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Gevrey-hypoelliptic operators which are not C^{∞}-hypoelliptic

By

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1. Introduction

The purpose of this paper is to show that the hypoellipticity in the Gevrey class is compatible with the non-hypoellipticity in C^{∞} class. For a subset ω of $\boldsymbol{R}^n$, we denote the Gevrey class with index s by $\gamma^s(\omega)$. We call the operator P γ^s-hypoelliptic {resp. C^{∞}-hypoelliptic} at the point x if there is some neighborhood ω of x such that if $u\in\mathcal{E}'(\omega)$ and for any $\omega'\subset\omega$, $Pu\in\gamma^s(\omega')$ {resp. $C^{\infty}(\omega')$}, then $u\in\gamma^s(\omega')$ {resp. $C^{\infty}(\omega')$}.

Let Ω be a subset of $\boldsymbol{R}^n$, Σ a real analytic conic submanifold of codimension 2 of $T^*(\Omega)$ and ρ a point on Σ. We define $M_k^m(\Sigma, \rho)$ to be the class of germs of homogeneous analytic symbols $p(x, \xi)\in\gamma^1\text{-}S_{1,0}^m$ at ρ which has the property that in some conic neighborhood Γ of ρ, $p(x, \xi)=0$ exactly on Σ and for some $z\in\boldsymbol{C}$, $zp=a+ib$, where a, b are real-valued, $d_{\xi}a\neq 0$ in Γ and $H_a^j b=0$ on $\Sigma\cap\Gamma$ if $j<k$ but $H_a^k b\neq 0$ in Γ. Here $H_a=\sum_j \partial a/\partial\xi_j\partial/\partial x_j-\partial a/\partial x_j\partial/\partial\xi_j$. We denote $H_f g$ by $\{f, g\}$.

We consider a classical analytic pseudo-differential operator P with symbol $p(x, \xi)\sim\sum_{j=0}^{\infty} p_{m-j}(x, \xi)$ and suppose that

1) $p_m(x, \xi)=0$ exactly on Σ,

2) for each $\rho\in\Sigma$ there is some conic neighborhood of ρ in which $p_m(x, \xi)=q^2(x, \xi)$ where $q\in M_2^{m/2}(\Sigma, \rho)$, and

3) $p_{m-1}^s(x, \xi)\neq 0$ on Σ.

Then, we have

Theorem 1. *Under the assumption* 1)-3), *P is γ^s-hypoelliptic in Ω if* $2\leqq s<4$. *Moreover, if for some $\rho\in\Sigma$, $\overline{z(q)}^2 p_m^s(\rho)\notin\bar{\boldsymbol{R}}_-$, then P is not C^{∞}-hypoelliptic at $\pi(\rho)$, where $z(q)=\{\bar{q}, \{q, \bar{q}\}\}$, and π is a projection on the base space.*

Theorem 2. *Let k be a positive even integer, c a non-zero complex number and $P=(D_1+ix_1^kD_2)^2+cD_2$, where $D_j=-i\partial/\partial x_j$. Then, P is γ^s-hypoelliptic at the origin for $1\leqq s<2k/(k-1)$ but P is not C^{∞}-hypoelliptic at the origin.*

We shall prove these results by constructing a parametrix which is viewed as vector-valued pseudo-differential operator of infinite order in the Gevrey class.

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2. Gevrey pseudo-differential operators of infinite order

In [6], some class of Gevrey pseudo-differential operators of infinite order has been already introduced. In this section, we introduce another class of them. Our class is a generalization of the class of analytic pseudo-differential operators of finite order given by G. Métivier.

Let Ω be an open set of $\boldsymbol{R}^n$ and s, ρ, δ be real numbers such that $s\geqq 1$, $0<\rho\leqq 1$ and $0\leqq\delta<1$. We shall denote by $\gamma^s\text{-}S^{\infty}_{\rho\delta}(\Omega)$ the space of all functions $p(x, \xi)\in C^{\infty}(\Omega\times\boldsymbol{R}^n)$ satisfying the following condition: for every compact set $K\subset\Omega$, there exist constants C and R, and for every $\varepsilon>0$, there exists a constant C_ε such that

$$|\partial_x^\alpha\partial_\xi^\beta p(x, \xi)|\leqq C_\varepsilon C^{|\alpha+\beta|}(|\beta|^s/|\xi|)^{\rho|\beta|}(|\alpha|^s+|\alpha|^{s(1-\delta)}|\xi|^\delta)^{|\alpha|}\exp(\varepsilon|\xi|^{1/s})$$

for every α, β, and $x\in K$, $\xi\in\boldsymbol{R}^n$ with $R|\beta|^s\leqq|\xi|$.

Replacing $\exp(\varepsilon|\xi|^{1/s})$ by $|\xi|^m$ in the above definition, we obtain $\gamma^s\text{-}S^m_{\rho\delta}(\Omega)$, the class of Gevrey pseudo-differential operators of finite order.

For $p\in\gamma^s\text{-}S^{\infty}_{\rho\delta}(\Omega)$, the operator $Op(p)$ {or $p(x, D)$}, with kernel $\int\exp\{i(x-y, \xi)\}\times p(x, \xi)d\xi$ is well defined and maps $\gamma_0^s(\Omega)$ in $\gamma^s(\Omega)$ and $\gamma^{(s)\prime}(\Omega)$ in $\gamma_0^{(s)\prime}(\Omega)$, where $\gamma^{(s)\prime}(\Omega)$ and $\gamma_0^{(s)\prime}(\Omega)$ are the duals of $\gamma^s(\Omega)$ and $\gamma_0^s(\Omega)$, respectively, and $d\xi=(2\pi)^{-n}d\xi$.

For the conic sets $\Omega\times\Gamma\subset\Omega\times\boldsymbol{R}^n$, we also define $\gamma^s\text{-}S^{\infty}_{\rho\delta}(\Omega\times\Gamma)$ by an obvious way: replacing $\xi\in\boldsymbol{R}^n$ by $\xi\in\Gamma$. We often call p a symbol of type (ρ, δ) when there is no ambiguity.

Now, we introduce the formal symbol: let μ_j be a sequence of non-negative real number such that for some $\kappa>0$, $\sum_j\exp(-\kappa\mu_j)<+\infty$. We shall say $\Sigma p_j(x, \xi)$ a formal symbol if the following condition is satisfied: for every compact subset $K\subset\Omega$, there exist constants C and R and for every $\varepsilon>0$, there exists a constant C_ε such that

$$|\partial_x^\alpha\partial_\xi^\beta p_j(x, \xi)|\leqq C_\varepsilon C^{|\alpha+\beta|}(C\mu_j)^{\mu_j}|\xi|^{-\mu_j}(|\beta|^s/|\xi|)^{\rho|\beta|}$$
$$\times(|\alpha|^s+|\alpha|^{s(1-\delta)}|\xi|^\delta)^{|\alpha|}\exp(\varepsilon|\xi|^{1/s})$$

for all j, α, β, $x\in K$, $\xi\in\boldsymbol{R}^n$ with $R(|\beta|+\mu_j+1)\leqq|\xi|$.

Next, we introduce the equivalence relation: let Σp_j, Σq_j be formal symbols. We say these two symbols be equivalent, $(\Sigma p_j\sim\Sigma q_j)$ if for every compact set $K\subset\Omega$ there exist constants C and R and for every $\varepsilon>0$ there exists a constant C_ε such that

$$|\partial_x^\alpha\partial_\xi^\beta\sum_{j<N}(p_j(x, \xi)-q_j(x, \xi))|\leqq C_\varepsilon C^{|\alpha+\beta|}(C\mu_N)^{\mu_N}|\xi|^{-\mu_N}$$
$$\times(|\beta|^s/|\xi|)^{\rho|\beta|}(|\alpha|^s+|\alpha|^{s(1-\delta)}|\xi|^\delta)^{|\alpha|}\exp(\varepsilon|\xi|^{1/s})$$

for all N, α, β, $x\in K$, $\xi\in\boldsymbol{R}^n$ with $R(|\beta|+\mu_N+1)\leqq|\xi|$.

For a conic neighborhood $\omega\times\Gamma$ of ρ {resp. a neighborhood ω of x}, we consider the equivalence relation $\Sigma p_j\sim\Sigma q_j$ at ρ {resp. at x}: replacing $\Omega\times\boldsymbol{R}^n$ by $\omega\times\Gamma$ {resp. $\omega\times\boldsymbol{R}^n$} in the above definition.

Now, we introduce the auxiliary functions: it is well known that if $\Omega_1\Subset\Omega_2$ are two open sets, one can find a sequence of functions $\Psi_N\in C_0^\infty(\Omega_2)$ and a constant C such that $\Psi_N=1$ on Ω_1 and for any N, α, $|\alpha|\leqq N$

$$|\partial^\alpha\Psi_N|\leqq(C|\alpha|^\rho N^{1-\rho})^{|\alpha|}, \tag{2.1}$$

where $\rho\in[0, 1)$ is a given parameter. Take $\Psi_N\in C_0^\infty(\boldsymbol{R}^n)$ satisfying (2.1) with $\Omega_1=\{|\xi|\leqq 1\}$ and $\Omega_2=\{|\xi|\leqq 2\}$. We define

$$\chi_j^\rho(\xi)=1-\Psi_{2j}(\xi/j^s).$$

Then, $\chi_j^\rho(\xi)=0$ for $|\xi|<j^s$, $\chi_j^\rho(\xi)=1$ for $|\xi|>2j^s$, and for some costant C

$$|\partial^\alpha\chi_j^\rho(\xi)|\leqq C^{|\alpha|}(|\alpha|^s/|\xi|)^{\rho|\alpha|/s}(1/j^{s-1})^{|\alpha|}$$

for any j, α, ξ, with $|\alpha|^s\leqq|\xi|$. By the similar way to lemma 3.1 in [4], we have

Lemma. *Given two cone $\Gamma_1\Subset\Gamma_2\subset\boldsymbol{R}^n$ and $\rho\in[0, 1)$, there exist $g\in C^\infty(\boldsymbol{R}^n)$ and a constant C such that $g(\xi)=0$ for $\xi\notin\Gamma_2$ or for $|\xi|<1$, $g(\xi)=1$ for $\xi\in\Gamma_1$ with $|\xi|>2$ and for any α, ξ with $|\alpha|^s\leqq|\xi|$, $|\partial^\alpha g(\xi)|\leqq C^{|\alpha|+1}(|\alpha|^s/|\xi|)^{\rho|\alpha|}$.*

We state the results on the calculus in this class.

Proposition 2.1. *Let Σp_j be a formal symbol. Set*

$$p(x, \xi)=\sum_j\chi^\rho_{[\mu j]+1}(\xi/\lambda)p_j(x, \xi)$$

where $[\mu]$ denotes the greatest integer less than μ. Then, for a sufficiently large λ, $p(x, \xi)$ belongs to $\gamma^s\text{-}S^\infty_{\rho\delta}(\Omega)$. Moreover p is uniquely determined up to the equivalence.

Here we call p a realization of Σp_j.

Proposition 2.2. *If $p(x, \xi)\sim 0$, then for any $u\in\mathcal{E}'(\Omega)$, $p(x, D)u\in\gamma^s(\Omega)$.*

Proposition 2.3. *Let $p\in\gamma^s\text{-}S^\infty_{\rho\delta}(\Omega)$. Then for $u\in\mathcal{E}'(\Omega)$, $WF_s(p(x, D)u)\subset WF_s(u)$. Furthermore if Γ is an opon cone in $\boldsymbol{R}^n$ and if $p\sim 0$ in $\omega\times\Gamma$ then for any $u\in\mathcal{E}'(\omega)$, $WF_s(p(x, D)u)\cap\omega\times\Gamma=\emptyset$.*

Proposition 2.4. *Let $a\in\gamma^s\text{-}S^\infty_{\rho'\delta}(\Omega)$, $b\in\gamma^s\text{-}S^\infty_{\rho\delta}(\Omega)$. If $\rho'>\delta$, the symbol $c(x, \xi)\sim\sum(1/\alpha!)\partial_\xi^\alpha a(x, \xi)D_x^\alpha b(x, \xi)$ is a formal symbol in $\gamma^s\text{-}S^\infty_{\rho''\delta}(\Omega)$ for $\rho''=\min(\rho', \rho)$. Moreover for any $\phi\in C_0^\infty(\omega)$ such that $\phi=1$ in a neighborhood of $\bar{\omega}_1\subset\omega$, and for any realization c of the formal symbol, we have $op(c)\sim op(a)\phi op(b)$ on ω_1.*

Here, the operator $A\sim 0$ on ω_1 means that for any $u\in\mathcal{E}'(\omega)$, $Au\in\gamma^s(\omega_1)$.

Proposition 2.5. *Let Σa_j and Σb_j be formal symbols in $\omega\times\Gamma$ respectively in $\gamma^s\text{-}S^{\infty}_{\rho'\delta}$ and in $\gamma^s\text{-}S^{\infty}_{\rho\delta}$. Define*

$$c_{j,k,\alpha}(x, \xi)=(1/\alpha!)(\partial_\xi^\alpha a_j)(x, \xi)(D_x^\alpha b_k)(x, \xi).$$

Then if $\rho'>\delta$, $c_{j,k,\alpha}$ is a formal symbol of type (ρ'', δ) with $\rho''=\min(\rho', \rho)$. Furthermore, for any realization a, b, c of these symbols, for any $\phi\in C_0^\infty(\omega)$, $\phi=1$ in a neighborhood of $\bar{x}\in\omega$, and for any $g(\xi)$ with support in Γ given by lemma with parameter ρ_1, $\delta<\rho_1<1$ and such that $g(\xi)=1$ for $|\xi|>2$, in a conic neighborhood of $\bar{\xi}\in\Gamma$, we have

$$op(gc)\sim op(ga)\phi op(gb) \quad at \quad (\bar{x}, \bar{\xi}).$$

Here, the operator $A\sim 0$ at $(\bar{x}, \bar{\xi})$ means that for any $u\in\mathcal{E}'(\omega)$, $WF_s(Au)\cap \omega\times\Gamma=\emptyset$ for some conic neighborhood $\omega\times\Gamma$ of $(\bar{x}, \bar{\xi})$.

These results are proved by the analogous arguments as [4]. The most different point is in the presence of the factor $\exp(\varepsilon|\xi|^{1/s})$. Roughly speaking, the arguments consist of two type: One is about the equivalence relations. It is easily seen that $p\sim 0$ implies $|p|\leqq C\exp(-\varepsilon_0|\xi|^{1/s})$ with some $\varepsilon_0>0$ and C. Another type of arguments is to show the pseudo-local property. Since $\exp(\varepsilon|\xi|^{1/s})=\sum_{k\geqq 0}(\varepsilon|\xi|^{1/s})^k/k!$, we need to proceed the integration by part in more times as k is large. But the careful treatment in this process gives us the above results.

3. Gevrey hypoellipticity

If $p\in M_k^m(\Sigma, \rho)$, p is microlocally equivalent to Mizohata operator. ([7]). Considering a canonical transformation and a elliptic Fourier integral operator, we may assume that in a conic neighborhood $\omega\times\Gamma$ of $\rho=(0, \xi_n)$,

$$p_m(x, \xi)=c_{m-2}(\xi_1+ix_1^k\xi_n)^2,$$

where $c_{m-2}\neq 0$ in $\omega\times\Gamma$. Dividing p by the elliptic factor and using Weierstrass' preparation theorem we have

$$p(x, \xi)=(\xi_1+ix_1^k\xi_n)^2+a_0(x, \xi')(\xi_1+ix_1^k\xi_n)+a_1(x, \xi')+\sum_{j=0}^{\infty}a_j(x, \xi')$$

where $a_j(x, \xi')$ are the classical analytic symbol in some neighborhood of ρ of order j and $\xi'=(\xi_2, \cdots, \xi_n)$. By the assumption that $p_{m-1}^s\neq 0$ on Σ, we have

$$a_1(x, \xi')\neq 0 \qquad \text{for} \quad (x, \xi')\in\omega\times\Gamma\,(\text{if necessary, shrinking } \omega\times\Gamma).$$

This enables us to factorize p as follows:

$$p(x, \xi)\sim p_1(x, \xi)\circ p_2(x, \xi),$$

$$p_l(x, \xi)\sim\xi_1+ix_1^k\xi_n+\sum_{j=-1}^{\infty}b_{l,-j}(x, \xi')$$

where $\circ$ denotes the composition and $b_{l,-j}$ satisfies that for some constants C_0

and C,

(3.1) $|\partial_x^\beta\partial_\xi^\alpha b_{l,-j}(x,\xi)|\leqq C_0C^{|\alpha+\beta|+j}(j^2/|\xi|)^{j/2}\beta!(|\alpha|/|\xi|)^{|\alpha|}$ for any α, β, j.

Therefore, to prove a half of theorem 1, it suffices to show that $p_l(x, D)$ is γ^s-hypoelliptic at ρ. From now on, we drop the index l and denote $\partial_x^\beta\partial_\xi^\alpha p$ by $p_{(\beta)}^{(\alpha)}$.

Theorem 3.1. *Let* $p(x,\xi)\sim\xi_1+ix_1^k\xi_n+\sum_{j=-1}^{\infty}b_{-j}(x,\xi')$, *where* b_{-j} *satisfies* (3.1). *Then, if* $k=2$, $p(x, D)$ *is* γ^s*-hypoelliptic at* ρ *with* $2\leqq s<4$.

Proof. By considering the adjoint of p, it suffices to construct the right parametrix of $p(x, D)$. Let $\omega=[-2T, 2T]\times\omega'$, $T>0$ and we seek this parametrix in the follwing form: denoting x_1 by t and x' by x,

$$Ku=\int e^{ix\xi}\int_{T_0}^{t}K(t,t',x,\xi)\hat{u}(t',\xi)dtd\xi, \tag{3.2}$$

where we take $T_0=-T$ if $\xi_n<0$ and $T^0=T$ if $\xi_n>0$, and $\hat{u}$ stands for the Fourier transform in x.

Hereafter, we assume $\xi_n<0$. In the contrary case, we can prove the result in a similar way. Let $K(t,t',x,\xi)\sim\Sigma K_j(t,t',x,\xi)$, where K_j is a formal symbol in γ^s-$S_{\rho\delta}^{\infty}$ uniformly in t, t'. Then $p(t,x,D_t,D_x)K\sim Id$ implies

$$(D_t+(1/\alpha!)p^{(\alpha)}(t,x,\xi)D_x)K(t,t',x,\xi)\sim 0$$

and

$$K(t,t,x,\xi)=1.$$

In view of this, we define

$$K_0(t,t',x,\xi)=\exp\left[-\int_{t'}^{t}\{s^2|\xi_n|+ib_1(s,x,\xi)+ib_0(s,x,\xi)\}ds\right],$$

$$K_j(t,t',x,\xi)=\sum_{l=0}^{j-1}\int_{t'}^{t}K_0(t,s,x,\xi)\mathcal{P}_l(s,x,\xi,D_x)K_{j-l-1}(s,t',x,\xi)ds$$

where

$$i\mathcal{P}_l(s,x,\xi,D_x)=\sum_{\substack{|\alpha|+j=0\\ j\neq 0}}\alpha!^{-1}b_{-j}^{(\alpha)}(s,x,\xi)D_x^\alpha+\delta_{1,l}\ is^2D_x$$
$$+\sum_{|\alpha|=l+1}\alpha!^{-1}b_0^{(\alpha)}(s,x,\xi)D_x^\alpha.$$

Here, $\delta_{1,l}$ is a Kronecker's delta.

Set $\Lambda=\int_{t'}^{t}s^2|\xi_n|ds$ and $Q=\Lambda+\Lambda^{1/3}|\xi_n|^{1/6}$. Then we see that for some constant C,

$$|\Lambda_{(\beta)}^{(\alpha)}|\leqq C^{|\alpha+\beta|+1}\alpha!\beta!Q|\xi|^{-|\alpha|}\qquad\text{for any}\quad\alpha,\ \beta,$$

since $\Lambda\geqq\lambda|t-t'|^3|\xi_n|$ for some $\lambda>0$. This inequality and the formula for the derivatives of a composition of functions (Faa di Bruno) give us: For some constants C and C'

$$|K_{0(\beta)}^{(\alpha)}(t, t', x, \xi)| \leqq C^{|\alpha+\beta|+1} \sum_{\mathcal{M}} (\alpha+\beta)!/(i_1! \cdots i_k!) Q^{|I|} |\xi|^{-|\alpha|}$$

$$\times \exp(C'|t-t'||\xi_n|^{1/2} - \Lambda/2)$$

where $\mathcal{M} = \left\{ \sum_{j=1}^{k} j_j \gamma_j = \alpha+\beta,\ \gamma_j \neq 0,\ \gamma_j \in N^n,\ i_j \in N \right\}$ and $I=(i_1, \cdots, i_k)$.

It is easily seen that for some constant A, we have

$$Q^{|I|} |\xi_n|^{-|\alpha+\beta|/6} \leqq A^{|I|} \Lambda^{|I|-|\alpha+\beta|/6} \quad \text{if} \quad |I| \geqq |\alpha+\beta|/6$$

and

$$\leqq A^{|I|} \quad \text{if} \quad |I| < |\alpha+\beta|/6.$$

Hence, using the inequalities: $y^N e^{-y} \leqq N!$ for all $y>0$ and for some constant B,

$$\sum_{\mathcal{M}} I!/(i_1! \cdots i_k!) \leqq B^{|\alpha+\beta|},$$

we obtain the estimate for K_0.

Proposition 3.2. *$\theta \Subset \omega' \times \Gamma$ be a conic subset. Then, there exist constant C_j ($j=1, 2, 3$) and R such that*

$$|K_{0(\beta)}^{(\alpha)}(t, t', x, \xi)| \leqq C_1^{|\alpha+\beta|+1} |\alpha|^{|\alpha|} |\xi|^{-\rho|\alpha|} (|\beta| + |\beta| |\xi|^{1-\rho})^{|\beta|}$$

$$\times \exp(-C_2 \Lambda + C_3 |t-t'| |\xi|^{1/2})$$

for $t \geqq t'$, $(x, \xi) \in \theta$, any α, β, with $R|\alpha| \leqq |\xi|$. Here $\rho=5/6$.

As for the estimate for K_j, we have

Proposition 3.3. *There exist constant C_0, C, C' and R such that*

$$|K_{j(\beta)}^{(\alpha)}(t, t', x, \xi)| \leqq C_0 C^{|\alpha+\beta|+j} (|\alpha|^2/|\xi|)^{-\rho|\alpha|} (|\beta|^2 + |\beta|^{2\rho} |\xi|^{1-\rho})^{|\beta|}$$

$$\times (j^2/|\xi|)^{j/2} \exp\left(-\frac{1}{2} C_2 \Lambda + C' |\xi|^{1/4}\right)$$

for $t \geqq t'$, $(x, \xi) \in \theta$, any α, β with $R(|\alpha|+j+1)^2 \leqq |\xi|$.

Proof. By Leibniz'rule, we see that

$$(\alpha+\beta)!^{-1} K_{j(\beta)}^{(\alpha)}(t, t', x, \xi) = \sum_{l=0}^{j-1} \sum \int_{t'}^{t} \prod_{k=1}^{3} (\alpha_k! \beta_k!)^{-1} K_{0(\beta_1)}^{(\alpha_1)}(t, s, x, \xi)$$

$$\times \mathcal{P}_{l(\beta_2)}^{(\alpha_2)}(s, t', x, \xi, D_x) K_{j-l-1(\beta_3)}^{(\alpha_3)}(s, t', x, \xi) ds,$$

where the sum is taken over all α_j, β_j such that $\sum \alpha_j = \alpha$ and $\sum \beta_j = \beta$.

Since

$$\int_{t'}^{t} s^2 |\xi_n| \exp\left(-\frac{1}{2} C_2 \Lambda(t, s)\right) ds \leqq 4/C_2,$$

the induction on j shows us that there exist constants C_0, C and R such that

$$|K_{j(\beta)}^{(\alpha)}(t, t', x, \xi)| \leqq C_0 C^{|\alpha+\beta|+j} (|\alpha+\beta|+j)! |\xi|^{-\rho|\alpha|-j/2}$$

$$\times (1+|\xi|^{1-\rho})^{|\beta|} (\Lambda^{1/3} |\xi|^{-1/6} + 1)^j$$

$$\times\exp\left(-\frac{1}{2}C_2\Lambda(t, t')+C_3|t-t'||\xi|^{1/2}\right)$$

for $t\geqq t'$, $(x, \xi)\in\theta$, any α, β, j with $R(|\alpha|+j+1)^2\leqq|\xi|$. The result follows from this since there exist constants C'' and R' such that

$$\Lambda^{N/3}|\xi|^{-N/6}e^{-C_2\Lambda/2}\leqq C''^N \quad \text{if} \quad |\xi|\geqq Rj^2,$$

and

$$\Lambda(t, t')-c|t-t'||\xi|^{1/2}\geqq\{(\lambda|t-t'||\xi|^{1/4})^3-c|t-t'||\xi|^{1/4}\}|\xi|^{1/4}.$$

Therefore, K_j is the formal symbol in $\gamma^s\text{-}S^{\infty}_{5/6,1/6}(\omega\times\Gamma)$ if $2\leqq s<4$. Let $K(t, t', x, \xi)$ be a realization of this symbol. Then we have $p(x, D)K\sim Id$ at ρ. Similarly, we can also construct the left parametrix of $p(x, D)$: $Lp(x, D)\sim Id$ at ρ. Then $L\sim K$ at ρ. Now take $\chi\in op(\gamma^s\text{-}S^0_{10}(\boldsymbol{R}^n))$, properly supported such that $\chi\sim Id$ in a conic neighborhood of ρ and $WF_s(\chi)$ is in a small conic neighborhood of ρ. Then, it is easily seen that $(t, \tau, x, \xi, t', \tau', x', \xi')\notin WF_s'(\chi K\chi)$ if $(x, \xi)\neq(x'\xi')$. Since either $(t, \tau)\neq(0, 0)$ or $(t', \tau')\neq(0, 0)$ if $(t, \tau)\neq(t', \tau')$, the ellipticity of p at $(t, \tau)\neq(0, 0)$ implies that if $(t, \tau)\neq(t', \tau')$

$$(t, \tau, x, \xi, t', \tau', x', \xi')\notin WF_s'(\chi K\chi).$$

Therefore, we conclude that

$$WF_s'(\chi K\chi)\subset\text{the diagonal of } T^*(\omega)\setminus 0.$$

Here, WF_s' stands for the analogy of WF' in the space $\gamma_0^{(s)\prime}(\boldsymbol{R}^n)$. This is well-defined in virtue of the kernel theorem ([3]). Q.E.D. of theorem 1.

Next, we consider the operator in theorem 2.

Proof of theorem 2. In this case, we have the factorization of p: $p\sim p_1\circ p_2$,

$$p_j(x, \xi)=\xi_1+ix_1^k\xi_2+b_j(x_1, \xi_2),$$

where $b_j\in\gamma^1\text{-}S^{1/2}_{1,0}(\omega\times\Gamma)$. We keep the same notation as before. Then, there exist constants $C_j(j=1, 2, 3)$ and R such that

$$|K_0^{(\alpha)}(t, t', \xi)|\leqq C_1^{|\alpha|+1}(|\alpha|/|\xi|)^{-\rho|\alpha|}\exp(-C_2\Lambda+C_3|t-t'||\xi|^{1/2})$$

for $t\geqq t'$, $\xi_2<0$, any α with $R|\alpha|\leqq|\xi|$, where $\rho=\frac{1}{2}+1/(k+1)$ and $\Lambda=\int_{t'}^t s^k|\xi_2|ds$. This is easily verified because $^\exists C$ and R,

$$(\Lambda+\Lambda^{1/(k+1)}|\xi|^{1-\rho})^{|I|}|\xi|^{-(1-\rho)|\alpha|}\exp(-\Lambda)$$
$$\leqq C^{|I|}|\alpha|^{|I|-\rho|\alpha|}$$

if $R|\alpha|\leqq|\xi|$. Therefore K_0 belongs to $\gamma^1\text{-}S_{\rho,1-\rho}(\omega'\times\Gamma)$ uniformly in $t\geqq t'$. This and the same argument as before give us the result. Q.E.D.

4. Non-C^∞-hypoellipticity

Let us consider the operator P with symbol

$$p(x, \xi)=c(x, \xi)\{q^{m}(x, \xi)+\sum_{|\alpha|+j\leq m-1} x_1^{l(\alpha, j)} b_{\alpha, j}(x, \xi')q^{j}(x, \xi)\}$$

where $q(x, \xi)=\xi_1+ix_1^k a(x, \xi')$, a, $b_{\alpha j}$, c are the classical elliptic symbol in a conic neighborhood of $\rho=(0, \bar{\xi}')$ of order 1, $|\alpha|$, 0, respectively, especially a is real-valued and $l(\alpha, j)$ is a non-negative integer. Let

$$\mathcal{M}=\{(\alpha, j): |\alpha|+j\leq m-1, l(\alpha, j)<(k+1)|\alpha|+j-m\}$$

$$\sigma=\min_{\mathcal{M}} \sigma(\alpha, j), \sigma(\alpha, j)=(m-j-|\alpha|)/(km-kj-l(\alpha, j)),$$

and

$$\mathcal{M}_0=\{(\alpha, j)\in\mathcal{M}: \sigma=\sigma(\alpha, j)\}.$$

For $(y, \eta)\in \boldsymbol{R}^2$, we introduce the function

$$P^{*}(y, \eta; \rho)=\eta^{m}+\sum_{\mathcal{M}_0} y^{l(\alpha, j)} b_{\alpha, j}(\rho)\eta^{j}.$$

Then we have

Theorem 4.1. *Suppose* $\mathcal{M}\neq\emptyset$ *and for each* y *with* $ya(\rho)<0$, *the equation* $P^{*}(y, \eta; \rho)=0$ *has a simple root* $\eta(y)$ *such that* $\operatorname{Im}\eta(y)<0$. *Then* P^{*} *is not solvable at* ρ.

Here, the operator P is solvable at $\rho\in T^{*}(\Omega)\setminus 0$ iff there is an integer N such that for every $f\in H_N^{loc}(\Omega)$ we have

$$WF(Pu-f)\not\ni\rho \quad \text{for some} \quad u\in\mathcal{D}'(\Omega).$$

The results for non-C^{∞}-hypoellipticity in theorem 1 and 2 are the straight forward consequance of this theorem since the C^{∞}-hypoellipticity of P at the origin implies the solvability of P^{*} at every $\rho=(0, \xi)$, $\xi\neq 0$.

Proof of theorem 4.1. We may assume that $c(x, \xi)=1$. Set $y_1=\lambda^{\sigma}x_1$, and $y'=\lambda x'$. Then,

$$P(\exp(iy\xi)v(x))\sim\exp(iy\xi)\sum\alpha!^{-1}p^{(\alpha)}(\lambda^{-\sigma}y_1, \lambda^{-1}y', \lambda^{\sigma}\xi_1, \lambda\xi')D_x^{\alpha}v.$$

By considering Taylor expansion at $x=0$, this becomes

$$\exp(iy\xi)\lambda^{\sigma m}\{z^{m}+\sum_{\mathcal{M}_0} y_1^{l(\alpha, j)} b_{\alpha, j}(\rho)\lambda^{\delta(m-j)}z^{j}+\sum C_{\beta', j}(y, \xi')$$
$$\times\lambda^{\delta(m-j)-|\beta'|-\varepsilon(\beta', j)}z^{j}D_{y'}^{\beta'}\}v(y),$$

where

$$z=\xi_1+D_{y_1}+iy_1^k a(\rho)\lambda^{\delta}, \delta=\{|\bar{\alpha}|(k+1)-m+\bar{j}-l(\bar{\alpha}, \bar{j})\}/)km-k\bar{j}-l(\bar{\alpha}, \bar{j}))$$

with $(\bar{\alpha}, \bar{j})\in\mathcal{M}_0$, $c_{\beta, j}$ are smooth and $\varepsilon(\beta', j)\geq\varepsilon_0>0$.

Therefore the same argument as in [5] enables us to show that for any $M>0$ there exists the asymptotic solution $u_{\lambda}^{M}(x)$ of $Pu=0(\lambda^{-M})$ such that

$$u_{\lambda}^{M}(x)=\exp w(y, \lambda)\cdot\sum_{j=0}^{J(M)}\lambda^{-\nu_j}u_j(y), \quad \nu_j\to\infty \quad \text{as} \quad j\to\infty,$$

where u_j and w are C^{∞}-function in a open set Ω in $\boldsymbol{R}^n$, and

$$\mathrm{Im}\, w(y, \lambda) \geqq (-h(y, \lambda)+|y'|^2)\lambda^{\delta}, \quad |\mathrm{Re}\, w(y, \lambda)-y'\lambda\xi'|=0(\lambda^{\delta})$$

Here the smooth function h has a minimum at $y_1=s(\lambda)$ which satisfies: $\lim_{\lambda\to\infty} s(\lambda) = s_0 \neq 0$, and $(s_0, 0)\in \boldsymbol{\Omega}$.

Then by the standard argument, it is seen that for every C, N, ν, every neighborhood U of the origin and every properly supported pseudo-differential operator A with $WFA \not\ni \rho$ the following inequality

$$\|v\|_{-N} \leqq C\{\|Pv\|_{\nu}+\|v\|_{-N-n}+\|Av\|_0\} \tag{4.1}$$

does not hold for $v=\varphi_{\lambda} u_{\lambda}^M \in C_0^{\infty}(U)$ if λ and M are large enough. Here, $\phi_{\lambda}=\chi(\lambda^{\sigma} x_1 - s(\lambda), \lambda x')$ with $\chi \in C_0^{\infty}(\boldsymbol{R}^n)$ whose suport is sufficiently small and contains the origin. $\|\ \|_{\theta}$ stands for the Sobolev norm.

Therefore, by lemma 26.4.5 in [2], we conclude that P^* is not solvable at ρ.

Q. E. D.

5. Appendix

In this section, we shall give some remarks on the solvability for the operator L:

$$L=(D_1+ix_2^k D_2)^2+bx_1^l D_2$$

where, k and l are non-negative integers and $b\in \boldsymbol{C}\setminus 0$.

In [8], the following result has been announced without proof:

Theorem. *L^* is locally solvable at the origin if and only if k is even and $l \geqq k-1$.*

The proof of this result is given in [1], in more general form, k is odd. When k is even, one can find it in [5] except for the case that l is odd and $b>0$.

In this section, we shall give the proof for this exceptional case:

Theorem A.1. *If k is even, l is odd and $b>0$, then L^* is not solvable at* $(0, 0, 0, 1)$.

Now, we consider the case that k is odd, in more detail. By the argument in [1], it is easily seen that L is not solvabe at $(0, 0, 0, 1)$ if k is odd. We shall show the following result:

Theorem A.2. *If k is odd, L is solvable at* $(0, 0, 0, -1)$ *and, more precisely, L^* is hypoelliptic at* $(0, 0, 0, -1)$.

*Proof of theorem A.*1. For simplicity, we assume that $b=1$ and denote $\xi_2=\lambda>0$. Taking Fourier transformation in x_2, we have

$$\hat{L}=(D_1+ix_1^k\lambda)^2+x_1^l\lambda.$$

We seek the solution u of the equation $\hat{L}u=0$ in the form:

$$u=\exp(x_1^{k+1}\lambda/(k+1)-x_2^2\lambda^{2\mu})v(x, \lambda)$$

where μ is a small positive number determined later.

Let $\sigma=1/(2k-l)$, $\delta=(k-1-l)/(2k-l)$, $t=x_1\lambda^\sigma$, and $y=x_2\lambda^\mu$. Then for a positive number $\mu<1-2\delta$, there is $\nu>0$ such that

$$\{-\partial_t^2+t^l\lambda^{2\delta}+\lambda^{-\nu}F(t,\ y,\ \lambda,\ D_t,\ D_y)\}v(t,\ y,\ \lambda)=0,$$

where F is the differential operator of order 2 with smooth coefficients and satisfies that $F(t,\ y,\ \lambda,\ \lambda^\delta,\ \lambda^\mu)$ are bounded if $(t,\ y)$ is bounded and λ tends to infinity.

From the asymptotic expansion theory of ordinary differential equations, it follows that there exist $V_\pm(t)$ such that for some constants $c_j^\pm$,

$$(-\partial_t^2+t^l)V_\pm(t)=0,$$

$$V_\pm(t)\sim t^{-1/4}\exp\left(\pm\int_0^t s^{l/2}ds\right) \quad \text{as} \quad t\to\infty,$$

and

$$\sim t^{-1/4}\left\{c_1^\pm\exp\left(-\int_0^t s^{l/2}ds\right)+c_2^\pm\exp\left(\int_0^t s^{l/2}ds\right)\right\} \quad \text{as} \quad t\to-\infty.$$

Set $v_\pm(t,\ \lambda)=V_\pm(t\lambda^{2\delta/(l+2)})$ and denote the Wronskian of $\{v_+(t,\ \lambda),\ v_-(t,\ \lambda)\}$ by W. Then $W=C\lambda^{2\delta/(l+2)}$ with some nonzero constant C.

We define the operator K by

$$Kf=W^{-1}\left\{\int_0^t v_-(t,\ \lambda)v_+(s,\ \lambda)f(s)ds+\int_t^1 v_+(t,\ \lambda)v_-(s,\ \lambda)f(s)ds\right\}.$$

Now, we define V_j by

$$V_0(t,\ \lambda)=v_-(t,\ \lambda)$$

and

$$V_j(t,\ y,\ \lambda)=-K\{\lambda^{-\nu}F(t,\ y,\ \lambda,\ D_t,\ D_y)V_{j-1}(t,\ y,\ \lambda)\} \quad \text{if} \quad j>0.$$

Then, it is easily shown that

$$|\partial_t^m\partial_y^n V_j(t,\ y,\ \lambda)|\leqq C_{m,n}\lambda^{-\nu j}\lambda^{\delta' m+\mu n}\exp(-2(l+2)^{-1}Y(t)t^{(l+2)/2}\lambda^\delta),$$

where $C_{m,n}$ is some constant, $Y(t)=1$ if $t>0$, $Y(t)=0$ if $t<0$ and $\delta'=2\delta/(l+2)$.

Set $u_M=\sum_{j=0}^M\lambda^{-\nu j}V_j(t,\ y,\ \lambda)\exp((k+1)^{-1}x_1^{k+1}\lambda-x_2^2\lambda^{2\mu})$. Then we have

$$|\partial_{x_1}^m\partial_{x_2}^n u_M(x,\ \lambda)|\leqq C_{m,n}\lambda^{(\sigma+\delta')m+2\mu n}\exp w(x_1\lambda^\sigma,\ x_2\lambda^\delta,\ \lambda)$$

$$|\partial_{x_1}^m\partial_{x_2}^n \hat{L}u_M(x,\ \lambda)|\leqq C_{m,n}\lambda^{(\sigma+\delta')m+2\mu n-M}\exp(x_1\lambda^\sigma,\ x_2\lambda^\delta,\ \lambda),$$

where $w(t,\ y,\ \lambda)=(-2Y(t)(l+2)^{-1}t^{(l+2)/2}+(k+1)^{-1}t^{k+1})\lambda^\delta-y^2$. Here we note that $\sigma+\delta'+(l+2)^{-1}<1$.

Let ω be a neighborhood of the origin in which $(l+2)^{-1}\times|t|^{(l+2)/2}\leqq(k+1)|t|^{k+1}$ and take $\chi\in C_0^\infty(\omega)$ such that $\chi=1$ near the origin. Put

$$U_\lambda^M(x)=\chi(\lambda^\sigma x_1,\ \lambda^\mu x_2)u_M(x,\ \lambda)e^{ix_2\lambda}.$$

Then the standard argument shows that the Hörmander inequality (4.1) does not hold for U_λ^M as $\lambda\to\infty$, if N is taken large enough as compared with N and ν. Therefore we conclude that L^* is not solvable at (0, 0, 0, 1). Q. E. D.

Proof of theorem A.2. Considering the adjoint of L^* we are going to construct the right parametrix of L. This parametrix has the different form dependent on arg b and l. We only consider the typical case. Let l be even, $b=-1$, and $\xi_2=-\eta<0$. Then it is easily seen that there exist the solutions $V_\pm(t, \eta)$ of

$$\hat{L}V=\{-(\partial_t+t^k\eta)^2+t^l\eta\}V(t, \eta)=0$$

such that $V_\pm(t, \eta)\sim(t\eta^{1/(l+1)})^{-1/4}\exp w_\pm(t, \eta)$ as $t\eta^{1/(l+1)}\to\mp\infty$, where

$$w_\pm(t, \eta)=-t^{k+1}\eta/(k+1)\pm\int_0^t|\tau^l\eta|^{1/2}d\tau.$$

With the same σ, δ as before, let $s=t\eta^\sigma$. Then

$$w_\pm(t, \eta)=\tilde{w}_\pm(s, \eta)=\left(-s^{k+1}/(k+1)\pm\int_0^s\tau^{l/2}d\tau\right)\eta^\delta.$$

For each $\pm$, this function $\tilde{w}_\pm$ has only one maximum at $s=s_\pm$ where $s_+>0$ and $s_-<0$. We define the operator $E(t, \eta)$ by

$$E(t, \eta)f=\int_{s_+\eta^{-\sigma}}^t e_+(t, t', \eta)f(t')dt'-\int_{s_-\eta^{-\sigma}}^t e_-(t, t', \eta)f(t')dt'$$

where $e_\pm(t, t', \eta)=V_\pm(t, \eta)V_\mp(t', \eta)/W(t', \eta)$, and $W(t', \eta)$ is the Wronskian of $\{V_+, V_-\}(=\exp(-2t^{k+1}\eta/(k+1))$. Then, it is easily shown that

$$|\partial_\eta^\alpha e_\pm(t, t', \eta)|\leqq C_\alpha\eta^{-\rho\alpha}\exp\left\{-2^{-1}\int_{t'}^t(\tau^k\eta\pm(\tau^l\eta)^{1/2}d\tau)\right\}$$

if t' is between $s\pm\eta^{-\sigma}$ and t, where C_α is some constant and

$$\rho=\min\{1-2^{-1}\delta, 2^{-1}+2^{-1}(k+1)^{-1}(l+2)\}.$$

Define the operator E by

$$Ef=\int e^{i(x_2-x_2', \eta)}E(x_1, \eta)g(\eta)f(x)dxd\eta$$

where $g(\eta)\in C^\infty(\boldsymbol{R}^n)$, $g(\eta)=1$ if $\eta>2$ and $g(\eta)=0$ if $\eta<1$. Then direct computations show that

$$E^*L^*\sim Id \quad\text{and}\quad L^*E^*\sim Id+R \quad\text{at}\quad (0, 0, 0, -\eta)$$

for some operator R with $WF'(R)\subset\{(0, x_2, 0, \xi_2, x_1, x_2, \xi_1, \xi_2)\}$. Here, $A\sim B$ at ρ iff there exists a conic neighborhood θ of ρ such that $WF'(A-B)\cap\operatorname{diag}\theta=\emptyset$.

From this, we conclude that there exists a conic neighborhood θ of $(0, 0, 0, -\eta)$ such that

$$WFu\cap\theta\subset WFL^*u\cap\theta \qquad \text{for any} \quad u\in\mathcal{E}'$$

since L^* is elliptic at (x, ξ) if $(x_1, \xi_1)\neq 0$.

For the other case of arg b and l, there exist solutions $V_\pm$ of $LV=0$ such that $\operatorname{Re}\log V$ has only one maximum. So, by choosing appropriately the lower bounds of the integral in the definition of $E(t, \eta)$, the similar argument show the hypoellipticity of L^* at $(0, 0, 0, -\eta)$. Q.E.D.

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