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Degeneration of K3 surfaces

By

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Introduction

Let $\pi: X \to \Delta$ be a proper surjective holomorphic map of a three-dimentional complex manifold X to a disk $\Delta = \{t \in \boldsymbol{C} \mid |t| < \varepsilon\}$ with connected fibers. Assume that π is smooth at each point of $\pi^{-1}(\Delta^*)$, $\Delta^* = \Delta - \{0\}$. We callsu ch a holomorphic map $\pi: X \to \Delta$ a degeneration of surfaces (a degeneration, for short). By the singular fiber X_0, we mean a divisor on X defined by $t=0$. A smooth surface $X_t = \pi^{-1}(t) (t \neq 0)$ is called a general fiber. We call $\pi: X \to \Delta$ a degeneration of K3 surfaces if a general fiber X_t is a K3 surface.

A degeneration $\pi': X' \to \Delta$ is called a modification of a degeneration $\pi: X \to \Delta$, if there exists a bimeromorphic map $\Phi: X \cdots\to X'$ such that the diagram

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi'} & \\ & \Delta & \end{array}$$

is commutative, and $\mathrm{res}\, \Phi: \pi^{-1}(\Delta^*) \to \pi'^{-1}(\Delta^*)$ is biholomorphic over Δ^*.

In this paper, we shall study degenerations of K3 surfaces up to modifications.

A degeneration $\pi: X \to \Delta$ is called semi-stable, if the singular fiber X_0 is a reduced divisor with simple normal crossings. Note that by Mumford's theorem, every degeneration can be made semi-stable after base change and modification.

Kulikov [7] and Persson-Pinkham [17] studied a semi-stable degeneration $\pi: X \to \Delta$ of K3 surfaces under the assumption that π is projective, or under the weaker assumption that every component of the singular fiber X_0 is algebraic. They showed that under the above assumption there exists a modification $\pi': X' \to \Delta$ of $\pi: X \to \Delta$ such that π' is also semi-stable and the canonical bundle of X' is trivial.

A main purpose of the present paper is to study semi-stable degenerations of K3 surfaces, in general, without the above assumption. Some results of this paper have already been announced in Nishiguchi [13].

In § 1, we shall generalize the theorem proved by Kulikov and Persson-Pinkham (see Theorem 1.1). § 2 will give a classification of semi-stable degenerations of K3 surfaces with trivial canonical bundles. In these sections, results about surfaces of class VII due to Enoki [1], Nakamura [9, 10] and Nishiguchi [14, 15] play a fundamental role. In § 3, we shall discuss which surface can be a

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component in the singular fiber of a semi-stable degeneration of K3 surfaces with the trivial canonical bundle. In connection with this problem, we also consider the smoothing problem of simple elliptic singularities and cusp singularities. This section has a deep relation with the work of Looijenga [8].

§ 4 begins with the construction of the easiest example of a semi-stable degeneration of K3 surfaces for which no semi-stable modification has the trivial canonical bundle. Namely, without assuming the algebraicity or Kählerity on the degenerations, the result of Kulikov and Persson-Pinkham does not necessarily hold. Degenerations like this example contain a blown-up Hopf surface or a blown-up (CB)-surface in their singular fibers. § 4 also treats a problem, which blown-up Hopf surface or blown-up (CB)-surface can be a component in the singular fiber of a semi-stable degeneration of K3 surfaces. This problem is connected with the existence problem of K3 surfaces with a certain configuration of curves. This section ends with a criterion of smoothability of some Gorenstein normal surface singularities with geometric genus two.

In § 5, we shall systematically construct examples of semi-stable degenerations of K3 surfaces which contain some series of (CB)-surfaces in their singular fibers. We make use of a result about the canonical bundles of (CB)-surfaces, which was studied in Nishiguchi [14, 15]. We also give a sufficient condition that such a (CB)-surface become a component in the singular fiber of a semi-stable degeneration of K3 surfaces. This condition is reduced to the existence problem of a surface with certain configuration of curves, as considered in § 3 and § 4.

The deformation theory due to Friedman [2] is fundamental and plays an important role throughout this paper.

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Notations and Conventions

We use the following notations for a compact complex manifold M.

$b_i(M)$: i-th Betti number
$a(M)$: algebraic dimension
$\kappa(M)$: Kodaira dimension
$P_m(M)$: m-th plurigenus
$p_g(M)$: geometric genus
$q(M)$: irregularity.

By a surface (resp. curve), we mean a connected reduced compact analytic space of dimension two (resp. one). Unless otherwise mentioned, we assume that it is smooth. In the configuration of curves on a surface, an integer attached to a curve indicates its self-intersection number, a small circle indicates a point to be blown-up, and a dotted line means an exceptional curve of the first kind.

Let $A=A_1+\cdots+A_n$ be a linear chain or a cycle of curves on a complex manifold of dimension two. Then Zykel(A) is a sequence of integers $(a_1, \cdots, a_n)$ when the self-intersection number of A_i is $-a_i$. Let C and D be linear chains or cycles of curves on a complex manifold of dimension two. Then $C+D$ has the type $(p_1, q_1, p_2, q_2, \cdots, q_{n-1}, p_n)$ when the self-intersection numbers of components of C and D are given as follows:

(1) if $p_1 \geqq 3$, then

$$\mathrm{Zykel}(C)=(p_1, \underbrace{2, \cdots, 2}_{q_1-3}, p_2, \underbrace{2, \cdots, 2}_{q_2-3}, \cdots, p_n),$$

$$\mathrm{Zykel}(D)=(\underbrace{2, \cdots, 2}_{p_n-2}, q_{n-1}, \underbrace{2, \cdots, 2}_{p_{n-1}-3}, q_{n-2}, \cdots, \underbrace{2, \cdots, 2}_{p_1-3}),$$

for certain positive integers $n(\geqq 2)$, p_j $(\geqq 3)$, q_j $(\geqq 3)$ $(1 \leqq j \leqq n-1)$, and $p_n \geqq 2$.

(1)′ if $p_1 \geqq 3$, and $n=1$, then

$$\mathrm{Zykel}(C)=(p_1-2),$$

$$\mathrm{Zykel}(D)=(\underbrace{2, \cdots, 2}_{p_1-2})$$

for a certain positive integer $p_1(\geqq 3)$.

(2) if $p_1=2$, then

$$\mathrm{Zykel}(C)=(\underbrace{2, \cdots, 2}_{q_1-2}, p_2, \underbrace{2, \cdots, 2}_{q_2-3}, p_3, \cdots, p_n),$$

$$\mathrm{Zykel}(D)=(\underbrace{2, \cdots, 2}_{p_n-2}, q_{n-1}, \underbrace{2, \cdots, 2}_{p_{n-1}-3}, q_{n-2}, \cdots, \underbrace{2, \cdots, 2}_{p_2-3}),$$

for certain positive integers $n(\geqq 2)$, p_j $(\geqq 3)$, q_j $(\geqq 3)$ $(2 \leqq j \leqq n-1)$, $q_1(\geqq 3)$, and $p_n(\geqq 2$, and $\geqq 3$ if $n=2)$.

A rational cycle means either a rational curve with a node or a cycle of non-singular rational curves on a complex manifold of dimension two.

By a surface of class VII, we mean a smooth surface with $b_1=1$. If a surface of class VII has no exceptional curves of the first kind, we call it a surface of class VII_0. Note that a surface of class VII_0 is, in fact, minimal. By a (CB)-surface, we mean a surface of class VII_0 with (CB), where a (CB) is an effective reduced divisor consisting of a cycle of curves and some trees of curves sprouting from the cycle. A (CB)-surface contains just one (CB) and the cycle in the (CB) is a rational cycle and the trees in the (CB) consist of non-singular rational curves. See Nishiguchi [14, 15] for more about a (CB)-surface. We refer to Nakamura [10] for definitions of several classes of surfaces of class VII_0, i.e., a Hopf surface, a hyperbolic Inoue surface, and a parabolic Inoue surface.

By a Hirzebruch surface Σ_n of degree n, we mean a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ of degree n, i.e., $\Sigma_n=\boldsymbol{P}(\mathcal{O}_{P^1}\oplus\mathcal{O}_{P^1}(n))$.

We confuse a divisor on a complex manifold with the associated line bundle, and write tensor products of line bundles additively. We denote by K_X the canonical bundle of a complex manifold X.

A variety X_0 with only normal crossings is denoted by $X_0=V_1\underset{E}{\cup}V_2$, when X_0 has two irreducible components V_1 and V_2 which intersect each other transversally along a divisor E. A one-point union of two topological spaces T_1 and T_2 along a point P is written as $T_1\underset{P}{\vee}T_2$.

§ 1. Classification of degenerations of K3 surfaces

Let $\pi: X\rightarrow\Delta$ be a semi-stable degeneration of K3 surfaces. We shall look into it without assuming that X is Kähler. Then we have the following theorem which is a generalization of the result of Kulikov [7] and Persson-Pinkham [17].

Theorem 1.1. *Let $\pi: X\rightarrow\Delta$ be as above. Then this satisfies one of the following conditions.*

(i) *There exists a modification $\pi': X'\rightarrow\Delta$ of $\pi: X\rightarrow\Delta$ such that π' is also semi-stable and $K_{X'}=0$.*

(ii) *In the singular fiber X_0 of π, there exists a component which is a Hopf surface or a surface obtained by blowing up a Hopf surface.*

(iii) *In the singular fiber X_0 of π, there exists a component which is a (CB)-surface or a surface obtained by blowing up a (CB)-surface (see Notations and Coventions for the definition of a (CB)-surface).*

Remark. (1) If every component of the singular fiber X_0 is algebraic (or Kähler), only the case (i) occurs. This is nothing but a result of Kulikov and Persson-Pinkham.

(2) In § 2, we shall classify semi-stable degenerations of K3 surfaces with trivial canonical bundles.

(3) There are examples in each case (i), (ii) or (iii); see §§ 2, 3 and 4.

(4) We have examples which satisfy both (i) and (ii); see § 3. There are also examples which satisfy (ii) but not (i). These examples show that the result of Kulikov and Persson-Pinkham does not necessarily hold without the assumption of projectivity or Kählerity.

(5) The cases (i) and (iii) are disjoint from each other, as seen in the proof of this theorem.

(6) Theorem 1.1 holds for a semi-stable degeneration of surfaces with trivial canonical bundles. Namely it holds for a semi-stable degeneration whose general fiber is either a complex torus of dimension two or a Kodaira surface. But the author does not know any examples satisfying (iii) with a complex torus or a Kodaira surface as a general fiber.

(7) The canonical bundle of each component in the singular fiber X_0 has a non-zero meromorphic section, as seen in the proof below.

Proof of Theorem 1.1. Let $X_0=V_1+\cdots+V_N$ be the irreducible decomposition of the singular fiber X_0, and $C_{ij}=V_i\cap V_j$ a double curve. Since the canonical bundle on a general fiber X_t is trivial, the canonical bundle K_X of X can be written in the form $\sum r_iV_i(r_i\in \boldsymbol{Z})$ as a divisor. Here r_i's are uniquely determined up to addition of a constant, i.e., $\sum r_iV_i$ is linearly equivalent to $\sum(r_i+s)V_i$, where $s\in \boldsymbol{Z}$. We call r_i the multiplicity of a component V_i. A component V_j with maximal r_j among r_i's is called a maximal component. Moreover, if a maximal component intersects a non-maximal component, we call it a strictly maximal component.

By the adjunction formula, we have

$$K_{V_i}=\sum_{j\neq i}(r_j-r_i-1)C_{ij}.$$

(See remark (7) above.) Hence a maximal component has an effective anti-canonical divisor. If it is a strictly maximal component, then the anti-canonical divisor has a component with multiplicity greater than one.

We shall look into a strictly maximal component which is not a Kähler surface. We have the following:

Proposition 1.2. *Let S be a non-Kähler surface with a non-zero effective anti-canonical divisor D, i.e.,*

$$K_S=-D.$$

Then S is a surface of class VII. *Moreover, if D has a component of multiplicity greater than one, then S is a Hopf surface, a (CB)-surface, or a surface obtained by blowing up such surfaces.*

Proof. First we remark that a non-Kähler surface has a minimal model and that every divisor on it has only normal crossings. This was proved by Kodaira [6] in case of non-Kähler surfaces which are not of class VII, and by Kato and Nakamura [10] in case of surfaces of class VII. Let V be the minimal model of S. Then the canonical divisor K_V of V is written as

$$K_V=-D_0$$

where D_0 is also a non-zero effective divisor. By the classification of analytic surfaces due to Kodaira [6], we know that V is a surface of class VII_0. Moreover if D has a component of multiplicity greater than one, then so does D_0 because D_0 has a divisor with normal crossings by the above remark. Therefore, by virtue of Theorem 2.1 of Nishiguchi [15], V is a Hopf surface or a (CB)-surface. Q.E.D.

The next step of the proof of Theorem is the following:

Proposition 1.3. *If every strictly maximal component is a Kähler surface, then one can find a generically contractible component of $\pi: X\rightarrow \Delta$ after modifications of type* I *and* II *(called Mod*I *and Mod*II *briefly). For the definition of generic* *contraction, see Persson-Pinkham* [17], *and for those of Mod* I *and Mod* II, *see Kulikov* [7].

Proof. This is a main result of Kulikov [7].

Let us now return to the proof of Theorem 1.1. If there is no strictly maximal component, then K_X is already trivial; so this is the case (i). We assume that there is still a strictly maximal component. If none of them is a surface of class VII, then we may assume the existence of generically contractible components by Proposition 1.3. Therefore, by the induction on the number of irreducible components of the singular fiber, the theorem follows from Proposition 1.2.

Concluding this section, we prepare the following result for §2.

Proposition 1.4. *Let* $\pi: X \to \Delta$ *be a semi-stable degeneration of K*3 *surfaces. If* S *is a maximal component and a surface of class* VII, *then* S *can be made minimal in the degeneration by means of Mod* I, *Mod* II *and generic contractions.*

Proof. Since S is a maximal component, $-K_S$ is an effective divisor and the support R of $-K_S$ is the union of all double curves on S. Let C be an exceptional curve of the first kind on S if it exists at all. We have

$$(-K_S)\cdot C=1.$$

Case 1. We assume that C is not a component of R. Then C intersects R in only one point which is not a double point. Hence one can move C off S to another component of X_0 by means of Mod I.

Case 2. We assume that C is a component of R.

2-1) We consider the case where C intersects $R-C$ in only one point. Let S_1 be a component of X_0 which intersects S along C. Then we have $(C^2)_{S_1}=0$ by the triple point formula (see Kulikov [7]). Hence S_1 is a ruled surface and C is its minimal fiber. Thus, by definition, S_1 is generically contractible and the generic contraction blows down C to a point.

2-2) When C intersects $R-C$ in exactly two points, we can contract C to a point on S by means of Mod II along C.

2-3) We consider the case where C intersects $R-C$ in more than two points. Recall that any divisor on a surface of class VII has only normal crossings in the support (see Nakamura [10]). But the surface obtained from S by contracting C to a point has a divisor with non-normal crossings. This is a contradiction, and this case does not occur. Q.E.D.

§2. Classification of degenerations of K3 surfaces with trivial canonical bundles

This section gives the classification of degenerations of K3 surfaces in case (i) of Theorem 1.1:

Theorem 2.1. *Let $\pi: X\to\Delta$ be a semi-stable degeneration of K3 surfaces. We assume that the canonical bundle K_X is trivial. Then after suitable Mod* I *and Mod* II *performed on X, the singular fiber X_0 becomes one of the following*:

I. *X_0 is a (non-singular) K3 surface.*

II. *$X_0=V_1+\cdots+V_N(N\geqq 2)$, where V_1 and V_N are rational surfaces and $V_2, \cdots, V_{N-1}$ are relatively minimal elliptic ruled surfaces. The double curves are elliptic curves. The dual graph $\Pi(X_0)$ is given as follows.*

$$\underset{V_1}{\cdot}\text{———}\underset{V_2}{\cdot}\cdots\cdot\text{———}\underset{V_N}{\cdot}$$

II′. *$X_0=V_1+\cdots+V_N(N\geqq 2)$, where V_1 and V_N are rational surfaces and $V_2, \cdots, V_{N-1}$ are relatively minimal elliptic ruled surfaces or Hopf surfaces. The double curves are elliptic curves. The dual graph $\Pi(X_0)$ is given as follows.*

$$\underset{V_1}{\cdot}\text{———}\underset{V_2}{\cdot}\cdots\cdot\text{———}\underset{V_N}{\cdot}$$

III. *$X_0=V_1+\cdots+V_N$, where all V_i's are rational surfaces. The double curves form a rational cycle on each surface V_i. The dual graph $\Pi(X_0)$ is a triangulation of a 2-sphere S^2.*

III′. *$X_0=V_1+\cdots+V_N+V_0^{(1)}+\cdots+V_0^{(h)}$, where V_i's are rational surfaces and $V_0^{(j)}$'s are hyperbolic Inoue surfaces. The double curves form a rational cycle on each rational surface V_i and exactly two rational cycles on each hyperbolic Inoue surface $V_0^{(j)}$. The dual graph $\Pi(X_0)$ is a triangulation of a one-point union of $h+1$ S^2's, where a point P_j joining two S^2's like $S^2\underset{P_j}{\vee}S^2$ corresponds to the component $V_0^{(j)}$ and points in S^2 other than P_j correspond to components V_i.*

II′+III′. *$X_0=V_1+\cdots+V_N$, where V_i is a rational surface, a relatively minimal elliptic surface, a hyperbolic Inoue surface or a parabolic Inoue surface. The dual graph $\Pi(X_0)$ is a triangulation of a one-point union of several spheres S^2 and several line segments L, where L must appear as an edge of the dual graph $\Pi(X_0)$. A point P_i joining two S^2's like $S^2\underset{P_i}{\vee}S^2$ corresponds to a component V_i which is a hyperbolic Inoue surface, and a point P_j joining L and S^2 like $S^2\underset{P_j}{\vee}L$ corresponds to a component V_j which is a parabolic Inoue surface. A point on S^2 of the triangulation other than P_i and P_j above corresponds to a component which is a rational surface, and a point on L other than P_j corresponds to a component which is a relatively minimal elliptic ruled surface, unless it is an edge point of the triangulation of L which then corresponds to a rational surface. The double curves on each component consist of a single elliptic curve or form a rational cycle.*

Proof. If every component of X_0 is a Kähler surface, then Persson [16] and Kulikov [7] proved that one of the cases I, II and III occurs for X_0 after suitable Mod I and Mod II performed on X. So we may assume that X_0 contains a non-Kähler component. First note that every component S of X_0 has the canonical divisor $K_S=-D$, where D is a reduced effective divisor with only simple normal crossings consisting of all double curves on S. We shall study a surface like this.

Proposition 2.2. *Let S be a surface with the canonical divisor $K_S=-D$, where D is a non-zero reduced effective divisor with only simple normal crossings. Then we have:*

(i) *If S is a Kähler surface, then S is either*

(1) *a rational surface with an elliptic curve or a rational cycle as an anti-canonical divisor D,*

(2) *an elliptic ruled surface with two disjoint sections as an anti-canonical divisor D.*

(ii) *If S is not a Kähler surface, then the minimal model V of S is one of the following:*

(1) *a Hopf surface, where D consists of two disjoint elliptic curves,*

(2) *a parabolic Inoue surface, where D consists of an elliptic curve and a rational cycle,*

(3) *a hyperbolic Inoue surface, where D consists of two rational cycles.*

Proof. The case (i) is proved in Kulikov [7]. So we consider the case (ii). By Proposition 1.2, S is a surface of class VII. As seen in the proof of Proposition 1.2, S has a minimal model V, and the canonical divisor K_V of V can be written as

$$K_V=-D_0,$$

where D_0 is a non-zero reduced effective divisor. Then this Proposition follows from Proposition 3.1 in Nishiguchi [15].

Next, for a general semi-stable degeneration $\pi: X\to\Delta$ of K3 surfaces, we have

Proposition 2.3. *Let $X_0=\sum V_i$ be the irreducible decomposition of the singular fiber X_0, $C_{ij}=V_i\cap V_j$ double curves, and T the number of triple points in X_0. Then we have*

$$2=\sum\chi(\mathcal{O}_{V_i})-\sum\chi(\mathcal{O}_{C_{ij}})+T$$

For the proof, see Kulikov [7].

We return to the proof of Theorem 2.1. First we take a component S of a non-Kähler surface. Then, applying Proposition 1.4, we may assume that the component S is minimal after suitable Mod I and Mod II. Here we do not need a generic contraction as required in Proposition 1.4, because every component has a reduced anti-canonical divisor and such a component cannot be generically contractible. Now note that in the singular fiber X_0, any component which meets S is a Kähler surface except for the case where S is a Hopf surface and the component which meets S is also a (blown-up) Hopf surface whose exceptional curves of the first kind do not intersect the double curve. This is easily shown by the triple point formula (cf. Kulikov [7]) and the property of self-intersection numbers of curves on a surface of class VII (cf. Nakamura [10]). Therefore, we can make another component S' of a non-Kähler surface minimal by suitable Mod I and Mod II, while keeping S minimal. Hence we do such modifications inductively, and we may assume that any component of non-Kähler surfaces is minimal after suitable Mod I and Mod II. Then, combining Propositions 2.1 and 2.2, this Theorem is proved as in Persson [16] and Kulikov [7] for Kähler case.

§ 3. Smoothing of simple elliptic and cusp singularities

In this section, we shall study a semi-stable degeneration $\pi: X \to \Delta$ of K3 surfaces with trivial canonical bundle, which is classified in the previous section. Especially, we discuss which type of surfaces can be a component in the singular fiber of such a degeneration $\pi: X \to \Delta$.

First we shall give examples of the degenerations $\pi: X \to \Delta$ which contain non-Kähler surfaces in their singular fibers, i.e., the cases II′, III′ and II′+III′ in Theorem 2.1 actually occur.

Example 3.1. (1) K. Ueno constructed an example of the case II′ with the singular fiber $X_0 = V_1 + V_2 + V_3$, where V_1 and V_3 are rational surfaces and V_2 is an elliptic Hopf surface. Roughly speaking, his construction is as follows: First one takes an elliptic Hopf surface, blows it up at two points which lie on distinct elliptic curve, and obtains a normal surface with two simple elliptic singularities of degree one (see Pinkham [18] for the definition). Next one proves that this normal surface is deformed to a K3 surface. (We can prove this fact by using Propositions 3.3, 3.4 and Lemma 3.5 below.) Finally a semi-stable degeneration we need is obtained by performing base change and Mod I.

(2) In a way similar to the construction in (1), K. Ueno also constructed an example of the case II′+III′ which contains a parabolic Inoue surface.

(3) Friedman-Miranda [3] constructed an example of the case III′ with a component of a hyperbolic Inoue surface with a small number of curves. They used the deformation theory of varieties with normal crossings as developed by Friedman [2]. Moreover, by using such semi-stable degenerations, they studied the problem of smoothability of cusp singularities, and proved Looijenga's conjecture in the case of small length (cf. Looijenga [8] and Problem 3.6 below).

Remark. The constructions by Ueno and Friedman-Miranda are converse to each other in the sense that Ueno used the smoothability of certain simple elliptic singularities to construct semi-stable degenerations and Friedman-Miranda used certain semi-stable degenerations to show the smoothability of some cusp singularities. These two methods will become a main theme of this section.

Now we shall discuss which type of surfaces can be a component in the singular fiber of a semi-stable degeneration of K3 surfaces with trivial canonical bundle. In connection with this problem, we shall consider also the smoothability of simple elliptic and cusp singularities.

Let S be a rational surface and D a reduced effective divisor on S with only simple normal crossings. We assume that D is an anti-canonical divisor on S, i.e., the canonical divisor K_S of S is given as $K_S=-D$. We call such a pair (S, D) an anti-canonical rational surface, and simply say that D is on an anti-canonical rational surface S. By Proposition 2.2, if (S, D) is an anti-canonical rational surface, D is either an elliptic curve or a rational cycle. Accordingly, we call (S, D) an anti-canonical rational surface of elliptic type or cusp type. If (S, D) is a component in the singular fiber of a semi-stable degeneration of surfaces with D the double curves on S, we simply say that (S, D) is a component of the degeneration. With this terminology, we can raise the following problem in the case of degenerations of type II (the numbering of type accords with that in Theorem 2.1) and simple elliptic singularities.

Problem 3.2. (1) Let (S, D) be an anti-canonical rational surface of elliptic type. When can (S, D) be a component of a semi-stable degeneration of K3 surfaces of type II?

(2) Let (V, P) be a simple elliptic singularity of degree k. When is (V, P) smoothable?

These problems have already been answered (cf. Friedman [2] and Pinkham [18]). Namely, with the same notations as in Problem 3.2, we have

Proposition 3.3. (1) *(S, D) is a component of a semi-stable degeneration of type* II *of $K3$ surfaces if and only if*

$$(O) \qquad -9 \leqq (D^2)_S \leqq 9.$$

(2) *A simple ellipic singularity (V, P) of degree k is smoothable if and only if $k \leqq 9$.*

Remark. In Problem 3.2 and Proposition 3.3, the statement (2) is considered as the local version of (1). Actually, in order to prove Proposition 3.3, we use the following proposition which connects the local deformation (i.e., the smoothing of a germ of a singularity) with the global one (i.e., the smoothing of a compact normal surface).

Proposition 3.4. *Let S' be a compact normal surface with exactly one singularity P. We assume that the germ (V, P) of the singularity P is smoothable. Then a smoothing of (V, P) can be extended to that of S', if we have*

$$H^2(S', \Theta_{S'})=0.$$

For the proof, see Wahl [19].

Next, if a normal surface is smoothable, then one would like to know what the general fiber of a smoothing is. In fact, we can see it in the following lemma, where we also compute a cohomological invariant of a normal surface. These results are made use of in the proof of Proposition 3.3.

Lemma 3.5. *Let S' be a compact normal surface, and S a non-singular model with an exceptional divisor E with only normal crossings. Then we have:*

(i) *If $H^0(E, \Omega_E^1 \otimes \mathcal{O}_S(mE))=0$ for any $m \geqq 1$ and $H^0(S, \Omega_S^1)=1$, then we have*

$$H^0(S', \Omega_{S'}^1)=0.$$

(ii) *We assume that the normal surface S' has a smoothing $\pi': X' \to \Delta$. If S' is a Gorenstein surface with trivial dualizing sheaf $\omega_{S'} \cong \mathcal{O}_{S'}$ and $H^0(S', \Omega_{S'}^1)=0$, then the general fiber X_t of π' is a $K3$ surface.*

Proof. (i) It is enough to prove that

$$(\text{☆}) \quad H^0(S, \Omega_S^1(mE))=0 \qquad \text{for any} \quad m \geqq 1.$$

We show this by induction on m. We consider the following two exact sequences (cf. Kodaira-Spencer [6]):

$$(*) \quad 0 \longrightarrow K \longrightarrow \Omega_S^1(mE) \longrightarrow \Omega_E^1 \otimes \mathcal{O}_S(mE) \longrightarrow 0$$

$$(**) \quad 0 \longrightarrow \Omega_S^1((m-1)E) \longrightarrow K \longrightarrow \mathcal{O}_E((m-1)E) \longrightarrow 0.$$

First we put $m=1$. Then the connecting homomorphism δ

$$\boldsymbol{C} \simeq H^0(\mathcal{O}_E) \xrightarrow{\delta} H^1(\Omega_S^1)$$

derived from (**) is nothing but the one which maps the fundamental class E into $H^1(\Omega_S^1)$. Hence the class E is not numerically trivial, so δ is injective. On the other hand, recall that $H^0(E, \Omega_E^1 \otimes \mathcal{O}_S(E))=H^0(S, \Omega_S^1)=0$. Therefore, using the cohomology exact sequences derived from (*) and (**) for $m=1$, we obtain

$$H^0(S, \Omega_S^1(E))=0.$$

Next we assume that (☆) holds for the case $m-1$ ($m \geqq 2$), namely $H^0(S, \Omega_E^1((m-1)E)=0$. By the hypothesis, we have $H^0(E, \Omega_E^1 \otimes \mathcal{O}_S(mE))=0$ for any $m \geqq 1$. Note that $H^0(E, \mathcal{O}_E((m-1)E))=0$ for any $m \geqq 2$, because the intersection matrix of E is negative definite. Therefore, as before, we obtain

$$H^0(S, \Omega_S^1(mE))=0.$$

(ii) The proof is straightforward, by the upper semi-continuity of numerical invariants (irregularity etc.).

Now we shall return to Proposition 3.3. By using Proposition 3.4 and Lemma 3.5, one can prove Proposition 3.3. In fact, the subsequent argument was given essentially by Friedman [2] and Friedman-Miranda [3]. So we shall only give a sketch of the proof of Proposition in the "generic" case to be explained later:

(1) First we assume that (S, D) is a component of a semi-stable degeneration of K3 surfaces of type II. Let (T, E) be a rational surface appearing on the other end of the singular fiber. Then note that E is isomorphic to D, and we identify them (denoted still by E). By the triple point formula, we have

$$(E^2)_S+(E^2)_T=0.$$

On the other hand, the anti-canonical divisor $-K=E$ on a rational surface has self-intersection number,

$$(E^2)_T\leqq 9, \qquad (E^2)_S\leqq 9,$$

which follows from the Noether formula. Therefore we have (O).

Conversely, we assume that D satisfies (O). Then we can find an anti-canonical rational surface (T, E), where E is isomorphic to D, which we identify with E and denote by E, and

$$N_{E/T}\simeq(N_{E/S})^*.$$

Let $X_0=S\bigcup_E T$ be a two-dimensional variety with only normal crossings along E. By the deformation theory due to Friedman [2], one can show X_0 is smoothable to a K3 surface under deformation. More precisely, X_0 can be the singular fiber of a semi-stable degeneration of K3 surfaces, which is of type II.

(2) Let $(\tilde{V}, E)\to(V, P)$ be a resolution of the singularity P with exceptional curve E. We embed $(\tilde{V}, E)$ into an anti-canonical rational surface (S, E) (cf. Pinkham [18]). Let (S', P) be a surface obtained by blowing down E on S. First we assume that $k\leqq 9$, i.e., $-9\leqq(E^2)_S\leqq -1$. Then, by the proof of (1), one can find a semi-stable degeneration $\pi: X\to\Delta$ of K3 surfaces with singular fiber $X_0=S\bigcup_E T$, where (T, E) is also an anti-canonical rational surface. Note that $N_{T/X}\simeq(\mathcal{O}_T(E))^*$. We have $(E^2)_T=-(E^2)_S>0$ by the triple point formula, and hence $(E\cdot C)_T\geqq 0$ for every curve C on T. We assume that $(E\cdot C)_T>0$ for every curve C on T. We call this case "generic". Then $N_{T/X}$ is a negative line bundle and T is contractible in X, by a theorem of Grauert. Let

$$\begin{array}{ccc} \pi: X & \longrightarrow & X' \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

be the contraction of T. The singular fiber X'_0 of π' is nothing but the normal surface (S', P). Therefore X' gives a smoothing of the singularity P. If there is a curve C with $(E\cdot C)_T=0$, we cannot prove this part of Proposition in this way (see Pinkham [18] for this case).

Conversely, we assume that the singularity (V, P) is smoothable. By the above arguments, embed (V, P) into a normal surface (S', P) which is obtained by contracting E of an anti-canonical rational surface (S, E). Note then that the dualizing sheaf $\omega_{S'}$ of S' is isomorphic to $\mathcal{O}_{S'}$. Therefore, by the Serre duality, we have

$$H^2(S', \Theta_{S'}) \simeq (H^0(S', \Omega^1_{S'}))^*.$$

On the other hand, the cohomological conditions of Lemma 3.5 (i) for Ω^1_S and E are easily verified, and so we obtain

$$H^0(S', \Omega^1_{S'}) = 0.$$

Hence a smoothing of (V, P) can be extended to that of (S', P) by Proposition 3.4, and the general fiber X'_t of a smoothing $\pi' : X' \to \Delta$ of (S', P) is a K3 surface by Lemma 3.5 (ii). After performing (if necessary) resolutions of singularities, base changes and Kulikov modifications (i. e., generic contractions, Mod I and Mod II), we obtain a semi-stable degeneration $\pi : X \to \Delta$ of K3 surfaces of type II which has a component (S, E) in its singular fiber (cf. Friedman-Miranda [3]). Therefore we have $(E^2)_S \geqq -9$, i. e., $k \leqq 9$. Hence we gave a sketch of the proof of Proposition 3.3.

We can raise a problem similar to Problem 3.2 for degenerations of type III and smoothings of cusp singularities. In the rest of this section we discuss this problem.

Problem 3.6. (1) Let (R, D) be an anti-canonical rational surface of cusp type. When can (R, D) be a component of a semi-stable degeneration of type III of K3 surfaces?

(2) Let (V, P) be a cusp singularity. When is (V, P) smoothable?

These problems, especially (2), have been studied by many people (Looijenga [8], Friedman-Miranda [3], etc.), but have not been solved yet. We have a conjecture due to Looijenga [8] concerning Problem 3.6 (2). To prove this Looijenga's conjecture, Friedman-Miranda [3] proposed a conjecture which is a global version of Looijenga's one in the sense that it treats the smoothing of compact surfaces. These conjectures require the notion of the dual cycle of a rational cycle, so we define it first.

Definition. Let D be a rational cycle with

$$\mathrm{Zykel}(D) = (p_1, \underbrace{2, \cdots, 2}_{q_1-3}, p_2, \underbrace{2, \cdots, 2}_{q_2-3}, \cdots, p_n, \underbrace{2, \cdots, 2}_{q_n-3})$$

where $p_i \geqq 3$ and $q_i \geqq 3\,(1 \leqq i \leqq n)$; hence the intersection matrix of D is negative definite. Then the dual cycle of D is defined to be a rational cycle D^* with

$$\mathrm{Zykel}(D^*) = (\underbrace{2, \cdots, 2}_{p_1-3}, q_1, \underbrace{2, \cdots, 2}_{p_2-3}, q_2, \cdots, q_n).$$

A rational cycle and its dual cycle appear in the following context:

Proposition 3.7. (1) *A hyperbolic Inoue surface has exactly two rational cycles with negative definite intersection matrices, which are dual to each other.*

(2) *Any rational cycle with negative definite intersection matrix can be realized*

on a hyperbolic Inoue surface.

See Nakamura [10] for the proof.

Now we fix the notations as follows: Let (V, P) be a cusp singularity, and $(\tilde{V}, D)$ its minimal resolution. Let D^* be the dual cycle of D. By Proposition 3.7, we can take a hyperbolic Inoue surface with two cycles D and D^*. Let S' be a normal surface obtained from S by blowing down D to the cusp singularity P.

We shall state Looijenga's conjecture about the smoothability of cusp singularities.

Conjecture 1. With the above notations, a cusp singularity (V, P) is smoothable if and only if

(#) the dual cycle D^* of D lies on an anti-canonical rational surface T with $K_T = -D^*$.

Looijenga [8] proved that (#) is a necessary condition. Namely we have

Proposition 3.8. *We assume that (V, P) is smoothable. Then the condition (#) is satisfied. More precisely, a smoothing of (V, P) can be extended to a smoothing $\pi' : X' \to \Delta$ of the normal surface S' such that a general fiber of π' is a rational surface T with the anti-canonical cycle D^*.*

Friedman-Miranda modified the smoothing $\pi' : X' \to \Delta$ of S' obtained above by Looijenga to a semi-stable degeneration and also proved that the converse process is possible as follows:

Proposition 3.9. (i) *Let $\pi' : X' \to \Delta$ be as in Proposition 3.8. Then with resolutions of singularities, base changes and Kulikov modifications applied suitably, the smoothing π' can be made a semi-stable degeneration $\pi : X \to \Delta$ of the rational surface (T, D^*) such that its singular fiber X_0 is described as follows.*

(##) *$X_0 = \bigcup_{i \geqq 1} V_i$ is a variety with normal crossings such that*

(0) $V_1 = S$,

(1) *the dual graph of X_0 is a triangulation of S^2,*

(2) *the double curves on $V_1 = S$ form D, and those on V_i $(i \geqq 2)$ form a rational cycle D_i,*

(3) *$V_i (i \geqq 2)$ is a rational surface with $K_{V_i} = -D_i$,*

(4) *X_0 satisfies the triple point formula.*

(Such a variety X_0 is called a variety satisfying (##) with (S, D).)

(ii) *Conversely, let $\pi : X \to \Delta$ be a semi-stable degeneration of the anti-canonical rational surface (T, D^*) whose singular fiber satisfies (##) with (S, D). Then the divisor $\bigcup_{i \geqq 2} V_i$ in the threefold X can be contracted to a point, and after the contraction π becomes a smoothing $\pi' : X' \to \Delta$ of S' whose general fiber is (T, D^*).*

Furthermore, Friedman-Miranda [3] proved that a variety X_0 satisfying

(##) is smoothable, using the deformation theory due to Friedman [2]. Namely we have:

Proposition 3.10. *Let S, D and D^* be as above. Then a variety X_0 satisfying (##) with (S, D) can be the singular fiber of a semi-stable degeneration of an anti-canonical rational surface (T, D^*).*

Combining Propositions 3.9 and 3.10, we obtain the following conjecture which is equivalent to Conjecture 1.

Conjecture 1′ (Friedman-Miranda). Let S be a hyperbolic Inoue surface with two cycles D and D^*. If (#) is satisfied, then there exists a variety X_0 satisfying (##) with (S, D).

Remark. This conjecture is true if the number of components of D is less than 4 by virtue of Friedman-Miranda [3].

Next we shall consider when an anti-canonical rational surface of cusp type can be a component of a semi-stable degeneration of K3 surfaces of type III (i.e., Problem 3.6 (1)). We do not have any answers, even a conjecture, to cover the general case. However, in a special case, we have

Conjetcure 2. Let (R, D) be an anti-canonical rational surface of cusp type where D has a negative definite intersection matrix. Then (R, D) is a component of a semi-stable degeneration of K3 surfaces if and only if the dual cycle D^* of D satisfies (#).

Similarly we can ask when a hyperbolic Inoue surface can be a component of a degeneration of type III′:

Conjecture 3. Let S be a hyperbolic Inoue surface with two cycles D and D^*. Then $(S, D+D^*)$ is a component of a semi-stable degeneration of K3 surfaces of type III′ with exactly one hyperbolic Inoue surface, if and only if

(#)′ both of cycles D and D^* lie on anti-canonical rational surfaces T and T^* with $K_T=-D$ and $K_{T^*}=-D^*$, respectively.

In Conjectures 2 and 3, the "only if" part is true as in Conjecture 1. In fact, we have

Proposition 3.11. (i) *Let (R, D) be as in Conjecture* 2. *If (R, D) is a component of a semi-stable degeneration of K3 surfaces, then the dual cycle D^* satisfies (#) in Conjecture* 1.

(ii) *Let S, D and D^* be as in Conjecture* 3. *If $(S, D+D^*)$ is a component of a semi-stable degeneration of K3 surfaces, then D and D^* satisfy (#)′ in Conjecture* 3.

We can prove this proposition by using Proposition 3.10. In fact, from the semi-stable degeneration given in Proposition 3.11 (i), we obtain a variety which has only normal crossings and satisfies (##) with (S, D), where S is a hyperbolic Inoue surface realizing D. In Proposition 3.11 (ii), from the semi-stable degeneration given there, we obtain two varieties which have only normal crossings and satisfy (##) with (S, D) and (S, D^*) respectively.

As in Proposition 3.10, the deformation theory due to Friedman [2] implies the following

Proposition 3.12. (i) *Let (R, D) be an anti-canonical rational surface of cusp type. Then a variety X_0 satisfying (##) with (R, D) can be the singular fiber of a degeneration of type* III *of K3 surfaces.*

(ii) *Let S be a hyperbolic Inoue surface with two cycles D and D^*. We assume that there exists a variety X_0 (resp. X_0^*) satisfying (##) with (S, D) (resp. (S, D^*)). Then the variety $X_0 \bigcup_S X_0^*$ which has only normal crossings can be the singular fiber of a degeneration of type* III′ *of K3 surfaces.*

Remark. By using Propositions 3.8, 3.9 and 3.12, one can prove that Conjectures 2 and 3 follow from Conjecture 1 (or equivalently from Conjecture 1′). Moreover, from Propositions 3.11 and 3.12, we obtain the following conjectures which are equivalent to Conjectures 2 and 3, respectively.

Conjecture 2′. Let (R, D) be as in Conjecture 2. If (#) is satisfied, then there exists a variety X_0 satisfying (##) with (R, D).

Conjecture 3′. Let S, D and D^* be as in Conjecture 3. If D (resp. D^*) satisfies (#), then there exists a variety X_0 (resp. X_0^*) satisfying (##) with (S, D) (resp. (S, D^*)).

Remark. For Problem 3.2 concerning the elliptic type, we can treat it similarly as we did in Conjectures 1, 2 and 3, interpreting the meaning of "dual cycle" as follows: Let (S, D) be an anti-canonical rational surface of elliptic type. The dual cycle D^* on a surface S^* is, by definition, an elliptic curve isomorphic to D satisfying $N_{D^*/S^*} \cong (N_{D/S})^*$. Then Proposition 3.3 can be rephrased as follows:

(1) (S, D) is a component of a degeneration of type II of K3 surfaces if and only if

$(\#)_{\mathrm{ellip}}$: the dual cycle D^* of D is on an anti-canonical rational surface S^* with $K_{S^*} = -D^*$.

(2) Let (V, P) and (S, D) be as in Proposition 3.3 and its proof. Then P is smoothable if and only if D^* satisfies $(\#)_{\mathrm{ellip}}$.

§4. Examples of non-Kähler degenerations of K3 surfaces and smoothing of certain singularities

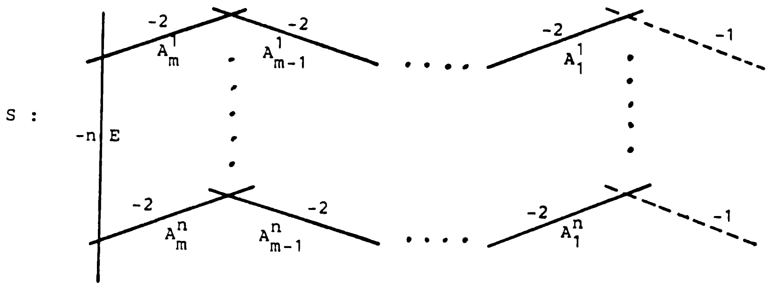
In this section, we shall give two typical examples of semi-stable degenerations of K3 surfaces which have no semi-stable modifications with trivial canonical bundles. These two do not belong to the case (i) in Theorem 1.1, but one belongs to (ii) and the other to (iii). Moreover, as in §3, we shall consider which Hopf surfaces or (CB)-surfaces can be a component of a semi-stable degeneration of K3 surfaces, and also study the smoothability of certain Gorenstein singular points with geometric genus two.

First we shall construct an example of a semi-stable degeneration of K3 surfaces which contains a Hopf surface in its singular fiber.

Example 4.1. Let T be a Hopf surface defined as follows: $T=\boldsymbol{C}^2-\{0\}/\langle g\rangle$, where g is the automorphism of $\boldsymbol{C}^2-\{0\}$ in the form:

$$g:(z_1, z_2)\longmapsto(\alpha_1 z_1+z_2^m,\ \alpha_2 z_2)\,;\ \alpha_1, \alpha_2\in\boldsymbol{C}, \qquad 0<|\alpha_1|,\ |\alpha_2|<1,\ \alpha_2^m=\alpha_1.$$

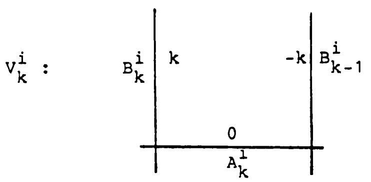
Let E be an elliptic curve on T defined by $z_2=0$. E is isomorphic to $\boldsymbol{C}^*/\langle\alpha_1\rangle$ and $K_T=-(m+1)E$ (see Kodaira [6]). Let S be a surface obtained from T by the blowing-up indicated as follows:



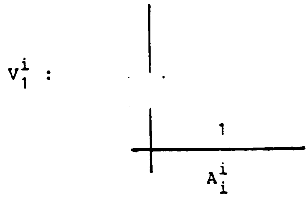
It is easy to see that

$$K_S=-(m+1)E-\sum_{i=1}^{n}\sum_{k=1}^{m} kA_k^i.$$

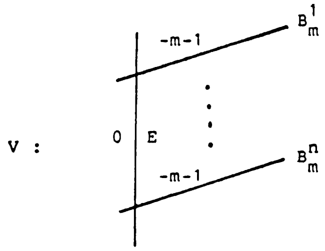
Let V_k^i ($2\leqq k\leqq m$, $1\leqq i\leqq n$) be a Hirzebruch surface Σ_k of degree k with two sections and a fiber named as follows:



where the canonical divisor is written as $K_{V_k^i}=(k-2)A_k^i-2B_k^i$. Let V_1^i ($1\leqq i\leqq n$) be a projective plane $\boldsymbol{P}^2$ with two curves named as follows:

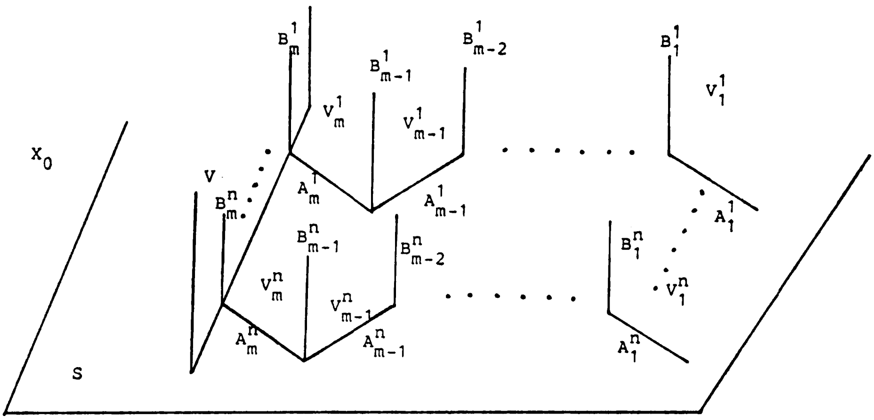


where $K_{V_1^i}=-A_1^i-2B_1^i$. We assume that there exists a minimal elliptic surface V over $\boldsymbol{P}^1$ with exactly one multiple fiber E of multiplicity m and with n multi-sections B_m^i $(1\leqq i\leqq n)$ isomorphic to $\boldsymbol{P}^1$ as shown below:



where $K_V=(m-1)E$. Note that if $m=1$ then V is an elliptic K3 surface, and that if $m\geqq 2$ then $\kappa(V)=1$ and V can be obtained by m times logarithmic transformations from an elliptic K3 surface which has mutually disjoint n sections.

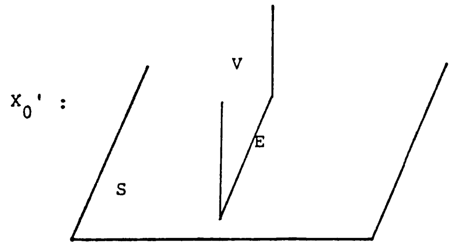
We construct a variety X_0 with only normal crossings, by glueing S, V_k^i $(1\leqq k\leqq m,\ 1\leqq i\leqq n)$ and V along the corresponding curves as follows:



Proposition 4.2. *Under the above hypotheses and notations, X_0 can be the singular fiber of a semi-stable degeneration $\pi: X\rightarrow\Delta$ of K3 surfaces. The canonical divisor K_X of X is written as*

$$K_X=(m+1)S+V+\sum_{i=1}^{n}\sum_{k=1}^{m}(m-k+2)V_k^i.$$

Moreover, by suitable modifications we can make $\pi': X'\rightarrow\Delta$ whose singular fiber X_0' is given by $X_0'=V\underset{E}{\cup}S$.

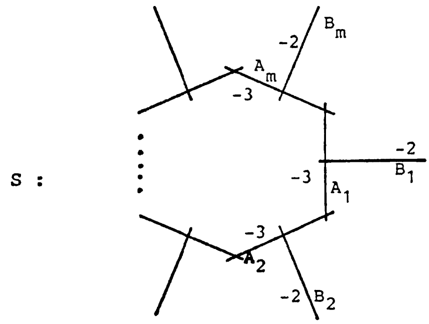


In fact, we start with the variety X_0'. Then, as in Nishiguchi [12], one can prove that X_0' is the singular fiber of a semi-stable degeneration $\pi': X' \to \Delta$, by virtue of the deformation theory (cf. Friedman [2]). Next, by blowing-up and Mod I, we obtain $\pi: X \to \Delta$ as in Proposition.

Remark. In the above construction, n must be less than 20, for V is obtained from a K3 surface by the logarithmic transformations as seen above. But we do not know a more precise upper bound of n for the existence of V (see Remark after Theorem 4.4).

Next we shall study an example of a semi-stable degeneration of K3 surfaces which contains a (CB)-surface in its singular fiber, i.e., an example of the case (iii) of Theorem 1.1.

Example 4.3. Let S be a (CB)-surface with the following configuration of non-singular rational curves $\boldsymbol{P}^1$:

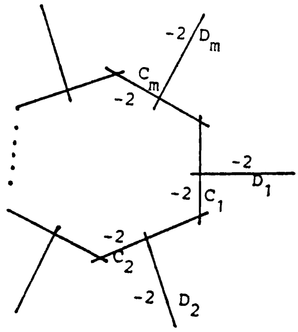


We assume that the canonical divisor K_S of S is written as

$$K_S = -2(A_1 + \cdots + A_m) - (B_1 + \cdots + B_m).$$

In fact, one can construct such a surface S containing a global spherical shell (GSS for short), as in Kato [4]. Such a surface S containing a GSS is also obtained as a deformation of the blown-up Hopf surface in Example 4.1 with $m=1$ (cf. Nakamura [10]).

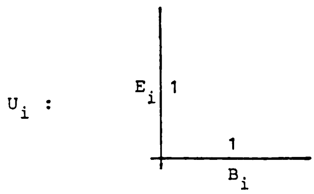
We consider the following configuration of non-singular rational curves C_i ($1 \leqq i \leqq m$) and D_j ($1 \leqq j \leqq m$) on a surface.



Then we have

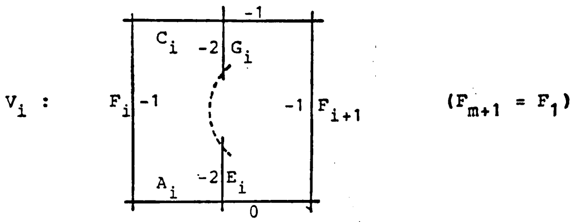
Theorem 4.4. *With the above notations, we assume that the divisor* $C_1+\cdots+C_m+D_1+\cdots+D_m$ *on a surface is realized on a K3 surface* V. *Then* S *can be a component in the singular fiber of a semi-stable degeneration of K3 surfaces.*

Proof. Let U_i be a projective plane $\boldsymbol{P}^2$ with the following configuration of lines:

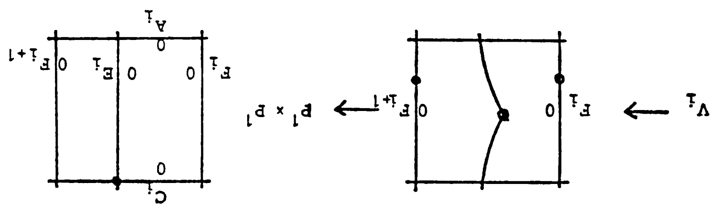


where we have $K_{U_i}=-B_i-2E_i$.

Let V_i be a rational surface with the following configuration of $\boldsymbol{P}^1$'s:

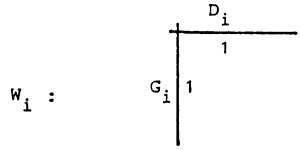


where we have $K_{V_i}=-F_i-F_{i+1}-G_i-2C_i$. Such a surface V_i is obtained by blowing up $\boldsymbol{P}^1\times\boldsymbol{P}^1$ at the points indicated by $\circ$ in the following picture:



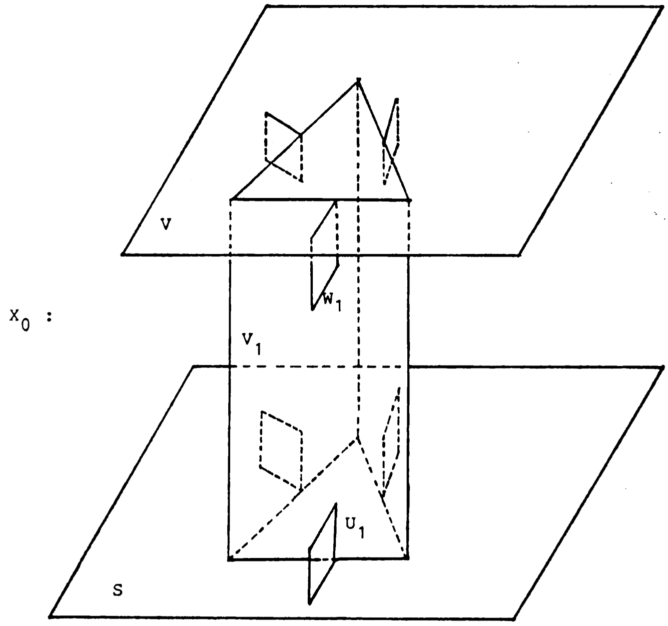
(We use the same symbol for a curve and its proper transform.)

Let W_i be $\boldsymbol{P}^2$ with the following configuration of lines:



where we have $K_{W_i} = -G_i - 2D_i$.

We construct a variety X_0 with only normal crossings by glueing S, V_i, W_i ($1 \leqq i \leqq m$) and V along the corresponding curves. We give a picture of X_0 only in the case $m=3$:



Then we may assume that X_0 is d-semi-stable (see Lemma 5.14 of Friedman [2]), and we can prove, as in Theorem 5.10 of [2], that the variety X_0 is smoothable to a K3 surface, more precisely, that X_0 is the singular fiber of a semi-stable degeneration of K3 surfaces. Q. E. D.

Remark. (i) Let $\pi : X \to \Delta$ be a semi-stable degeneration of K3 surfaces whose singular fiber is isomorphic to X_0 above. Then the canonical divisor K_X of X is written as

$$K_X = 3S + 3\sum_{i=1}^{m} U_i + 2\sum_{i=1}^{m} V_i + 2\sum_{i=1}^{m} W_i + V.$$

(ii) We consider a K3 surface V with the configuration of curves $D = C_1 + \cdots + C_m + D_1 + \cdots + D_m$ as above. Since the intersection matrix of D has signature $(1, 2m-1)$, $2m$ is not more than the Picard number $\rho(V)$ of V, where $\rho(V) \leqq 20$. Hence we have $m \leqq 10$. Professor Masa-Hiko Saito showed the author that for $m \leqq 9$, there exists a K3 surface with D, by virtue of the lattice theory due to Nikulin [11]. But it is still an open problem whether there exists a K3 surface containing D with $m=10$.

To conclude this section, we consider the smoothing of singularities obtained from the above surfaces of class VII by blowing down curves. In Example 4.1 (resp. Example 4.3), the divisor $E+\sum_{i=1}^{n}\sum_{k=1}^{m}B_k^i$ on the blown-up Hopf surface S $\left(\text{resp. } \sum_{i=1}^{m}A_i+\sum_{i=1}^{m}B_i \text{ on the (CB)-surface } S\right)$ can be blown down to a normal singular point. The singularity obtained from Example 4.3 is the "degenerate" case of that obtained from Example 4.1 with $m=1$. We study these two cases at the same time, and use the same notation for Examples 4.1 and 4.3 in the following. Let (S', P) be a normal surface with the singular point obtained by the blowing-down. One can easily show that P is a Gorenstein singular point with geometric genus $p_g=2$ (cf. Nishiguchi [15]). It is natural to ask when the singularity is smoothable under deformation. First we have the following result about the globalization of a smoothing:

Proposition 4.5. *Under the above hypotheses and notations, we assume that P is smoothable. Then a smoothing of P can be extended to a smoothing $\pi': X' \to \Delta$ of the normal surface S'. Moreover, a general fiber X_t' of π' is a K3 surface.*

This is a straightforward consequence of Proposition 3.4 and Lemma 3.5.

Finally, we give a sufficient condition for the singularity P to be smoothable:

Proposition 4.6. *For the singularity P on S', we assume that there exists an elliptic surface V with the curves $D=E+\sum_{i=1}^{n}B_m^i$ as in Example 4.1 $\left(\text{resp. a K3 surface } V \text{ with the curves } D=\sum_{i=1}^{m}C_i+\sum_{i=1}^{m}D_i \text{ as in Example 4.3}\right)$. Furthermore, we also assume that the line bundle $-D$ on V is negative, i.e., D is ample. Then P is smoothable under deformation.*

Proof. We show both cases at the same time. By virtue of Proposition 4.2 and Theorem 4.4, one obtains a semi-stable degeneration $\pi: X\to\Delta$ whose singular fiber X_0 contains the surface S as a component. Let W be the union of components of the singular fiber X_0 other than S. Then, by the assumption, the divisor W is negative in the sense of Grauert, and W can be blown down to a point in X, by a theorem of Grauert. Let $\pi': X'\to\Delta$ be a deformation obtained from $\pi: X\to\Delta$ by blowing down W. Then π' gives a smoothing of the normal surface S' and a fortiori a smoothing of the singular point P.

§ 5. Construction of degenerations of K3 surfaces

In this section, we shall systematically construct examples of semi-stable degenerations of K3 surfaces containing (CB)-surfaces in their singular fibers, i.e., examples of the case (iii) of Theorem 1.1. All known (CB)-surfaces contain global spherical shells (GSS). So we restrict ourselves to (CB)-surfaces containing a GSS. Then the canonical bundles of such surfaces are given explicitly provided they have exactly one branch (cf. Nishiguchi [14, 15]).

We shall freely use the notions and results in Nishiguchi [14, 15] concerning surfaces of class VII_0. Let S be a (CB)-surface containing a GSS. We assume that the (CB) on S has exactly one branch. Let $C=C_1+D_1$ be the (CB), where $C_1=\sum A_i$ forms a cycle and $D_1=\sum B_j$ forms a branch. Nakamura [9] proved that there are no curves other than A_i's and B_j's, and that $C=C_1+D_1$ has the type $(p_1, q_1, p_2, \cdots, p_n)$, where the first component of C_1 meets the first one of D_1. By Remark (7) after Theorem 1.1, the canonical bundle K_S of S has a meromorphic section, i.e., is a divisor, provised S is a component in the singular fiber of a semi-stable degeneration of K3 surfaces. Then we may assume that C has only simple normal crossings, i.e., $n\geqq 2$. Hence, by virtue of Theorem 6.1 in Nishiguchi [15], we know that the type of C is one of the following:

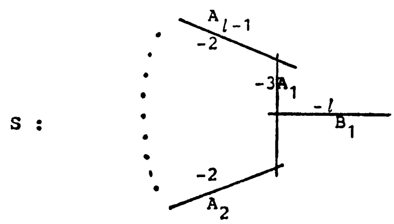
(1) $(3, l, 2)$

(2) $p_1=2$.

First we consider the case (1):

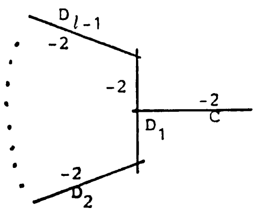
Example 5.1. Let S be a surface as above of type $(3, l, 2)$. Then we know that the canonical bundle K_S of S is numerically a divisor by Nishiguchi [15]. We assume that K_S is a divisor. Such a surface exists as shown in Example 4.3. Then we have

$$K_S=-B_1-2A_1-\cdots-2A_{l-1}$$



(cf. Proposition 6.3 of [15]). Then we have the following sufficient condition for S to be a component of a semi-stable degeneration of K3 surfaces.

Theorem 5.2. *Let S be as above. We consider the following configuration of non-singular rational curves P^1 on a surface:*

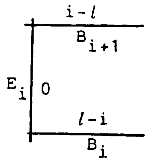


We assume that there exists a K3 surface V with the divisor $C+D_1+\cdots+D_{l-1}$ as above. Then S can be made a component of the singular fiber of a semi-stable degeneration of K3 surfaces.

Proof. The proof is very similar to that of Theorem 4.4. So we only describe each component which is in the singular fiber.

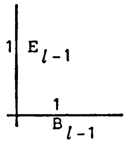
Let U_i $(1 \leqq i \leqq l-2)$ be a Hirzebruch surface Σ_{l-i} with the configuration of P^1's as follows:

U_i :



where $K_{U_i} = -B_i - B_{i+1} - 2E_i$. Let U_{l-1} be a projective plane P^2 as follows:

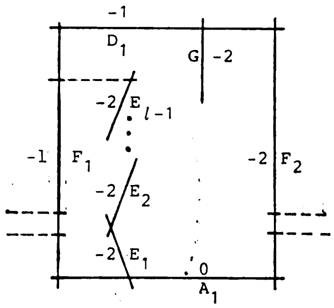
U_{l-1} :



where $K_{U_{l-1}} = -B_{l-1} - 2E_{l-1}$.

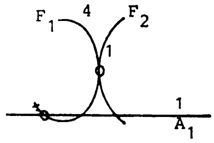
Let V_1 be a rational surface with the configuration of P^1's as follows:

V_1 :



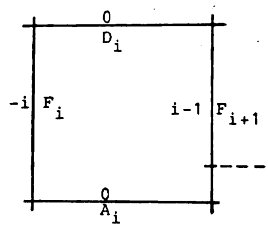
where $K_{V_1} = -F_1 - F_2 - 2D_1 - G$. Such a surface V_1 is obtained by blowing up P^2 at points indicated as follows by $\circ$:

P^2 :



where A_1 and F_2 are lines and F_1 is a conic which meets A_1 transversally at two points and tangents F_2 at a point. Let V_i $(2 \leqq i \leqq l-1)$ be a rational surface with the configuration of P^1's as follows:

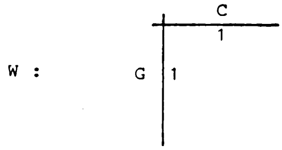
V_i :



$(F_l = F_1)$

where $K_{V_i}=-2D_i-F_i-F_{i+1}$. Such a surface V_i $(2\leqq i\leqq l-1)$ can be obtained by blowing up a Hirzebruch surface Σ_i at a point on a positive section F_{i+1}.

Finally let W be a projective plane $\boldsymbol{P}^2$ with the configration of lines as follows:



where $K_W=-G-2C$. Now glue these surfaces together along the corresponding curves. Q.E.D.

Remark. Let $\pi: X\to\Delta$ be the semi-stable degeneration of K3 surfaces constructed in the above proof. Then we have

$$K_X=3S+3U_1+\cdots+3U_{l-1}+2V_1+\cdots+2V_{l-1}+2W+V.$$

Next we consider the case (2) $p_1=2$ with the notation explained first in this section. First we treat the easiest case $n=2$, i.e., the case with type $(2, q_1, p_2)$. Then we have

$$\mathrm{Zykel}(C_1)=(\underbrace{2, \cdots, 2}_{q_1-2}, p_2)$$

$$\mathrm{Zykel}(D_1)=(\underbrace{2, \cdots, 2}_{p_2-2}).$$

By virtue of Example in Nishiguchi [15, §6], the canonical bundle K_S of S is numerically a divisor if and only if the type is

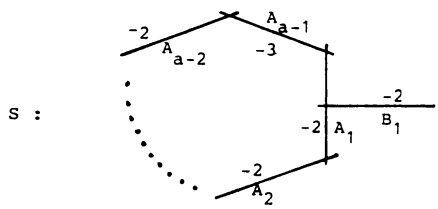
$$(2, (p-2)(a-1)+1, p) \qquad a\geqq 2,\ p\geqq 3.$$

In this case, K_S is written as

$$K_S\equiv-(a-1)B_1-2(a-1)B_2-\cdots-(p-2)(a-1)B_{p-2}$$
$$-(p-1)(a-1)A_1-((p-1)(a-1)-1)A_2-\cdots-aA_{N-p+2},$$

where $N=b_2(S)=(p-2)a$.

Example 5.3. With the above notations, we put $p=3$, i.e., the type is $(2, a, 3)$ $(a\geqq 3)$. Then the configuration of curves on S is given as follows:

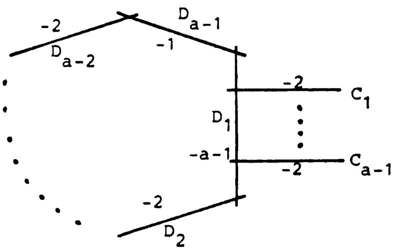


We assume that the canonical divisor K_S has a non-zero meromorphic section. Then

$$K_S=-(a-1)B_1-2(a-1)A_1-\cdots-aA_{a-1}.$$

We have the following sufficient condition for S to be a component of a semi-stable degeneration of K3 surfaces.

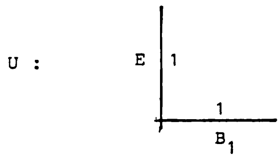
Theorem 5.4. *Let S be as above. We consider the following configuration of P^1's on a surface:*



We assumme that there exists a blown-up K3 surface V with the divisor $C_1+\cdots+C_{a-1}+D_1+\cdots+D_{a-1}$ as above. Then S can be made a component of the singular fiber of a semi-stable degeneration of K3 surfaces.

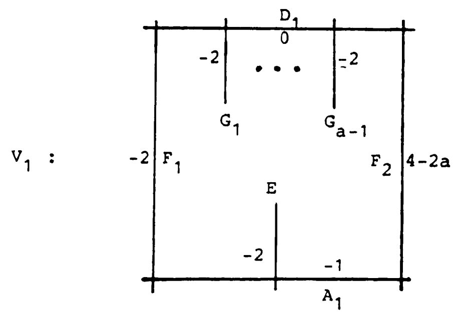
Proof. As in the proof of Theorem 5.2, we only describe each component which is to be in the singular fiber.

Let U be a projective plane P^2 with the following configuration of lines:

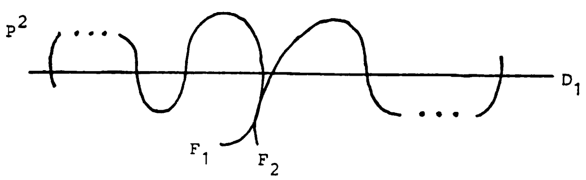


where we have $K_U=(a-3)B_1-aE$.

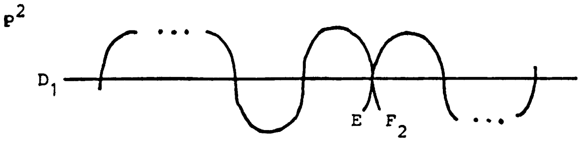
Let V_1 be a rational surface with the configuration of P^1's as follows:



where $K_{V_1}=2(a-2)A_1+(a-2)E+(a-3)F_1-2D_1-G_1-\cdots-G_{a-1}$. Such a rational surface V_1 is obtained as follows: In case a is even, we take curves on P^2 as indicated below:

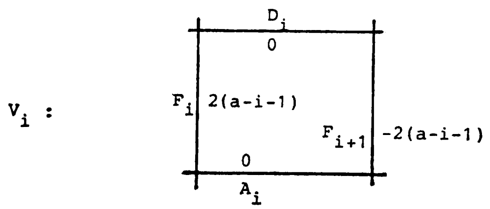


where D_1 is a (singular) curve of degree $a/2$ with only nodes, F_1 is a line and F_2 is a conic, F_1 and F_2 being tangent to each other ; hence $K_{P^2}=-2D_1+(a-3)F_1$. Then the surface V_1 is obtained by blowing up $\boldsymbol{P}^2$ suitably. We omit the detail of this process. In case a is odd, we take curves D_1, E and F_2 on $\boldsymbol{P}^2$ as follows:

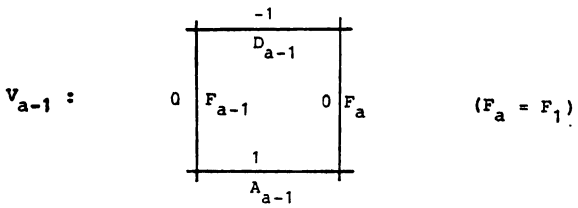


where D_1 is a (singular) curve of degree $(a+1)/2$ with only nodes, E is a line and F_2 is a conic, E and F_2 being tangent to each other at a point of D_1 ; hence $K_{P^2}=-2D_1+(a-2)E$. Then the surface V_1 is obtained by blowing up $\boldsymbol{P}^2$ suitably. We omit the detail as before.

Next let V_i $(2\leqq i\leqq a-2)$ be a Hirzebruch surface $\Sigma_{2(a-i-1)}$ with the following configuration of $\boldsymbol{P}^1$'s:

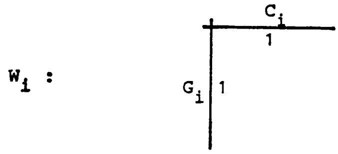


where $K_{V_i}=-2F_i-(i+1)D_i+(2a-3-i)A_i$. Let V_{a-1} be a Hirzebruch surface Σ_1 with the following configuration of $\boldsymbol{P}^1$'s:



where $K_{V_{a-1}}=-2F_{a-1}-(a-1)F_a+(a-2)A_{a-1}-aD_{a-1}$.

Let W_i $(1\leqq i\leqq a-1)$ be a projective plane $\boldsymbol{P}^2$ with the following configuration of lines:

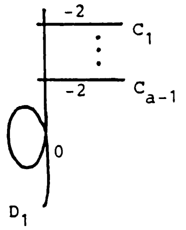


where $K_{W_i}=-G_i-2C_i$.

Note that if there exists a blown-up K3 surface V with a divisor $C_1+\cdots+C_{a-1}+D_1+\cdots+D_{a-1}$ as described in Theorem 5.4, then we have

$$K_V=D_2+2D_3+\cdots+(a-2)D_{a-1}$$

and the minimal model of V has the following configuration of curves:



where the proper transform D_1' of D_1 is a rational curve with node. Q. E. D.

Remark. In Example 5.3 and Theorem 5.4, we gave a sufficient condition for a (CB)-surface of type (2, a, 3) to be a component of a semi-stable degeneration of K3 surfaces. However the author does not know whether or not a similar result holds, in general, for a (CB)-surface of type (2, $(p-2)(a-1)+1$, p), $a\geqq 2$ and $p\geqq 3$.

Next we consider a (CB)-surface S in case $p_1=2$ and $n=3$. We restrict ourselves to the type (2, q_1, 3, q_2, 3), where $q_1, q_2\geqq 3$. Then we have

$$C=(\underbrace{2,\cdots,2}_{q_1-2},3,\underbrace{2,\cdots,2}_{q_2-3},3)$$

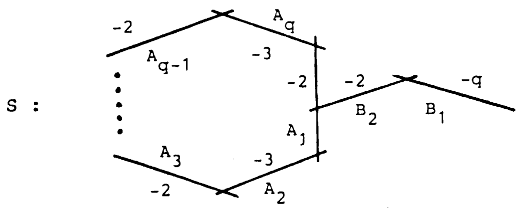
$$D=(2,q_2)$$

By virtue of Theorem 6.4 in Nishiguchi [15], the canonical bundle K_S of S is numerically a divisor if and only if the type is

$$(2,\ 2(q-2)(a-2)+2a-1,\ 3,\ q,\ 3) \qquad \text{with} \quad a\geqq 2,\ q\geqq 3,$$

where $N=b_2(S)=2(q-2)(a-2)+2a+q-2$.

Example 5.5. With the above notations we put $a=2$, i.e., the type is $(2, 3, 3, q, 3)$ with $q\geqq 3$. Then the configuration of curves on S is given as follows:

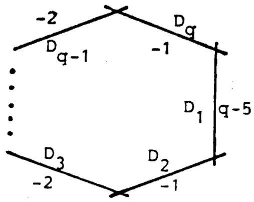


We assume that the canonical bundle K_S of S has a non-zero meromorphic section. Then we have

$$K_S=-B_1-2B_2-3A_1-2A_2-2A_3-\cdots-2A_q$$

and furthermore, we have the following

Theorem 5.6. *Let S be as above. Consider the following configuration of curves on a surface:*

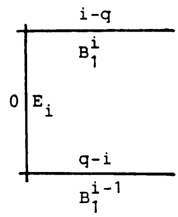


where D_1 is a non-singular elliptic curve and D_i is a non-singular rational curve for $2 \leqq i \leqq q$. We assume that there exists a rational surface with a divisor $D_1 + \cdots + D_q$ and $K_V = -D_1$. Then S can be a component of the singular fiber of a semi-stable degeneration of K3 surfaces.

Proof. As usual, we only describe each component which is to be contained in a singular fiber.

Let U_1^i $(1 \leqq i \leqq q-2)$ be a Hirzebruch surface Σ_{q-i} with the following configuration of $\boldsymbol{P}^1$'s:

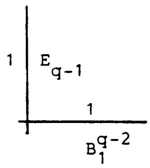
U_1^i :



$(B_1^0 = B_1)$

where $K_{U_1^i} = -B_1^{i-1} - B_1^i - 2E_i$. Let U_1^{q-1} be a projective plane $\boldsymbol{P}^2$ with the following configuration of lines:

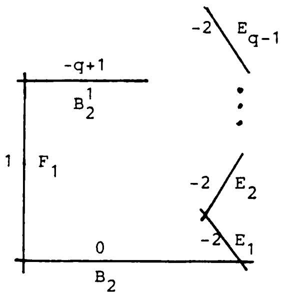
U_1^{q-1}



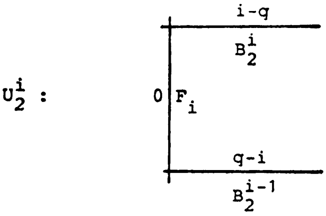
where $K_{U_1^{q-1}} = -B_1^{q-2} - 2E_{q-1}$.

Let U_2^1 be a rational surface with the following configuration of $\boldsymbol{P}^1$'s:

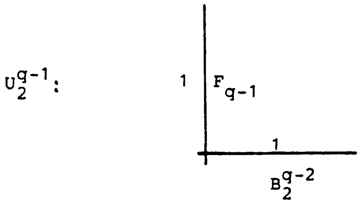
U_2^1:



where $K_{U_2^1} = -B_2^1 - 2F_1$. Such a surface is easily constructed by blowing up $\boldsymbol{P}^2$. Let U_2^i $(2 \leqq i \leqq q-2)$ be a Hirzebruch surface Σ_{q-i} with the following configuration of $\boldsymbol{P}^1$'s:

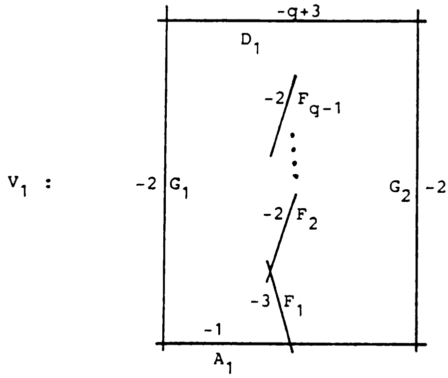


where $K_{U_2^i}=-B_2^{i-1}-B_2^i-2F_i$. Let U_2^{q-1} be a projective plane $\boldsymbol{P}^2$ with the following configuration of lines:

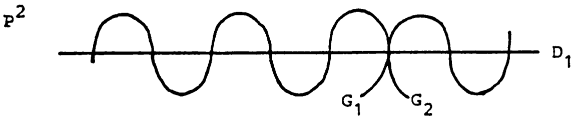


where $K_{U_2^{q-1}}=-2F_{q-1}-B_2^{q-2}$.

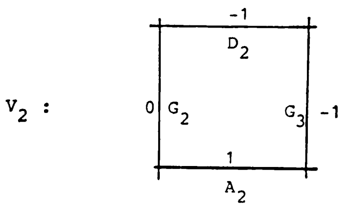
Let V_1 be a rational surface with the following configuration of curves:



where D_1 is a non-singular elliptic curve and the other curves are non-singular rational curves; hence $K_{V_1}=A_1-D_1$. Such a surface V_1 is obtained by blowing up $\boldsymbol{P}^2$ suitably from the following configuration of curves:

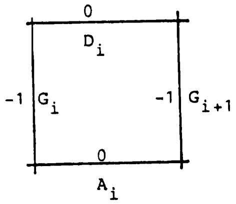


where G_1 is a conic, G_2 is a line, and D_1 is a non-singular cubic, G_1 and G_2 being tangent to each other at a point of D_1. Let V_2 be a rational surface with the following configuration of $\boldsymbol{P}^1$'s:



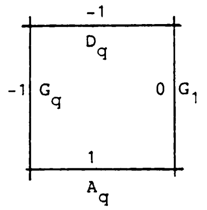
where $K_{V_2}=-2G_2-2D_2-G_3$. Let V_i $(3\leqq i\leqq q-1)$ be a rational surface with the following configuration of $\boldsymbol{P}^1$'s:

V_i :



where $K_{V_i}=-2D_i-G_i-G_{i+1}$. Let V_q be a rational surface with the following configuration of $\boldsymbol{P}^1$'s:

V_q :



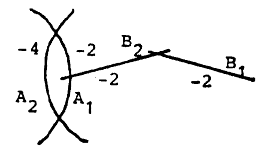
where $K_{V_q}=-G_q-2D_q-2G_1$. The surfaces V_2 and V_q (resp. $V_3, \cdots, V_{q-1}$) are obtained by blowing up Σ_1 (resp. Σ_0).

Let V be a rational surface with the divisor $D_1+\cdots+D_q$ described in Theorem 5.6 and $K_V=-D_3$. Q. E. D.

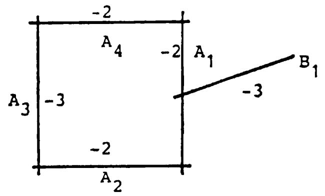
Similarly, we obtain an example for other (CB)-surfaces. Here we state the following results, without proof, only for (CB)-surfaces of four types.

Example 5.7. Let S_1, S_2, S_3 and S_4 be (CB)-surfaces of type (2, 3, 4), (2, 4, 3, 3, 2), (3, 3, 2)$\oplus$(3) and (2, 4, 3, 3, 3, 3, 2), respectively. For the notaion $\oplus$, see Nakamura [9] (S_3 has two branches). Then S_i has the following configuration of curves:

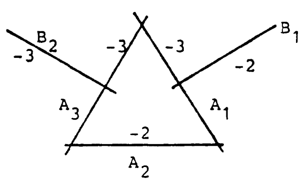
S_1 :



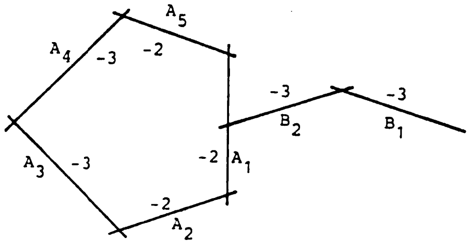
S_2 :



S_3 :



S_4 :



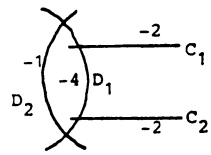
By virtue of Theorem 6.4 in Nishiguchi [15] or by a direct computation, the canonical bundle K_{S_i} ($i=1, 2, 3, 4$) is numerically a divisor. We assume that K_{S_i} has a non-zero meromorphic section. Then we have

$$K_{S_1}=-B_1-2B_2-3A_1-2A_2,$$
$$K_{S_2}=-2B_1-5A_1-4A_2-3A_3-4A_4,$$
$$K_{S_3}=-B_1-B_2-2A_1-2A_2-2A_3,$$
$$K_{S_4}=-B_1-2B_2-4A_1-3A_2-2A_3-2A_4-3A_5.$$

Each S_i ($i=1, 2, 3, 4$) can be made a component of a semi-stable degeneration of K3 surfaces provided

(i) for S_1, there exists a blown-up K3 surface V_1 with the following configuration of $\boldsymbol{P}^1$'s

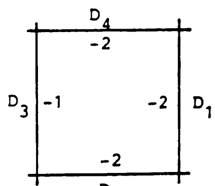
V_1 :



where $K_{V_1}=D_2$;

(ii) for S_2, there exists a rational surface V_2 with the following configuration of curves:

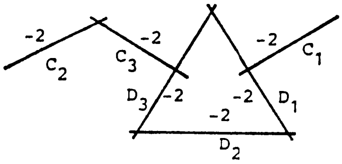
V_2 :



where D_1 is a non-singular elliptic curve, D_2, D_3 and D_4 are non-singular rational curves, and we have $K_{V_2}=-D_1+D_3$;

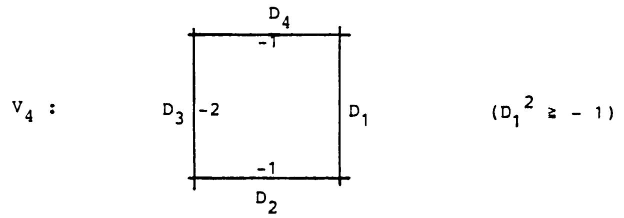
(iii) for S_3, there exists a K3 surface V_3 with the following configuration of $\boldsymbol{P}^1$'s:

V_3 :



where $K_{V_3}=0$;

(iv) for S_4, there exists a rational surface V_4 with the following configuration of curves:



where D_1 is a non-singular elliptic curve, D_2, D_3 and D_4 are non-singular rational curves, and we have $K_{V_4}=-D_1$.

Finally, we consider a (CB)-surface with small second Betti number. Let S be a (CB)-surface containing a GSS whose canonical bundle has a non-zero meromorphic section. We assume that the second Betti number $b_2(S)$ is less than six. Then, by virtue of Theorems 6.2 and 6.4 in Nishiguchi [15], the type of S is one of the following:

(i) in case $b_2(S)=2$, (3);

(ii) in case $b_2(S)=3$, (3, 3, 2), (2, 3, 3);

(iii) in case $b_2(S)=4$, (3, 4, 2), (2, 4, 3), (2, 3, 4), (3)⊕(3);

(iv) in case $b_2(S)=5$, (3, 5, 2), (2, 5, 3), (2, 3, 3, 3, 3), (2, 4, 3, 3, 2), (3, 3, 2)⊕(3).

All these cases have already been treated in Examples 4.3, 5.1, 5.3, 5.5 and 5.7, where the sufficient conditions are obtained for them to be a component of a semi-stable degeneration of K3 surfaces. Moreover, it is easy to see that these conditions are satisfied, i.e., one can find K3 surfaces or rational surfaces required in the sufficient conditions in Theorems 4.4, 5.2, 5.4, 5.6 and Example 5.7. So we have the following

Corollary 5.8. *Let S be a (CB)-surface containing a GSS whose canonical bundle has a non-zero meromorphic section. We assume that* $b_2(S)\leqq 5$. *Then S can be made a component of the singular fiber of a semi-stable degeneration of K3 surfaces.*

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