, the operator $Op(p)$, $P(x, D)$, with kernel

$$\int e^{i\langle x-y,\xi\rangle}p(x, \xi)d\xi\,, \qquad (d\xi=(2\pi)^{-d}d\xi)$$

is well-defined and maps $C_0^\infty(\Omega)$ in $C^\infty(\Omega)$ and $\mathcal{E}'(\Omega)$ in $\mathcal{D}'(\Omega)$.

We introduce the formal symbols in $\mathcal{L}^{m,\rho\delta}_{\tau\theta}(\Omega)$: Let μ_j be a sequence of non-negative real numbers such that for some $\kappa>0$,

$$\sum e^{-\kappa\mu_j}<\infty\,.$$

We shall say $\Sigma p_j(x, \xi)$ be a formal symbol in $\mathcal{L}^{m,\rho\delta}_{\tau\theta}(\Omega)$ if the following condition is satisfied:

When $\tau>0$ and $\theta>0$, for every compact subset $K\subset\Omega$, there exists contants C and $r>0$ and for every $\varepsilon>0$, there exist constants C_ε, R_ε such that

$$\begin{aligned}(2.1)\qquad |D_x^\beta D_\xi^\alpha p_j(x, \xi)|\leqq & C_\varepsilon C^{|\alpha+\beta|}(C\varepsilon\mu_j)^{\mu_j}(\log\langle\xi\rangle)^{-\mu_j}\langle\xi\rangle^{m-\rho|\alpha|}\\ &\times(\varepsilon|\alpha|/\log\langle\xi\rangle)^{\tau|\alpha|}\{|\beta|+|\xi|^\delta(\varepsilon|\beta|/\log\langle\xi\rangle)^\theta\}^{|\beta|}\end{aligned}$$

for any α, β, j, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $\varepsilon|\alpha|+r\mu_j+R_\varepsilon\leqq\log\langle\xi\rangle$.

When $\theta=0$, $\tau>0$ and $\delta<1$, for every compact subset $K\subset\Omega$, every β, there exist constants C and $r>0$ and for every $\varepsilon>0$, there exist constants C_ε and R_ε such that

$$\begin{aligned}|D_x^\beta D_\xi^\alpha p_j(x, \xi)|\leqq & C_\varepsilon C^{|\alpha|}(C\varepsilon\mu_j)^{\mu_j}(\log\langle\xi\rangle)^{-\mu_j}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\\ &\times(\varepsilon|\alpha|/\log\langle\xi\rangle)^{\tau|\alpha|}\end{aligned}$$

for any α, j, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $\varepsilon|\alpha|+r\mu_j+R_\varepsilon\leqq\log\langle\xi\rangle$.

When $\tau=0$, $\theta>0$ and $\rho>0$, for every compact subset $K\subset\Omega$, every α, there exist constants C and $r>0$ and for every $\varepsilon>0$ there exists a constant C_ε such that

$$\begin{aligned}|D_x^\beta D_\xi^\alpha p_j(x, \xi)|\leqq & C_\varepsilon C^{|\beta|}(C\varepsilon\mu_j)^{\mu_j}(\log\langle\xi\rangle)^{-\mu_j}\langle\xi\rangle^{m-\rho|\alpha|}\\ &\times(|\beta|+|\xi|^\delta(\varepsilon|\beta|/\log\langle\xi\rangle)^\theta)^{|\beta|}\end{aligned}$$

for every β, j, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $r(\mu_j+1)\leqq\log\langle\xi\rangle$.

When $\tau=\theta=0$, $\rho>0$ and $\delta<1$, for every compact subset $K\subset\Omega$, every α, β, there exist constants C and $r>0$ and for every $\varepsilon>0$, there exists a constant C_ε such that

$$|D_x^\beta D_\xi^\alpha p_j(x, \xi)|\leqq C_\varepsilon C(C\varepsilon\mu_j)^{\mu_j}(\log\langle\xi\rangle)^{-\mu_j}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for every j, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $r(\mu_j+1)\leqq\log\langle\xi\rangle$.

Next, we introduce the equivalence relation: When $\tau>0$ and $\theta>0$, for the formal symbol $\sum_{j\geq0}p_j$ and $\sum_{j\geq0}q_j$ in $\mathcal{L}^{m,\rho\delta}_{\tau\theta}(\Omega)$ we say these symbols be equivalent ($\Sigma p_j\sim\Sigma q_j$) if for every compact subset $K\subset\Omega$, there exist constant C and $r>0$ and for every $\varepsilon>0$, there exist constants C_ε and R_ε such that

$$\left|D_x^\beta D_\xi^\alpha \sum_{j<N} \{p_j(x,\xi)-q_j(x,\xi)\}\right| \leqq C_\varepsilon C^{|\alpha+\beta|}(C\varepsilon\mu_N)^{\mu_N}(\log\langle\xi\rangle^{-\mu_N}$$
$$\times(\varepsilon|\alpha|/\log\langle\xi\rangle)^{-\tau|\alpha|}(|\beta|+|\xi|^\delta(\varepsilon|\beta|/\log\langle\xi\rangle)^\theta)^{|\beta|}\langle\xi\rangle^{m-\theta|\alpha|}$$

for every α, β, N, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $\varepsilon|\alpha|+r\mu_N+R_\varepsilon\leqq\log\langle\xi\rangle$. When $\tau>0$ and $\delta<1$, $\Sigma p_j\sim\Sigma q_j$ in $\mathcal{L}_{\tau 0}^{m\rho\delta}(\Omega)$ iff for every compact subset $K\subset\Omega$ every β, there exist constants C and $r>0$ and for every $\varepsilon>0$, there exist constant C_ε and R_ε such that

$$\left|D_x^\beta D_\xi^\alpha \sum_{j<N}(p_j-q_j)\right| \leqq C_\varepsilon C^{|\alpha|}(C\varepsilon\mu_N)^{\mu_N}(\log\langle\xi\rangle)^{-\mu_N}$$
$$\times(\varepsilon|\alpha|/\log\langle\xi\rangle)^{-\tau|\alpha|}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for every α, N, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $\varepsilon|\alpha|+r\mu_N+R_\varepsilon\leqq\log\langle\xi\rangle$. When $\theta>0$ and $\rho>0$, $\Sigma p_j\sim\Sigma q_j$ in $\mathcal{L}_{0\theta}^{m\rho\delta}(\Omega)$ iff for every compact subset $K\subset\Omega$ every α, there exist constants C and $r>0$ and for every $\varepsilon>0$, there exists a constant C_ε such that

$$\left|D_x^\beta D_\xi^\alpha \sum_{j<N}(p_j-q_j)\right| \leqq C_\varepsilon C^{|\beta|}(C\varepsilon\mu_N)^{\mu_N}(\log\langle\xi\rangle)^{-\mu_N}\langle\xi\rangle^{m-\rho|\alpha|}$$
$$\times(|\beta|+|\xi|^\delta(\varepsilon|\beta|/\log\langle\xi\rangle)^\theta)^{|\beta|}$$

for every β, N, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $r(\mu_N+1)\leqq\log\langle\xi\rangle$. When $\rho>0$ and $\delta<1$, $\Sigma p_j\sim\Sigma q_j$ in $\mathcal{L}_{00}^{m\rho\delta}(\Omega)$ iff for every compact subset K, α, β. there exist constant C and $r>0$ and for every $\varepsilon>0$, there exists a constant C_ε such that

$$\left|D_x^\beta D_\xi^\alpha \sum_{j<N}(p_j-q_j)\right| \leqq C_\varepsilon(C\varepsilon\mu_N)^{\mu_N}(\log\langle\xi\rangle)^{-\mu_N}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for every N, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $r(\mu_N+1)\leqq\log\langle\xi\rangle$.

For a conic set $\omega\times\Gamma\subset T^*(\boldsymbol{R}^d)$, we also define $\mathcal{L}_{\tau\theta}^{m\rho\delta}(\omega\times\Gamma)$, the formal symbol in $\mathcal{L}_{\tau\theta}^{m\rho\delta}(\omega\times\Gamma)$ and the equivalence relation in $\mathcal{L}_{\tau\theta}^{m}(\omega\times\Gamma)$ by an obvious way: replacing $\xi\in\boldsymbol{R}^d$ by $\xi\in\Gamma$.

If A, B: $C_0^\infty(\boldsymbol{R}^d)\to\mathcal{D}'(\boldsymbol{R}^d)$ are continuous linear operators with $WF'(A)\cup WF'(B)\subset\mathrm{diag}\,(T^*(\boldsymbol{R}^d)\setminus 0)$ and $\mathcal{G}\subset T^*(\boldsymbol{R}^d)\setminus 0$ is a conic open set, we say that $A\sim B$ in $\mathcal{G}$ if $WF'(A-B)\cap\mathrm{diag}\,(\mathcal{G})=\emptyset$. Moreover, we say $A\sim B$ in a open set $\omega\subset\boldsymbol{R}^d$ if the kernel of $A-B$ is C^∞ in $\omega\times\omega$.

Now, we introcuce the auxiliarly function $\chi_j^\rho(\xi)$: It is well-known that if $\Omega_1\Subset\Omega_2$ are two open sets, one can find a sequence of functions $\psi_N\in C_0^\infty(\Omega_2)$ and a constant C such that

$$\psi_N=1 \text{ on } \Omega_1 \quad\text{and}\quad |D^\alpha\psi_N|\leqq(C|\alpha|^\rho N^{1-\rho})^{|\alpha|} \tag{2.2}$$

for every $N, \alpha, |\alpha|\leqq N$, where $\rho\in[0, 1)$ is a given number. ([5], [1]) Take $\psi_N\in C^\infty(\boldsymbol{R}^d)$ satisfying (2.2) with $\Omega_1=\{\xi\in\boldsymbol{R}^d: |\xi|\leqq1\}$ and $\Omega_2=\{\xi\in\boldsymbol{R}^d: |\xi|\leqq2\}$. Define

$$\chi_j^\rho(\xi)=1-\psi_{[\log 2j]+1}(\xi/j)\,,$$

where [ ] stands for the Gauss'symbol. Then, for any $\varepsilon>0$, there exists a constant C_ε such that

$$|D^\alpha \chi_j^\rho(\xi)| \leqq C_\varepsilon(\varepsilon|\alpha|/(|\xi|\log\langle\xi\rangle))^{\rho|\alpha|}$$

for any α and $\xi \in \boldsymbol{R}^d$ with $|\alpha| \leqq \log\langle\xi\rangle$. Moreover we have

Lemma A. *Given two cones $\Gamma_1 \Subset \Gamma_2 \subset \boldsymbol{R}^d$ and $\rho \in [0, 1)$, there exists $g \in C^\infty(\boldsymbol{R}^d)$ and for every $\varepsilon > 0$ there exists a constant C_ε such that $g(\xi) = 0$ if $\xi \notin \Gamma_2$ or $|\xi| \leqq 1$, $g(\xi) = 1$ if $\xi \in \Gamma_1$ and $|\xi| \geqq 2$, and for any $\alpha, \xi \in \boldsymbol{R}^d$ with $|\alpha| \leqq \log\langle\xi\rangle$,*

$$|D^\alpha g(\xi)| \leqq C_\varepsilon(\varepsilon|\alpha|/|\xi|)^{\rho|\alpha|}.$$

This result follows from lemma 3.1 in [10].

As for the calculus in $\mathcal{L}_{\tau\theta}^{m\theta\delta}$ we have

Theorem 1. *Let Σp_j be a formal symbol in $\mathcal{L}_{\tau\theta}^{m\rho\delta}(\Omega)$. Set*

$$p(x, \xi) = \sum_j \chi^\rho_{[\exp(\tau\mu_j)]+1}(\xi) p_j(x, \xi).$$

Then, $p(x, \xi) \in \mathcal{L}_{\tau\theta}^{m\rho\delta}(\Omega)$. Moreover p is uniquely determined up to the equivalence.

We call p a realization of Σp_j.

Theorem 2. *If $p \in \mathcal{L}_{\tau\theta}^{m\rho\delta}(\Omega) \sim 0$, then for any $u \in \mathcal{E}'(\Omega)$,*

$$P(x, D)u \in C^\infty(\Omega).$$

Theorem 3. *Let $p \in \mathcal{L}_{\tau\theta}^{m\rho\delta}(\Omega)$. Then, for any $u \in \mathcal{E}'(\Omega)$,*

$$WF(P(x, D)u) \subset WFu.$$

These theorem are shown by the standard way. The argument is close to that for the pseudo-differential operators of infinitely order. The key point is the following result.

Lemma B. *Let $u \in \mathcal{D}'(\Omega)$. Then, $(x, \xi) \notin WFu$ if and only if there exist $\varphi \in C_0^\infty(\Omega)$ with $\varphi = 1$ near x and a conic neighborhood Γ of ξ and for every $\varepsilon > 0$ there exists a constant C_ε such that*

$$|\widehat{\varphi u}(\xi)| \leqq C_\varepsilon(\varepsilon N/\log\langle\xi\rangle)^N$$

for any N, $\xi \in \Gamma$ with $\varepsilon N \leqq \log\langle\xi\rangle$.

This is a consequence of the fact that

$$\langle\xi\rangle^M \widehat{\varphi u}(\xi) = \sum_{k \geqq 0} (M \log\langle\xi\rangle)^k / k! \cdot \widehat{\varphi u}(\xi).$$

Now, we consider the composition. To clarify this, we introduce a subclass $\tilde{\mathcal{L}}_{\tau\theta}^{m\rho\delta}$ of $\mathcal{L}_{\tau\theta}^{m\rho\delta}$. For $\tau > 0$ and $\theta > 0$, we define $\tilde{\mathcal{L}}_{\tau\theta}^{m\rho\delta}(\Omega) = \mathcal{L}_{\tau\theta}^{m\rho\delta}(\Omega)$. For $\theta = 0$ and $\delta < 1$, we define $\tilde{\mathcal{L}}_{\tau 0}^{m\rho\delta}(\Omega)$ by $\{p(x, \xi) \in C^\infty(\Omega \times \boldsymbol{R}^d) : {}^\forall K \Subset \Omega\ {}^\exists C$ and ${}^\forall \varepsilon > 0$ ${}^\exists C_\varepsilon\ {}^\exists R_\varepsilon$ such that

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leqq C_\varepsilon C^{|\alpha+\beta|} \langle\xi\rangle^{m-\rho|\alpha|} (\varepsilon|\alpha|/\log\langle\xi\rangle)^{\tau|\alpha|}$$
$$\times(|\beta|+|\beta|^{1-\delta}|\xi|^{\delta})^{|\beta|}$$

for every α, β, $x\in K$ and $\xi\in \boldsymbol{R}^d$ with $\varepsilon|\alpha|+R_\varepsilon\leqq\log\langle\xi\rangle$. For $\tau=0$ and $\rho>0$, $\tilde{\mathcal{L}}_{0\theta}^{m\rho\delta}(\Omega)=\{p(x,\xi)\in C^\infty(\Omega\times\boldsymbol{R}^d): {}^\forall K\Subset\Omega\ {}^\exists C\ {}^\exists R$ and ${}^\forall\varepsilon>0\ {}^\exists C_\varepsilon$,

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leqq C_\varepsilon C^{|\alpha+\beta|} \langle\xi\rangle^{m} (|\alpha|/|\xi|)^{\rho|\alpha|} (|\beta|+|\xi|^{\delta}(\varepsilon|\beta|/\log\langle\xi\rangle)^{\theta})^{|\beta|}$$

for any α, β, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $R|\alpha|\leqq|\xi|\}$. For $\rho>0$ and $\delta<1$, $\tilde{\mathcal{L}}_{00}^{m\rho\delta}(\Omega)$ $=\gamma^1\text{-}S_{\rho\delta}^m(\Omega)=\{p(x,\xi)\in C^\infty(\Omega\times\boldsymbol{R}^d): {}^\forall K\Subset\Omega\ {}^\exists C\ {}^\exists R$ such that

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leqq C^{|\alpha+\beta|+1} (|\alpha|/|\xi|)^{\rho|\alpha|} (|\beta|+|\beta|^{(1-\delta)}|\xi|)^{|\beta|}$$

for any α, β, $x\in K$ and $\xi\in\boldsymbol{R}^d$ with $R|\alpha|\leqq|\xi|\}$.

We also define the formal symbols, equivalence relation in $\mathcal{L}_{\tau\theta}^{m\rho\delta}(\Omega)$ by the analogous way. We use the notation: $(\mu,\nu)\geqq(\mu',\nu')$ iff $\mu>\mu'$ or $\mu=\mu'$ and $\nu\geqq\nu'$. Then the similar argument to that in $\gamma^1\text{-}S_{\tau\delta}^\infty(\Omega)$ ([12]) gives us the following results:

Theorem 4. *Let $a\in\tilde{\mathcal{L}}_{\tau\rho}^{m\rho\delta}(\Omega)$ and $b\in\tilde{\mathcal{L}}_{\tau\theta}^{m\rho\delta}(\Omega)$. If either $\rho>\delta'$ or $\rho=\delta'<1$ and $\tau+\theta'>0$, then the symbol*

$$c(x,\xi)\sim\sum_\alpha(\alpha!)^{-1}(\partial_\xi^\alpha a)(x,\xi)(D_x^\alpha b)(x,\xi)$$

is a formal symbol in $\tilde{\mathcal{L}}_{\tau''\theta''}^{m+m\rho''\delta''}(\Omega)$ for $(\rho'',\tau'')=\min((\rho,\tau),(\rho',\tau'))$ and $(\delta'',-\theta'')$ $=\max((\delta,-\theta),(\delta',-\theta'))$. Furthermore, for any $\phi\in C_0^\infty(\Omega)$ such that $\phi=1$ in a neighborhood of $\bar{\Omega}_1\subset\Omega$ and for any realization c, we have

$$op(c)\sim op(a)\phi op(b) \qquad on\ \Omega_1 .$$

Theorem 5. *Let Σp_j and Σb_j be the formal symbols in $\tilde{\mathcal{L}}_{\tau\theta}^{m\rho\delta}(\Omega\times\Gamma)$ and $\tilde{\mathcal{L}}_{\tau'\theta'}^{m'\theta'\delta'}(\Omega\times\Gamma)$, respectively. Define*

$$c_{j,k,\alpha}(x,\xi)=(\alpha\,!)^{-1}\partial_\xi^\alpha p_j(x,\xi)D_x^\alpha q_k(x,\xi)\,.$$

If either $\rho>\delta'$ or $1>\rho=\delta'$ and $\tau+\theta'>0$, $\sum_{j,k,\alpha}c_{j,k,\alpha}$ is a formal symbol in $\tilde{\mathcal{L}}_{\tau''\theta''}^{m+m'\rho''\delta''}(\Omega\times\Gamma)$ with $(\rho'',\tau'')=\min\{(\rho,\tau),(\rho',\tau')\}$ and $(\delta'',-\theta'')=\max\{(\delta,-\theta),(\delta',-\theta')\}$. Furthermore, for any realization a, b, c of these symbols, for any $\phi\in C_0^\infty(\Omega)$, $\phi=1$ in a neighborhood of x_0, and for any $g(\xi)$ with support in Γ, given by lemma A with parameter ρ_1, $\delta<\rho_1<1$ and such that $g(\xi)=1$ for $|\xi|\geqq2$, ξ in a conic neighborhood of ξ_0, we have

$$op(gc)\sim op(ga)\phi op(gb) \qquad at\ (x_0,\xi_0)\,.$$

We remark that the composition can be defined from $\mathcal{L}_{\tau\theta}^{m\rho\delta}(\Omega)\times\mathcal{L}_{\tau'\theta'}^{m'\rho'\delta'}(\Omega)$ to a class of symbol which has the pseudo-local property if $\rho>\delta'$ or $\rho=\delta'$ and $\tau+\theta'>0$, but this class larger than $\mathcal{L}_{\tau''\theta''}^{m+m'\rho''\delta''}(\Omega)$.

3. Infinitely degenerate parobolic operators

Let $I=(-2T, 2T)$, Ω be an open set in $\boldsymbol{R}^d$ and m be a positive even integer. We consider the operator L given by

$$L=\partial_t-P(x, t, D_x)$$

$$P(x, t, \xi)=\sum_{j=0}^{m} p_{m-j}(x, t, \xi) \qquad \text{and}$$

$$p_{m-j}(x, t, \xi)=\sum_{|\alpha|=m-j} a_\alpha(x, t)\xi^\alpha$$

with $a_\alpha(x, t)\in C^\infty(I\times\Omega)$. We assume the following condition: Let τ and θ be non-negative real numbers and

$$l(\alpha, \beta, j)=|\alpha|\tau+|\beta|\theta+j(\tau+\theta).$$

For any compact set $K\subset\Omega$, there exists a constant C such that

$$|D_\xi^\alpha D_x^\beta p_{m-j}(x, t, \xi)|\leqq C^{|\alpha+\beta|+1}|\mathrm{Re}\, p_m|^{1-l(\alpha,\beta,j)}|\xi|^{ml(\alpha,\beta,j)-|\alpha|-j} \tag{3.1}$$

for any $t\in I$, $x\in K$, $\xi\in\boldsymbol{R}^d$ and α, β, j with $l(\alpha, \beta, j)\leqq 1$ and

$$|D_\xi^\alpha D_x^\beta p_{m-j}(x, t, \xi)|\leqq C^{|\alpha+\beta|+1}\alpha!\beta!|\xi|^{m-|\alpha|-j} \tag{3.2}$$

for any $t\in I$, $x\in K$, $\xi\in\boldsymbol{R}^d$, α, β, j with $l(\alpha, \beta, j)>1$.

Moreover, we impose the following condition on $\mathrm{Re}\, p_m$: for any compact set $K\subset\Omega$, there exist positive number h, r and constant $\lambda\in\boldsymbol{R}\setminus 0$ such that

$$\begin{cases} \lambda\,\mathrm{Re}\, p_m(x, t, \xi)\geqq 0 & \text{for } (x, t, \xi)\in K\times I\times\boldsymbol{R}^d \quad \text{and} \\ \displaystyle\int_{t'}^{t}\mathrm{Re}\, p_m(x, s, \xi)ds\geqq(t-t')(\log\langle\xi\rangle)^{2h} \end{cases} \tag{3.3}$$

for $t-t'\geqq r(\log\langle\xi\rangle)^{-h}$ and $(t, t', x, \xi)\in I\times I\times K\times\boldsymbol{R}^d$. Then, we have

Theorem 3.1. *Under the assnmption* (3.1)-(3.3), *L is hypoelliptic in $I\times\Omega$ if $m(\tau+\theta)\leqq 1$ and $h(\tau+\theta)>1$.*

Example. Let

$$L_1=\partial_t-(\exp(-|t|^{-n})D_{x_1}^2+D_{x_2}^2) \qquad \text{and}$$

$$L_2=\partial_t-(\exp(-|t|^{-n})+x_1^2)D_{x_1}^2$$

If $n<1/2$, then these operators are hypoelliptic at the origin. In fact, for L_1, $\tau=1/2$, $\theta=0$, $h=1/n$ and for L_2, $\tau=0$, $\theta=1/2$, $h=1/n$.

When the coefficients of L are independent of x, we have more sharp result.

Theorem 3.2. *Suppose that $p_{m-j}=0$ for $j>0$, the coefficients of p_m are independent of x and* (3.1)-(3.3). *Then L is hypoelliptic in $I\times\Omega$ if $\tau h/(1-\tau)>1$.*

Whee $m(\tau+\theta)<1$, our result intersects with [4] and [9].

Proof of theorem 3.1. For simplicity, we shall show the hypoellipticity of L in $(-T, T)\times\Omega$. We are going to construct a left parametrix of L in the following form:

$$Ku=\int e^{ix\xi}\int_{T'}^{t} K(t, t', x, \xi)\hat{u}(t', \xi)dt'd\xi ,$$

where $T'=T$ if $\lambda>0$, $T'=-T$ if $\lambda<0$, and $\hat{u}$ stands for the Fouier transform in x of u. Hereafter, we only consider the case $\lambda<0$. $KL\sim Id$ implies

$$\begin{cases} -\partial_{t'}K(t, t', x, \xi)-\sum_{\alpha}(\alpha !)^{-1}\partial_{\xi}^{\alpha}K(t, t', x, \xi)D_{x}^{\alpha}P(x, t', \xi)\sim 0 \\ K(t, t, x, \xi)=1 . \end{cases}$$

Let $K(t, t', x, \xi)=\sum_{j\geq 0}K_{j}(t, t', x, \xi)$. Define K_j by

$$K_{0}(t, t', x, \xi)=\exp\left(\int_{t'}^{t}p_{m}(x, s, \xi)ds\right)$$

$$K_{j}(t, t', x, \xi)=\sum_{l=1}^{j}\int_{t'}^{t}K_{0}(s, t', x, \xi)\mathcal{P}_{l}(s, x, \xi, \partial_{\xi})K_{j-l}(t, s, x, \xi)ds ,$$

where $\mathcal{P}_{l}(s, x, \xi, \partial_{\xi})=\sum_{|\alpha|+j=l}(\alpha !)^{-1}(D_{x}^{\alpha}p_{m-j})(x, s, \xi)\partial_{\xi}^{\alpha}$.

Then, we have

Lemma 3.3. *For some constant* C

$$|D_{\xi}^{\alpha}D_{x}^{\beta}K_{0}(t, t', x, \xi)|\leq C^{|\alpha+\beta|+1}|\alpha+\beta|\,!|\xi|^{(m\tau-1)|\alpha|+m\theta|\beta|} \tag{3.4}$$
$$\times(\log\langle\xi\rangle)^{-h(\tau|\alpha|+\theta|\beta|)}\exp\left(\frac{1}{2}\int_{t'}^{t}\mathrm{Re}\, p_{m}(x, s, \xi)ds\right)$$

for any $t\geq t'$, α, β, $x\in K$ *and* $\xi\in \boldsymbol{R}^{d}$.

Proof. We see that

$$\int_{t'}^{t}|\mathrm{Re}\, p_{m}|^{1-l}(x, s, \xi)ds\leq|t-t'|^{l}\left(\int_{t'}^{t}|\mathrm{Re}\, p_{m}|(x, s, \xi)ds\right)^{1-l}.$$

Therefore, denoting $\int_{t'}^{t}|\mathrm{Re}\, p_{m}|(x, s, \xi)ds$ by Λ, by the formula for the derivatives of the composition of functions, we see that the left hand side of (3.4) is less than

$$\Sigma'|\alpha+\beta|\,!/(i_{1}!\cdots i_{k}!)|t-t'|^{\tau|\alpha|+\theta|\beta|}\Lambda^{|I|-\tau|\alpha|-\theta|\beta|}|\xi|^{(m\tau-1)|\alpha|+m\theta|\beta|}e^{-\Lambda}$$
$$+\Sigma''|\alpha+\beta|\,!/(i_{1}!\cdots i_{k}!)|t-t'|^{|I|}|\xi|^{(m\tau-1)|\alpha|+m\theta|\beta|-\tau|\alpha|-\theta|\beta|+|I|}e^{-\Lambda},$$

where $I=(i_{1}, \cdots, i_{k})$, $|I|\leq|\alpha+\beta|$, in Σ', $|I|\geq\tau|\alpha|+\theta|\beta|$, in Σ'', $|I|<\tau|\alpha|+\theta|\beta|$, and $m-|\alpha|\leq(m\tau-1)|\alpha|+m\theta|\beta|$, and

$$(\Sigma'+\Sigma'')|I|\,!/(i_{1}!\cdots i_{k}!)\leq C_{0}^{|\alpha+\beta|}$$

for some universal constant C_0. The assumption (3.3) implies

$$|t-t'|\leqq(\Lambda(\log\langle\xi\rangle)^{-h}+1)(\log\langle\xi\rangle)^{-h},$$

so that the inequality:

$$y^N e^{-y/2}\leqq C_1^N N! \qquad \text{if } y\geqq 0,$$

gives us (3.4). Q. E. D.

Lemma 3.4. *There exist constants* C_0, C *and* R *such that*

$$(3.5)\qquad |D_\xi^\alpha D_x^\beta K_j(t, t', x, \xi)|\leqq C_0 C^{|\alpha+\beta|+j}(|\alpha+\beta|+j)!$$
$$\times(\log\langle\xi\rangle)^{-h(\tau|\alpha|+\theta|\beta|)-h(\tau+\theta)j}\langle\xi\rangle^{(m\tau-1)|\alpha|+m\theta|\beta|}\exp(-\Lambda/4)$$

for any α, , β, j, $x\in K\Subset\Omega$, $t\geqq t'$ *and* $\xi\in\boldsymbol{R}^d$ *with* $R(|\alpha|+j)\leqq\log\langle\xi\rangle$.

Proof. It is seen that for some constant A

$$\int_{t'}^{t}|\mathrm{Re}\, p_m|^{1-l}(x, s, \xi)\exp\Big(-\frac{1}{2}\int_{t'}^{s}|\mathrm{Re}\, p_m|(x, s', \xi)ds'\Big)ds$$

$$\leqq|t-t'|^l\Big\{\int_{t'}^{t}|\mathrm{Re}\, p_m|(x, s, \xi)\exp\Big(-(2(1-l))^{-1}$$

$$\times\int_{t'}^{s}|\mathrm{Re}\, p_m|(x, s', \xi)ds'\Big)\Big\}^{1-l}$$

$$\leqq|t-t'|^l A \qquad \text{if } 1>l\geqq 0 \text{ and } t\geqq t'.$$

By the induction on j, we observe that there exist constants C_0, C_1 and R such that the left hand side of (3.5) is less than

$$C_0 C_1^{|\alpha+\beta|+j}(|\alpha+\beta|+j)!\{\Lambda^\theta(\log\langle\xi\rangle)^{-h\theta}+1\}^j$$
$$\times(\log\langle\xi\rangle)^{-h(\tau+\theta)j-h(\tau|\alpha|+\theta|\beta|)}\langle\xi\rangle^{-(m\tau-1)|\alpha|+m\theta|\beta|}e^{-(1/2)\Lambda}.$$

From this, (3.5) follows since

$$(\Lambda^\theta(\log\langle\xi\rangle^{-h\theta}+1)^j e^{-(1/4)\Lambda}\leqq C_2^j \qquad \text{if } (\log\langle\xi\rangle)^h\geqq Rj. \qquad \text{Q. E. D.}$$

Let $\sigma>1$, then for every $\varepsilon>0$, there exist C_ε and R_ε such that for any N

$$N!(\log\langle\xi\rangle)^{-\sigma N}\leqq C_\varepsilon(\varepsilon N/\log\langle\xi\rangle)^{\sigma N} \qquad \text{and}$$

$$(\log\langle\xi\rangle)^\sigma\geqq RN \qquad \text{if } \log\langle\xi\rangle\geqq\varepsilon N+R_\varepsilon.$$

From this, $\sum_j K_j(t, t', x, \xi)$ is a formal symbol in $\mathcal{I}^{m,1-m\tau,1-m\theta}_{h\tau,h\theta}(\Omega)$ if $d(\tau+\theta)<1$, uniformly in (t, t'). Let $K(t, t', x, \xi)$ be a realization of this symbol. We note that if $|t-t'|\geqq\nu>0$, $K(t, t', x, \xi)$ is rapidly decreasing as $|\xi|\to\infty$.

Let $\Gamma_1=\{(\sigma, \xi)\in\boldsymbol{R}^{d+1}: |\sigma|\leqq|\xi|^m\}$, $\Gamma_2=\{(\sigma, \xi)\in\boldsymbol{R}^{d+1}: |\sigma|\geqq|\xi|^m/2\}$ and $\omega\Subset(-T, T)\times\Omega$ be an open set. Then,

$$KL\sim Id \text{ in } \omega\times\Gamma_1 \quad\text{and}\quad WF_{(m,1)}u\cap\omega\times\Gamma_1=\emptyset$$

if $Lu \in C^\infty(\omega)$. Here WF_M stands for the quasi-homogeneous wave front set ([8]). Since L is elliptic in $\omega \times \Gamma_2$, we can construct the parametrix Q of L such that

$$QL \sim Id \qquad \text{in } \omega \times \Gamma_2 .$$

From this, we have

$$WF_{(m,1)} u \cap \omega \times \Gamma_2 = \emptyset \qquad \text{if } Lu \in C^\infty(\omega) .$$

Therefore, we conclude that L is hypoelliptic in $(-T, T) \times \Omega)$.

Q. E. D. of theorem 3.1

Proof of theorem 3.2. The result follows from the fact: that there exist constants C and R such that

$$|\partial_\tau^\alpha K_0(t, t', \xi)| \leqq C^{|\alpha|+1} \alpha !^{1-\tau} (\log \langle \xi \rangle)^{-h\tau|\alpha|} \langle \xi \rangle^{(m\tau-1)|\alpha|}$$

for any α, $\xi \in \boldsymbol{R}^d$ with $R|\alpha| \leqq (\log \langle \xi \rangle)^h$.

4. Double characteristics operators with infinitely degenerate coefficients

We consider the operator P on a subset Ω of $\boldsymbol{R}^2$:

$$P = (D_1 + ia(x)D_2)^2 + b(x)(D_1 + ia(x)D_2) + c(x)D_2 + d(x) ,$$

where $D_j = -i\partial/\partial x_j$, $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are in $G^s_{x_2}(\Omega)$ with some $s \geqq 1$. Here $G^s_{x_2}(\Omega) = \{f \in C^\infty(\Omega) : {}^\forall K\, {}^\exists C\ |\partial_2^k f(x)| \leqq C^{k+1} k!^s$ for any k and $x \in K\}$.

We suppose that for some constants C and positive real numbers r, h,

$$\int_{t'}^t a(y, x_2)dy \geqq |t-t'| \langle \xi \rangle^{-1} (\log \langle \xi \rangle)^{2h} \qquad \text{if } t-t' \geqq r(\log \langle \xi \rangle)^{-h} ,$$

$$a(x) \geqq 0 \qquad \text{and } |c(x)| \leqq Ca(x) \qquad \text{for } x \in \Omega .$$

Toeorem 4.1. *P is hypoelliptic in Ω.*

Proof. For simplicity, we assume that $\Omega \ni 0$ and show the hypoellipticity of P at the origin. The equation

$$-(D_t + ig(t))^2 u = f(t)$$

has a solution

$$u = \int_T^t (t-s) \exp\left(\int_s^t g(\sigma)d\sigma\right) f(s)ds .$$

With this observation, the same argument as that in the previous section gives us to construct the left parametrix K of P:

$$Ku = \int \exp(ix_2\xi_2) \int_{T(\xi_2/|\xi_2|)}^{x_1} K(x_1, x_1', x_2, \xi_2) \hat{u}(x_1', \xi_2) dx_1' d\xi_2 ,$$

where $T(\pm 1)$ is a small constant with the same sign as that of ± 1, and $K \sim \sum_{j \geq 0} K_j(x_1, x_1', x_2, \xi_2)$. Here K_j satisfy that there exist constant C and R

such that

$$|\partial_{\xi_2}^{\alpha}\partial_{x_2}^{\beta}K_j(x_1, x_1', x_2, \xi_2)| \leqq C^{\alpha+\beta+j+1}(\log\langle\xi_2\rangle)^{-hj}(\alpha^s/|\xi_2|)^{\alpha}$$

$$\times(\beta^s+|\xi_2|^{1/2}(\beta^s/(\log\langle\xi\rangle)^{h/2})^{\beta}\exp\left(-\frac{1}{4}\int_{x_1'}^{x_1}a(y, x_2, \xi_2)dy\right)$$

for any j, α, β, $x\in K\Subset\Omega$ with $(x_1-x_1')\xi_2\leqq 0$ and $R(j+\alpha)\leqq(\log\langle\xi_2\rangle)^h$.

Therefore, for a small positive $\varepsilon>0$, there exist τ, θ such that $K(x_1, x_1', x_2, \xi_2)\in\mathcal{L}_{\tau\theta}^{0,1,(1/2)+\varepsilon}$ with respect to (x_2, ξ_2). By the same argument as before, we conclude that P is hypoelliptic at the origin. Q. E. D.

Remark. If $0<\rho\leqq 1$ and $0\leqq\delta<1$, then

$$\gamma^s\text{-}S_{\rho\delta}^m(\Omega)\subset\bigcap_{\varepsilon>0}\bigcup_{\tau\theta}\mathcal{L}_{\tau\theta}^{m,\rho-\varepsilon,\delta+\varepsilon}(\Omega).$$

Example. $a(x)=\exp(-|x|^{-n})$ or $\exp(-|x_1|^{-n})$ with $n>0$.

If $c(x)$ does not satisfy the above condition, in general, P is not hypoelliptic at the origin. In fact, we have

Theorem 4.2. *Let* $a(x)=\exp(-|x_1|^{-n})$ $c(x)=\exp(-A|x|^{-l})$ *and* $b(x)$, $d(x)$ *be independent of* x_2. *Then,* P^* *is not solvable at the origin if either* $0<A<1$ *and* $l=n$ *or* $l<n$.

Proof. We only consider the case $l=n$. When $l<n$, the similar argument gives us the result. Let

$$w(t, \rho)=-\int_0^t\{a(y)\rho-(c(y)\rho)^{1/2}\}dy \qquad \text{and}$$

$$s=(\log\rho)^{1/n}t.$$

Then $w(t, \rho)$ is written by

$$-\int_0^{s(\log\rho)^{-n}}(\rho^{1-y^{-n}}-\rho^{2^{-1}-Ay^{-n}})dy(\log\rho)^{-1/n}.$$

From this, we deduce that in $s>0$, $w(t, \rho)$ has only one maximum $M(\rho)$ at $s=\bar{s}(\rho)$ such that $M(\rho)\to\infty$ as $\rho\to\infty$.

On the other hand, by the iteration, for any N we can construct the asymptotic solution u_N of $P(e^{ix_2\rho}u_N(x, \rho))\sim O(\rho^{-N})$ which essentially behaves like $\exp(w(x_1, \rho)-x_2^2\rho^{1/2})$.

Take the intervals $I\Subset I'\subset\boldsymbol{R}_+$ such that for some $\delta>0$ and $\varepsilon>0$, $\bar{s}(\rho)\in I$ and $2^{-1}-As^{-n}\geqq\delta$ if $s\in I'$. Let $\psi(s)\in C_0^\infty(\boldsymbol{R})$, $\phi(s)\in C_0^\infty(\boldsymbol{R})$ and $F(z)\in C_0^\infty(\boldsymbol{R}^2)$ such that $\psi=1$ near the origin, $\operatorname{supp}\phi\subset I'$, $\phi=1$ on I and $\int e^{iz_2}F(z)dz=1$. Then, it is easily shown that for

$$f(x)=F(\rho(x_1(\log\rho)^{1/n}-\bar{s}(\rho)), \rho x_2) \qquad \text{and}$$

$$v(x, \rho)=\phi(x_1(\log\rho)^{1/n})\psi(x_2)\exp(ix_2\rho)\cdot u(x, \rho),$$

the Hörmander's inequality

$$\left|\int f(x)v(x)dx\right| \leqq C(|f|_N + |Pv|_M)$$

does not hold as $\rho \to \infty$ for any given C, N and M. Here, $|\cdot|_l$ stands for the norm of $C^l(\bar{\Omega})$. This proves the non-solvability of P^* at the origin. Q.E.D.

Taking into consideration of the connection formula for the solutions of the equation

$$-u'' + t^k u = 0$$

the similar argument shows us

Theorem 4.3. *Let a, b and d be the same as in theorem 4.2. Let $c(x) = cx_1^k$, where $c \in \boldsymbol{C} \setminus 0$ and k is a non-negative integer. Then, P^* is not solvable at the origin.*

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