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# Some limit theorems of almost periodic function systems under the relative measure

By

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## 0. Introduction.

Kac-Steinhaus [6] obtained the following central limit theorem.

**Theorem A.** *If a real sequence* $\{\lambda_j\}$ *is algebraically independent,*

$$\lim_{n\to\infty}\mu_R\left\{x\,;\,\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\sqrt{2}\cos\lambda_j x<\alpha\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha}e^{-\xi^2/2}d\xi\,,\qquad \textit{for all}\quad \alpha\in\boldsymbol{R}^1.$$

Here $\mu_R$ denotes the *relative measure*: $\mu_R(E)=\lim_{T\to\infty}\frac{1}{2T}\mu(E\cap[-T,\,T])$ whenever the limit exists, where $\mu$ is the Lebesgue measure. Theorem A implies, in particular, the relative measures of sets of the form $\left\{x\,;\,\frac{1}{\sqrt{n}}\sum_{j=1}^{n}a_j\cos\lambda_j x<\alpha\right\}$ exists if $\{\lambda_j\}$ is algebraically independent. Note that the family of sets whose relative measures is well defined does not constitute a finite field and the relative measure itself does not satisfy the countable additivity. So the space $(\boldsymbol{R},\,\mu_R)$ is not a probability space in the usual sense. But the central limit theorem holds for $(\sqrt{2}\cos\lambda_j x\}$ on $(\boldsymbol{R},\,\mu_R)$ as Kac-Steinhaus assert.

This theorem was extended to the case of weighted sums in the following theorem due to Salem-Zygmund [15] which is a famous paper on the lacunary trigonometric series.

**Theorem B.** *If a real sequence* $\{\lambda_j\}$ *is algebraically independent and if a real sequence* $\{a_j\}$ *satisfies*

(0.1) $\qquad a_n=o(A_n)$ *and* $A_n\uparrow\infty$, *where* $A_n^2=a_1^2+\cdots+a_n^2$,

*then*

$$\lim_{n\to\infty}\mu_R\left\{x\,;\,\frac{1}{A_n}\sum_{j=1}^{n}a_j\sqrt{2}\cos\lambda_j x<\alpha\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha}e^{-\xi^2/2}d\xi\,,\qquad \textit{for all}\quad \alpha\in\boldsymbol{R}^1.$$

The purpose of this paper is to obtain more general limit theorem, for example, the law of large numbers (LLN), functional central limit theorem (FCLT) and the law of the iterated logarithm (LIL) for $\{\sqrt{2}\cos\lambda_j x\}$ under $\mu_R$. For this purpose we use the idea of Salem-Zygmund (Lemma 1) and

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construct a probability space and random variables whose laws are the same as those of almost periodic functions under $\mu_R$. Of course, a well-known construction of such a probability space is to use the Bohr compactification of $\boldsymbol{R}$ which is a compact Abelian group and the Haar probability measure on this group. Here we give a much simpler construction in section 2: we construct a probability measure $\boldsymbol{P}$ on the product space $\boldsymbol{R}^{\boldsymbol{B}}$ where $\boldsymbol{B}$ is the set of all almost periodic functions, such that whose finite coodinate variables $(\xi_{f_1}, \cdots, \xi_{f_n})$ $(f_1, \cdots, f_n \in \boldsymbol{B})$ have the same joint law as $f_1, \cdots, f_n$ under $\mu_R$. So if we want to study about $\{\sqrt{2}\cos\lambda_j x\}$ under $\mu_R$, we can apply usual probabilistic methods for random variables $\{\xi_{\sqrt{2}\cos\lambda_j x}\}$ on $(\boldsymbol{R}^{\boldsymbol{B}}, \boldsymbol{P})$.

The first aim is to weaken the condition on $\{\lambda_j\}$. We introduce a weaker condition than algebraic independence which we call the *signed sum codition* ("SS-condition", Definition 1). We prove in Lemma 2 that $\{\xi_{\sqrt{2}\cos\lambda_j x}\}$ is i.i.d. if any only if $\{\lambda_j\}$ is algebraically independent, and that $\{\xi_{\sqrt{2}\cos\lambda_j x}\}$ is a *equi-normed multiplicative system* (EMS) in the sense of Definition 2 if and only if $\{\lambda_j\}$ satisfies the SS-condition. EMS is a type of *multiplicative system* (MS) which belongs to a category of weakly dependent random variables.

Next we prove the law of large number (LLN). We know that LLN holds for i.i.d. and MS. The results are translated for $\{\sqrt{2}\cos\lambda_j x\}$ under $\mu_R$ to obtain LLN in a weak form (Theorem 1).

The third attempt is to prove the functional central limit theorem (FCLT). FCLT was first proved for i.i.d. by Donsker and was extended by Prohorov to the case of independent random variables satisfying the Lindeberg condition. FCLT of Donsker type for MS was studied by Kôno [7]. Here we prove FCLT of Prohorov type for EMS (Theorem 2,3). Now we translate these results into a theorem for $\{\sqrt{2}\cos\lambda_j x\}$ under $\mu_R$ (Theorem 4) and, from this, we derive other limit theorems (Theorem 5,6).

Finally we prove the law of the iterated logarithms (LIL). LIL was studied by Kolmogorov for independent random variables and, for MS, by Hungarian school. Using these theorems we can derive a weak form of LIL under $\mu_R$. We also prove here the functional law of the iterated logarithms (i.e. Strassen type theorem) for EMS.

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## 1. Preliminary.

First we define the "SS-condition".

**Definition 1.** We say that $\{\lambda_j\}$ satisfies the signed sum condition (SS condition), if

$$r \in \boldsymbol{N},\ n_1 < \cdots < n_r \quad \text{implies} \quad \pm\lambda_{n_1} \pm \cdots \pm \lambda_{n_r} \neq 0.$$

Next we introduce several classes of multiplicative systems (1), (2) and (4) are due to Alexits [1]). These notions will play an important role in this paper.

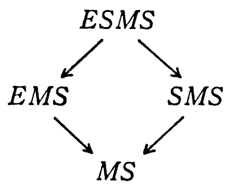


Diagram 1.

**Definition 2.**

Let $\{\xi_j\}$ be a sequence of random variables.

(1) $\{\xi_j\}$ is called a multiplicative system (MS) if

$$\boldsymbol{E}(\xi_{n_1}\cdots\xi_{n_r})=0 \qquad \text{for any} \quad n_1<\cdots<n_r .$$

(2) $\{\xi_j\}$ is called a strongly multiplicative system (SMS) if

$$\boldsymbol{E}(\xi_{n_1}^{\alpha_1}\cdots\xi_{n_r}^{\alpha_r})=0$$

for any $n_1<\cdots<n_r$ and $\alpha_j\in\{1, 2\}$ but at least one of $\alpha_j$ is 1.

(3) MS $\{\xi_j\}$ is called a equinormed multiplicative system (EMS) if

$$\boldsymbol{E}(\xi_{n_1}^2\cdots\xi_{n_r}^2)=1 \qquad \text{for any} \quad n_1<\cdots<n_r .$$

(4) SMS $\{\xi_j\}$ is called a equinormed strongly multiplicative system (ESMS) if

$$\boldsymbol{E}(\xi_{n_1}^2\cdots\xi_{n_r}^2)=1 \qquad \text{for any} \quad n_1<\cdots<n_r .$$

Implications among these are shown in the diagram 1.

Finally we give the following definition.

**Definition 3.** We first define a probability measure $\boldsymbol{P}_T$ on $\boldsymbol{R}^1$ as follows. For a measurable set $E$,

$$\boldsymbol{P}_T(E)=\frac{1}{2T}\mu(E\cap[-T, T]).$$

Next we define the upper relative measure $\bar{\mu}_R(E)$ and the lower relative measure $\underline{\mu}_R(E)$ for a mesurable set $E$ by

$$\bar{\mu}_R(E)=\limsup_{T\to\infty}\boldsymbol{P}_T(E) \qquad \text{and} \qquad \underline{\mu}_R(E)=\liminf_{T\to\infty}\boldsymbol{P}_T(E).$$

$\mu_R(E)=\bar{\mu}_R(E)=\underline{\mu}_R(E)$ if the upper and lower relative measure coincide.

## 2. Main results.

Next lemma is essentially due to Salem-Zygmund [15].

**Lemma 1.** *Let $f_1, \cdots, f_n$ be almost periodic functions. We define a mapping $(f_1, \cdots, f_n)$ from $\boldsymbol{R}^1$ to $\boldsymbol{R}^n$ by*

$$(f_1, \cdots, f_n)(s)=(f_1(s), \cdots, f_n(s)).$$

*Then there exists a probability measure $\boldsymbol{P}_{f_1,\cdots,f_n}$ on $\boldsymbol{R}^n$ such that*

$$\boldsymbol{P}_T^{(f_1,\cdots,f_n)} \xrightarrow{w} \boldsymbol{P}_{f_1,\cdots,f_n} \quad \text{as} \quad T\to\infty, \tag{2.1}$$

*where $\boldsymbol{P}_T^{(f_1,\cdots,f_n)}$ is a image measure of $\boldsymbol{P}_T$ by $(f_1, \cdots, f_n)$.*

We consider the following family of probability measures.

$$\{\boldsymbol{P}_{f_1,\cdots,f_n}\}_{n\in\boldsymbol{N},\, f_1,\cdots,f_n\in\boldsymbol{B}}.$$

Since this family satisfies the Kolmogorov's consistency condition, we can apply the Kolmogorov's extention theorem to obtain a probability measure $\boldsymbol{P}$ on $\boldsymbol{R}^{\boldsymbol{B}}$ such that

$$\boldsymbol{P}_{\pi^{-1}_{f_1,\cdots,f_n}}=\boldsymbol{P}_{f_1,\cdots,f_n} \qquad n\in\boldsymbol{N},\ f_1, \cdots, f_n\in\boldsymbol{B}.$$

Now we define coodinate variables. Let $f$ be an almost periodic function and we write $\xi_f$ as an $f$-th coodinate of the space $(\boldsymbol{R}^{\boldsymbol{B}}, \boldsymbol{P})$. $\xi_f$ is a random variable on $(\boldsymbol{R}^{\boldsymbol{B}}, \boldsymbol{P})$ and $n$-dimensional distribution $\boldsymbol{P}^{\xi_{f_1},\cdots,\xi_{f_n}}$ of $n$ random variables $\xi_{f_1}, \cdots, \xi_{f_n}$ coincides with $\boldsymbol{P}_{f_1,\cdots,f_n}$ for every $n\in\boldsymbol{N}$ and $f_1, \cdots, f_n\in\boldsymbol{B}$ and furthermore, it holds

$$\boldsymbol{P}_T^{f_1,\cdots,f_n} \xrightarrow{w} \boldsymbol{P}^{\xi_{f_1},\cdots,\xi_{f_n}}(T\to\infty) \qquad \text{for} \quad n\in\boldsymbol{N},\ f_1, \cdots, f_n\in\boldsymbol{B}. \tag{2.2}$$

From this we can say that the law of the almost periodic function undes $\mu_R$ is roughly equal to the law of the coodinate variable of the probability space $(\boldsymbol{R}^{\boldsymbol{B}}, \boldsymbol{P})$.

**Lemma 2.**

(1) *$\{\xi_{\sqrt{2}\cos\lambda_j x}\}$ is i.i.d. if and only if $\{\lambda_j\}$ is algebraically independent.*

(2) *$\{\xi_{\sqrt{2}\cos\lambda_j x}\}$ is uniformly bounded EMS if and only if $\{\lambda_j\}$ satisfy the* SS-*condition.*

(3) *For all $\lambda\in\boldsymbol{R}^1$ and $\alpha\in[-\sqrt{2}, \sqrt{2}]$,*

$$\boldsymbol{P}\{\xi_{\sqrt{2}\cos\lambda_j x}<\alpha\}=\mu\{x\in[0, 1];\ \sqrt{2}\cos 2\pi x<\alpha\}.$$

A weak LLN for $\{\sqrt{2}\cos\lambda_j x\}$ is stated as follows.

**Theorem 1.** *Let $\{\lambda_j\}$ be algebraically independent or satisfy the SS-condition. Let $\{a_j\}$ be a real seaquence such that*

$$B_n=a_1+\cdots+a_n\uparrow\infty\ (n\to\infty), \qquad a_n=o(B_n).$$

*Then*

$$\lim_{n\to\infty}\bar{\mu}_R\left\{x:\left|\frac{1}{B_n}\sum_{j=1}^{n}a_j\sqrt{2}\cos\lambda_j x\right|\geqq\varepsilon\right\}=0 \qquad \textit{for all} \quad \varepsilon>0.$$

Next two theorems are FCLT for EMS. Denote by $\boldsymbol{C}$ the space of all continuous functions on [0, 1] with sup norm and $\sigma[\boldsymbol{C}]$ is its topological $\sigma$-field. Denote by $\boldsymbol{D}$ the space of all discontinuous functions of the first kind with Skorohod metric and $\sigma[\boldsymbol{D}]$ is its topological $\sigma$-field. (Cf. Billingsley [3])

**Theorem 2.** *Let $\{\xi_j\}$ be a uniformly bounded EMS and $\{a_j\}$ be a real sequence such that* (0.1) *holds. Put $S_j=a_1\xi_1+\cdots+a_j\xi_j$. We define a $\boldsymbol{C}$-valued random variable $\boldsymbol{X}_n$ by*

$$\text{(2.3)} \qquad \boldsymbol{X}_n\left(\frac{A_j^2}{A_n^2}\right)=\frac{S_j}{A_n} \qquad \textit{and is linear in} \quad \left[\frac{A_j^2}{A_n^2},\ \frac{A_{j+1}^2}{A_n^2}\right].$$

*Then we have*

$$\boldsymbol{X}_n \xrightarrow{\mathscr{D}} \boldsymbol{W} \quad (n\to\infty).$$

*where $\boldsymbol{W}$ is the Wiener measure on $\boldsymbol{C}$. Here $\mathscr{D}$ denotes the convergence in distribution, i.e. the law $\boldsymbol{P}^{X_n}$ of $\boldsymbol{X}_n$ converges weakly to $\boldsymbol{W}$.*

**Theorem 3.** *Under the condition of Theorem 2, we define a $\boldsymbol{D}$-valued random variable $\boldsymbol{Y}_n$ by*

$$\text{(2.4)} \qquad \boldsymbol{Y}_n(t)=\frac{S_j}{A_n} \qquad \textit{if} \quad t\in\left[\frac{A_j^2}{A_n^2},\ \frac{A_{j+1}^2}{A_n^2}\right).$$

*Then we have*

$$\boldsymbol{Y}_n \xrightarrow{\mathscr{D}} \boldsymbol{W} \ (n\to\infty) \qquad \textit{in} \quad \boldsymbol{D}.$$

Using FCLT for EMS, we derive the following theorems.

**Theorem 4.** *Denote $A_n$ and $S_n$ by*

$$A_n^2=a_1^2+\cdots+a_n^2 \qquad \textit{and} \qquad S_n=a_1\sqrt{2}\cos\lambda_1 x+\cdots+a_n\sqrt{2}\cos\lambda_n x$$

*respectively. We define a $\boldsymbol{C}$-valued random variable $\boldsymbol{X}_n$ by* (2.3) *and a $\boldsymbol{D}$-valued random variables $\boldsymbol{Y}_n$ by* (2.4). *Suppose $\{a_j\}$ satisfy* (0.1) *and $\{\lambda_j\}$ satisfies the SS-condition or the condition of algebraic independence. Then for $A\in\sigma[\boldsymbol{C}]$ such that $\boldsymbol{W}(\partial A)=0$, we have*

$$\lim_{n\to\infty}\bar{\mu}_R\{\boldsymbol{X}_n\in A\}=\lim_{n\to\infty}\underline{\mu}_R\{\boldsymbol{X}_n\in A\}=\boldsymbol{W}(A),$$

*and for $A\in\sigma[\boldsymbol{D}]$ such that $\boldsymbol{W}(\partial A)=0$, we have*

$$\lim_{n\to\infty}\bar{\mu}_R\{\boldsymbol{Y}_n\in A\}=\lim_{n\to\infty}\underline{\mu}_R\{\boldsymbol{Y}_n\in A\}=\boldsymbol{W}(A).$$

**Theorem5.** *Suppose* $\{a_j\}$ *satisfies* (0.1).

$$\lim_{n\to\infty}\mu_R\left\{\frac{1}{A_n}\max_{j=1}^{n} S_j\leqq\alpha\right\}=\frac{2}{\sqrt{2\pi}}\int_0^\alpha e^{-u^2/2}du\,,$$

$$\lim_{n\to\infty}\mu_R\left\{\frac{1}{A_n}\max_{j=1}^{n}|S_j|\leqq\alpha\right\}=1-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}\exp\left(\frac{-\pi^2(2k+1)^2}{8\alpha^2}\right)$$

*hold for all* $\alpha>0$ *under the condition of algebraic independence. Under the SS-condition the above formulas also true except at most countable values of* $\alpha$ *and if* $\mu$ *is replaced by* $\bar{\mu}_R$ *or* $\underline{\mu}_R$, *then the above formulas hold for all* $\alpha$.

**Theorem 6.** *Under the condition of algebraic independence and* (0.1), *we have*

$$\lim_{n\to\infty}\mu_R\left\{\frac{1}{A_n^2}\sum_{\substack{j\leq n\\ S_j>0}} a_j^2\leqq\alpha\right\}=\frac{2}{\pi}\arcsin\sqrt{\alpha}\,.$$

We should guess that, under the SS-condition, Theorem 5 holds without exceptional values of $\alpha$ and also Theorem 6 holds. But we could not prove these conjectures.

Next theorem is the functional law of the iterated logarithms (FLIL) for EMS.

**Theorem 7.** *Let* $\{\xi_j\}$ *be a uniformly bounded EMS and* $\{a_j\}$ *be a real sequence such that*

$$A_n^2=a_1^2+\cdots+a_n^2\uparrow\infty \quad \text{and} \quad a_n^2=o\left(\frac{A_n^2}{\log\log A_n^2}\right). \tag{2.5}$$

*Put* $X_n$ *as* (2.3). *Then we have*

(1) $\{X_n/\sqrt{2\log\log A_n^2}\}$ *is relatively compact in* $C[0, 1]$ *a.s. and*

(2) $P(\{$*The cluster of* $\{X_n/\sqrt{2\log\log A_n^2}\}$ *in* $C[0, 1]\}\subset K)=1$. *Moreover if we suppose*

$$A_n^2=a_1^2+\cdots+a_n^2\uparrow\infty \quad \text{and} \quad a_n=o(A_n^{1-\delta}) \ \text{for some} \ \delta>0, \tag{2.6}$$

*then we have*

(3) $P(\{$*The cluster of* $\{X_n/\sqrt{2\log\log A_n^2}\}$ *in* $C[0, 1]\}=K)=1$, *where* $K=\{x\in C[0, 1];\ x(0)=0,$ *x is absolutely continuous and* $\int_0^1\left(\frac{dx}{dt}\right)^2 dt\leqq 1\}$.

We derive from this theorem a weak form of LIL for $\{\sqrt{2}\cos\lambda_j x\}$ under $\mu_R$.

**Theorem 8.** *Under the condition* (2.5),

$$\lim_{n\to\infty}\lim_{m\to\infty}\underline{\mu}_R\left\{\max_{n\leq j\leq m}\frac{S_j}{\sqrt{2A_j^2\log\log A_j^2}}<1+\varepsilon\right\}=1 \qquad \forall\varepsilon>0,$$

*and under the condition* (2.6),

$$\mu_R\left\{\sup_{j\geq n}\frac{S_j}{\sqrt{2A_j^2\log\log A_j^2}}>1-\varepsilon\right\}=1 \qquad \forall\varepsilon>0,\ \forall n\in\boldsymbol{N},$$

*where* $S_j(x)=a_1\sqrt{2}\cos\lambda_1 x+\cdots+a_j\sqrt{2}\cos\lambda_j x$.

### 3. Proof of Lemma 1 and 2.

*Proof of Lemma* 1. Since an almost periodic function is bounded, a range of $(f_1, \cdots, f_n)$ is compact. This compact set is a support of $\boldsymbol{P}_T^{(f_1,\cdots,f_n)}$ for all $T$. Thus $\{\boldsymbol{P}_T^{(f_1,\cdots,f_n)}\}$ is tight. So, to prove the weak convergence, we only have to show the pointwise convergence of the characteristic functions.

$$\begin{aligned}\hat{\boldsymbol{P}}_T{}^{(f_1,\cdots,f_n)}(\gamma_1, \cdots, \gamma_n)&=\int_{R^d}\exp\Big(i\sum_{j=1}^n z_j\gamma_j\Big)\boldsymbol{P}_T^{(f_1,\cdots,f_n)}(\mathrm{d}z)\\ &=\frac{1}{2T}\int_{-T}^{T}\exp\Big(i\sum_{j=1}^n f_j(s)\gamma_j\Big)\mathrm{d}s\end{aligned}$$

Since the integrand of the last integral is an almost periodic function, by the existence theorem of the mean value of the almost periodic function (Cf. Bohr. [4]), this integral converges as $T$ tends to infinity.

The proof of the lemma 2 is based on the idea of Kac-Steinhaus [6].

*Proof of Lemma* 2. Proof of 3). By (2.1), with at most countably many exceptional values of $\alpha$,

$$\lim_{T\to\infty}\boldsymbol{P}_T^{\sqrt{2}\cos\lambda x}[-\sqrt{2}, \alpha]=\boldsymbol{P}^{\xi\sqrt{2}\cos\lambda x}[-\sqrt{2}, \alpha]$$

holds. On the other hand,

$$\begin{aligned}\lim_{T\to\infty}\boldsymbol{P}_T^{\sqrt{2}\cos\lambda x}[-\sqrt{2}, \alpha]&=\lim_{T\to\infty}\frac{1}{2T}\mu([-T, T]\cap\{x\,;\ \sqrt{2}\cos\lambda x\in[-\sqrt{2}, \alpha]\})\\ &=\mu\{x\in[0, 1]\,;\ \sqrt{2}\cos 2\pi x\in[-\sqrt{2}, \alpha]\}.\end{aligned}$$

Since the right-hand side is continuous in $\alpha$, we can conclude that there is no exceptional values of $\alpha$. Now we proceed to the proof of 1) and 2). We first prove,

$$\begin{aligned}(3.1)\qquad &\boldsymbol{E}(\xi_{\sqrt{2}\cos\lambda_1}^{r_1}\cdots\xi_{\sqrt{2}\cos\lambda_n x}^{r_n})\\ &=\sum_{p_1=0}^{r_1}\cdots\sum_{p_n=0}^{r_n}{}_{r_1}\boldsymbol{C}_{p_1}\cdots{}_{r_n}\boldsymbol{C}_{p_n}2^{-1/2(r_1+\cdots+r_n)}\delta_{0,\sum_{j=1}^n(2p_j-r_j)\lambda_j}\end{aligned}$$

where $\delta_{i,j}$ is the Kronecker's delta. By (2.2)

$$\begin{aligned}&\boldsymbol{E}(\xi_{\sqrt{2}\cos\lambda_1}^{r_1}\cdots\xi_{\sqrt{2}\cos\lambda_n x}^{r_n})\\ &=\lim_{T\to\infty}\int x_1^{r_1}\cdots x_n^{r_n}\boldsymbol{P}_T^{(\xi\sqrt{2}\cos\lambda_1,\cdots,\xi\sqrt{2}\cos\lambda_n)}(dx)\end{aligned}$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(\sqrt{2}\cos\lambda_1 x)^{r_1}\cdots(\sqrt{2}\cos\lambda_n x)^{r_n}dx\,.$$

Expanding this formula by substituting $\cos x$ with $(e^{ix}+e^{-ix})/2$ and calculating the limitation, we get (3.1). Let $\{\lambda_j\}$ be algebraically independent. Then,

$$\delta_{0,\sum_{j=1}^{n}(2p_j-r_j)\lambda_j}=\delta_{0,2p_1-r_1}\cdots\delta_{0,2p_n-r_n}$$

holds and we get by (3.1)

$$E(\xi^{r_1}_{\sqrt{2}\cos\lambda_1}\cdots\xi^{r_n}_{\sqrt{2}\cos\lambda_n x})=E(\xi^{r_1}_{\sqrt{2}\cos\lambda_1 x})\cdots E(\xi^{r_n}_{\sqrt{2}\cos\lambda_n x})$$

$$(n\in N,\ r_1,\ \cdots,\ r_n\in N).$$

This implies that $\{\xi_{\sqrt{2}\cos\lambda_j x}\}$ is independent and 1) is proved. Now we assume the SS-condition. Let $r_1=\cdots r_n=1$ in (3.1). The summation in the Kronecker's delta is

$$\sum_{j=1}^{n}(2p_j-1)\lambda_j \qquad (p_j=0,\ 1,\ j=1,\ \cdots,\ n)$$

and by the SS-condition on $\{\lambda_j\}$, this summation never vanish. Thus we can conclude that

$$E(\xi_{\sqrt{2}\cos\lambda_1 x}\cdots\xi_{\sqrt{2}\cos\lambda_n x})=0\,.$$

Let $r_1=\cdots r_n=2$ in (3.1). Summation is

$$2\sum_{j=1}^{n}(p_j-1)\lambda_j \qquad (p_j=0,\ 1,\ 2,\ j=1,\ \cdots,\ n).$$

By the SS-condition this summation equals to 0 if and only if

$$p_j=1\,, \qquad j=1,\ \cdots,\ n\,.$$

This proves

$$E(\xi^2_{\sqrt{2}\cos\lambda_1 x}\cdots\xi^2_{\sqrt{2}\cos\lambda_n x})=1\,.$$

Thus the assertion of 2) is also proved.

## 4. Proof of Theorem 2, 3 and 7.

In the proof of Theorem 2, we use following inequalities. (Cf. Azuma [2], Révész [13] and Takahashi [16].)

**Theorem C.** (*Azuma's inequality and Révész-Takahashi's inequality.*) *Let* $\{\xi_n\}$ *be a uniformly bounded* ($|\xi_n|\leqq K$) *MS and* $\{a_n\}$ *be a real sequence. Put* $A_n^2=a_1^2+\cdots+a_n^2$ *and* $S_n=a_1\xi_1+\cdots+a_n\xi_n$. *Then Azuma's inequality*

$$E(\exp\{\lambda S_n\})\leqq\exp\Big(\frac{1}{2}\lambda^2A_n^2K^2\Big)$$

*holds and this implies Révész-Takahashi's inequality*

$$P(|S_i|\geqq yKA_i\sqrt{2})\leqq 2e^{-y^2} \qquad \forall y\geqq 0,\ \forall i\in \boldsymbol{N}. \tag{4.1}$$

*Proof of Theorem* 2.

[Part 1. Weak convergence of finite dimensional distributions] We use the next theorem due to D. L. McLeish [9].

**Theorem D.** *Let* $\{\zeta_{n,j};\ 1\leqq j\leqq k_n\}$ *be a given triangular array of random variables and put* $T_n=\prod_{j\leqq k_n}(1+it\zeta_{n,j})$. *Suppose for all real* $t$,

$$\text{(a)}\quad \boldsymbol{E}(T_n)\longrightarrow 1, \qquad \text{(b)}\quad \{T_n\} \text{ is uniformly integrable,}$$

$$\text{(c)}\quad \sum_{j\leqq k_n}\zeta_{n,j}^2 \xrightarrow{p} 1 \quad \text{and} \quad \text{(d)}\quad \max_{j\leqq k_n}|\zeta_{n,j}| \xrightarrow{p} 0.$$

*Then we have*

$$\sum_{j\leqq k_n}\zeta_{n,j} \xrightarrow{\mathcal{D}} N(0, 1) \qquad (n\to\infty).$$

Now we put $k_n=n$ and $\zeta_{n,j}=\dfrac{a_j}{A_n}\xi_j$. Then we have,

$$\boldsymbol{E}(T_n)=1, \quad |T_n|\leqq e^{t^2K^2/2} \qquad \text{and} \qquad \max|\zeta_{n,j}|\leqq \frac{K}{A_n}\max_{j\leqq n}|a_j| \longrightarrow 0.$$

Thus we only have to check (c). Making use of (0.1) and the orthogonality of $\{\xi_j^2-1\}$, we have

$$\frac{1}{A_n^2}\sum_{j\leqq n}a_j^2(\xi_j^2-1)\xrightarrow{p} 0 \qquad (n\to\infty).$$

Thus (c) is proved. Now we have proved the 1-dimensional CLT. And we can prove the multi-dimensional one using Cramér-Wold theorem (Cf. Billingsley [3] Th 7.7). Thus we have proved the weak convergence of finite dimensional distributions.

[Part 2. Tightness]

We prove here

$$\boldsymbol{P}\{|\boldsymbol{X}_n(t)-\boldsymbol{X}_n(s)|\geqq\lambda\}\leqq 6\exp\left(-\frac{\lambda^2}{6K^2|t-s|}\right) \qquad (\lambda>0). \tag{4.2}$$

It is standard that (4.2) implies the tightness of $\{\boldsymbol{X}_n\}$ (Cf. Billingsley [3]). Let $t>s$ and

$$s\in(A_i^2/A_n^2,\ A_{i+1}^2/A_n^2], \qquad t\in(A_j^2/A_n^2,\ A_{j+1}^2/A_n^2].$$

Then

$$\boldsymbol{X}_n(t)-\boldsymbol{X}_n(s)=\frac{A_n}{a_{i+1}}\left(\frac{A_{i+1}^2}{A_n^2}-s\right)\xi_{i+1}+\frac{1}{A_n}(a_{i+2}\xi_{i+2}+\cdots+a_j\xi_j)$$
$$+\frac{A_n}{a_{j+1}}\left(t-\frac{A_j^2}{A_n^2}\right)\xi_{j+1}.$$

Now we put $p, q, r\geqq 0$ by

$$p=\frac{A_{i+1}^2}{A_n^2}-s, \qquad q=\frac{1}{A_n^2}(A_j^2-A_{i+1}^2), \qquad r=t-\frac{A_i^2}{A_n^2}$$

It is obvious that $p+q+r=t-s$. Let $\lambda>0$, then

$$\begin{aligned} \boldsymbol{P}\{|\boldsymbol{X}_n(t)-\boldsymbol{X}_n(s)|\geqq\lambda\} &\leqq \boldsymbol{P}\left\{\frac{A_n}{|a_{i+1}|}p|\xi_{i+1}|\geqq\frac{\sqrt{p}\,\lambda}{\sqrt{p}+\sqrt{q}+\sqrt{r}}\right\} \\ &+\boldsymbol{P}\left\{\frac{1}{A_n}|a_{i+2}\xi_{i+2}+\cdots+a_j\xi_j|\geqq\frac{\sqrt{q}\,\lambda}{\sqrt{p}+\sqrt{q}+\sqrt{r}}\right\} \\ &+\boldsymbol{P}\left\{\frac{A_n}{|a_{j+1}|}r|\xi_{j+1}|\geqq\frac{\sqrt{r}\,\lambda}{\sqrt{p}+\sqrt{q}+\sqrt{r}}\right\}. \end{aligned}$$

By (4.1),

$$\begin{aligned} &\leqq 2\exp\left(-\frac{a_{i+1}^2}{2A_n^2p}\frac{\lambda^2}{K^2(\sqrt{p}+\sqrt{q}+\sqrt{r})^2}\right)+2\exp\left(-\frac{\lambda^2}{2K^2(\sqrt{p}+\sqrt{q}+\sqrt{r})^2}\right) \\ &+2\exp\left(-\frac{a_{j+1}^2}{2A_n^2r}\frac{\lambda^2}{K^2(\sqrt{p}+\sqrt{q}+\sqrt{r})^2}\right). \end{aligned}$$

Making use of $p<\dfrac{a_{i+1}^2}{A_n^2}$ and $r<\dfrac{a_{j+1}^2}{A_n^2}$,

$$\boldsymbol{P}\{|\boldsymbol{X}_n(t)-\boldsymbol{X}_n(s)|\geqq\lambda\}\leqq 6\exp\left(-\frac{\lambda^2}{2K^2(\sqrt{p}+\sqrt{q}+\sqrt{r})^2}\right).$$

By $(\sqrt{p}+\sqrt{q}+\sqrt{r})^2\leqq 3(p+q+r)$ (4.2) is proved. □

Now we proceed to Theorem 3. Let $d$ be the Prohorov metric on space $\boldsymbol{D}$ and $\boldsymbol{X}_n$, $\boldsymbol{Y}_n$ be defined as (2.3), (2.4). Then,

$$\begin{aligned} d(\boldsymbol{X}_n, \boldsymbol{Y}_n) &\leqq \sup_t|\boldsymbol{X}_n(t)-\boldsymbol{Y}_n(t)| \\ &\leqq\frac{1}{A_n}\max_{j=1}^{n}|a_j\xi_j|. \end{aligned}$$

Thus under the condition of Theorem 2 and 3,

$$d(\boldsymbol{X}_n, \boldsymbol{Y}_n)\longrightarrow 0 \quad \text{a.s.} \quad (n\to\infty).$$

This proves that Theorem 3 can be derived from Theorem 2.

**Remark.** In the proof of Theorem 2 and 3, we use only the orthogonality of $\{\xi_j^2-1\}$, the uniform boundedness and the multiplicativity of $\{\xi_j\}$.

In the proof of the first part of Theorem 7, we need the next theorem due to Móricz.

**Theorem E.** (*Móricz* [10]). *Let $\{\zeta_j\}$ be a sequence of random variables and put*

$$\boldsymbol{E}(\zeta_j^2)=\sigma_j^2,\quad S(b,m)=\sum_{j=b+1}^{b+m}\zeta_j,\quad M(b,m)=\max_{j\leq m}|S(b,m)|\quad \textit{and}$$

$$g(b,m)=A\sum_{j=b+1}^{b+m}\sigma_j^2.$$

*Suppose*

$$\boldsymbol{P}\{|S(b,m)|\geqq\lambda\}\leqq C\exp\Bigl(-\frac{\lambda^2}{g(b,m)}\Bigr)\qquad \forall\lambda>0,\ \forall b,\ m\in\boldsymbol{N}.$$

*Then for some constant* $C_1$

$$\boldsymbol{P}\{M(b,m)\geqq\lambda\}\leqq C_1\exp\Bigl(-\frac{\lambda^2}{2g(b,m)}\Bigr)\qquad \forall\lambda>0.$$

Putting $\zeta_j=a_j\xi_j$ and making use of this theorem, by (4.1) we have

$$\text{(4.3)}\qquad \boldsymbol{P}\Bigl\{\max_{p<j\leq q}\Bigl|\sum_{i=p+1}^{j}a_j\xi_j\Bigr|\geqq\lambda\Bigr\}\leqq C_1\exp\Bigl(-\frac{\lambda^2}{4K^2(A_q^2-A_p^2)}\Bigr).$$

*Proof of Theorem* 7 1).

Let a sequence $\{p(k)\}$ satisfy $A_{p(k)-1}^2<\theta^k\leqq A_{p(k)}^2$ then we have

$$\sup_{p(r-1)\leq n\leq p(r)}\ \sup_{|t-s|\leq\delta}|\boldsymbol{X}_n(t)-\boldsymbol{X}_n(s)|\leqq\theta\sup_{|t-s|\leq\delta}|\boldsymbol{X}_{p(r)}(t)-\boldsymbol{X}_{p(r)}(s)|.$$

We denote $A_r(\varepsilon,\delta)$ for $\varepsilon>0$ and $\delta>0$ by

$$A_r(\varepsilon,\delta)=\Bigl\{\sup_{|t-s|\leq\delta}\frac{|\boldsymbol{X}_{p(r)}(t)-\boldsymbol{X}_{p(r)}(s)|}{\sqrt{\log\log A_{p(r)}^2}}>\varepsilon\Bigr\}$$

We prove here that

$$\text{(4.4)}\qquad \forall\varepsilon>0,\ \exists\delta>0\quad\text{such that}\quad \sum_{r=1}^{\infty}\boldsymbol{P}(A_r(\varepsilon,\delta))<\infty.$$

Once it is proved, by the Ascoli-Arzela theorem, the relative compactness becomes clear. Now we prove (4.4). Taking $n$ large enough and fix it. By (2.5), for all $\delta>0$ there exists a sequence of integer $0=q(0)<q(1)<\cdots<q(r)=n$ such that

$$2\delta\geqq t_i-t_{i-1}\geqq\delta\quad i=1,\cdots,r,\qquad\text{where}\quad t_i=\frac{A_{q(i)}^2}{A_n^2}.$$

This implies (Billingsley [3] p. 56 Cor),

$$\boldsymbol{P}(\sup_{|t-s|\leq\delta}|\boldsymbol{X}_n(t)-\boldsymbol{X}_n(s)|\geqq\eta)\leqq\sum_{j=1}^{r}\boldsymbol{P}\Bigl(\sup_{t_{i-1}\leq s\leq t_i}|\boldsymbol{X}_n(s)-\boldsymbol{X}_n(t_{i-1})|\geqq\frac{\eta}{3}\Bigr)$$

$$=\sum_{j=1}^{r}\boldsymbol{P}\Bigl(\max_{q(j-1)<k\leq q(j)}\Bigl|\sum_{i=q(j-1)+1}^{k}a_i\xi_i\Bigr|\geqq\frac{\eta}{3}A_n\Bigr)$$

Making use of (4.3), we have

$$\leqq C_1\sum_{j=1}^{r}\exp\Bigl(-\frac{A_n^2\eta^2}{36K^2(A_{q(j)}^2-A_{q(j-1)}^2)}\Bigr)$$

$$\leqq C_1\frac{1}{\delta}\exp\left(-\frac{\eta^2}{72K^2\delta}\right).$$

Now we put $\eta=\varepsilon\sqrt{2\log\log A_{p(r)}^2}$, we have

$$\boldsymbol{P}(A_r(\varepsilon, \delta))\leqq\frac{C_1}{\delta}(r-1)^{-\varepsilon^2/36K^2\delta}.$$

Taking $\delta$ small enough we have $\sum_{r=1}^{\infty}\boldsymbol{P}(A_r(\varepsilon, \delta))<\infty$. Thus (4.4) is proved.

□

In the proof of the later part of Theorem 7, we use the following theorems.

**Theorem F** (*Révész* [13]). *Let* $\{\xi_n\}$ *be a uniformly bounded MS and* $\{b_n\}$ *be a real sequence satisfying*

$$B_n\uparrow\infty \qquad \text{and} \qquad b_n=o\left(\frac{B_n}{\log\log B_n}\right).$$

*Then we have*

$$\frac{1}{B_n}\sum_{j=1}^{n}b_j\xi_j\longrightarrow 0 \quad a.s. \quad (n\to\infty).$$

**Theorem G** (*Kuelbs* [8]). *Assume that*

$$\boldsymbol{P}(\{\boldsymbol{X}_n/\sqrt{2\log\log A_n^2}\} \text{ is relatively compact in } \boldsymbol{C}[0, 1\})=1$$

*and, for all signed measure* $\nu$ *with bounded variation on* [0, 1],

$$\boldsymbol{P}\left(\limsup_{n\to\infty}\frac{\int_0^1\boldsymbol{X}_n(t)\mathrm{d}\nu}{\sqrt{2\log\log A_n^2}}\leqq K_{\nu,1}\right)=1.$$

*Then we have*

$$\boldsymbol{P}(\{\text{The cluster of } \{\boldsymbol{X}_n/\sqrt{2\log\log A_n^2}\} \text{ in } \boldsymbol{C}[0, 1]\}\subset K)=1.$$

*Furthermore suppose that*

$$\boldsymbol{P}\left(\limsup_{n\to\infty}\frac{\int_0^1\boldsymbol{X}_n(t)\mathrm{d}\nu}{\sqrt{2\log\log A_n^2}}=K_{\nu,1}\right)=1.$$

*Then we have*

$$\boldsymbol{P}(\{\text{The bluster of } \{\boldsymbol{X}_n/\sqrt{2\log\log A_n^2}\} \text{ in } \boldsymbol{C}[0, 1]\}=K)=1,$$

*where*

$$K_{\nu,\theta}^2=\boldsymbol{E}\left[\left(\int_0^1\boldsymbol{W}(t\wedge\theta^{-1})\mathrm{d}\nu(t)\right)^2\right]=\int_0^1\int_0^1 t\wedge s\wedge\theta^{-1}\mathrm{d}\nu(t)\mathrm{d}\nu(s).$$

($\boldsymbol{W}(t)$ *denotes the standard Brownian motion.*)

*Proof of Theorem* 7 2) *and* 3).

Put $N=|\nu|([0, 1])$,

$$\phi_j^{(n)}(t)=\begin{cases} 0 & \text{for } t\in\left[0, \dfrac{A_{j-1}^2}{A_n^2}\right] \\ \dfrac{A_n^2}{a_j^2}\left(t-\dfrac{A_{j-1}^2}{A_n^2}\right) & \text{for } t\in\left[\dfrac{A_{j-1}^2}{A_n^2}, \dfrac{A_j^2}{A_n^2}\right] \text{ and} \\ 1 & \text{otherwise} \end{cases}$$

$$c_j^{(n)}=\int_0^1\phi_j^{(n)}(t)d\nu(t).$$

We have

$$\boldsymbol{X}_n(t)=\frac{1}{A_n}\sum_{j=1}^n a_j\phi_j^{(n)}\xi_j \quad \text{and} \quad \int_0^1\boldsymbol{X}_n(t)d\nu(t)=\frac{1}{A_n}\sum_{j=1}^n a_jc_j^{(n)}\xi_j.$$

Weak convergence of $\boldsymbol{X}_n$ implies

$$\lim_{n\to\infty}\boldsymbol{E}\left[\left(\int_0^1 X_n(t\wedge\theta^{-1})d\nu(t)\right)^2\right]=\boldsymbol{E}\left[\left(\int_0^1\boldsymbol{B}(t\wedge\theta^{-1})d\nu(t)\right)^2\right]=K_{\nu,\theta}^2.$$

Thus we have

$$\lim_{n\to\infty}\frac{1}{A_n^2}\sum_{j=1}^n(a_jc_j^{(n)})^2=K_{\nu,1}^2. \tag{4.5}$$

Since $|c_j^{(n)}|\leqq N$, (4.5) and (2.5) implies

$$a_jc_j^{(n)}=o\left(\frac{B_n}{\sqrt{\log\log B_n}}\right), \quad \text{where} \quad B_n^2=\sum_{j=1}^n(a_jc_j^{(n)})^2.$$

This implies (Theorem F)

$$\frac{1}{B_n^2}\sum_{j=1}^n(a_jc_j^{(n)})^2(\xi_j^2-1)\longrightarrow 0 \qquad \text{a.s.}.$$

Thus by (4.5) we have

$$\frac{1}{A_n^2}\sum_{j=1}^n(a_jc_j^{(n)})^2\xi_j^2\longrightarrow K_{\nu,1}^2 \qquad \text{a.e.}.$$

Now we use the method due to Takahashi. (Takahashi [16]) Put $\lambda_n=K_{\nu,1}^{-1}\sqrt{2\log\log A_n^2}$. Making use of $e^x\leqq(1+x)\exp\left(\frac{x^2}{2}+|x|^3\right)$ ($|x|\leqq 1$), taking large enough $r$,

$$\begin{aligned}
&\boldsymbol{E}\left[\exp\left(\frac{\lambda_{p(r)}}{A_{p(r)}}\sum_{j=1}^{p(r)}c_j^{(p(r))}a_j\xi_j-\frac{\lambda_{p(r)}^2}{2A_{p(r)}^2}\sum_{j=1}^{p(r)}\{c_j^{(p(r))}a_j\xi_j\}^2-(1+2\varepsilon)\frac{K_{\nu,1}^2\lambda_{p(r)}^2}{2}\right)\right] \\
&\leqq\exp\left(\frac{\lambda_{p(r)}^3K^3}{A_{p(r)}^3}\sum_{j=1}^{p(r)}|c_j^{(p(r))}a_j|^3-(1+2\varepsilon)\frac{K_{\nu,1}^2\lambda_{p(r)}^2}{2}\right) \\
&\leqq\exp\left(\frac{\lambda_{p(r)}^3K^3}{A_{p(r)}}N^3\max_{j\leq p(r)}|a_j|-(1+2\varepsilon)\frac{K_{\nu,1}^2\lambda_{p(r)}^2}{2}\right)
\end{aligned}$$

$$=\exp(\log\log A^2_{p(r)}o(1)-(1+2\varepsilon)\log\log A^2_{p(r)})$$
$$\leqq K' r^{-1-\varepsilon}.$$

Since this is a term of convergent series, by the Beppo-Levi's theorem we have

$$\lim_{r\to\infty}\lambda^2_{p(r)}\Big(\frac{1}{\lambda_{p(r)}}\int_0^1 X_{p(r)}d\nu-(1+\varepsilon)K^2_{\nu,1}\Big)=-\infty.$$

Thus we have

$$\limsup_{r\to\infty}\frac{\int_0^1 X_{p(r)}d\nu}{\sqrt{2\log\log A^2_{p(r)}}}\leqq K_{\nu,1}\qquad \text{a.s.}$$

For given $n$, take $r$ as $p(r-1)<n\leqq p(r)$. Then

$$\int_0^1\frac{X_n(t)}{\sqrt{2\log\log A_n^2}}\nu(dt)-\int\frac{X_{p(r)}(t)}{\sqrt{2\log\log A^2_{p(r)}}}\nu(dt)$$
$$=\int_0^1\frac{X_n(t)-X_{p(r)}(t)}{\sqrt{2\log\log A_n^2}}\nu(dt)$$
$$+\Big(\frac{1}{\sqrt{2\log\log A_n^2}}-\frac{1}{\sqrt{2\log\log A^2_{p(r)}}}\Big)\int_0^1 X_{p(r)}\nu(dt).$$

This latter term clealy tends to 0 a.s. as $n\to\infty$.

|The former term|

$$\leqq\frac{1}{\sqrt{2\log\log A^2_{p(r-1)}}}\Big(\frac{A_{p(r)}}{A_n}\Big|\int_0^1\Big\{X_{p(r)}\Big(\frac{A_n^2}{A^2_{p(r)}}t\Big)-X_{p(r)}(t)\Big\}\nu(dt)\Big|$$
$$+\Big(\frac{A_{p(r)}}{A_n}-1\Big)\Big|\int_0^1 X_{p(r)}\nu(dt)\Big|\Big).$$

The first part tends to 0 a.s. as $\theta\downarrow 1$ by equi-continuity and the second part also tends to 0 a.s. clearly. Thus we have proved

$$\limsup_{n\to\infty}\frac{\int_0^1 X_n d\nu}{\sqrt{2\log\log A_n^2}}\leqq K_{\nu,1}\qquad \text{a.s.}$$

Next we prove 3) under the condition (2.6) by the method of Révész [18]. First we put

$$Z_r=\sum_{j=1}^{p(r)}a_j c_j^{(p(r+1))}\xi_j.$$

Then

$$\frac{Z_r}{A_{p(r+1)}}=\int_0^1 X_{p(r+1)}\Big(t\wedge\frac{A^2_{p(r)}}{A^2_{p(r+1)}}\Big)d\nu(t),$$

and we have

$$\lim_{r\to\infty}\frac{1}{A^2_{p(r+1)}}\sum_{j=1}^{p(r)}(a_j c_j^{p(r+1)})^2=K^2_{\nu,0}.$$

Making use of this and calculating in the same way as before, we can prove

$$\limsup_{n\to\infty}\frac{Z_n}{\sqrt{2A_{p(n+1)}^2\log\log A_{p(n+1)}^2}}\leqq K_{\nu,\theta} \qquad \text{a.s.}$$

Now we prove for any $\varepsilon>0$,

$$(4.6)\qquad \frac{\sum_{j=p(n)+1}^{p(n+1)} a_j c_j^{p(n+1)}\xi_j}{\sqrt{(2-\varepsilon)A_{p(n+1)}^2\log\log A_{p(n+1)}^2}}\geqq\sqrt{K_{\nu,1}^2-K_{\nu,\theta}^2} \qquad \text{i.o. a.s.}$$

These two formulas imply that for any $\varepsilon>0$,

$$\sum_{j=1}^{p(n+1)} a_j c_j^{p(n+1)}\xi_j\geqq\left(\sqrt{(K_{\nu,}^2-K_{\nu,\theta}^2)(2-\varepsilon)}-\sqrt{K_{\nu,\theta}^2(2+\varepsilon)}\right)$$

$$\times\sqrt{A_{p(n+1)}^2\log\log A_{p(n+1)}^2} \qquad \text{i.o. a.s.}$$

For any $\delta>0$,

$$\sqrt{(K_{\nu,1}^2-K_{\nu,\theta}^2)(2-\varepsilon)}-\sqrt{K_{\nu,\theta}^2(2+\varepsilon)}\geqq\sqrt{(2-\delta)}\,K_{\nu,1}$$

by taking $\theta$ large enough and $\varepsilon$ small enough. Consequently we have

$$\limsup_{n\to\infty}\frac{\int_0^1 X_n d\nu}{\sqrt{2\log\log A_n^2}}\geqq K_{\nu,1} \qquad \text{a.s.}$$

The last part of Theorem 7 is proved. Now we prove (4.6). We introduce the following notations.

$$D_n^2=\sum_{j=p(n)+1}^{p(n+1)}(a_j c_j^{(p(n+1))})^2$$

$$\eta_n=\frac{1}{D_n}\sum_{j=p(n)+1}^{p(n+1)} a_j c_j^{(p(n+1))}\xi_j$$

$$\alpha_n=\prod_{j=p(n)+1}^{p(n+1)}\left(1+\frac{it\,\alpha_j c_j^{p(n+1)}\xi_j}{D_n}\right)$$

$$\beta_n=\frac{1}{D_n^2}\sum_{j=p(n)+1}^{p(n+1)}(a_j c_j^{(p(n+1))}\xi_j)^2$$

$$\varphi_{n,m}(s,t)=\boldsymbol{E}(\exp\{is\eta_n+it\eta_{n+m}\})$$

$$F_{n,m}(x,y)=\boldsymbol{P}\{\eta_n<x,\ \eta_{n+m}<y\}.$$

The next lemma which is the generalization of lemma 1 in [18] will be proved later.

**Lemma 3.** *Suppose that*

$$\frac{|s|^3+|t|^3}{\theta^{n\delta}}\leqq\zeta \qquad (n\geqq N_1)$$

*for some* $\delta\in\left(0,\frac{1}{3}\right)$ *where* $\zeta>0$ *and* $n\in\boldsymbol{N}$ *are constants depending only on* $\theta$.

*Then we have*

$$\left|\varphi_{n,\, m}(s,\, t)-\exp\left(-\frac{s^2+t^2}{2}\right)\right|\leqq C\frac{|s|^3+|t|^3+1}{\theta^{n\delta}}.$$

*where C is a constant depending only on* $\theta$.

We can derive (4.6) by making use of Lemma 3 in the same way as Révész. We state a summary of the method of Révész for convenience. Lemma 3 implies

$$\left|F_{n,m}(x,\, y)-\frac{1}{2\pi}\int_{-\infty}^{x}\int_{-\infty}^{y}\exp\left(-\frac{u^2+v^2}{2}\right)dudv\right|\leqq\frac{a}{\theta^{Bn}} \tag{4.7}$$

for some constant $a>0$ and $B>0$. It is an application of the next theorem due to Sadikova.

**Theorem H** (*Sadikova* [14]). *Let* $F(x, y)$ *and* $G(x, y)$ *be two dimensional distribution functions. Denote the corresponding characteristic functions by* $f(s, t)$ *and* $g(s, t)$. *Suppose G has a bounded density function. Furthermore, set*

$$\tilde{f}(s,\, t)=f(s,\, t)-f(s,\, 0)f(0,\, t)$$

*and*

$$\tilde{g}(s,\, t)=g(s,\, t)-g(s,\, 0)g(0,\, t).$$

*Then*

$$\sup_{x,y}|F(x,\, y)-G(x,\, y)|\leqq C_1\int_{-T}^{T}\int_{-T}^{T}\left|\frac{\tilde{f}(s,\, t)-\tilde{g}(s,\, t)}{st}\right|dsdt$$
$$+C_2\int_{-T}^{T}\left|\frac{f(s,\, 0)-g(s,\, 0)}{s}\right|ds$$
$$+C_3\int_{-T}^{T}\left|\frac{f(0,\, t)-g(0,\, t)}{t}\right|dt+\frac{C_4}{T}$$

*for any* $T>0$ *where* $C_1$, $C_2$, $C_3$ *and* $C_4$ *are positive constants.*

Now setting $A_n=\{\eta_n\geqq\sqrt{(2-\varepsilon)\log\log D_n^2}\}$, (4.6) can be derived from (4.7) by making use of the following extension of second Borel-Cantelli lemma.

**Theorem I** (*Rényi* [11]). *Suppose that events* $A_1$, $A_2$, $\cdots$ *satisfy*

$$\sum_{n=1}^{\infty}\boldsymbol{P}(A_n)=\infty \quad \textit{and} \quad \liminf_{n\to\infty}\frac{\sum_{k=1}^{n}\sum_{j=1}^{n}\boldsymbol{P}(A_k\cap A_j)}{\left(\sum_{j=1}^{n}\boldsymbol{P}(A_j)\right)^2}=1.$$

*Then we have*

$$\boldsymbol{P}(\limsup_{n\to\infty}A_n)=1.$$

Now we only have to prove Lemma 3. We first recall a basic formula.

$$e^{ix}=(1+ix)\exp\left(-\frac{x^2}{2}+r(x)\right) \quad \text{and} \quad |r(x)|\leqq|x|^3 \quad \forall x.$$

Put

$$R_n(t)=\sum_{j=p(n)+1}^{p(n+1)} r\left(t\frac{a_j c_j^{p(n+1)}\xi_j}{D_n}\right).$$

Then we have

$$\begin{aligned}|R_n(t)| &\leqq \frac{|t|^3K^3}{D_n^3}\sum_{j=p(n)+1}^{p(n+1)}|a_j c_j^{\{p(n+1)\}}|^3 \\ &\leqq \frac{|t|^3K^3N}{D_n}\sum_{j=p(n)+1}^{p(n+1)}|a_j| \\ &=|t|^3K^3N\frac{o(A_{p(n+1)}^{1-\delta})}{D_n} \qquad \text{(by (2.6))}.\end{aligned}$$

Thus for large $n$,

$$|R_n(t)|\leqq|t|^3K^3N\frac{1}{A_{p(n+1)}^{\delta}}\leqq\frac{K^3N|t|^3}{\theta^{\delta n}}.$$

Since

$$\exp\{is\eta_n+it\eta_{n+m}\}=\alpha_n(s)\alpha_{n+m}(t)\exp\left(-\frac{s^2\beta_n^2+t^2\beta_{n+m}^2}{2}+R_n(s)+R_{n+m}(t)\right),$$

we have

$$\begin{aligned}&\left|\varphi_{n,m}(s,t)-\exp\left(-\frac{s^2+t^2}{2}\right)\right| \\ &=\left|\boldsymbol{E}\left[\alpha_n(s)\alpha_{n+m}(t)\left(\exp\left\{-\frac{s^2\beta_n^2+t^2\beta_{n+m}^2}{2}+R_n(s)+R_{n+m}(t)\right\}-\exp\left\{-\frac{s^2+t^2}{2}\right\}\right)\right]\right|.\end{aligned}$$

Making use of $|\alpha_n(s)|\leqq\exp(s^2\beta_n^2/2)$, we have

$$\begin{aligned}&\leqq\boldsymbol{E}\left[\left|\exp(R_n(s)+R_{n+m}(t))-\exp\left(\frac{s^2(\beta_n-1)+t^2(\beta_{n+m}-1)}{2}\right)\right|\right] \\ &\leqq\boldsymbol{E}[|\exp(R_n(s)+R_{n+m}(t))-1|] \\ &\quad+\boldsymbol{E}\left[\left|1-\left(\exp\frac{s^2(\beta_n-1)+t^2(\beta_{n+m}-1)}{2}\right)\right|\right].\end{aligned}$$

If $\frac{K^2N}{\theta^{\delta n}}(|t|^3+|s|^3)\leqq1$, then $|R_n(s)+R_{n+m}(t)|\leqq1$. Making use of $|e^x-1|\leqq2|x|$ $(x\leqq1)$, we have

$$\begin{aligned}\boldsymbol{E}[|\exp(R_n(s)+R_{n+m}(t))-1|]&\leqq2(R_n(s)+R_{n+m}(t)) \\ &\leqq\frac{2K^2N}{\theta^{\delta n}}(|t|^3+|s|^3).\end{aligned}$$

Theorem C implies that

$$\boldsymbol{P}\left(|\beta_n-1|\geqq\frac{\sqrt{2}(K^2+1)}{D_n^{2/3}}\right)\leqq2\exp(-D_n^{2/3}).$$

Using this estimate we have

$$E\left[\left|1-\exp\left(\frac{s^2(\beta_n-1)+t^2(\beta_{n+m}-1)}{2}\right)\right|\right]$$

$$\leqq\left|\exp\left(\frac{s^2(K^2+1)}{\sqrt{2}D_n^{2/3}}+\frac{t^2(K^2+1)}{\sqrt{2}D_{n+m}^{2/3}}\right)-1\right|$$

$$+2\left|\exp\left(\frac{(s^2+t^2)(K^2+1)}{2}\right)-1\right|(\exp(-D_n^{2/3})+\exp(-D_{n+m}^{2/3})).$$

There exists a constant $E>0$ and $N_1$ such that for $n>N_1$

$$\frac{s^2(K^2+1)}{\sqrt{2}D_n^{2/3}}+\frac{t^2(K^2+1)}{\sqrt{2}D_{n+m}^{2/3}}\leqq E\frac{s^2+t^2}{\theta^{n/3}}.$$

Thus

$$|\text{The former part}|\leqq 2E\frac{s^2+t^2}{\theta^{n/3}}\leqq 2E\frac{|t|^3+|s|^3+1}{\theta^{n/3}}.$$

On the other hand

$$|\text{The latter term}|\leqq 4\exp\left(\frac{(s^2+t^2)(K^2+1)}{2}-D_n^{2/3}\right)\leqq\frac{D'}{\theta^{n\delta}}.$$

Thus the proof is completed. □

## 5. Proof of Theorem 1, 4, 5, 6 and 8.

*Proof of Theorem* 1. Making use of (2.2), we have

$$P_T^{\frac{1}{B_n}\sum_{j=1}^n a_j\sqrt{2}\cos\lambda_j x}\xrightarrow{w}P^{\frac{1}{B_n}\sum_{j=1}^n a_j\xi_{\sqrt{2}\cos\lambda_j x}}\qquad(T\to\infty)$$

Since $(-\infty,-\varepsilon]\cup[\varepsilon,\infty)$ is a closed set, it holds that

$$\bar{\mu}_R\left\{x\,;\,\left|\frac{1}{B_n}\sum_{j=1}^n a_j\sqrt{2}\cos\lambda_j x\right|\geqq\varepsilon\right\}$$

$$=\limsup_{T\to\infty}P_T\left\{x\,;\,\left|\frac{1}{B_n}\sum_{j=1}^n a_j\sqrt{2}\cos\lambda_j x\right|\geqq\varepsilon\right\}$$

$$\leqq P\left\{\left|\frac{1}{B_n}\sum_{j=1}^n a_j\xi_{\sqrt{2}\cos\lambda_j x}\right|\geqq\varepsilon\right\}$$

Since $\{\lambda_j\}$ satisfies the SS-condition, $\{\xi_{\sqrt{2}\cos\lambda_j x}\}$ is a uniformly bounded EMS. By the weak law of the large number for orthogonal sequence we have

$$\lim_{n\to\infty}P\left\{\left|\frac{1}{B_n}\sum_{j=1}^n a_j\xi_{\sqrt{2}\cos\lambda_j x}\right|\geqq\varepsilon\right\}=0.$$

This completes the proof. □

Put $S_i^*=a_1\xi_{\sqrt{2}\cos\lambda_1 x}+\cdots+a_i\xi_{\sqrt{2}\cos\lambda_i x}$ and define a $C$-valued random variable $X_n^*$ and a $D$-valued random variable $Y_n^*$ as the same as (2.3) and using

$S_i^*$ instead of $S_i$.

*Proof of Theorem* 4. Convergence in $\boldsymbol{C}$ and $\boldsymbol{D}$ are proved in the same way. So we prove only the convergence in $\boldsymbol{C}$. First we proved the following,

$$(5.1)\qquad \boldsymbol{P}_T^{X_n} \xrightarrow{w} \boldsymbol{P}^{X_n^*} \qquad (T\to\infty)$$

A mapping from $\boldsymbol{R}^n$ to $\boldsymbol{C}$ to make a linear interpolation of the subsums of $n$ variables is a continuous mapping. So by (2.2), (5.1) is proved. By the Theorem 2, we hahe

$$(5.2)\qquad \boldsymbol{P}^{X_n^*} \xrightarrow{w} \boldsymbol{W} \qquad (n\to\infty).$$

By (5.1) and (5.2) we have for $A\in\sigma[\boldsymbol{C}]$

$$\boldsymbol{P}^{X_n^*}(A^i)\leqq\liminf_{T\to\infty}\boldsymbol{P}_T^{X_n}(A^i), \qquad \boldsymbol{W}(A^i)\leqq\liminf_{n\to\infty}\boldsymbol{P}^{X_n^*}(A^i).$$

By the definition of the lower relative measure,

$$\liminf_{T\to\infty}\boldsymbol{P}_T^{X_n}(A^i)=\underline{\mu}_R\{\boldsymbol{X}_n\in A^i\}.$$

Thus we have

$$\boldsymbol{W}(A^i)\leqq\liminf_{n\to\infty}\underline{\mu}_R\{\boldsymbol{X}_n\in A^i\}\leqq\liminf_{n\to\infty}\underline{\mu}_R\{\boldsymbol{X}_n\in A\}.$$

Thinking about $\bar{A}^i$, we have

$$\boldsymbol{W}(A^c)\geqq\limsup_{n\to\infty}\bar{\mu}_R\{\boldsymbol{X}_n\in A^c\}\geqq\limsup_{n\to\infty}\bar{\mu}_R\{\boldsymbol{X}_n\in A\}.$$

If $\boldsymbol{W}(\partial A)=0$, we have

$$\lim_{n\to\infty}\underline{\mu}_R\{\boldsymbol{X}_n\in A\}=\lim_{n\to\infty}\bar{\mu}_R\{\boldsymbol{X}_n\in A\}=\boldsymbol{W}(A). \qquad \square$$

*Proof of Theorem* 5. We prove only the part of $\max S_j$. Rest is proved in the same way. We denote by sup the mapping from $\boldsymbol{C}$ to $\boldsymbol{R}$ defined by $\sup(x)=\sup_{t\in I} x(t)$, $x\in\boldsymbol{C}$ $(I=[0, 1])$. Since sup is a cotinuous mapping, we have

$$(5.3)\qquad \boldsymbol{P}_T^{\sup X_n} \xrightarrow{w} \boldsymbol{P}^{\sup X_n^*} \qquad (T\to\infty)$$

and

$$(5.4)\qquad \boldsymbol{P}^{\sup X_n^*} \xrightarrow{w} \boldsymbol{W}^{\sup} \qquad (n\to\infty).$$

Since $\boldsymbol{W}^{\sup}$ has a continuous distribution, we have

$$(5.5)\qquad \lim_{n\to\infty}\boldsymbol{P}^{\sup X_n^*}[0, \alpha)=\lim_{n\to\infty}\boldsymbol{P}^{\sup X_n^*}[0, \alpha]=\boldsymbol{W}^{\sup}[0, \alpha].$$

And by (5.3)

$$(5.6)\qquad \boldsymbol{P}^{\sup X_n^*}[0, \alpha)\leqq\liminf_{T\to\infty}\boldsymbol{P}_T^{\sup X_n}[0, \alpha)$$

$$\leqq \limsup_{T\to\infty} P_T^{\sup X_n}[0, \alpha] \leqq P^{\sup X_n^*}[0, \alpha].$$

By (5.5) and (5.6) we can conclude the last part of the theorem. If $P^{\sup X_n^*}\{\alpha\}=0$ (it is true except at most countably many exception $\alpha$ for all $n$),

$$\lim_{T\to\infty} P_T^{\sup X_n}[0, \alpha] = P^{\sup X_n^*}[0, \alpha].$$

So we have to prove that $P^{\sup X_n^*}\{\alpha\}=0$ for all $\alpha$ under the condition of algebraic independence.

$$P^{\sup X_n^*}\{\alpha\} = P\left\{\max_{j=1}^{n} S_j^* = \alpha A_n\right\} \leqq \sum_{j=1}^{n} P\{S_j^* = \alpha A_n\}.$$

Thus we have to prove that $S_j^*$ has the continuous law, but it is clear because this law is a convolution of continuous laws of $\xi_{\sqrt{2}\cos\lambda_j x}$. □

In the proof of Theorem 6 we use the following Lemma.

**Lemma** 4 (*Cf. Billingsley* [3]). *Let* $P$ *be a probability measure on* $(D, \sigma[D])$ *and* $P_{\pi_t^{-1}}\{0\}=0$ *for* $\mu$*-a.e.* $t$. *Then*

1) $\mathrm{h}: D\to R: \mathrm{h}(x)=\mu\{t\in[0, 1]; x(t)>0\}$ *is* $\sigma[D]/\mathcal{B}$*-measurable.*
2) *Discontinuity set of* h *is* $P$*-null set.*

Proofs of Theorem 6 and 8 are obtained by the same method as others. So we omit the details.

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