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Finite multiplicity theorems for induced representations of semisimpmle Lie groups II

—Applications to generalized Gelfand-Graev representations—

By

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Introduction

This paper is the sccond part of our work on finite multiplicity property for induced representations. We give in this article finite multiplicity theorems for generalized Gelfand-Graev representations of semisimple Lie groups, applying the results of the first part [32] (referred as [I] later on).

Let G be a connected semisimple Lie group with finite center. In [I], we generalized the result of van den Ban [1] on finiteness of multiplicities in the Plancherel formula associated to a semisimple symmetric space, developing the theory of spherical functions in a much more general setting. Furthermore, we gave there nice sufficient conditions for an induced representation of G to have finite multiplicity property. These criterions enable us to understand, in a unified manner, many finite multiplicity theorems for induced representations, obtained in different situations ([2], [8], [26], etc.).

Among others, we are interested in the following important example. Let $G=KA_pN_m$ be an Iwasawa decomposition of G. Then N_m is a maximal unipotent subgroup of G. We showed in [I] that the induced representation $\mathrm{Ind}_{N_m}^G(\xi)$ (differentiably ($=C^\infty$-) or unitarily ($=L^2$-) induced) has finite multiplicity property for any one-dimensional representation (=character) ξ of N_m (see [I, 4.2]). As wes suggested in Appendix of [I], the study of such an induced representation is reduced, to a large extent, to that of $\mathrm{Ind}_{N_m}^G(\xi)$ with a non-degenerate character ξ. The latter representation is called a *Gelfand-Graev representation* (=GGR for short). According to Shalika [26], the GGRs have a remarkable property: the unitarily induced GGRs are of multiplicity one if G is linear and quasi-split (cf. [I, Theorem 4.5]). These GGRs have been playing an important role not only in the representation theory itself but also in the theory of automorphic forms. (For the historical background of the study of GGRs, we refer to [31, 0.1].)

But, in the representation theoretical point of view, GGRs are not large enough to understand all the irreducible representations through them. In other words, there exist numbers of irreducible representations of G that never "occur"

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in GGRs (see e.g., [9]). It is by this defect that a good generalization of GGRs had been desired.

In this direction, N. Kawanaka [13] constructed, for reductive algebraic groups over a finite (or a local) field, a series of induced representations parametrized by the set of nilpotent classes of the Lie algebras, by making use of the Dynkin-Kostant theory on the nilpotent classes. Such a representation is now called *generalized* Gelfand-Graev representation (=GGGR), since Kawanaka's representations associated with the regular nilpotent classes coincide with the original GGRs. He himself has been studying mainly the GGGRs of finite reductive groups using their characters, and has obtained nice results (e.g., a complete proof of the Ennola conjecture) that show the utility of GGGRs in the theory of group representations (see [13], [14], [15]).

We think that these GGGRs are the desired generalization of the original GGRs which enables us to understand all the irreducible representations in connection with the nilpotent orbits. But, unfortunately, the GGGRs of semisimple Lie groups are in general far from being of multiplicity finite, contrary to the case of GGRs. This comes from the fact that the GGGRs are induced from representations of unipotent subgroups generally smaller than N_m.

So, in order to reduce infinite multiplicities to be finite (or to be one) if possible, we consider here a variant of GGGRs which we call *reduced* GGGRs (=RGGGRs). In the present article, we give finite multiplicity theorems for some important classes of RGGGRs (Theorems 6.5 and 6.6).

To be more precise, suppose that G be a simple Lie group of hermitian type. Put $l=\mathrm{rank}(G/K)$. Then we can construct explicitly $(l+1)$-number of nilpotent $\mathrm{Ad}(G)$-orbits ω_i $(\omega_i=-\omega_{l-i})$ in $\mathfrak{g}=\mathrm{Lie}\, G$ with an important property: the direct sum $\bigoplus_{0\leq i\leq l} L^2\text{-}\Gamma_i$ of the unitarily induced GGGRs $L^2\text{-}\Gamma_i$ associated with ω_i is quasi-equivalent to the regular representation of G (see Theorem B below). We prove finite multiplicity property for RGGGRs coming from these Γ_i's. Every discrete series representation D of G is embedded into some $L^2\text{-}\Gamma_i$. We can call this embedding a "Whittaker model" for D.

Among these GGGRs Γ_i $(0\leqq i\leqq l)$, Γ_0 and Γ_l are closely related to the holomorphic and anti-holomorphic discrete series representations respectively. We have multiplicity one theorem (Theorem 6.9) for RGGGRs associated with ω_i $(i=0, l)$. These two topics will be studied in the subsequent paper [33].

Now let us explain the contents of this article in more detail.

In §1, following Kawanaka, we define the GGGRs of semisimple Lie groups. Then we clarify the relationship between the original GGRs and the generalized ones.

Let G be a semisimple Lie group as before. Let X be a non-zero nilpotent element of the Lie algebra $\mathfrak{g}$ of G. Denote by ω the $\mathrm{Ad}(G)$-orbit through X: $\omega=\mathrm{Ad}(G)X$. We want to define the GGGR Γ_ω associated with ω. To do so, it is convenient to proceed as follows. First, by virtue of the Jacobson-Morozov theorem, we can take an $\mathfrak{sl}_2$-triplet (X, H, Y) in $\mathfrak{g}$ containing X:

$$[H, X]=2X, \quad [H, Y]=-2Y, \quad [X, Y]=H. \tag{0.1}$$

Assume that H is a dominant element of $\mathfrak{a}_p \equiv \mathrm{Lie}\, A_p$ with respect to a fixed order of the root system of $(\mathfrak{g}, \mathfrak{a}_p)$. Then H is determined uniquely by the nilpotent G_C-orbit $\mathfrak{o} \equiv G_C \cdot X \subseteq \mathfrak{g}_C$ through X, where G_C denotes the adjoint group of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$. So we denote H by $H(\mathfrak{o})$. One has a canonical gradation of $\mathfrak{g}$ by $\mathrm{ad}\, H(\mathfrak{o})$: $\mathfrak{g} = \bigoplus_{s \in Z} \mathfrak{g}(s)_{\mathfrak{o}}$, where $\mathfrak{g}(s)_{\mathfrak{o}}$ is the s-eigenspace for $\mathrm{ad}\, H(\mathfrak{o})$. Put $\mathfrak{p}_{\mathfrak{o}} = \bigoplus_{s \geq 0} \mathfrak{g}(s)_{\mathfrak{o}}$, $\mathfrak{l}_{\mathfrak{o}} = \mathfrak{g}(0)_{\mathfrak{o}}$ and $\mathfrak{n}_{\mathfrak{o}} = \bigoplus_{s \geq 1} \mathfrak{g}(s)_{\mathfrak{o}}$. Then $\mathfrak{p}_{\mathfrak{o}}$ is a parabolic subalgebra of $\mathfrak{g}$, and $\mathfrak{p}_{\mathfrak{o}} = \mathfrak{l}_{\mathfrak{o}} \oplus \mathfrak{n}_{\mathfrak{o}}$ gives its Levi decomposition. Denote by $P_{\mathfrak{o}} = L_{\mathfrak{o}} N_{\mathfrak{o}}$ with $L_{\mathfrak{o}} = \theta P_{\mathfrak{o}} \cap P_{\mathfrak{o}}$, the corresponding decomposition on the group level, where θ is a Cartan involution of G compatible with the decomposition $G = KA_pN_m$.

Define a linear form X^* on $\mathfrak{n}_{\mathfrak{o}}$ via

$$\langle X^*, Z \rangle = B(Z, \theta X) \qquad \text{for} \quad Z \in \mathfrak{n}_{\mathfrak{o}}, \tag{0.2}$$

where B is the Killing form of $\mathfrak{g}$. Let ξ_X denote the irreducible unitary representation of the nilpotent Lie group $N_{\mathfrak{o}}$ corresponding to X^* through the Kirillov correspondence (see 1.3). We consider the induced representation $\Gamma_{(X,H,Y)} \equiv \mathrm{Ind}_{N_{\mathfrak{o}}}^{G}(\xi_X)$ (unitarily or differentiably induced). Then the equivalence class of $\Gamma_{(X,H,Y)}$ depends only on the nilpotent $\mathrm{Ad}(G)$-orbit $\omega \equiv \mathrm{Ad}(G)X \subseteq \mathfrak{o}$ through X. So we may express it without any confusion as

$$\Gamma_{\omega} \equiv \Gamma_{(X,H,Y)} = \mathrm{Ind}_{N_{\mathfrak{o}}}^{G}(\xi_X). \tag{0.3}$$

We can thus attach to each nilpotent class ω in $\mathfrak{g}$ an induced representation Γ_{ω}, which is called the *generalized Gelfand-Graev representation* associated to ω. Note that $\mathfrak{o} \cap \mathfrak{g}$ does not necessarily consist of a single $\mathrm{Ad}(G)$-orbit. Nevertheless the unipotent subgroup $N_{\mathfrak{o}}$ is common for any $\mathrm{Ad}(G)$-orbit contained in $\mathfrak{o} \cap \mathfrak{g}$. This fact is useful for later duscussion.

In § 2, keeping in mind the Mackey theory for the representations of group extension, we construct reduced GGGRs, a variant of GGGRs. Consider the GGGR $\Gamma_{\omega} = \mathrm{Ind}_{N_{\mathfrak{o}}}^{G}(\xi_X)$ constructed above. The Levi subgroup $L_{\mathfrak{o}}$ acts on $N_{\mathfrak{o}}$, hence it acts also on the unitary dual $\hat{N}_{\mathfrak{o}}$ of $N_{\mathfrak{o}}$ in a canonical way. Let $H_{\mathfrak{o}}(X)$ denote the stabilizer of the equivalence class $[\xi_X] \in \hat{N}_{\mathfrak{o}}$ of ξ_X in $L_{\mathfrak{o}}$. Then $H_{\mathfrak{o}}(X)$ is a reductive subgroup of $L_{\mathfrak{o}}$, and it coincides with the centralizer $Z_{L_{\mathfrak{o}}}(\theta X)$ of θX in $L_{\mathfrak{o}}$ (Lemma 2.1). The representation ξ_X can be extended canonically to a unitary (projective, in general) representation $\hat{\xi}_X$ of the semidirect product subgroup $H_{\mathfrak{o}}(X) N_{\mathfrak{o}} \subseteq P_{\mathfrak{o}}$ acting on the same Hilbert space.

We give in Proposition 2.2 a good sufficient condition for ξ_X to be extendable to a genuine (not just projective) representation $\tilde{\xi}_X$. Thanks to this criterion, we find out that ξ_X admits an extension to a genuine $\tilde{\xi}_X$ for any case studied in §§ 3-6. So, we assume here such property for ξ_X for the sake of simplicity.

For an irreducible representation c of $H_{\mathfrak{o}}(X)$, consider the induced representation

$$\Gamma_{\omega}(c) \equiv \mathrm{Ind}_{H_{\mathfrak{o}}(X)N_{\mathfrak{o}}}^{G}(\tilde{c} \otimes \tilde{\xi}_X) \qquad \text{with} \quad \tilde{c} = c \otimes 1_{N_{\mathfrak{o}}}, \tag{0.4}$$

where $1_{N_{\mathfrak{o}}}$ is the trivial character of $N_{\mathfrak{o}}$. We call $\Gamma_{\omega}(c)$ the *reduced* GGGR

(=RGGGR) associated with (ω, c). (See [15, 2.5].) The unitary GGGR $L^2\text{-}\Gamma_\omega$ is decomposed into a direct integral of RGGGRs $L^2\text{-}\Gamma_\omega(c)$, $c\in H_o^\wedge(X)$=the unitary dual of $H_o(X)$ (see (2.8)).

In case where the parabolic subgroup $P_o=L_oN_o$ is maximal, it often happens that $(L_o, H_o(X))$ has a structure of so-called reductive symmetric pair (at least on the level of Lie algebras). In such a case, one can apply the results of [I] to RGGGRs $\Gamma_\omega(c)$. In §§3-6, we give the most important example of such case for simple Lie groups of hermitian type, and prove finite multiplicity theorems for RGGGRs.

In §3, we construct explicitly nilpotent orbits ω_i $(0\leqq i\leqq l=\dim\mathfrak{a}_p=\operatorname{rank}(G/K))$ that define important kind of GGGRs $\Gamma_i=\Gamma_{\omega_i}$, closely related to the regular representation of G (see supra). Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ determined by θ. From now on, we always assume that G be a connected simple Lie group of hermitian type, with finite center. Then $\mathfrak{g}$ has a compact Cartan subalgebra $\mathfrak{t}\subseteqq\mathfrak{k}$. Σ denotes the root system of $(\mathfrak{g}_C, \mathfrak{t}_C)$. Choose a positive system Σ^+ of Σ compatible with the G-invariant complex structure on G/K. For each $\gamma\in\Sigma$, we can select a non-zero root vector X_γ of γ in such a way that

$$X_\gamma-X_{-\gamma},\quad \sqrt{-1}(X_\gamma+X_{-\gamma})\in\mathfrak{k}\oplus\sqrt{-1}\mathfrak{p},\quad\text{and}\quad [X_\gamma, X_{-\gamma}]=H'_\gamma.$$

Here H'_γ is the element of $\sqrt{-1}\mathfrak{t}$ corresponding to the co-root $\gamma^\vee=2\gamma/\langle\gamma,\gamma\rangle$ through the Killing form of $\mathfrak{g}_C$.

Let $(\gamma_1, \gamma_2, \cdots, \gamma_l)$ be a fundamental sequence of non-compact positive roots (see 3.1 for the precise definition), and take $\mathfrak{a}_p$ as $\mathfrak{a}_p=\sum_{1\leq k\leq l}\boldsymbol{R}H_k$ with $H_k=X_{\gamma_k}+X_{-\gamma_k}$.

Define nilpotent orbits ω_i $(0\leqq i\leqq l)$ in $\mathfrak{g}$ as follows. For $1\leqq k\leqq l$, set $E_k=2^{-1}\sqrt{-1}(H'_{\gamma_k}-X_{\gamma_k}+X_{-\gamma_k})$. Then E_k is a nilpotent element of $\mathfrak{g}$ corresponding to $-\sqrt{-1}X_{\gamma_k}$ through the Cayley transform $\mu\in G_C$ in (3.8). For $0\leqq i\leqq l$, ω_i is defined to be the nilpotent $\operatorname{Ad}(G)$-orbit through $A[i]\equiv-\sum_{k\leq i}E_k+\sum_{m>i}E_m$: ω_i $\operatorname{Ad}(G)\cdot A[i]$. Then, among these orbits ω_i, there is the following relationship.

Theorem A (Theorem 3.13). *Let* $\mathfrak{o}$ *denote the nilpotent* G_C*-orbit through* $\sum_{1\leq k\leq l}X_{\gamma_k}$. *Then, the intersection* $\mathfrak{o}\cap\mathfrak{g}$ *splits into a disjoint union of nilpotent* $\operatorname{Ad}(G)$*-orbits* ω_i $(0\leqq i\leqq l)$: $\mathfrak{o}\cap\mathfrak{g}=\coprod_{0\leq i\leq l}\omega_i$.

In the proof of this theorem, we utilize reductive symmetric pairs $(\mathfrak{l}_o, \mathfrak{h}^i)$ with $\mathfrak{h}^i\equiv\operatorname{Lie}(H_o(A[i]))$ $(0\leqq i\leqq l)$ that play important roles also in the succedding discussion.

In §4, we study fundamental properties for (reduced) generalized Gelfand-Graev representations associated with ω_i $(0\leqq i\leqq l)$. Consider the GGGRs

$$\Gamma_i=\Gamma_{\omega_i}=\operatorname{Ind}_N^G(\xi_i)\qquad\text{with}\quad \xi_i\equiv\xi_{A[i]},\tag{0.5}$$

where $N\equiv N_o$. Using the results of Duflo-Raïs [5] and Nomura [20] on the Plancherel theorem for certain classes of exponential solvable Lie groups (including the Iwasawa subgroup A_pN_m of our G), we can prove the following equivalence of unitary representations of G.

Theorem B (Theorem 4.2). *The left regular representation λ_G of G is decomposed as*

$$\lambda_G \cong \bigoplus_{i=0}^{l} [\infty] \cdot L^2\text{-}\Gamma_i, \tag{0.6}$$

where $[\infty] \cdot L^2\text{-}\Gamma_i$ means the infinite multiple of the unitary GGGR $L^2\text{-}\Gamma_i$.

This theorem shows the importance of these kinds of GGGRs.

Now let us construct the RGGGRs associated with ω_i. For this purpose, we prove, using Proposition 2.2, that the irreducible representation ξ_i of N extends canonically to a genuine representation $\tilde{\xi}_i$ of the semidirect product subgroup $H^i N \subseteqq P_0$, where $H^i \equiv H_0(A[i])$ is the stabilizer of $[\xi_i] \in \hat{N}$ in L_0 (see Theorem 4.6). This enables us to define RGGGRs corresponding to ω_i as

$$\Gamma_i(c) = \Gamma_{\omega_i}(c) = \mathrm{Ind}_{H^i N}^{G}(\tilde{c} \otimes \tilde{\xi}_i) \qquad \text{with} \quad \tilde{c} = c \otimes 1_N \tag{0.7}$$

for irreducible representations c of H^i.

We list up here several facts which are useful for applying the results of [I] to RGGGRs $\Gamma_i(c)$.

(1) The subgroup N is an at most two-step nilpotent Lie group, and it is canonically diffeomorphic to the Šilov boundary of the Siegel domain which realizes G/K. Moreover, N is abelian if and only if G/K is holomorphically equivalent to a tube domain.

(2) The representations ξ_i $(0 \leqq i \leqq l)$ are all one-dimensional or all infinite-dimensional, according as N is abelian or not.

(3) $H^0 = H^l = K \cap L$, a maximal compact subgroup of L (Lemma 3.5(3)).

(4) For any $0 \leqq i \leqq l$, (L, H^i) has a structure of reductive symmetric pair at least on the level of Lie algebras (Lemma 3.15).

(5) H^i contains $M \equiv Z_K(\mathfrak{a}_p)$ for any i (which follows from Lemma 3.5(3)).

(6) Making use of another type of realization ρ_i of ξ_i, so-called Fock model, one can describe explicitly a genuine extension $\tilde{\xi}_i | (K \cap H^i) N \cong \tilde{\rho}_i | (K \cap H^i) N$ of $\xi_i \cong \rho_i$ (Theorem 4.13).

In § 5, we study the representations $\psi_i(c) \equiv (c | M) \otimes (\tilde{\rho}_i | M)$ of the compact group M, for irreducible representations of c of H^i. In order to apply the results of [I] to RGGGRs $\Gamma_i(c)$, we prove finite multiplicity property for $\psi_i(c)$, by examing separately each case of simple Lie groups of hermitian type. The results of § 5 are summarized in Theorem 5.17.

In the last section, § 6, we give finite multiplicity theorems for RGGGRs $\Gamma_i(c)$, using the results of [I] and Theorem 5.17.

Firstly, for RGGGRs $C^\infty\text{-}\Gamma_i(c)$ induced in C^∞-context, our main result is stated as follows.

Theorem C (Theorem 6.5). *Let G be a connected simple Lie group with* *finite center. Suppose that G/K is a hermitian symmetric space. Consider the nilpotent* $\mathrm{Ad}(G)$*-orbits* ω_i $(0\leqq i\leqq l=\mathrm{rank}(G/K))$ *in* $\mathfrak{g}$ *through the points* $A[i]$. *For an irreducible admissible representation* c *of* H^i, *let* C^∞-$\Gamma_i(c)=C^\infty$-$\mathrm{Ind}_{H^iN}^G$ $(\hat{c}\otimes\tilde{\xi}_i)$ *(induced in* C^∞*-context) be the RGGGR associated with* (ω_i, c). *Then one has*

(1) C^∞-$\Gamma_i(c)$ *has finite multiplicity property for any* $0\leqq i\leqq l$ *and any finite-dimensional* c.

(2) *Suppose that* $G=SU(p, l)$ $(p\geqq l)$. *Let* $c=c(\Phi)$ *be the irreducible admissible representation of* $H^i\cong U(l-i, i)\times SU(p-l)$ $(p>l)$; $H^i\cong SU(l-i, i)$ $(p=l)$, *with highest weight* $\Phi\in\sqrt{-1}\mathfrak{t}^{+*}$ *(see* 5.3.3 *for the precise definition). Then, the RGGGR* C^∞-$\Gamma_i(c)$ *is of multiplicity finite.*

Secondly, we obtain our main result for unitarily induced RGGGRs L^2-$\Gamma_i(c)$ as follows.

Theorem D (Theorem 6.6). (1) *If* $i=0$ *or* l, *then all the RGGGRs* L^2-$\Gamma_i(c)$ $(c\in(K\cap L)^\wedge)$ *have finite multiplicity property. Here* $(K\cap L)^\wedge$ *denotes the unitary dual of the compact group* $K\cap L=H^0=H^l$.

(2) *Suppose that* G/K *reduces to a tube domain. Then, the unitary representation* L^2-$\Gamma_i(c)$ *is of multiplicity finite for any* i *and any finite-dimensional unitary representation* c *of* H^i.

Theorems C and D are the main results of this paper.

The original non-reduced GGGR Γ_i is far from being of multiplicity finite, and its study is reduced to that of RGGGRs $\Gamma_i(c)$. For $i=0$ or l, every $\Gamma_i(c)$, $c\in(K\cap L)^\wedge$, is of multiplicity finite. Futhermore, some of these RGGGRs have multiplicity free property (Theorem 6.9), which will be proved in the subsequent paper [33].

If G is of real rank one, or equivalently $\mathfrak{g}\cong\mathfrak{su}(n, 1)$ $(n\geqq 1)$, then the left regular representation λ_G is decomposed into a direct sum of RGGGRs as

$$\lambda_G\cong[\infty]\cdot\{\bigoplus_{c\in(K\cap L)^\wedge}(L^2\text{-}\Gamma_0(c)\oplus L^2\text{-}L_0(c)^*)\}, \tag{0.8}$$

where L^2-$\Gamma_0(c)^*$ is the representation of G contragredient to L^2-$\Gamma_0(c)$. And the constituents L^2-$\Gamma_0(c)$ and L^2-$\Gamma_0(c)^*$ are all multiplicity finite.

In the present paper, we concentrate our attension only on finiteness of multiplicities in RGGGRs. Nevertheless, our method is applicable also for estimating multiplicities in RGGGRs. More precisely, developing the argument in §5, one can describe multiplicities in the representations $\psi_i(c)$ of M explicitly. Such a description together with the results of [I] gives us a nice upper bound for multiplicities in RGGGRs $\Gamma_i(c)$.

At last, we propose here a very interesting problem. Notice that our simple Lie groups admit discrete series representations. In view of Theorem B, any discrete series representation occurs in a GGGR L^2-Γ_i for some i. Our problem is as follows.

Problem EDS. Describe the embeddings of discrete series representations into the GGGRs Γ_i ($0 \leqq i \leqq l$).

In case of $i=0$ or l, this problem is reduced to the following

Problem EDSbis. Describe the embeddings of discrete series representations into reduced generalized Gelfand-Graev representations $\Gamma_i(c)$ ($i=0, l$; $c \in (K \cap L)^{\wedge}$).

We will study these problems in the forthcoming paper [33].

If this paper appears long, it is largely because we have tried to make it as self-contained as possible.

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§ 1. Generalized Gelfand-Graev representations

Let G be a connected real semisimple Lie group with finite center, and $\mathfrak{g}$ be the Lie algebra of G. The purpose of this section is to define, after Kawanaka, a series of induced representations Γ_ω of G indexed by the nilpotent $\mathrm{Ad}(G)$-orbits ω in $\mathfrak{g}$. Γ_ω is called the generalized Gelfand-Graev representation (=GGGR) associated with ω.

Kawanaka introduced in [13] (see also [14], [15]) the GGGRs of reductive algebraic groups over various fields. Since his articles are intended to study the GGGRs for finite reductive groups, the definition of GGGR is given for that case exactly. Therefore, for our real case, we need to (and do) translate his construction of the GGGRs, and give the precise definition.

For this purpose, we prepare in 1.1 and 1.2 the Dynkin-Kostant theory and some of its consequences about the classification of nilpotent classes of a complex semisimple Lie algebra or a real form of it. In 1.3, a brief survey is given on the Kirillov orbit method for representations of nilpotent Lie groups. After these preparations, we give in 1.4 the definition of the GGGRs of G exactly. The subsections 1.5 and 1.6 are devoted to clarifying the relation between the original Gelfand-Graev representations and the generalized ones.

1.1. The Dynkin-Kostant theory on nilpotent classes.

Here, let $\mathfrak{g}_C$ be a complex semisimple Lie algebra. Denote by G_C the adjoint group of $\mathfrak{g}_C$. Then, according to Dynkin and Kostant, the nilpotent G_C-orbits in $\mathfrak{g}_C$ are parametrized by the weighted Dynkin diagrams. As our survey, let us begin with the following

Definition 1.1. If three non-zero elements X, H and Y in $\mathfrak{g}_C$ satisfy the bracket relations

$$[H, X]=2X, \quad [H, Y]=-2Y, \quad [X, Y]=H, \tag{1.1}$$

then $\mathcal{T}=(X, H, Y)$ is said to be an $\mathfrak{sl}_2$-*triplet* in $\mathfrak{g}_C$. Moreover, H (resp. X, Y) is called the *semisimple* (resp. *nilpositive, nilnegative*) element of $\mathcal{T}$.

For any $\mathfrak{sl}_2$-triplet $\mathcal{T}=(X, H, Y)$, it follows from (1.1) that $\mathfrak{g}_C(\mathcal{T})\equiv \boldsymbol{C}X\oplus\boldsymbol{C}H\oplus\boldsymbol{C}Y$ forms a Lie subalgebra of $\mathfrak{g}_C$ isomorphic to $\mathfrak{sl}_2(\boldsymbol{C})$. Taking into account the adjoint representation of $\mathfrak{g}_C(\mathcal{T})$ on $\mathfrak{g}_C$, one finds out that H is a semisimple element of $\mathfrak{g}_C$ and that the eigenvalues of ad H on $\mathfrak{g}_C$ are all integers. Moreover, X and Y are nilpotent elements of $\mathfrak{g}_C$. Conversely, for any given nilpotent element of $\mathfrak{g}_C$, the Jacobson-Morozov theorem tells us

Theorem 1.2 (Jacobson-Morozov, Kostant [17]). *Let $X\in\mathfrak{g}_C$ be a non-zero nilpotent element. Then there exists an $\mathfrak{sl}_2$-triplet (X, H, Y) in $\mathfrak{g}_C$ containing X as nilpositive element. Furthermore, if X lies in a real form $\mathfrak{g}$ of $\mathfrak{g}_C$, both H and Y can be chosen from $\mathfrak{g}$.*

Let $\mathcal{T}=(X, H, Y)$ be an $\mathfrak{sl}_2$-triplet in $\mathfrak{g}_C$. Set $\mathfrak{g}_C^X=(\operatorname{ad} X)\mathfrak{g}_C\cap\mathfrak{z}_{\mathfrak{g}_C}(X)$, where $\mathfrak{z}_{\mathfrak{g}_C}(X)$ is the centralizer of X in $\mathfrak{g}_C$. If one denotes by $\mathfrak{g}_C(s)$ $(s\in\boldsymbol{Z})$ the s-eigenspace for ad H on $\mathfrak{g}_C$, $\mathfrak{g}_C^X$ is expressed as

$$\mathfrak{g}_C^X=\mathfrak{z}_{\mathfrak{g}_C}(X)\cap\{\bigoplus_{s\geq 1}\mathfrak{g}_C(s)\} \tag{1.2}$$

thanks to the finite-dimensional representation theory of $\mathfrak{g}_C(\mathcal{T})\cong\mathfrak{sl}_2(\boldsymbol{C})$. This implies that $\mathfrak{g}_C^X$ has a structure of nilpotent Lie sugalgebra of $\mathfrak{g}_C$. Let G_C^X denote the analytic subgroup of G_C corresponding to $\mathfrak{g}_C^X$. The set of $\mathfrak{sl}_2$-triplets containing X as nilpositive element are parametrized as follows.

Proposition 1.3 [17, Theorem 3.6]. *Let $\mathcal{T}=(X, H, Y)$ be an $\mathfrak{sl}_2$-triplet in $\mathfrak{g}_C$. Then the map*

$$G_C^X\ni g\longmapsto g\mathcal{T}\equiv(X, gH, gY) \tag{1.3}$$

sets up a bijective correspondence from G_C^X to the set of all $\mathfrak{sl}_2$-triplets in $\mathfrak{g}_C$ containing X as nilpositive element.

Now let $\mathfrak{j}_C$ be a Cartan subalgebra of $\mathfrak{g}_C$. Denote by $\Delta=\Delta(\mathfrak{g}_C, \mathfrak{j}_C)$ the root system of $\mathfrak{g}_C$ with respect to $\mathfrak{j}_C$. Take a positive system Δ^+ of Δ. $\tilde{\Pi}$ will denote the set of simple roots in Δ^+.

Theorem 1.4 ([6], [17]). *Let $\mathfrak{o}$ be a non-trivial nilpotent orbit in $\mathfrak{g}_C$. Then,*

(1) *there exists a unique dominant (with respect to Δ^+) element $H(\mathfrak{o})\in\mathfrak{j}_C$ such that $H(\mathfrak{o})$ is the semisimple element of an $\mathfrak{sl}_2$-triplet $(X, H(\mathfrak{o}), Y)$ with $X\in\mathfrak{o}$.*

(2) *Define a function $f_\mathfrak{o}$ on $\tilde{\Pi}$ by $f_\mathfrak{o}(\alpha)=\alpha(H(\mathfrak{o}))$ $(\alpha\in\tilde{\Pi})$. Then, $f_\mathfrak{o}(\alpha)=0$, 1 or 2 for any $\alpha\in\tilde{\Pi}$.*

(3) *The map $\mathfrak{o}\mapsto f_\mathfrak{o}$ gives a one-to-one correspondence from the set of non-trivial nilpotent classes of $\mathfrak{g}_C$ into the set of functions on $\tilde{\Pi}$ with values in $\{0, 1, 2\}$.*

Let D be the Dynkin diagram of the root system $(\Delta, \tilde{\Pi})$. For a nilpotent class o, the weighted Dynkin diagram $D(\mathrm{o})$ is defined as the graph D with the number $f_{\mathrm{o}}(\alpha)$ attached to the node corresponding to $\alpha \in \tilde{\Pi}$. Theorem 1.4(3) tells us that the weighted Dynkin diagrams $D(\mathrm{o})$ parametrize the set of nilpotent G_C-orbits o $(\neq (0))$.

For the later use, we prepare some additional notations and a lemma. Now let

$$\mathfrak{g}_C = \bigoplus_{s\in \boldsymbol{Z}} \mathfrak{g}_C(s)_{\mathrm{o}} \tag{1.4}$$

be the eigenspace decomposition of $\mathfrak{g}_C$ with respect to $\mathrm{ad}\, H(\mathrm{o})$, where $\mathfrak{g}_C(s)_{\mathrm{o}}$ denotes the s-eigenspace. Then we see easily that

$$[\mathfrak{g}_C(s)_{\mathrm{o}}, \mathfrak{g}_C(t)_{\mathrm{o}}] \subseteq \mathfrak{g}_C(s+t)_{\mathrm{o}} \qquad \text{for} \quad s, t \in \boldsymbol{Z}, \tag{1.5}$$

which implies in particular that $\mathfrak{g}_C(0)_{\mathrm{o}}$ is a Lie subalgebra of $\mathfrak{g}_C$. Furthermore, if $G_C(0)_{\mathrm{o}}$ denotes the analytic subgroup of G_C with Lie algebra $\mathfrak{g}_C(0)_{\mathrm{o}}$, then $\mathfrak{g}_C(s)_{\mathrm{o}}$ is stable under $G_C(0)_{\mathrm{o}}$ for every $s \in \boldsymbol{Z}$.

Define a subset $\hat{\mathfrak{g}}_C(2)_{\mathrm{o}}$ of $\mathfrak{g}_C(2)_{\mathrm{o}}$ as

$$\hat{\mathfrak{g}}_C(2)_{\mathrm{o}} = \{V \in \mathfrak{g}_C(2)_{\mathrm{o}};\ [V, \mathfrak{g}_C(0)_{\mathrm{o}}] = \mathfrak{g}_C(2)_{\mathrm{o}}\}. \tag{1.6}$$

If $(X, H(\mathrm{o}), Y)$ is an $\mathfrak{sl}_2$-triplet, then X belongs to $\hat{\mathfrak{g}}_C(2)_{\mathrm{o}}$. Moreover one has

Lemma 1.5 [17, §4]. (1) *$\hat{\mathfrak{g}}_C(2)_{\mathrm{o}}$ is an open, dense and connected subset of $\mathfrak{g}_C(2)_{\mathrm{o}}$, and it forms a single $G_C(0)_{\mathrm{o}}$-orbit in $\mathfrak{g}_C(2)_{\mathrm{o}}$.*

(2) *The nilpotent class* o *is expressed as* $\mathrm{o} = G_C \hat{\mathfrak{g}}_C(2)_{\mathrm{o}}$.

1.2. Nilpotent classes of real semisimple Lie algebras.

Let G be a connected semisimple Lie group with finite center. In this subsection, we concern ourselves with the classification of nilpotent $\mathrm{Ad}(G)$-orbits in $\mathfrak{g} = \mathrm{Lie}\, G$ in connection with that of such G_C-orbits in $\mathfrak{g}_C \equiv \mathfrak{g} \otimes_{\boldsymbol{R}} \boldsymbol{C}$. The latter was expounded in 1.1.

Now let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_p \oplus \mathfrak{n}_m$ be an Iwasawa decomposition of $\mathfrak{g}$. Take a maximal abelian subalgebra $\mathfrak{h}^+$ of $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}_p)$. Here, for a subgroup H of G and Lie subalgebras $\mathfrak{h}$ and $\mathfrak{r}$ of $\mathfrak{g}$, we denote by $Z_H(\mathfrak{r})$ (resp. $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{r})$) the centralizer of $\mathfrak{r}$ in H (resp. in $\mathfrak{h}$). Then, $\mathfrak{j} \equiv \mathfrak{h}^+ \oplus \mathfrak{a}_p$ is a maximally split Cartan subalgebra of $\mathfrak{g}$. Let $\Lambda = \Lambda(\mathfrak{g}, \mathfrak{a}_p)$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}_p$. Choose a positive system Λ^+ of Λ so that $\mathfrak{n}_m = \sum_{\lambda \in \Lambda^+} \mathfrak{g}(\mathfrak{a}_p; \lambda)$, where $\mathfrak{g}(\mathfrak{a}_p; \lambda)$ is the root space of a root λ. Π denotes the set of simple roots in Λ^+. We keep to the notations in 1.1 for the complexification $\mathfrak{g}_C$ and $\mathfrak{j}_C$ of $\mathfrak{g}$ and $\mathfrak{j}$ respectively, provided that the positive system Δ^+ of $\Delta = \Delta(\mathfrak{g}_C, \mathfrak{j}_C)$ is chosen to be compatible with Λ^+: $(\Delta^+ | \mathfrak{a}_p) \cup (0) = \Lambda^+ \cup (0)$.

Lemma 1.6. *For any nilpotent G_C-orbit in $\mathfrak{g}_C$, the element $H(\mathrm{o}) \in \mathfrak{j}_C$ in Theorem* 1.4(1) *lies in $\sqrt{-1}\,\mathfrak{h}^+ \oplus \mathfrak{a}_p$. Furthermore, $H(\mathrm{o}) \in \mathfrak{a}_p$ if and only if the orbit* o *intersects with* $\mathfrak{g}$.

Proof. The first assertion follows from Theorem 1.4(2) and the fact that $\sqrt{-1}\mathfrak{h}^{+}\oplus\mathfrak{a}_p=\{H\in\mathfrak{j}_C;\ \alpha(H)\in \boldsymbol{R}$ for all $\alpha\in\Delta\}$.

Let us prove the second statement. Suppose that $\mathfrak{g}\cap\mathfrak{o}$ is non-empty, and take an $X\in\mathfrak{g}\cap\mathfrak{o}$. From Theorem 1.2 there exists an $\mathfrak{sl}_2$-triplet (X, H, Y) in $\mathfrak{g}$ which contoins X as nilpositive element. Keeping in mind the adjoint representation of $\boldsymbol{R}X\oplus\boldsymbol{R}H\oplus\boldsymbol{R}Y\cong\mathfrak{sl}_2(\boldsymbol{R})$ on $\mathfrak{g}$, we find out that $\operatorname{ad} H$ is diagonarizable over $\mathfrak{g}$. Such an $H\in\mathfrak{g}$ is conjugate, under the adjoint action of G, to some $H'\in\mathfrak{a}_p$ dominant with respect to Λ^{+}. Thanks to our choice of Λ^{+}, H' is dominant with respect to Δ^{+}, too. We thus conclude that $H(\mathfrak{o})=H'\in\mathfrak{a}_p$ because of the uniqueness of $H(\mathfrak{o})$ in Theorem 1.4(1).

Conversely, suppose that $H(\mathfrak{o})\in\mathfrak{a}_p$. Then $\mathfrak{g}$ has a structure

$$\mathfrak{g}=\bigoplus_{s\in\boldsymbol{Z}}\mathfrak{g}(s)_{\mathfrak{o}}\quad \text{with}\quad \mathfrak{g}(s)_{\mathfrak{o}}=\mathfrak{g}_C(s)_{\mathfrak{o}}\cap\mathfrak{g}\,. \tag{1.7}$$

Let $\mathcal{M}$ be the complex vector space consisting of linear maps $\pi(X)\equiv(\operatorname{ad} X)|\mathfrak{g}_C(0)_{\mathfrak{o}}$ $(X\in\mathfrak{g}_C(2)_{\mathfrak{o}})$ from $\mathfrak{g}_C(0)_{\mathfrak{o}}$ to $\mathfrak{g}_C(2)_{\mathfrak{o}}$. Then $\mathcal{M}_{\boldsymbol{R}}=\pi(\mathfrak{g}(2)_{\mathfrak{o}})$ is a real form of $\mathcal{M}$. Set $\hat{\mathcal{M}}=\pi(\hat{\mathfrak{g}}_C(2)_{\mathfrak{o}})\subseteqq\mathcal{M}$. Then, for an $L\in\mathcal{M}$, L lies in $\hat{\mathcal{M}}$ if and only if the map $L:\mathfrak{g}_C(0)_{\mathfrak{o}}\to\mathfrak{g}_C(2)_{\mathfrak{o}}$ is surjective. The latter condition is equivalent to $F_i(L)\neq 0$ for some $1\leqq i\leqq N$. Here $F_i(L)$ $(1\leqq i\leqq N)$ are the minors of L of degree $\dim\mathfrak{g}_C(2)_{\mathfrak{o}}$. Note that $F_i: L\to F_i(L)$ are polynomial functions on $\mathcal{M}$. Since $\hat{\mathcal{M}}$ is not empty by Lemma 1.5(1), F_i is, for some i, not identically zero on $\mathcal{M}$, whence so is on the real form $\mathcal{M}_{\boldsymbol{R}}$. This means that $\mathfrak{g}_C(2)_{\mathfrak{o}}\cap\hat{\mathfrak{g}}\neq\emptyset$. From Lemma 1.5(2), one deduces $\mathfrak{o}\cap\mathfrak{g}\supseteqq\hat{\mathfrak{g}}_C(2)_{\mathfrak{o}}\cap\mathfrak{g}\neq\emptyset$ as desired. Q.E.D.

Lemma 1.7. *If* $\mathfrak{o}$ *is a nilpotent* G_C*-orbit in* $\mathfrak{g}_C$, *then one has* $\mathfrak{o}\cap\mathfrak{g}=\operatorname{Ad}(G)\hat{\mathfrak{g}}(2)_{\mathfrak{o}}$ *with* $\hat{\mathfrak{g}}(2)_{\mathfrak{o}}=\hat{\mathfrak{g}}_C(2)_{\mathfrak{o}}\cap\mathfrak{g}$.

Proof. In view of Lemma 1.5(2), we may assume that $\mathfrak{o}\cap\mathfrak{g}\neq\emptyset$. Let $X\in\mathfrak{o}\cap\mathfrak{g}$. Then, as in the proof of Lemma 1.6, there exists an $\mathfrak{sl}_2$-triplet $(X', H(\mathfrak{o}), Y)$ containing a certain $\operatorname{Ad}(G)$-conjugate X' of X as nilpositive element. Then it follows that $[X', \mathfrak{g}_C(0)_{\mathfrak{o}}]=\mathfrak{g}_C(2)_{\mathfrak{o}}$, or equivalently, $X'\in\hat{\mathfrak{g}}(2)_{\mathfrak{o}}$. We thus proved the inclusion $\mathfrak{o}\cap\mathfrak{g}\subseteqq\operatorname{Ad}(G)\hat{\mathfrak{g}}(2)_{\mathfrak{o}}$. The converse inclusion is clear from Lemma 1.5(2), which completes the proof. Q.E.D.

Now suppose that $\mathfrak{o}$ intersects with $\mathfrak{g}$. Clearly, $\mathfrak{o}\cap\mathfrak{g}$ is stable under $\operatorname{Ad}(G)$, so it is expressed as a disjoint union of some number of nilpotent classes in $\mathfrak{g}$. We shall establish in Proposition 1.9 a one-to-one correspondence between these $\operatorname{Ad}(G)$-orbits and the set of all orbits in $\hat{\mathfrak{g}}(2)_{\mathfrak{o}}$ under the adjoint action of a reductive subgroup $L_{\mathfrak{o}}\subseteqq G$.

For this purpose, we need some more notations. Set

$$\mathfrak{p}_{\mathfrak{o}}=\bigoplus_{s\geqq 0}\mathfrak{g}(s)_{\mathfrak{o}},\quad \mathfrak{l}_{\mathfrak{o}}=\mathfrak{g}(0)_{\mathfrak{o}}\quad\text{and}\quad \mathfrak{n}_{\mathfrak{o}}=\bigoplus_{s\geqq 1}\mathfrak{g}(s)_{\mathfrak{o}}. \tag{1.8}$$

Then $\mathfrak{p}_{\mathfrak{o}}=\mathfrak{l}_{\mathfrak{o}}\oplus\mathfrak{n}_{\mathfrak{o}}$ gives a Levi decomposition of a parabolic subalgebra $\mathfrak{p}_{\mathfrak{o}}$. Let $P_{\mathfrak{o}}=N_G(\mathfrak{n}_{\mathfrak{o}})$, the normalizer of $\mathfrak{n}_{\mathfrak{o}}$ in G, then it is the parabolic subgroup of G corresponding to $\mathfrak{p}_{\mathfrak{o}}$. It holds that $P_{\mathfrak{o}}=L_{\mathfrak{o}}N_{\mathfrak{o}}$ (a Levi decomposition) with $L_{\mathfrak{o}}=$

$Z_G(H(\mathrm{o}))$ and $N_{\mathrm{o}}=\exp \mathfrak{n}_{\mathrm{o}}$. Here exp denotes the exponential mapping from $\mathfrak{g}$ to G. The subspaces $\mathfrak{g}(0)_{\mathrm{o}}$ and $\mathfrak{g}(2)_{\mathrm{o}}$ are $\mathrm{Ad}(L_{\mathrm{o}})$-stable, whence so is $\hat{\mathfrak{g}}(2)_{\mathrm{o}}$ by definition.

The following lemma is crucial for the succeeding Proposition 1.9.

Lemma 1.8. *For two elements $X_1, X_2\in\hat{\mathfrak{g}}(2)_{\mathrm{o}}$, X_1 is conjugate to X_2 under G if and only if they are conjugate under L_{o}.*

Proof. Since $X_i\in\hat{\mathfrak{g}}(2)_{\mathrm{o}}\subseteqq\hat{\mathfrak{g}}_C(2)_{\mathrm{o}}$ for $i=1, 2$, by Lemma 1.5(1) there exists an $\mathfrak{sl}_2$-triplet $(X_i, H(\mathrm{o}), Y_i)$ in $\mathfrak{g}_C$ containing X_i and $H(\mathrm{o})$ as nilpositive and semisimple elements respectively. Suppose that $\mathrm{Ad}(g)X_2=X_1$ for some $g\in G$. Then $(X_1, \mathrm{Ad}(g)H(\mathrm{o}), \mathrm{Ad}(g)Y_2)$ as well as $(X_1, H(\mathrm{o}), Y_1)$ is an $\mathfrak{sl}_2$-triplet containing nilpositive element X_1 in common. Hence, by Proposition 1.3, they are conjugate under $G_C^{X_1}$, the unipotent radical of $Z_{G_C}(X_1)$. Especially $z\cdot\mathrm{Ad}(g)H(\mathrm{o})=H(\mathrm{o})$ for some $z\in G_C^{X_1}$, which implies that $z\cdot\mathrm{Ad}(g)(\mathfrak{n}_{\mathrm{o}})_C=(\mathfrak{n}_{\mathrm{o}})_C$. It follows from (1.2) that the Lie algebra $\mathfrak{g}_C^{X_1}$ of $G_C^{X_1}$ is contained in $(\mathfrak{n}_{\mathrm{o}})_C$. Therefore we have $\mathrm{Ad}(g)(\mathfrak{n}_{\mathrm{o}})_C=(\mathfrak{n}_{\mathrm{o}})_C$, which means that $g\in P_{\mathrm{o}}$.

Express g as $g=nl$ with $n\in N_{\mathrm{o}}$ and $l\in L_{\mathrm{o}}$. Then it is easily seen that

$$\mathrm{Ad}(g)X_2\equiv\mathrm{Ad}(l)X_2 \qquad (\mathrm{mod.} \bigoplus_{s\geqq 3}\mathfrak{g}(s)_{\mathrm{o}}).$$

Keeping in mind the assumption $\mathrm{Ad}(g)X_2=X_1\in\mathfrak{g}(2)_{\mathrm{o}}$, we conclude that $\mathrm{Ad}(l)X_2=X_1$. Consequently, X_1 and X_2 are $\mathrm{Ad}(L_{\mathrm{o}})$-conjugate.

The "if" part of the statement is obvious, thus we complete the proof.

Q. E. D.

By Lemmas 1.7 and 1.8, we obtain a parametrization of nilpotent $\mathrm{Ad}(G)$-orbits in $\mathrm{o}\cap\mathfrak{g}$ by means of $\mathrm{Ad}(L_{\mathrm{o}})$-orbits in $\hat{\mathfrak{g}}(2)_{\mathrm{o}}$ as follows.

Proposition 1.9. *Let o be a nilpotent G_C-orbit in $\mathfrak{g}_C$ which intersects with $\mathfrak{g}$. Then,*

(1) *every $\mathrm{Ad}(G)$-orbit ω in $\mathrm{o}\cap\mathfrak{g}$ intersects with $\hat{\mathfrak{g}}(2)_{\mathrm{o}}$.*

(2) *The mapping $\omega\mapsto\omega\cap\hat{\mathfrak{g}}(2)_{\mathrm{o}}$ gives a bijective correspondence from the set of $\mathrm{Ad}(G)$-orbits in $\mathrm{o}\cap\mathfrak{g}$ to the set of $\mathrm{Ad}(L_{\mathrm{o}})$-orbits in $\hat{\mathfrak{g}}(2)_{\mathrm{o}}$.*

1.3. The Kirillov orbit method. In order to construct the GGGRs of G, one needs irreducible unitary representations of nilpotent subgroups N_{o}. For this purpose, we shall survey briefly the Kirillov theory on representations of nilpotent Lie groups.

In this subsection, let N be a connected and simply connected nilpotent Lie group. Denote by $\mathfrak{n}$ the Lie algebra of N. Let $f\in\mathfrak{n}^*$, the dual space of $\mathfrak{n}$. A subalgebra $\mathfrak{h}$ of $\mathfrak{n}$ is said to be a *real polarization* at f if $\mathfrak{h}$ satisfies following two conditions:

(1) $f([\mathfrak{h}, \mathfrak{h}])=(0)$, (2) for an $X\in\mathfrak{n}$, $f([X, \mathfrak{h}])=(0)$ implies $X\in\mathfrak{h}$.

For any $f\in\mathfrak{n}^*$, a real polarization $\mathfrak{h}$ at f always exists, moreover it holds that

$2\dim\mathfrak{h}=\dim\mathfrak{n}+\dim\mathfrak{n}(f)$, where $\mathfrak{n}(f)=\{X\in\mathfrak{n};\ f([X,\mathfrak{n}])=(0)\}$.

For a real polarization $\mathfrak{h}$ at $f\in\mathfrak{n}^*$, put $H=\exp\mathfrak{h}$. Then, we see from (1) that

$$\eta_f(\exp Z)=\exp\{\sqrt{-1}f(Z)\}\qquad(Z\in\mathfrak{h})\tag{1.9}$$

defines a unitary character η_f of H. Let us consider the unitarily induced representation $\xi(f,\mathfrak{h})=L^2\text{-Ind}_H^N(\eta_f)$. Then $\xi(f,\mathfrak{h})$ is irreducible. Moreover, its unitary equivalence class $\sigma(f,\mathfrak{h})$ is independent of a choice of the real polarization $\mathfrak{h}$. So one may express this as $\sigma(f)$ instead of $\sigma(f,\mathfrak{h})$. We thus get a mapping σ from $\mathfrak{n}^*$ to the set $\hat{N}$ of equivalence classes of irreducible unitary representations of N.

Let Ad^* denote the coadjoint representation of N on $\mathfrak{n}^*$. Then we see easily that $\sigma(\mathrm{Ad}^*(n)f)=\sigma(f)$ for any $n\in N$ and $f\in\mathfrak{n}^*$. Thus σ induces canonically a mapping, denoted again by σ, from $\mathfrak{n}^*/N$ to $\hat{N}$, where $\mathfrak{n}^*/N$ is the coadjoint orbit space for $\mathfrak{n}^*$. On this map σ, the Kirillov theory tells us

Theorem 1.10 [16]. *The map σ sets up a bijective correspondence from the orbit space $\mathfrak{n}^*/N$ to the unitary dual $\hat{N}$.*

This bijection $\sigma:\mathfrak{n}^*/N\to\hat{N}$ is called the Kirillov correspondence.

1.4. Definition of generalized Gelfand-Graev representation (=GGGR). Let G be a connected semisimple Lie group with finite center again. Keep to the notations in 1.2. Using the facts in the provious sections, we can now construct, after N. Kawnaka, the GGGRs of G exactly.

Let ω be a non-trivial nilpotent $\mathrm{Ad}(G)$-orbit in $\mathfrak{g}$. We define the GGGR Γ_ω in the following procedure. Set $\mathfrak{o}=\mathfrak{o}(\omega)\equiv G_C\omega$, the G_C-orbit containing ω. For this $\mathfrak{o}$, let $\mathfrak{g}(2)_{\mathfrak{o}}$, $H(\mathfrak{o})$, $L_{\mathfrak{o}}$, $N_{\mathfrak{o}}$, $\cdots$ be as in 1.2. Take an X from $\omega\cap\hat{\mathfrak{g}}(2)_{\mathfrak{o}}$, which is assured to be non-empty thanks to Proposition 1.9(1). Let us define $X^*\in\mathfrak{n}_{\mathfrak{o}}^*$ by

$$\langle X^*,Z\rangle=B(Z,\theta X)\qquad\text{for}\quad Z\in\mathfrak{n}_{\mathfrak{o}},\tag{1.10}$$

where B denotes the Killing form of $\mathfrak{g}$, and θ the Cartan involution of $\mathfrak{g}$ corresponding to the decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}_p\oplus\mathfrak{n}_m$. We note that $\theta\hat{\mathfrak{g}}(s)_{\mathfrak{o}}=\mathfrak{g}(-s)_{\mathfrak{o}}$ ($s\in\boldsymbol{Z}$) because $\theta H(\mathfrak{o})=-H(\mathfrak{o})$ by Lemma 1.6. Since $N_{\mathfrak{o}}$ is a connected and simply connected nilpotent Lie group, we may attach $\sigma([X^*])\in\hat{N}_{\mathfrak{o}}$ to the $\mathrm{Ad}^*(N_{\mathfrak{o}})$-orbit $[X^*]$ through X^*, via the Kirillov correspondence σ.

Take a concrete irreducible unitary representation ξ_X of class $\sigma([X^*])$. Let us consider the induced representation $\Gamma_X\equiv\mathrm{Ind}_{N_{\mathfrak{o}}}^{G}(\xi_X)$. Here, Ind means either the unitary induction ($=L^2$-Ind) or the C^∞-induction ($=C^\infty$-Ind) defined in §§2 and 3 of the first part [32] of this paper (referred as [I] later on). Clearly the equivalence class $[\Gamma_X]$ of Γ_X does not depend on a realization ξ_X. Furthermore, it is independent of a choice of $X\in\omega\cap\hat{\mathfrak{g}}(2)_{\mathfrak{o}}$, because $\omega\cap\hat{\mathfrak{g}}(2)_{\mathfrak{o}}$ is a single $\mathrm{Ad}(L_{\mathfrak{o}})$-orbit in $\hat{\mathfrak{g}}(2)_{\mathfrak{o}}$ by Lemma 1.8. So we may express Γ_X as Γ_ω without any confusion.

Definition 1.11. For each nilpotent $\mathrm{Ad}(G)$-orbit ω $(\neq(0))$ in $\mathfrak{g}$, the induced representation $\Gamma_\omega=\mathrm{Ind}_{N_0}^G(\xi_X)$ $(X\in\omega\cap\hat{\mathfrak{g}}(2)_0)$ is called the *generalized Gelfand-Graev representation* (=GGGR) associated with ω. (For $\omega=(0)$, the GGGR $\Gamma_{(0)}$ is defined to be the left regular representation of G.)

For the later use, it is convenient to realize ξ_X explicitly as a monomial representation. This is done in the following way (see [13, (1.3.8)]). Set $\mathfrak{n}(i)_0=\bigoplus_{s\geq i}\mathfrak{g}(s)_0$ for $i\geqq 1$. Then $\mathfrak{n}(i)_0$ is an ideal of $\mathfrak{n}_0=\mathfrak{n}(1)_0$. Let $B(X)$ be the alternating bilinear form on $\mathfrak{n}_0\times\mathfrak{n}_0$ defined by

$$B(X)(Y, Z)=\langle X^*, [Y, Z]\rangle \quad \text{for} \quad Y, Z\in\mathfrak{n}_0. \tag{1.11}$$

It is easily seen that $\mathfrak{n}(2)_0$ coincides with the radical of $B(X)$:

$$\mathfrak{n}(2)_0=\{Y\in\mathfrak{n}_0;\ B(X)(Y, \mathfrak{n}_0)=(0)\}. \tag{1.12}$$

This implies that the restriction $B(X)|(\mathfrak{g}(1)_0\times\mathfrak{g}(1)_0)$ is non-degenerate. Hence there exists a subspace $\mathfrak{v}(X)$ of $\mathfrak{g}(1)_0$ satisfying

$$B(X)([\mathfrak{v}(X), \mathfrak{v}(X)])=(0) \quad \text{and} \quad 2\dim\mathfrak{v}(X)=\dim\mathfrak{g}(1)_0.$$

Put $\mathfrak{n}(1.5, X)=\mathfrak{v}(X)\oplus\mathfrak{n}(2)_0$, then $\mathfrak{n}(1.5, X)$ is a real polarization at X^* by construction. Therefore, from the Kirillov theory in 1.3, ξ_X can be realized as

$$\xi_X=L^2\text{-}\mathrm{Ind}_{N(X)}^{N_0}(\eta_X), \tag{1.13}$$

where η_X is a unitary character of $N(X)\equiv\exp\mathfrak{n}(1.5, X)$ given as

$$\eta_X(\exp Z)=\exp\sqrt{-1}\langle X^*, Z\rangle \quad \text{for} \quad Z\in\mathfrak{n}(1.5, X). \tag{1.14}$$

1.5. Gelfand-Graev representations; quasi-split case.

Let ξ be a one-dimensional unitary representation of the maximal unipotent subgroup $N_m\subseteqq G$. The induced representation $\mathrm{Ind}_{N_m}^G(\xi)$ is called a *Gelfand-Graev represention* (=GGR) if the unitary character ξ is non-degenerate: $\xi|(\exp\mathfrak{g}(\mathfrak{a}_p;\lambda))\not\equiv 1$ for any $\lambda\in\Pi$.

The rest of this section is devoted to clarifying the relationship between these GGRs and GGGRs in 1.4. We deal with the case where G is quasi-split in this subsection, and then, the non quasi-split case in the next 1.6.

Firstly, we recall some basic facts on regular nilpotent classes. An element $X\in\mathfrak{g}$ is said to be regular if $\dim\mathfrak{z}_\mathfrak{g}(X)=\mathrm{rank}\,\mathfrak{g}$, where $\mathrm{rank}\,\mathfrak{g}$ denotes the dimension of a Cartan subalgebra of $\mathfrak{g}$. By a regular nilpotent orbit (or class), we mean an $\mathrm{Ad}(G)$-orbit in $\mathfrak{g}$ consisting of regular and nilpotent elements.

If $\mathfrak{g}$ has a structure of complex Lie algebra, then there exists a unique regular nilpotent orbit in $\mathfrak{g}$. Even if $\mathfrak{g}$ is not defined over $\boldsymbol{C}$, $\mathfrak{g}$ possibly admits a regular nilpotent class.

Lemma 1.12 [28, Corollary 3.5]. *A real semisimple Lie algebra* $\mathfrak{g}$ *contains a regular nilpotent* $\mathrm{Ad}(G)$*-orbit if and only if* $\mathfrak{g}$ *is quasi-split, that is, the centralizer* $\mathfrak{m}\equiv\mathfrak{z}_\mathfrak{k}(\mathfrak{a}_p)$ *is abelian.*

We now assume that $\mathfrak{g}$ be quasi-split, and $\mathfrak{o}_r$ denotes the unique regular nilpotent G_C-orbit in $\mathfrak{g}_C$. By Lemma 1.12, $\mathfrak{o}_r$ intersects with $\mathfrak{g}$. Moreover, the subset $\hat{\mathfrak{g}}(2)_{\mathfrak{o}_r} \subseteqq \mathfrak{g}(2)_{\mathfrak{o}_r}$ is described as follows.

Lemma 1.13. *One has under the notation of* 1.2,

$$\hat{\mathfrak{g}}(2)_{\mathfrak{o}_r} = \{X = \sum_{\lambda \in \Pi} X_\lambda \,;\ X_\lambda \in \mathfrak{g}(\mathfrak{a}_p\,;\lambda) \setminus (0)\}\,, \tag{1.15}$$

where $\mathfrak{g}(\mathfrak{a}_p\,;\lambda)$ *is the root space of a root* $\lambda \in \Lambda$.

In order to prove this lemma, let us describe beforehand the spaces $\mathfrak{g}(\mathfrak{a}_p\,;\lambda)$ $(\lambda \in \Pi)$ by means of simple root spaces $\mathfrak{g}_C(\mathfrak{j}_C\,;\alpha)$ for $\alpha \in \tilde{\Pi}$ with respect to $(\mathfrak{g}_C, \mathfrak{j}_C)$. Let σ be the conjugation of $\mathfrak{g}_C$ with respect to $\mathfrak{g}$. For a $\beta \in \mathfrak{j}_C^*$, set $(\sigma^*\beta)(H) = \overline{\beta(\sigma(H))}$ $(H \in \mathfrak{j}_C)$, where the bar means the complex conjugation. Then Δ is stable under σ^*. Since $\mathfrak{g}$ is quasi-split, any root in Δ does not vanish on $\mathfrak{a}_p$. The positive root system Δ^+ is defined to be compatible with Λ^+ (see 1.2). From these facts, we see that Δ^+ is σ^*-stable. Hence so is $\tilde{\Pi}$. Therefore, there exist non-negative integers k and m such that

$$\sigma^*\alpha_i = \alpha_i \quad (1 \leqq i \leqq k), \qquad \alpha^*\alpha_i = \alpha_{i+m} \quad (k < i \leqq k+m)$$

and $k+2m=t$. Here $\alpha_1, \alpha_2, \cdots, \alpha_t$ are the elements of $\tilde{\Pi}$. Since any simple root $\lambda \in \Pi$ is given by restriction of an $\alpha \in \tilde{\Pi}$ to $\mathfrak{a}_p$, it holds that $\Pi = \{\lambda_1, \lambda_2, \cdots, \lambda_r\}$ with $\lambda_i = \alpha_i | \mathfrak{a}_p$ for $1 \leqq i \leqq r \equiv k+m$. Moreover, the simple root space $\mathfrak{g}(\mathfrak{a}_p\,;\lambda_i)$ has a structure

$$\mathfrak{g}(\mathfrak{a}_p\,;\lambda_i) = \begin{cases} \mathfrak{g}_C(\mathfrak{j}_C\,;\alpha_i) \cap \mathfrak{g} & \text{if } 1 \leqq i \leqq k\,, \\ \{\mathfrak{g}_C(\mathfrak{j}_C\,;\alpha_i) \oplus \mathfrak{g}_C(\mathfrak{j}_C\,;\sigma^*\alpha_i)\} \cap \mathfrak{g} & \text{if } k < i \leqq r\,. \end{cases} \tag{1.16}$$

Proof of Lemma 1.13. Using (1.16) we can prove Lemma 1.13 in the following way. Take a non-zero element $X_i \in \mathfrak{g}(\mathfrak{a}_p\,;\lambda_i)$ for each $1 \leqq i \leqq r$. Then $X = \sum_{1 \leqq i \leqq r} X_i$ belongs to $\hat{\mathfrak{g}}(2)_{\mathfrak{o}_r}$. In fact, from (1.16), X admits an expression such as $X = \sum_{\alpha \in \tilde{\Pi}} Y_\alpha$ with $Y_\alpha \in \mathfrak{g}_C(\mathfrak{j}_C\,;\alpha) \setminus (0)$ for any $\alpha \in \tilde{\Pi}$. Hence one has

$$(\mathrm{ad}\, X)\mathfrak{g}_C(0)_{\mathfrak{o}_r} = [X, \mathfrak{j}_C] = \sum_{\alpha \in \tilde{\Pi}} \mathfrak{g}_C(\mathfrak{j}_C\,;\alpha) = \mathfrak{g}_C(2)_{\mathfrak{o}_r}$$

which means that $X \in \hat{\mathfrak{g}}_C(2)_{\mathfrak{o}_r}$ by definition. Consequently, $X \in \hat{\mathfrak{g}}_C(2)_{\mathfrak{o}_r} \cap \mathfrak{g} = \hat{\mathfrak{g}}(2)_{\mathfrak{o}_r}$. We thus proved the inclusion "$\supseteqq$" in (1.15). The converse inclusion clearly holds by the definition of $\hat{\mathfrak{g}}(2)_{\mathfrak{o}_r}$. Q.E.D.

Consequently, the set of open $\mathrm{Ad}(MA_p)$-orbits in $\sum_{\lambda \in \Pi} \mathfrak{g}(\mathfrak{a}_p\,;\lambda)$ parametrizes the regular nilpotent classes of $\mathfrak{g}$.

Now let ω be a regular nilpotent orbit in $\mathfrak{g}$, or equivalently, an $\mathrm{Ad}(G)$-orbit contained in $\mathfrak{g} \cap \mathfrak{o}_r$. Then the GGGR Γ_ω is realized as $\Gamma_\omega = \mathrm{Ind}_{N_m}^G(\xi_X)$ for some $X \in \hat{\mathfrak{g}}(2)_{\mathfrak{o}_r} \cap \omega$, where ξ_X is a unitary character of the maximal unipotent subgroup $N_m = \exp \mathfrak{n}_m$ of G given as

$$\xi_X(\exp Z) = \exp \sqrt{-1}\, B(Z, \theta X) \qquad \text{for} \quad Z \in \mathfrak{n}_m\,. \tag{1.17}$$

Thanks to Lemma 1.13, ξ_X is non-degenerate. Conversely, any non-degenerate unitary character of N_m is of the form ξ_X for some $X \in \hat{\mathfrak{g}}(2)_{\mathfrak{o}_r}$. Keeping Proposition 1.9 in mind, we can summarize this as follows.

Proposition 1.14. *Assume G be quasi-split. Then, for any regular nilpotent class ω of G, the GGGR Γ_ω is realized as a representation induced from a non-degenerate character of N_m, a member of GGRs. Conversely, any Gelfand-Graev representation is equivalent to Γ_ω for some regular nilpotent class ω of $\mathfrak{g}$.*

In conclusion, the GGGRs associated with regular nilpotent classes coincide with the GGRs of quasi-split semisimple Lie groups.

1.6. The GGRs; non quasi-split case. We now proceed to the case where $\mathfrak{g}$ is not necessarily quasi-split. Also in this case, do GGGRs give an extension of the GGRs? Although we have not obtained a completely affermative answer to this question, Wakimoto's result tells us the following proposition which generalizes the first half of Proposition 1.14.

Proposition 1.15. *The GGGRs associated with the principal nilpotent orbits in $\mathfrak{g}$ are necessarily the GGRs. Here, a nilpotent $\mathrm{Ad}(G)$-orbit ω in $\mathfrak{g}$ is said to be principal if $\dim \mathfrak{z}_{\mathfrak{g}}(X)$ $(X \in \omega)$, which is independent of $X \in \omega$, is minimal among those for all nilpotent classes.*

Proof. Let ω be a principal nilpotent class of $\mathfrak{g}$, and put $\mathfrak{o}_p = G_C\omega$. According to Wakimoto [28, Proposition 3.3], the corresponding semisimple element $H(\mathfrak{o}_p) \in \mathfrak{a}_p$ is characterized by

$$\lambda(H(\mathfrak{o}_p)) = 2 \qquad \text{for all} \quad \lambda \in \Pi. \tag{1.18}$$

This implies that the principal nilpotent classes of $\mathfrak{g}$ are all conjugate under G_C to each other. Moreover, one gets $N_{\mathfrak{o}_p} = N_m$ and $\mathfrak{g}(2)_{\mathfrak{o}_p} = \sum_{\lambda \in \Pi} \mathfrak{g}(\mathfrak{a}_p\,;\lambda)$. From the definition of $\hat{\mathfrak{g}}(2)_{\mathfrak{o}_p}$, we see easily that

$$\hat{\mathfrak{g}}(2)_{\mathfrak{o}_p} \subseteqq \{X = \sum_{\lambda \in \Pi} X_\lambda\,;\ X_\lambda \in \mathfrak{g}(\mathfrak{a}_p\,;\lambda) \setminus (0)\}. \tag{1.19}$$

Using this inclusion relation, we can prove the assertion just as in the proof of the first half of Proposition 1.14. Q.E.D.

Conversely, can any given GGR be realized as the GGGR associated with a principal nilpotent class? The answer is "Yes" or "No" according as the equality holds in (1.19) or not.

§ 2. Reduced generalized Gelfand-Graev representations

We introduce in this section a version of the GGGRs, which we call reduced generalized Gelfand-Graev representations (=RGGGRs). These RGGGRs are main objects in the present article. In the succeeding sections, we shall show that an important kind of RGGGRs have finite multiplicity property, making use of the results of [I].

2.1. Definition of RGGGRs. Let G be as in 1.4 and keep to the notations in § 1. Consider the unitarily induced GGGR $L^2\text{-}\Gamma_\omega = L^2\text{-Ind}_{N_o}^G(\xi_X)$, $\mathfrak{o}=\mathfrak{o}(\omega)$, associated with a nilpotent $\mathrm{Ad}(G)$-orbit ω in $\mathfrak{g}$.

The most fundamental problem in the study of group representations would be to decompose a given representation into irreducibles explicitly. So, we hope to settle this problem for the GGGRs Γ_ω in future. Toward this end, we make in this subsection an observation, keeping in mind the Mackey theory for representations of group extension. This leads us naturally to the notion of RGGGRs. Furthermore, the problem of irreducible decomposition for $L^2\text{-}\Gamma_\omega$ is reduced mainly to those for RGGGRs associated with ω.

Now, for an $X \in \hat{\mathfrak{g}}(2)_o \cap \omega$, let $\xi_X = L^2\text{-Ind}_{N(X)}^{N_o}(\eta_X)$ be the irreducible unitary representation of N_o in (1.13). Since the Levi subgroup L_o acts on N_o through the conjugation, it acts also on the unitary dual $\hat{N}_o$ via

$$l \cdot [\xi] = [l \cdot \xi], \quad (l \cdot \xi)(n) = \xi(l^{-1} n l) \quad (n \in N_o) \tag{2.1}$$

for $l \in L_o$ and $[\xi] \in \hat{N}_o$. Here $[\xi]$ is the equivalence class of an irreducible unitary representation ξ of N_o. Let $H_o(X)$ be the stabilizer of $[\xi_X]$ in L_o:

$$H_o(X) = \{l \in L_o;\ l \cdot [\xi_X] = [\xi_X]\}. \tag{2.2}$$

Then one has

Lemma 2.1. *The subgroup $H_o(X)$ is reductive. Moreover it coincides with the centralizer $Z_{L_o}(\theta X)$ of θX in L_o, where θ is the Cartan involution of $\mathfrak{g}$.*

Proof. First we show the latter assertion. For an $l \in L_o$, we see easily $l \cdot [\xi_X] = [\xi_{\mathrm{Ad}(\theta l)X}]$, or equivalently, $l \cdot \sigma([X^*]) = \sigma([(\mathrm{Ad}(\theta l)X)^*])$. Here, $\sigma : \mathfrak{n}_o^*/N_o \to \hat{N}_o$ is the Kirillov correspondence, and $[Y^*]$ denotes the the $\mathrm{Ad}^*(N_o)$-orbit through $Y^* \in \mathfrak{n}_o^*$ (see 1.3). This implies that $l \in H_o(X)$ if and only if $[X^*] = [(\mathrm{Ad}(\theta l)X)^*]$. On the other hand, according to [13, (3.1.2)], one has for any $Y \in \hat{\mathfrak{g}}(2)_o$,

$$[Y^*] = Y^* + \mathfrak{g}(1)_o^*. \tag{2.3}$$

Here the dual space $\mathfrak{g}(1)_o^*$ of $\mathfrak{g}(1)_o$ is canonically identified with the subspace of $\mathfrak{n}_o^*$ consisting of all $Z^* \in \mathfrak{n}_o^*$ identically zero on $\mathfrak{n}(2)_o$. We thus conclude that $l \in H_o(X)$ if and only if $\mathrm{Ad}(l)\theta X = \theta X$ as desired.

Secondly we prove that $H_o(X)$ is reductive. According to [4, Proposition 5.5.9], $\mathfrak{z}_{(\mathfrak{l}_o)_C}(\theta X)$ is a reductive Lie algebra, whence so is a real form $\mathfrak{z}_{\mathfrak{l}_o}(\theta X)$ of it. Consequently, $H_o(X)$ is reductive because of the latter assertion proved above. Q. E. D.

If $h \in H_o(X)$, then two irreducible unitary representations $h \cdot \xi_X$ and ξ_X are mutually equivalent. So, there exists a unique (up to scalar multiples) unitary operator $\tilde{\xi}_X(h)$ on the representation space $\mathcal{H}(\xi_X)$ of ξ_X satisfying

$$(h\cdot\xi_X)(n)=\tilde{\xi}_X(h)^{-1}\xi_X(n)\tilde{\xi}_X(h) \qquad \text{for all} \quad n\in N_o. \tag{2.4}$$

By putting $\tilde{\xi}_X(hn)=\tilde{\xi}_X(h)\xi_X(n)$ $(h\in H_o(X),\ n\in N_o)$, we thus obtain in general a projective unitary representation $\tilde{\xi}_X$ of the semidirect product group $S_o(X)\equiv H_o(X)\ltimes N_o$ which extends ξ_X. Nevertheless, under a certain kind of assumption on ω, we find out that ξ_X extends to a genuine (not just projective) unitary representation $\tilde{\xi}_X$.

Proposition 2.2. *Suppose that the subspace* $\mathfrak{g}(1)_o$ *admits an* $\mathrm{Ad}(H_o(X))$*-invariant complex structure* j, *that is, a real linear map* j *on* $\mathfrak{g}(1)_o$ *satisfying* (1) $j^2=-I$ ($I=$*the identity operator*) *and* (2) $(\mathrm{Ad}(h)\circ j\circ\mathrm{Ad}(h)^{-1})|\mathfrak{g}(1)_o=j$ *for all* $h\in H_o(X)$. *Then* ξ_X *is extendable to a genuine unitary representation* $\tilde{\xi}_X$ *of* $S_o(X)$ *acting on the same Hilbert space* $\mathcal{H}(\xi_X)$.

We prove this proposition in the succeeding subsection 2.3.
As a direct consequence of this proposition, one obtains

Proposition 2.3. *For an* $X\in\hat{\mathfrak{g}}(2)_o\cap\omega$, *the representation* ξ_X *can be extended to a genuine representation of* $S_o(X)$ *if either the followieg condition* (1) *or* (2) (*on* ω *or* $\mathfrak{g}$) *is satisfied.*

(1) *The nilpotent class* ω *is even* (*i.e.*, $\mathfrak{g}(1)_o=(0)$),
(2) $\mathfrak{g}$ *is a complex semisimple Lie algebra.*

Remark 2.4. Prof. Kawanaka kindly suggested us that the assertion of Proposition 2.3 (for the condition (2)) should be true. Thanks to his advice, we were able to obtain Proposition 2.2 which is very useful for later discussion (see Theorem 4.6).

As we saw above, extensions of ξ_X to genuine representations $\tilde{\xi}_X$ are possible for many number of ω. Moreover, for any GGGR Γ_ω which we discuss detailedly in later sections, the subspace $\mathfrak{g}(1)_o$ with $o=o(\omega)=G_C\omega$ always satisfies the assumption of Proposition 2.2. Therefore, we henceforth restrict ourselves to the case where ξ_X admits an extension to a genuine representation. (Even if ξ_X is not extendable to a genuine representation, one can construct RGGGRs analogously. But we do not treat it here. See [15, 2.5].)

Now we consider the induced representation $L^2\text{-}\mathrm{Ind}_{N_o}^{S_o(X)}(\xi_X)$, suggested by the isomorphism

$$L^2\text{-}\Gamma_\omega\cong L^2\text{-}\mathrm{Ind}_{S_o(X)}^{G}(L^2\text{-}\mathrm{Ind}_{N_o}^{S_o(X)}(\xi_X)). \tag{2.5}$$

Lemma 2.5. *One has an isomorphism of unitary representations*

$$L^2\text{-}\mathrm{Ind}_{N_o}^{S_o(X)}(\xi_X)\cong\tilde{\phi}\otimes\tilde{\xi}_X \tag{2.6}$$

where $\tilde{\phi}(hn)=\phi(h)$ $(h\in H_o(X),\ n\in N_o)$ *with the left regular representation* ϕ *of* $H_o(X)$.

Proof. Let $\mathcal{L}$ be the Hilbert space of $\mathcal{H}(\xi_X)$-valued Borel functions f on $H_o(X)$ such that

$$\|f\|_{\mathcal{L}}^2=\int_{H_o(X)}\|f(h)\|^2 dh<+\infty,$$

where dh denotes the left Haar measure on $H_o(X)$, and $\|\cdot\|$ the norm on $\mathcal{H}(\xi_X)$. Since the factor space $S_o(X)/N_o$ is canonically isomorphic to $H_o(X)$, the representation $L^2\text{-Ind}_{N_o}^{S_o(X)}(\xi_X)$ can be realized on $\mathcal{L}$ in the following way: the action λ of $S_o(X)$ on $\mathcal{L}$ is expressed as

$$\lambda(h)f(h')=f(h^{-1}h')=\phi(h)f(h'),$$
$$\lambda(n)f(h')=\tilde{\xi}_X(h'^{-1})\xi_X(n)\tilde{\xi}_X(h')f(h')$$

for $f\in\mathcal{L}$, $h.\ h'\in H_o(X)$ and $n\in N_o$.

Define a unitary operator ι on $\mathcal{L}$ by $\iota f(h)=\tilde{\xi}_X(h)f(h)$ $(h\in H_o(X))$. Identifying $\mathcal{L}$ with $L^2(H_o(X), dh)\otimes\mathcal{H}(\xi_X)$ in the canonical way, we may regard ι as an isometry from $\mathcal{L}$ onto $L^2(H_o(X), dh)\otimes\mathcal{H}(\xi_X)$. Then it is easily checked that ι gives a unitary intertwining operator from λ to $\tilde{\phi}\otimes\tilde{\xi}_X$. Q.E.D.

We now proceed to the GGGR $L^2\text{-}\Gamma_\omega$. Let

$$\phi\cong\int_Z^{\oplus} c_z d\sigma(z) \tag{2.7}$$

be a desintegration of ϕ over some measure space (Z, σ) (see 3.1 of [I]). Then, in view of Lemma 2.5, $L^2\text{-}\Gamma_\omega$ is decomposed as

$$L^2\text{-}\Gamma_\omega\cong\int_Z^{\oplus} L^2\text{-Ind}_{S_o(X)}^G(\tilde{c}_z\otimes\tilde{\xi}_X)d\sigma(z) \qquad \text{with} \quad \tilde{c}_z=c_z\otimes 1_{N_o}, \tag{2.8}$$

where 1_{N_o} is the trivial character of N_o. Suggested by (2.8), we introduce a variant of the GGGR Γ_ω as follows.

Definition 2.6. Let ω be a nilpotent $\mathrm{Ad}(G)$-orbit in $\mathfrak{g}$, and $X\in\omega\cap\hat{\mathfrak{g}}(2)_o$, $o=o(\omega)$. Suppose that the irreducible unitary representation ξ_X defined by (1.13) extends to a genuine unitary representation $\tilde{\xi}_X$ of the semidirect product group $S_o(X)=H_o(X)N_o$. Then, for an irreducible representation c of $H_o(X)$, the induced representation $\Gamma_\omega(c)\equiv\mathrm{Ind}_{S_o(X)}^G(\tilde{c}\otimes\tilde{\xi}_X)$ with $\tilde{c}=c\otimes 1_{N_o}$, is called the *reduced generalized Galfand-Graev representation* (=RGGGR) associated with (ω, c).

Remark 2.7. (1) In the above definition, Ind means either C^∞-Ind or L^2-Ind as before. When the unitary induction is considered, c is of course assumed to be unitary. But we need not (and do not) restrict c to be unitary so far as the C^∞-induction is in mind.

(2) In general, all the irreducible unitary representations of $H_o(X)$ are not necessary to decompose ϕ into irreducibles. Therefore, all the RGGGRs $L^2\text{-}\Gamma_\omega(c)$ $(c\in \hat{H_o}(X))$ are not needed in order to decompose $L^2\text{-}\Gamma_\omega$ as in (2.8). Nevertheless, we do not exclude here, to define RGGGRs, $L^2\text{-}\Gamma_\omega(c)$'s which do

not "occur" in (2.8).

At the end of this subsection, we refer to the problem of explicit irreducible decomposition of the GGGRs $L^2\text{-}\Gamma_\omega$. In view of (2.7) and (2.8), this problem is reduced to the following (P1) and (P2).

(P1) Decompose the regular representation ψ of $H_0(X)$ into irreducibles explicitly. In other words, describe an explicit Plancherel formula for the reductive group $H_0(X)$.

(P2) For any $c \in \hat{H_0}(X)$ (=the unitary dual of $H_0(X)$) which "occurs" in ψ, give the irreducible decomposition of the RGGGR $L^2\text{-}\Gamma_\omega(c)$.

As for (P1), Harish-Chandra established an explicit Plancherel formula for reductive Lie groups so-called of class $(\mathscr{H})$. (See [27] for the definition of groups of class $(\mathscr{H})$). Therefore, if $H_0(X)$ is of class $(\mathscr{H})$, one gets a complete answer for (P1) by applying Harish-Chandra's result. Although we do not know whether or not any $H_0(X)$ is of class $(\mathscr{H})$, this request would be a mild one. Thus our original problem for $L^2\text{-}\Gamma_\omega$ is reduced mainly to (P2).

2.2. Oscillator representations of unitary groups.

The rest of this section is devoted to proving Proposition 2.2 by reducing it to a basic fact of J. A. Wolf [30] on the oscillator representations of unitary groups. In this subsection, we explain the result of Wolf, crucial for the succeeding 2.3.

Let V be a finite-dimensional complex vector space with a non-degenerate hermitian form h (not necessarily positive or negative definite). Then $N_h \equiv V \times \boldsymbol{R}$ has a structure of two-step nilpotent Lie group by giving the product of two elements (z_1, x_1) and (z_2, x_2) as

$$(2.9) \qquad (z_1, x_1)(z_2, x_2) = (z_1+z_2,\ x_1+x_2+\operatorname{Im} h(z_1, z_2)).$$

N_h is called the Heisenberg group of dimension $2n+1$, where $n=\dim V$. Express by $V_{\boldsymbol{R}}$ the space V regarded canonically as a real vector space. Since $A=\operatorname{Im} h$ gives a non-degenerate alternating bilinear form on $V_{\boldsymbol{R}} \times V_{\boldsymbol{R}}$, we can define the real symplectic group $Sp(V_{\boldsymbol{R}}, A)$ with respect to A as follows:

$$(2.10) \qquad Sp(V_{\boldsymbol{R}}, A) = \{g \in GL(V_{\boldsymbol{R}});\ A(gv, gw) = A(v, w) \text{ for all } v, w \in V_{\boldsymbol{R}}\}.$$

The "unitary" group with respect to h:

$$(2.11) \qquad U(V, h) = \{g \in GL(V);\ h(gz_1, gz_2) = h(z_1, z_2) \text{ for all } z_1, z_2 \in V\},$$

forms a closed subgroup of $Sp(V_{\boldsymbol{R}}, A)$.

$Sp(V_{\boldsymbol{R}}, A)$ acts on N_h as a group of automorphisms via

$$(2.12) \qquad g\cdot(z, x) = (gz, x) \quad \text{for} \quad g \in Sp(V_{\boldsymbol{R}}, A) \quad \text{and} \quad (z, x) \in N_h.$$

So we may consider the semidirect product groups $Sp(V_{\boldsymbol{R}}, A) \ltimes N_h \supseteq U(V, h) \ltimes N_h$.

Let r be a non-zero real number. Then we see by the Kirillov orbit method that there exists a unique (up to equivalence) irreducible unitary representation ξ_r of N_h for which the central elements $(0, x)$ $(x\in\boldsymbol{R})$ are represented by scalar operators

$$\xi_r((0, x))=\exp\{\sqrt{-1}rx\}I \qquad (I=\text{the identity operator}). \tag{2.13}$$

These ξ_r with $r\neq 0$ are infinite-dimensional. Furthermore, they exhaust all the infinite-dimensional irreducible representations of N_h.

For $g\in Sp(V_{\boldsymbol{R}}, A)$, put $(g\cdot\xi_r)(n)=\xi_r(g^{-1}\cdot n)$ $(n\in N_h)$. Then $g\cdot\xi_r$ as well as ξ_r is an irreducible representation of N_h satisfying (2.13). Therefore, analogously to our construction of $\tilde{\xi}_X$ in 2.1, ξ_r can be extended to a projective unitary representation $\tilde{\xi}_r$ of $Sp(V_{\boldsymbol{R}}, A)\ltimes N_h$ in such a way that

$$(g\cdot\xi_r)(n)=\tilde{\xi}_r(g)^{-1}\xi_r(n)\tilde{\xi}_r(g), \qquad \tilde{\xi}_r(gn)=\tilde{\xi}_r(g)\xi_r(n)$$

for $g\in Sp(V_{\boldsymbol{R}}, A)$ and $n\in N_h$.

Remark 2.8. This $\tilde{\xi}_r$ can not be a genuine representation till it is lifted to the representation (denoted again by $\tilde{\xi}_r$) of $Mp(V_{\boldsymbol{R}}, A)\ltimes N_h$ in the canonical way. Here $Mp(V_{\boldsymbol{R}}, A)$ is the two-sheeted cover of $Sp(V_{\boldsymbol{R}}, A)$. The restriction of $\tilde{\xi}_r$ to the subgroup $Mp(V_{\boldsymbol{R}}, A)$ is called the *Weil*, *metaplectic*, or *harmonic oscillator representation*.

Contrary to the above case of Weil representation, a result of Wolf tells us the extension of ξ_r to $U(V, h)\ltimes N_h$.

Proposition 2.9 [30, Proposition 4.16]. *For any $r\in\boldsymbol{R}\setminus(0)$, ξ_r extends to a genuine representation of $U(V, h)\ltimes N_h$. Such an extension, if restricted to $U(V, h)$, is called an oscillator representation of $U(V, h)$.*

2.3. Proof of Proposition 2.2. We now prove Proposition 2.2 by using Proposition 2.9 above. Return to the notations in 2.1, and assume that $\mathfrak{g}(1)_o$ admits an $\mathrm{Ad}(H_o(X))$-invariant complex structure j. Our proof consists of four steps.

STEP I. If $\mathfrak{g}(1)_o$ reduces to (0), the assertion is clearly true because ξ_X is one-dimensional by construction. So, in the following we assume that $\mathfrak{g}(1)_o\neq(0)$. In this case, the representation $\xi_X=L^2\text{-}\mathrm{Ind}_{N(X)}^{N_o}(\eta_X)$ constructed in 1.4 is infinite-dimensional. Here $\mathfrak{n}(X)=\mathrm{Lie}\,N(X)$ is a real polarization at $X^*\in\mathfrak{n}_o^*$ containing $\mathfrak{n}(2)_o=\bigoplus_{s\geq 2}\mathfrak{g}(s)_o$, and η_X is the unitary character of $N(X)$ given as $\eta_X(\exp Y)=\exp\sqrt{-1}\langle X^*, Y\rangle$ $(Y\in\mathfrak{n}(X))$.

STEP II. Set $U=(\mathrm{Ker}\,\eta_X)\cap N(2)_o$ with $N(2)_o\equiv\exp\mathfrak{n}(2)_o$, then U is a connected normal subgroup of N_o with Lie algebra $\mathfrak{u}\equiv(\mathrm{Ker}\,X^*)\cap\mathfrak{n}(2)_o$. By the construction of ξ_X, we see $\mathrm{Ker}\,\xi_X\supseteqq U$. Therefore, ξ_X induces, through the canonical map $p: N_o\to N_o/U$, an irreducible unitary representation ξ'_X of the factor group $N_o/U: \xi'_X(p(n))=\xi_X(n)$ $(n\in N_o)$.

By Lemma 2.1, U is stable under $H_o(X)$. So $H_o(X)$ acts on N_o/U, hence it acts also on the unitary dual $(N_o/U)^\wedge$ in the canonical way. Moreover, the unitary equivalence class $[\xi'_X]$ of ξ'_X is fixed under the action of $H_0(X)$ because so is $[\xi_X]\in\hat{N}_0$. From this discussion, the proof of Proposition 2.2 is now reduced to showing that ξ'_X is extendable to a genuine representation of the semidirect product group $H_0(X)\ltimes(N_0/U)$.

STEP III. Now let V denote the complex vector space $\mathfrak{g}(1)_0$ equipped with the complex structure j. Set for $z_1, z_2\in V$

$$h(z_1, z_2)\equiv 2^{-1}\{X^*([jz_1, z_2])+\sqrt{-1}\,X^*([z_1, z_2])\}. \tag{2.14}$$

Then h gives a hermitian form on V, which is non-degenerate because so is the real bilinear form $\mathfrak{g}(1)_0\times\mathfrak{g}(1)_0\ni(x_1, x_2)\mapsto X^*([x_1, x_2])=2\,\mathrm{Im}\,h(x_1, x_2)$. For this V and h, we consider the Heisenberg group N_h and the unitary group $U(V, h)$ in 2.2.

For $n=\exp(x+y)\in N_0$ with $x\in\mathfrak{g}(1)_0$ and $y\in\mathfrak{n}(2)_0$, put

$$\tau_0(n)=(x, X^*(y))\in N_h.$$

Then $\tau_0: N_0\to N_h$ gives a surjective group homomorphism with $\mathrm{Ker}\,\tau_0=U$, which induces canonically an isomorphism $\tau: N_0/U\xrightarrow{\sim}N_h$. This extends to a group homomorphism $\tilde{\tau}$:

$$H_0(X)\ltimes(N_0/U)\longrightarrow U(V, h)\ltimes N_h$$

in the following way. If $l\in H_0(X)$, then $\pi(l)\equiv\mathrm{Ad}(l)|V$ is a complex linear map since j commutes with $\pi(l)$ by assumption. Moreover, it leaves h invariant, which means that $\pi(l)\in U(V, h)$. Set $\tilde{\tau}(l\bar{n})=\pi(l)\tau(\bar{n})$ ($l\in H_0(X)$, $\bar{n}\in N_0/U$). Then $\tilde{\tau}$ is the desired group homomorphism.

STEP IV. At last, we show that ξ'_X admits an extension to a genuine representation $\tilde{\xi}'_X$ of $H_0(X)\ltimes(N_0/U)$. Let ξ''_X be the representation of N_h given as $\xi''_X=\xi'_X\circ\tau^{-1}$. Thanks to Proposition 2.9, ξ''_X extends to a genuine representation $\tilde{\xi}''_X$ of $U(V, h)\ltimes N_h$. Then $\tilde{\xi}'_X=\tilde{\xi}''_X\circ\tilde{\tau}$ gives a desired extension of ξ'_X. This completes the proof. Q.E.D.

§3. Simple Lie groups of hermitian type and nilpotent classes ω_i

In the previous sections, we developed the generalities on (reduced) generalized Gelfand-Graev representations of semisimple Lie groups. We retain the abbreviation GGGR (or RGGGR) in order to express such a representation. In the sections starting with this §3, we shall concentrate on some important types of (reduced) GGGRs of simple Lie groups G of hermitian type, closely related to the regular representation of G, and prove finite multiplicity property for RGGGRs of such types.

Since the (reduced) GGGRs correspond to nilpotent classes of $\mathfrak{g}$, the first thing we should do is to determine the nilpotent classes which give the above important kinds of RGGGRs. This is the main purpose of this section. To be more precise, we recall in 3.1 refined structure theorems due to Moore [18], for simple groups of hermitian type. We construct in 3.2 the nilpotent G_C-orbit $o\subseteq\mathfrak{g}_C$ in question, and then describe completely the $\mathrm{Ad}(G)$ orbits contained in $o\cap\mathfrak{g}$ (Theorem 3.13).

3.1. Moore's restricted root theorem ([18], see also [19], [24]).

Let G be a connected simple Lie group with finite center, and K a maximal compact subgroup of G. Hereafter, we always assume that G/K has a structure of hermitian symmetric space. Then the center $\mathfrak{c}$ of the Lie algebra $\mathfrak{k}$ of K is one-dimensional. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, and θ the corresponding Cartan involution of G. The given G-invariant complex structure on G/K induces canonically an $\mathrm{Ad}(K)$-invariant complex structure J on $\mathfrak{p}$ by identifying $\mathfrak{p}$ with the tangent space of G/K at the origin K. Moreover, there exists a unique $Z_0\in\mathfrak{c}$ such that

$$J=(\mathrm{ad}\,Z_0)|\mathfrak{p}. \tag{3.1}$$

Extend J to a map on $\mathfrak{p}_C$ by complex linearity. Since $J^2=-I_{\mathfrak{p}_C}$ ($I_{\mathfrak{p}_C}$=the identity operator on $\mathfrak{p}_C$), $\mathfrak{p}_C$ is decomposed as

$$\mathfrak{p}_C=\mathfrak{p}_+\oplus\mathfrak{p}_-, \tag{3.2}$$

where $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are respectively the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces for J on $\mathfrak{p}_C$.

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Necessarily, $\mathfrak{t}$ contains $\mathfrak{c}$. Furthermore, $\mathfrak{t}$ itself is a (compact) Cartan subalgebra of $\mathfrak{g}$. Let $\Sigma=\Sigma(\mathfrak{g}_C, \mathfrak{t}_C)$ be the root system of $\mathfrak{g}_C$ with respect to $\mathfrak{t}_C$. For each $\alpha\in\Sigma$, the corresponding root space $\mathfrak{g}_C(\mathfrak{t}_C; \alpha)$ is contained either in $\mathfrak{k}_C$ or $\mathfrak{p}_C$. In the former case (resp. the latter case), the root α is said to be compact (resp. non-compact). $\Sigma_{\mathfrak{k}}$ (resp. $\Sigma_{\mathfrak{p}}$) will denote the set of compact (resp. non-compact) roots.

Let us define a positive system Σ^+ of Σ compatible with the decomposition (3.2) as follows. Consider the lexicographic order (see 1.3 of [I]) on $\sqrt{-1}\mathfrak{t}^*$ with respect to an ordered basis $(X_1=Z_0, X_2, \cdots, X_p)$ of $\mathfrak{t}$ containing Z_0 as the top constituent. Σ^+ is the collection of positive elements in Σ. Set $\Sigma_{\mathfrak{k}}^+=\Sigma_{\mathfrak{k}}\cap\Sigma^+$ and $\Sigma_{\mathfrak{p}}^+=\Sigma_{\mathfrak{p}}\cap\Sigma^+$. Then one sees easily

$$\mathfrak{g}_C=\mathfrak{p}_-\oplus\mathfrak{k}_C\oplus\mathfrak{p}_+, \tag{3.3}$$

$$\mathfrak{p}_\pm=\sum_{\alpha\in\Sigma_{\mathfrak{p}}^+}\mathfrak{g}_C(\mathfrak{t}_C; \pm\alpha), \quad \mathfrak{k}_C=\mathfrak{t}_C\oplus\sum_{\alpha\in\Sigma_{\mathfrak{k}}}\mathfrak{g}_C(\mathfrak{t}_C; \alpha). \tag{3.4}$$

Moreover, $\mathfrak{p}_\pm$ are abelian subalgebras of $\mathfrak{g}_C$ because $[\mathfrak{p}_\pm, \mathfrak{p}_\pm]\subseteq\mathfrak{k}_C\cap\mathfrak{p}_\pm=(0)$.

One may (and do) select, for every $\alpha\in\Sigma$, a root vector $X_\alpha\in\mathfrak{g}_C(\mathfrak{t}_C; \alpha)$ in such a way that

$$X_\alpha-X_{-\alpha},\quad \sqrt{-1}(X_\alpha+X_{-\alpha})\in\mathfrak{k}\oplus\sqrt{-1}\mathfrak{p} \quad\text{and}\quad [X_\alpha, X_{-\alpha}]=H'_\alpha, \tag{3.5}$$

where $H'_\alpha=2\alpha(H_\alpha)^{-1}H_\alpha$ with $H_\alpha\in\sqrt{-1}\mathfrak{t}$ determined by $B(H, H_\alpha)=\alpha(H)$ for all $H\in\mathfrak{t}_C$ (B=the Killing form of $\mathfrak{g}_C$).

A root α is said to be strongly orthogonal to $\beta\in\Sigma$ if neither $\alpha+\beta$ nor $\alpha-\beta$ belongs to $\Sigma\cup(0)$. Construct inductively a maximal family $(\gamma_1, \gamma_2, \cdots, \gamma_l)$ of mutually strongly orthogonal roots in $\Sigma_{\mathfrak{p}}^+$ in such a way that, for each j, γ_j

is the largest non-compact positive root strongly orthogonal to $\gamma_{j+1}, \cdots, \gamma_l$. Now we set $\mathfrak{t}^- = \sum_{1\le k\le l} \boldsymbol{R} H'_{\gamma_k} \subseteq \sqrt{-1}\mathfrak{t}$. Denote by $\mathfrak{t}^+$ the orthogonal complement of $\mathfrak{t}^-$ in $\mathfrak{t}$ with respect to B. For any $\gamma \in \mathfrak{t}_C^*$, put $\pi(\gamma) = \gamma | \mathfrak{t}_C^-$, the restriction of γ to $\mathfrak{t}_C^-$. If γ is identically zero on $\mathfrak{t}^+$, then one may express $\pi(\gamma)$ still as γ without any confusion. So are the cases $\gamma = \gamma_k$ $(1 \leqq k \leqq l)$.

Let us describe the restrictions $\pi(\gamma)$ $(\gamma \in \Sigma^+)$ explicitly, after [7] and [18]. For this purpose, set for $1 \leqq m < k \leqq l$

$$C_k = \{\gamma \in \Sigma_{\mathfrak{k}}^+ ;\ \pi(\gamma) = 2^{-1}\gamma_k\}, \quad P_k = \{\gamma \in \Sigma_{\mathfrak{p}}^+ ;\ \pi(\gamma) = 2^{-1}\gamma_k\},$$
$$C_{km} = \{\gamma \in \Sigma_{\mathfrak{k}}^+ ;\ \pi(\gamma) = 2^{-1}(\gamma_k - \gamma_m)\}, \quad P_{km} = \{\gamma \in \Sigma_{\mathfrak{p}}^+ ;\ \pi(\gamma) = 2^{-1}(\gamma_k + \gamma_n)\},$$
$$C_0 = \{\gamma \in \Sigma_{\mathfrak{k}}^+ ;\ \pi(\gamma) = 0\} \quad \text{and} \quad P_0 = \{\gamma_1, \gamma_2, \cdots, \gamma_l\}.$$

According to Harish-Chandra, one has the following

Proposition 3.1 [7, VI, §6]. (1) *The subsets $\Sigma_{\mathfrak{k}}^+$ and $\Sigma_{\mathfrak{p}}^+$ of Σ are expressed respectively as*

$$\Sigma_{\mathfrak{k}}^+ = C_0 \cup \Big(\bigcup_{1\le k\le l} C_k\Big) \cup \Big(\bigcup_{1\le m<k\le l} C_{km}\Big), \tag{3.6}$$

$$\Sigma_{\mathfrak{p}}^+ = \Big(\bigcup_{1\le k\le l} P_k\Big) \cup P_0 \cup \Big(\bigcup_{1\le m<k\le l} P_{km}\Big), \tag{3.7}$$

where the unions are disjoint.

(2) *For any $1 \leqq m < k \leqq l$ (resp. $1 \leqq k \leqq l$), the map $\gamma \mapsto \gamma_m + \gamma$ (resp. $\gamma \mapsto \gamma_k - \gamma$) gives a bijection from C_{km} to P_{km} (resp. from C_k to P_k).*

Based on this proposition, C. C. Moore determined which constituents in the right hand sides of (3.6) and (3.7) may be non-empty, and obtained the following restricted root theorem.

Proposition 3.2 [18, Theorem 2]. (1) *There are only two possibilities for $\pi(\Sigma^+)$ except for* (0):
(CASE I) $\pi(\Sigma^+) \cup (0) = \{2^{-1}(\gamma_k \pm \gamma_m) ;\ 1 \leqq m \leqq k \leqq l\}$,
(CASE II) $\pi(\Sigma^+) \cup (0) = \{2^{-1}(\gamma_k \pm \gamma_m) ;\ 1 \leqq m \leqq k \leqq l\} \cup \{2^{-1}\gamma_k ;\ 1 \leqq k \leqq l\}$.

(2) *All the γ_k $(1 \leqq k \leqq l)$ have the same length.*

We set $\mathfrak{a}_p = \sum_{1\le k\le l} \boldsymbol{R} H_k$ with $H_k \equiv X_{\gamma_k} + X_{-\gamma_k}$. Then $\mathfrak{a}_p$ is a maximal abelian subspace of $\mathfrak{p}$. Moreover, $\mathfrak{j} \equiv \mathfrak{t}^+ \oplus \mathfrak{a}_p$ is a maximally split Cartan subalgebra of $\mathfrak{g}$. Let μ denote the inner automorphism of $\mathfrak{g}_C$ defined by

$$\mu = \exp\Big\{\frac{\pi}{4} \sum_{k=1}^{l} \mathrm{ad}(X_{\gamma_k} - X_{-\gamma_k})\Big\} \qquad \text{(Cayley transform).} \tag{3.8}$$

The μ carries $\mathfrak{j}_C$ bijectively to $\mathfrak{t}_C$ in such a way that

$$\begin{cases} \mu(H_k) = H'_{\gamma_k} \ (1 \leqq k \leqq l), \text{ whence } \mu(\mathfrak{a}_p) = \mathfrak{t}^-, \\ \mu | \mathfrak{t}^+ = \text{the identity map on } \mathfrak{t}^+. \end{cases} \tag{3.9}$$

Let $\Delta = \Delta(\mathfrak{g}_C, \mathfrak{j}_C)$ denote the root system of $\mathfrak{g}_C$ with respect to $\mathfrak{j}_C$. Put

$\Delta^+=\{\gamma\circ(\mu|\mathfrak{j}_C);\ \gamma\in\Sigma^+\}$, then Δ^+ gives a positive system of Δ. Let $\Lambda=\Lambda(\mathfrak{g}, \mathfrak{a}_p)$ be the (restricted) root system of $(\mathfrak{g}, \mathfrak{a}_p)$. Select a positive system Λ^+ of Λ compatible with Δ^+: $\Delta^+|\mathfrak{a}_p\subseteqq\Lambda^+\cup(0)$. Π denotes the set of simple roots in Λ^+. The Weyl group of $(\mathfrak{g}, \mathfrak{a}_p)$ is denoted by W. Set $\lambda_k=\gamma_k\circ(\mu|\mathfrak{a}_p)$ for $1\leqq k\leqq l$. Then (λ_k) is an orthogonal basis of $\mathfrak{a}_p^*$. Thanks to Proposition 3.2, one may describe Λ^+, Π and W by means of this basis (λ_k) as follows.

Theorem 3.3 [18, Theorems 2′ and 3]. (1) *There are two possibilities for the positive system* Λ^+:

(Case I) $\Lambda^+=\{2^{-1}(\lambda_k-\lambda_m);\ 1\leqq m<k\leqq l\}\cup\{2^{-1}(\lambda_k+\lambda_m);\ 1\leqq m\leqq k\leqq l\}$,

(Case II) $\Lambda^+=\{2^{-1}(\lambda_k-\lambda_m);\ 1\leqq m<k\leqq l\}\cup\{2^{-1}\lambda_k;\ 1\leqq k\leqq l\}\cup\{2^{-1}(\lambda_k+\lambda_m);\ 1\leqq m\leqq k\leqq l\}$.

(2) *According as* (Case I) *or* (Case II) *above,* Π *is expressed as*

(Case I) $\Pi=\{\lambda_1, 2^{-1}(\lambda_2-\lambda_1), \cdots, 2^{-1}(\lambda_l-\lambda_{l-1})\}$,

(Case II) $\Pi=\{2^{-1}\lambda_1, 2^{-1}(\lambda_2-\lambda_1), \cdots, 2^{-1}(\lambda_l-\lambda_{l-1})\}$.

(3) *W consists of all transforms of the form $\lambda_k\mapsto\pm\lambda_{\sigma(k)}$, where σ is an arbitrary permutation of $1, 2, \cdots, l$. In this sense, W is naturally isomorphic to the semidirect product group $\mathfrak{S}_l\ltimes(\boldsymbol{Z}/2\boldsymbol{Z})^l$. Here the action of the symmetric group $\mathfrak{S}_l$ of degree l on $(\boldsymbol{Z}/2\boldsymbol{Z})^l\cong\{1, -1\}^l$ is given as $\sigma\cdot(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l)=(\varepsilon_{\sigma^{-1}(1)}, \varepsilon_{\sigma^{-1}(2)}, \cdots, \varepsilon_{\sigma^{-1}(l)})$ for $\sigma\in\mathfrak{S}_l$ and $\varepsilon_k\in\{1, -1\}$ $(1\leqq k\leqq l)$.*

The (Case I) happens if and only if G/K is holomorphically equivalent to a tube domain. Moreover, non-compact simple Lie algebras $\mathfrak{g}$ of hermitian type are classified (up to isomorphisms) as follows (see [11, Chap. X]). Under the notation of Cartan,

(Case I) $\begin{cases}\text{AIII}(\mathfrak{g}=\mathfrak{su}(p, q);\ p=q),\ \text{CI}(\mathfrak{sp}(n, \boldsymbol{R})),\ \text{BDI}(\mathfrak{so}(p, q);\ q=2),\\ \text{DIII}(\mathfrak{so}^*(2n);\ n \text{ even}) \text{ and EVII.}\end{cases}$

(Case II) AIII($p\neq q$), DIII(n odd) and EIII.

3.2. A nilpotent G_C-orbit $\mathfrak{o}$ in $\mathfrak{g}_C$. Keep to the notations of 1.1 and 1.2 under the situation in 3.1. Let $\mathfrak{o}$ be the nilpotent G_C-orbit through $\sum_{1\leq k\leq l}X_{\gamma_k}\in\mathfrak{p}_+$. We describe in this section the Ad(G)-orbits ω_i in $\mathfrak{o}\cap\mathfrak{g}$, and we will study in the succeeding sections the (reduced) GGGRs attached to these ω_i's.. These kinds of GGGRs have the following important property: the direct sum of these GGGRs L^2-Γ_{ω_i} (unitarily induced) is quasi-equivalent to the regular representation of G (Theorem 4.2). This is one of the reasons why we deal with this orbit $\mathfrak{o}$.

Now let us begin with the following

Lemma 3.4. (1) *The dominant element $H(\mathfrak{o})\in\sqrt{-1}\mathfrak{h}^+\oplus\mathfrak{a}_p$ in Theorem* 1.4 *is given by $H(\mathfrak{o})=\sum_{1\leq k\leq l}H_k$ ($\mathfrak{h}^+=\mathfrak{t}^+$ in the notation of* 3.1). *In particular, $H(\mathfrak{o})$ is in $\mathfrak{a}_p$.*

(2) *The orbit* $\mathfrak{o}$ *intersects with* $\mathfrak{g}$: $\mathfrak{o}\cap\mathfrak{g}\neq\emptyset$.

Proof. By the definition of X_γ, H'_γ $(\gamma\in\Sigma)$, $(X_\gamma, H'_\gamma, X_{-\gamma})$ is an $\mathfrak{sl}_2$-triplet in $\mathfrak{g}_C$ for any $\gamma\in\Sigma$. Since γ_k $(1\leqq k\leqq l)$ are mutually strongly orthogonal, $(\sum_k X_{\gamma_k}, \sum_k H'_{\gamma_k}, \sum_k X_{-\gamma_k})$ is an $\mathfrak{sl}_2$-triplet, too. Notice that the Cayley transform μ carries $\sum_k H_k$ to $\sum_k H'_{\gamma_k}$. We thus find out that $\sum_k H_k\in\mathfrak{a}_p$ is a semisimple element of an $\mathfrak{sl}_2$-triplet containing an element of $\mathfrak{o}$ as nilpositive element. Moreover $\sum_k H_k$ is dominant with respect to the positive system Δ^+. By virtue of Theorem 1.4(1), one gets $H(\mathfrak{o})=\sum_k H_k\in\mathfrak{a}_p$. We have proved (1). The assertion (2) follows from (1) and Lemma 1.6. Q. E. D.

3.3. Nilpotent $\mathrm{Ad}(G)$**-orbits** ω_i **in** $\mathfrak{o}\cap\mathfrak{g}$. Hereafter, we are concerned with this orbit $\mathfrak{o}$ in $\mathfrak{g}_C$ and $\mathrm{Ad}(G)$-orbits in $\mathfrak{o}\cap\mathfrak{g}$. So, we use the notations of § 1 without the subscript $\mathfrak{o}$ if there is no danger of confusion: write $\mathfrak{g}(s)$, $\mathfrak{l}$, $\mathfrak{n}$, $\cdots$ for $\mathfrak{g}(s)_\mathfrak{o}$, $\mathfrak{l}_\mathfrak{o}$, $\mathfrak{n}_\mathfrak{o}$, $\cdots$.

Now let $\mathfrak{g}=\bigoplus_{s\in Z}\mathfrak{g}(s)$ be the eigenspace decomposition of $\mathfrak{g}$ for $\mathrm{ad}\,H(\mathfrak{o})$. Then, in view of Theorem 3.3, the s-eigenspaces $\mathfrak{g}(s)$ $(s\in \boldsymbol{Z})$ are described as

$$(3.10)\quad \begin{cases} \mathfrak{g}(2)=\sum_{1\leq m\leq k\leq l}\mathfrak{g}(\mathfrak{a}_p\,;2^{-1}(\lambda_k+\lambda_m)), \quad \mathfrak{g}(1)=\sum_{1\leq k\leq l}\mathfrak{g}(\mathfrak{a}_p\,;2^{-1}\lambda_k)\ \text{(possibly (0))}, \\ \mathfrak{g}(0)=\mathfrak{l}=\mathfrak{m}\oplus\mathfrak{a}_p\oplus\sum_{1\leq m\neq k\leq l}\mathfrak{g}(\mathfrak{a}_p\,;2^{-1}(\lambda_k-\lambda_m)) \quad \text{with } \mathfrak{m}=\mathfrak{z}_\mathfrak{k}(\mathfrak{a}_p), \end{cases}$$

$$(3.11)\qquad \mathfrak{g}(s)=\theta\mathfrak{g}(-s)\ (s\in\boldsymbol{Z}), \quad \mathfrak{g}(s)=(0)\ \text{ if }\ |s|\geqq 3.$$

The parabolic subalgebra $\mathfrak{p}_\mathfrak{o}=\bigoplus_{s\geq 0}\mathfrak{g}(s)$ is maximal, and $\mathfrak{p}_\mathfrak{o}=\mathfrak{l}\oplus\mathfrak{n}$ gives a Levi decomposition of $\mathfrak{p}_\mathfrak{o}$. Let $P=LN$ with $L=P\cap\theta P=Z_G(H(\mathfrak{o}))$, denote the corresponding decomposition of the parabolic subgroup P of G with the Lie algebra $\mathfrak{p}_\mathfrak{o}$. Then N is an at most two-step nilpotent Lie group. And N is abelian if and only if $\mathfrak{g}(1)=(0)$. This happens just in the (CASE I) of Theorem 3.3.

Let $\hat{\mathfrak{g}}(2)=\hat{\mathfrak{g}}(2)_\mathfrak{o}$ be the open $\mathrm{Ad}(L)$-invariant subset of $\mathfrak{g}(2)$ defined in Lemma 1.7: $\hat{\mathfrak{g}}(2)=\{X\in\mathfrak{g}(2)\,;[X,\mathfrak{g}(0)]=\mathfrak{g}(2)\}$. Then, thanks to Proposition 1.9, the problem of describing $\mathrm{Ad}(G)$-orbits in $\mathfrak{o}\cap\mathfrak{g}$ is completely reduced to describing the $\mathrm{Ad}(L)$-orbits in $\hat{\mathfrak{g}}(2)$. So, in the following, we first settle the latter problem. For this purpose, we make some preparations.

For $1\leqq k\leqq l$, set

$$(3.12)\qquad E_k=2^{-1}\sqrt{-1}(H'_{\gamma_k}-X_{\gamma_k}+X_{-\gamma_k}).$$

Then it is easily seen that $E_k\in\mathfrak{g}(\mathfrak{a}_p\,;\lambda_k)$ and that $\mu^{-1}(X_{\gamma_k})=\sqrt{-1}E_k$. Define a nilpotent element $A\in\mathfrak{g}(2)$ by

$$(3.13)\qquad A=\sum_{1\leq k\leq l}E_k.$$

Lemma 3.5. (1) $A\in\hat{\mathfrak{g}}(2)$, *that is, it holds that* $[A,\mathfrak{g}(0)]=\mathfrak{g}(2)$.

(2) $2B(Z_0, X)=B(\theta A, X)$ *for all* $X\in\mathfrak{g}(2)$. *Here* B *is the Killing form of* $\mathfrak{g}$ *and* Z_0 *the central element of* $\mathfrak{k}$ *in* (3.1).

(3) *The centralizer $Z_L(A)$ of A in L coincides with $K\cap L$, a maximal compact subgroup of L.*

Proof. (1) As in the proof of Lemma 3.4, $(-\sqrt{-1}\sum_k X_{\gamma_k}, \sum_k H'_{\gamma_k}, \sqrt{-1}\sum_k X_{-\gamma_k})$ forms an $\mathfrak{sl}_2$-triplet in $\mathfrak{g}_C$. The inverse of the Cayley transform μ carries this triplet to an $\mathfrak{sl}_2$-triplet $(A, H(o), \mu^{-1}(\sqrt{-1}\sum_k X_{-\gamma_k}))$ in $\mathfrak{g}$ (not just in $\mathfrak{g}_C$) because $A, H(o)\in\mathfrak{g}$. This implies that $A\in\hat{\mathfrak{g}}(2)$ by virtue of the theory of finite-dimensional representarions of $\mathfrak{sl}_2(\boldsymbol{C})$.

(2) Let $X\in\mathfrak{g}(2)$. Then we have

$$\begin{aligned}2B(Z_0, X)&=B(Z_0, [H(o), X])=B([Z_0, \textstyle\sum_k(X_{\gamma_k}+X_{-\gamma_k})], X)\\ &=\sqrt{-1}\textstyle\sum_k B(X_{\gamma_k}-X_{-\gamma_k}, X).\end{aligned}$$

Here the last equality follows from the fact that $\mathrm{ad}(Z_0)|\mathfrak{p}_\pm=\pm\sqrt{-1}I_{\mathfrak{p}}$ ($I_{\mathfrak{p}_\pm}=$ the identity operator on $\mathfrak{p}_\pm$). Moreover, since $B(\mathfrak{k}_C, \mathfrak{p}_C)=(0)$, one gets

$$\begin{aligned}\sqrt{-1}\textstyle\sum_k B(X_{\gamma_k}-X_{-\gamma_k}, X)&=\sqrt{-1}\textstyle\sum_k B(X_{\gamma_k}-X_{-\gamma_k}, 2^{-1}(X-\theta X))\\ &=\sqrt{-1}\textstyle\sum_k B(-H'_{\gamma_k}+X_{\gamma_k}-X_{-\gamma_k}, 2^{-1}(X-\theta X))\\ &=B(A, \theta X-X).\end{aligned}$$

Notice that $B(\mathfrak{g}(2), \mathfrak{g}(2))=(0)$ and that $A, X\in\mathfrak{g}(2)$. Thus we conclude that $2B(Z_0, X)=B(\theta A, X)$ $(X\in\mathfrak{g}(2))$ as desired.

(3) Let $k\in K\cap L$. Then it follows from the assertion (2) proved above

$$B(\theta\,\mathrm{Ad}(k)A, X)=2B(Z_0, \mathrm{Ad}(k)^{-1}X)=2B(X, Z_0)=B(X, \theta A)$$

for all $X\in\mathfrak{g}(2)$. Since the Killing form B restricted to $\mathfrak{g}(2)\times\mathfrak{g}(-2)$ is non-degenerate, we have $\mathrm{Ad}(k)A=A$, which means that $K\cap L\subseteq Z_L(A)$.

Put $N_m=\exp\mathfrak{n}_m$ with $\mathfrak{n}_m=\sum_{\lambda\in\Delta^+}\mathfrak{g}(\mathfrak{a}_p;\lambda)$. Then $G=KA_pN_m$ and $L=(K\cap L)A_p(N_m\cap L)$ give respectively Iwasawa decompositions of G and L. From this decomposition of L, we see $Z_L(A)=(K\cap L)Z_{S_0}(A)$, where $S_0=A_p(N_m\cap L)$ is a so-called Iwasawa subgroup of L. Since S_0 is exponential solvable, the stabilizer $Z_{S_0}(A)$ in S_0 is connected (see [3, I-3.3]). So, in order to complete the proof, it sufficies to prove that $\mathfrak{z}_{\mathfrak{s}_0}(A)$ $(=\{Y\in\mathfrak{s}_0; [A, Y]=0\})=(0)$, where $\mathfrak{s}_0$ is the Lie algebra of S_0. This is done in the following way. From Proposition 3.1 and Theorem 3.3, we have $\dim\mathfrak{g}(2)=\dim\mathfrak{s}_0$. By virtue of the assertion (1), the map $\mathrm{Ad}(A)|\mathfrak{g}(0)\colon\mathfrak{g}(0)\to\mathfrak{g}(2)$ is surjective, which implies that

$$\begin{aligned}\dim\mathrm{Ker}(\mathrm{ad}(A)|\mathfrak{g}(0))&=\dim\mathfrak{g}(0)-\dim\mathfrak{g}(2)\\ &=\dim\mathfrak{g}(0)-\dim\mathfrak{s}_0=\dim\mathfrak{k}\cap\mathfrak{l}.\end{aligned}$$

On the other hand, there holds that $\mathfrak{k}\cap\mathfrak{l}\subseteq\mathrm{Ker}(\mathrm{ad}(A)|\mathfrak{g}(0))$ because $K\cap L\subseteq Z_L(A)$. We thus obtain $\mathrm{Ker}(\mathrm{ad}(A)|\mathfrak{g}(0))=\mathfrak{k}\cap\mathfrak{l}$, or $\mathfrak{z}_{\mathfrak{s}_0}(A)=(0)$. Q. E. D.

The subspace $\mathfrak{p}\cap\mathfrak{l}$ as well as $\mathfrak{g}(2)$ is stable under the adjoint action of $K\cap L$. From Lemma 3.5, one gets immediately the following

Lemma 3.6. *The linear map* $\mathrm{ad}\,A$ *restricted to* $\mathfrak{p}\cap\mathfrak{l}$ *gives a bijection from* $\mathfrak{p}\cap\mathfrak{l}$ *to* $\mathfrak{g}(2)$ *commuting with the adjoint action of* $K\cap L$.

We now describe the $\mathrm{Ad}(K\cap L)$-orbits in $\mathfrak{p}\cap\mathfrak{l}$. Let $\mathfrak{a}_p^+=\{\sum_{1\leq k\leq l}y_k H_k;\ y_1\leqq y_2\leqq\cdots\leqq y_l\}$ be a closed Weyl chamber of $\mathfrak{a}_p$ with respect to the root system of $(\mathfrak{l}, \mathfrak{a}_p)$.

Proposition 3.7. *For an* $X\in\mathfrak{p}\cap\mathfrak{l}$, *there exists a unique* $H\in\mathfrak{a}_p^+$ *such that* $X=\mathrm{Ad}(k)H$ *for some* $k\in K\cap L$. *Therefore,* $\mathfrak{a}_p^+$ *parametrizes the set of* $\mathrm{Ad}(K\cap L)$-*orbits in* $\mathfrak{p}\cap\mathfrak{l}$.

This proposition is well-known (see e.g. [2, Proposition 1.3]). Nevertheless, we prove it here because it plays an essetial role for our description of nilpotent classes in $\mathfrak{o}\cap\mathfrak{g}$. (Prof. T. Hirai kindly showed us the outline of the following proof.)

Proof of Proposition 3.7. Let $X\in\mathfrak{p}\cap\mathfrak{l}$ be arbitrary. Take a maximal abelian subspace $\mathfrak{a}'$ of $\mathfrak{p}\cap\mathfrak{l}$ containing X. Since maximal abelian subspaces of $\mathfrak{p}\cap\mathfrak{l}$ are all conjugate under $K\cap L$, X is conjugate under $K\cap L$ to some $X'\in\mathfrak{a}_p$. Moreover X' is conjugate under the action of the Weyl group $W'\equiv N_{K\cap L}(\mathfrak{a}_p)/Z_{K\cap L}(\mathfrak{a}_p)$ of $(\mathfrak{l}, \mathfrak{a}_p)$, to some $H\in\mathfrak{a}_p^+$. We thus conclude that $X=\mathrm{Ad}(k)H$ for some $k\in K\cap L$ and some $H\in\mathfrak{a}_p^+$.

Next we show the uniqueness of $H\in\mathfrak{a}_p^+$. Let $H^{(1)}$ and $H^{(2)}$ be two elements of $\mathfrak{a}_p^+$ such that $\mathrm{Ad}(k)H^{(1)}=H^{(2)}$ for some $k\in K\cap L$. Denote by

$$\mathfrak{l}=\bigoplus_{r\in\boldsymbol{R}}\mathfrak{l}^{(i)}(r)\qquad(i=1, 2)\tag{3.14}$$

the eigenspace decomposition of $\mathfrak{l}$ with respect to $\mathrm{ad}\,H^{(i)}$, where $\mathfrak{l}^{(i)}(r)$ is the r-eigenspace for $\mathrm{ad}\,H^{(i)}$ on $\mathfrak{l}$. Since $H^{(i)}$ $(i=1, 2)$ are dominant, $\mathfrak{q}^{(i)}\equiv\bigoplus_{r\geq 0}\mathfrak{l}^{(i)}(r)$ are parabolic subalgebras of $\mathfrak{l}$ containing a minimal parabolic subalgebra $\mathfrak{m}\oplus\mathfrak{a}_p\oplus(\mathfrak{n}_m\cap\mathfrak{l})$ in commom. On the other hand, we have $\mathrm{Ad}(k)\mathfrak{q}^{(1)}=\mathfrak{q}^{(2)}$ by assumption. These two facts imply that

$$\mathfrak{q}^{(1)}=\mathfrak{q}^{(2)}\quad\text{and}\quad k\in Q^{(1)}\cap K,\tag{3.15}$$

where $Q^{(1)}\equiv N_L(\mathfrak{q}^{(1)})$ is the parabolic subgroup of L corresponding to $\mathfrak{q}^{(1)}$. This is because any two different parabolic subgroups of L can never be conjugate under L if they contain a minimal parabolic subgroup in common (see [27, p. 284]).

Notice that the Levi subgroup $Q^{(1)}\cap\theta Q^{(1)}$ coincides with the centralizer $Z_L(\mathfrak{a}^{(1)})$, where $\mathfrak{a}^{(1)}$ is the intersection of $\mathfrak{p}\cap\mathfrak{l}$ with the center of $\mathfrak{l}^{(1)}(0)$. Since $H^{(1)}\in\mathfrak{a}^{(1)}$, we have

$$Q^{(1)}\cap\theta Q^{(1)}=Z_L(\mathfrak{a}^{(1)})\subseteqq Z_L(H^{(1)}).$$

(3.15) together with this relation implies that $k\in Z_L(H^{(1)})\cap K$. We thus conclude $H^{(2)}=\mathrm{Ad}(k)H^{(1)}=H^{(1)}$. Q.E.D.

For $y=(y_1, y_2, \cdots, y_l)\in\boldsymbol{R}^l$, set $A(y)=\sum_{1\leq k\leq l}y_k E_k\in\mathfrak{g}(2)$. Then, the subset

$\mathfrak{g}^+(2) \equiv -[A, \mathfrak{a}_p^+] \subseteq \mathfrak{g}(2)$ is described as

$$\mathfrak{g}^+(2) = \{A(y);\ y_1 \leqq y_2 \leqq \cdots \leqq y_l\}. \tag{3.16}$$

In view of Lemma 3.6, we can translate Proposition 3.7 to describe $\mathrm{Ad}(K \cap L)$-orbits in $\mathfrak{g}(2)$ as follows.

Proposition 3.8. *The subset $\mathfrak{g}^+(2) \subseteq \mathfrak{g}(2)$ parametrizes the $\mathrm{Ad}(K \cap L)$-orbits in $\mathfrak{g}(2)$: for an $X \in \mathfrak{g}(2)$, there exists a unique $A(y) \in \mathfrak{g}^+(2)$ such that $A(y) \in \mathrm{Ad}(K \cap L)X$.*

Among the $\mathrm{Ad}(K \cap L)$-orbits in $\mathfrak{g}(2)$, those contained in the $\mathrm{Ad}(L)$-invariant open subset $\hat{\mathfrak{g}}(2)$ are characterized as follows.

Lemma 3.9. *Fer $y = (y_1, y_2, \cdots, y_l) \in \boldsymbol{R}^l$, the representative $A(y) = \sum_{1 \leq k \leq l} y_k E_k$ is in $\hat{\mathfrak{g}}(2)$ if and only if $y_k \neq 0$ for every $1 \leqq k \leqq l$.*

Proof. Recall the root space decomposition of $\mathfrak{g}(0)$:

$$\mathfrak{g}(0) = \mathfrak{m} \oplus \mathfrak{a}_p \bigoplus_{k \neq m} \mathfrak{g}(\mathfrak{a}_p;\ 2^{-1}(\lambda_k - \lambda_m)).$$

For any $y \in \boldsymbol{R}^l$, we see easily that

$$[A(y), \sum_{k \neq m} \mathfrak{g}(\mathfrak{a}_p;\ 2^{-1}(\lambda_k - \lambda_m)] \subseteq \sum_{k > m} \mathfrak{g}(\mathfrak{a}_p;\ 2^{-1}(\lambda_k + \lambda_m))$$

and

$$[A(y), H_k] = -2y_k E_k \qquad (1 \leqq k \leqq l).$$

Moreover, Lemma 3.5(3) together with the fact $[\mathfrak{m}, \mathfrak{g}(\mathfrak{a}_p;\ \lambda_k)] \subseteq \mathfrak{g}(\mathfrak{a}_p;\ \lambda_k)$ implies that $[A(y), \mathfrak{m}] = (0)$. Therefore, if $A(y) \in \hat{\mathfrak{g}}(2)$, i.e., $[A(y), \mathfrak{g}(0)] = \mathfrak{g}(2)$, then $y_k \neq 0$ for all $1 \leqq k \leqq l$.

Conversely, suppose that $y_k \neq 0$ for all k. Then we easily find a (unique) $\mathfrak{sl}_2$-triplet containing $A(y)$ and $H(o)$ as nilpositive and semisimple elements respectively. Hence $A(y)$ is in $\hat{\mathfrak{g}}(2)$. Q.E.D.

Consequently, the set

$$\hat{\mathfrak{g}}(2) \cap \mathfrak{g}^+(2) = \{A(y);\ y_1 \leqq y_2 \leqq \cdots \leqq y_l,\ y_k \neq 0 \text{ for all } k\}$$

parametrizes the $\mathrm{Ad}(K \cap L)$-orbits in $\hat{\mathfrak{g}}(2)$.

We now proceed to the description of $\mathrm{Ad}(L)$-orbits in $\hat{\mathfrak{g}}(2)$. For this purpose, the following proposition plays an essential role.

Proposition 3.10. *For $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l)$ with $\varepsilon_k \in \{1, -1\}$ $(1 \leqq k \leqq l)$, let H^ε be the centralizer of $A(\varepsilon)$ in L: $H^\varepsilon = Z_L(A(\varepsilon))$, and H_0^ε its identity component. Then L admits a decomposition $L = (K \cap L) A_p H_0^\varepsilon$ for every fixed ε.*

We prove this proposition in the next subsection 3.4. Its proof is based on the fact that the pair $(\mathfrak{l}, \mathfrak{h}^\varepsilon)$ with $\mathfrak{h}^\varepsilon \equiv \mathrm{Lie}\, H^\varepsilon$ has a structure of so-called reductive symmetric pair. The above decomposition is the corresponding Cartan decomposition of L.

Under these preparations, we achieve the purpose of this section.

Theorem 3.11. *The subset* $\hat{\mathfrak{g}}(2)\subseteq\mathfrak{g}(2)$ *splits into* $(l+1)$*-number of* $\mathrm{Ad}(L)$*-orbits* $\tilde{\omega}_i$ $(1\leqq i\leqq l)$: $\hat{\mathfrak{g}}(2)=\amalg_{0\leqq i\leqq l}\tilde{\omega}_i$ *(disjoint union), where* $\tilde{\omega}_i\equiv\mathrm{Ad}(L)A[i]$ *with* $A[i]\equiv A(\varepsilon(i))$, $\varepsilon(i)\equiv(\underbrace{-1,\cdots,-1}_{i},\underbrace{1,\cdots,1}_{l-i})$.

Proof. First we show that each $E\in\hat{\mathfrak{g}}(2)$ is conjugate to some $A[i]$ under $\mathrm{Ad}(L)$. In fact, by Proposition 3.8 and Lemma 3.9, E is conjugate under $\mathrm{Ad}(K\cap L)$ to some $A(y)$ such that

$$y_1\leqq y_2\leqq\cdots\leqq y_l \quad\text{and}\quad y_k\neq 0 \quad\text{for every } 1\leqq k\leqq l.$$

Let i be the largest integer such that $y_i<0$. (If $y_k>0$ for all k, put $i=0$.) Then $A(y)$ is conjugate to $A[i]$ under $\mathrm{Ad}(A_p)$. We thus get $\hat{\mathfrak{g}}(2)=\bigcup_{0\leqq i\leqq l}\tilde{\omega}_i$.

Secondly, let us prove that the union is disjoint. It follows from Proposition 3.10 that $\tilde{\omega}_i=\mathrm{Ad}((K\cap L)A_p)A[i]$. On the other hand, the right hand side is expressed as

$$\mathrm{Ad}(K\cap L)\{A(y);\ y_1\leqq y_2\leqq\cdots\leqq y_i<0<y_{i+1}\leqq\cdots\leqq y_l\}.$$

In view of Proposition 3.8, we obtain $\tilde{\omega}_i\cap\tilde{\omega}_j=\emptyset$ if $i\neq j$, which completes the proof. Q.E.D.

Remark 3.12. The vector space $\mathfrak{g}(2)$ has a structure of *Jordan algebra* equipped with the product $X\cdot Y$ of elements $X, Y\in\mathfrak{g}(2)$ given by $X\cdot Y=2^{-1}[[X,\theta A], Y]$. Namely, the bilinear map $\mathfrak{g}(2)\times\mathfrak{g}(2)\ni(X,Y)\mapsto X\cdot Y\in\mathfrak{g}(2)$ satisfies the following conditions: $X\cdot Y=Y\cdot X$, $X^2\cdot(X\cdot Y)=X\cdot(X^2\cdot Y)$, where $X^2=X\cdot X$. (These are easily verified.) Moreover, this Jordan algebra $\mathfrak{g}(2)$ is formally real, i.e., $X^2+Y^2=0$ implies $X=Y=0$.

I. Satake [25, Proposition 4] described the orbits in the set $\mathfrak{J}^\times$ of invertible elements of $\mathfrak{J}$ under the action of $(\mathrm{Str}\,\mathfrak{J})_0$, for any simple, formally real Jordan algebra $\mathfrak{J}$. Here $(\mathrm{Str}\,\mathfrak{J})_0$ is the identity component of the structure group $\mathrm{Str}\,\mathfrak{J}$ (see [25, p. 618]). In case of $\mathfrak{J}=\mathfrak{g}(2)$, one has $\mathfrak{J}^\times=\hat{\mathfrak{g}}(2)$, and $(\mathrm{Str}\,\mathfrak{J})_0$ contains the identity component of the group $\mathrm{Ad}(L)|\mathfrak{g}(2)\subseteq GL(\mathfrak{g}(2))$. So, if one applies Satake's description to $\mathfrak{J}=\mathfrak{g}(2)$, one can get Theorem 3.11 more easily.

Nevertheless, we did not choose this way for the following two reasons. One is for the sake of self-containedness. The other is because, in our proof, we come to encounter reductive symmetric pairs $(\mathfrak{l},\mathfrak{h}^\varepsilon)$ that play important roles in later sections.

In view of Proposition 1.9, the nilpotent $\mathrm{Ad}(G)$-orbits in $\mathfrak{o}\cap\mathfrak{g}$ are now completely described as in

Theorem 3.13. *Let* $\mathfrak{o}$ *be the nilpotent* G_C*-orbit throvgh the point* $\sum_{1\leqq k\leqq l}X_{\gamma_k}$. *Then, the intersection* $\mathfrak{o}\cap\mathfrak{g}$ *splits into* $(l+1)$*-number of nilpotent* $\mathrm{Ad}(G)$*-orbits* ω_i $(0\leqq i\leqq l)$: $\mathfrak{o}\cap\mathfrak{g}=\amalg_{0\leqq i\leqq l}\omega_i$. *Here, for each* i, ω_i *denotes the* $\mathrm{Ad}(G)$*-orbit through*

$A[i]=-\sum_{k\leq i}E_k+\sum_{m>i}E_m$.

3.4. The stabilizers H^{ε}, $\mathfrak{h}^{\varepsilon}$. For $\varepsilon=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l)\in\{1, -1\}^l$, let $A(\varepsilon)=\sum_{1\leq k\leq l}\varepsilon_k E_k$ be as in 3.3. In this subsection, we clarify the structure of the stabilizer H^{ε} (resp. $\mathfrak{h}^{\varepsilon}$) of $A(\varepsilon)$ in L (resp. in $\mathfrak{l}$), and then prove Proposition 3.10. At first, one should notice that, if $\varepsilon=\pm\mathbf{1}$ with $\mathbf{1}=(1, 1, \cdots, 1)$, H^{ε} and $\mathfrak{h}^{\varepsilon}$ have been already described in Lemma 3.5:

$$H^{\pm 1}=K\cap L, \quad \mathfrak{h}^{\pm 1}=\mathfrak{k}\cap\mathfrak{l}. \tag{3.17}$$

This description for $\mathfrak{h}^{\pm 1}$ can be generalized for any $\mathfrak{h}^{\varepsilon}$ as follows.

Lemma 3.14. *For any $\varepsilon\in\{1, -1\}^l$, the stabilizer $\mathfrak{h}^{\varepsilon}$ has a structure as*

$$\mathfrak{h}^{\varepsilon}=\mathfrak{m}\oplus\sum_{1\leq m<k\leq l}\{X+\varepsilon_k\varepsilon_m\theta X;\ X\in\mathfrak{g}(\mathfrak{a}_p;\ 2^{-1}(\lambda_k-\lambda_m))\}. \tag{3.18}$$

Proof. Let $W\in\mathfrak{l}$. According as the root space decomposition (3.10) of $\mathfrak{l}=\mathfrak{g}(0)$, express W as

$$W=\sum_{k>m}X_{km}^{+}+H+Y+\sum_{k>m}X_{km}^{-},$$

where $X_{km}^{\pm}\in\mathfrak{g}(\mathfrak{a}_p;\ \pm 2^{-1}(\lambda_k-\lambda_m))$, $Y\in\mathfrak{m}$ and $H\in\mathfrak{a}_p$. Put $[W, A(\varepsilon)]=\sum_{k\geq m}Z_{km}$ with $Z_{km}\in\mathfrak{g}(\mathfrak{a}_p;\ 2^{-1}(\lambda_k+\lambda_m))$. Then, a direct calculation yields

$$Z_{km}=\begin{cases}\varepsilon_m[X_{km}^{+}, E_m]+\varepsilon_k[X_{km}^{-}, E_k] & (k>m),\\ \varepsilon_k([Y, E_k]+\lambda_k(H)E_k) & (k=m).\end{cases} \tag{3.19}$$

By virtue of Lemma 3.5(3), it holds that

$$[X_{km}^{+}, E_m]=-[\theta X_{km}^{+}, E_k] \quad (k>m) \qquad \text{and} \quad [Y, E_k]=0. \tag{3.20}$$

From (3.19) and (3.20), we obtain

$$Z_{km}=\begin{cases}\varepsilon_k[X_{km}^{-}-\varepsilon_k\varepsilon_m\theta X_{km}^{+}, E_k] & (k>m),\\ \varepsilon_k\lambda_k(H)E_k & (k=m).\end{cases} \tag{3.21}$$

Notice that $\mathrm{ad}(E_k)|\mathfrak{g}(\mathfrak{a}_p;\ -2^{-1}(\lambda_k-\lambda_m))$ $(k>m)$ gives a bijection from $\mathfrak{g}(\mathfrak{a}_p;\ -2^{-1}(\lambda_k-\lambda_m))$ to $\mathfrak{g}(\mathfrak{a}_p;\ 2^{-1}(\lambda_k+\lambda_m))$ (cf. Proposition 3.1(2)). Then we deduce from (3.21) that $W\in\mathfrak{h}^{\varepsilon}$ if and only if $X_{km}^{-}=\varepsilon_k\varepsilon_m\theta X_{km}^{+}$ and $H=0$. This proves (3.18) as desired. Q.E.D.

Define a linear map θ_{ε} on $\mathfrak{l}$ by

$$\theta_{\varepsilon}(X)=\begin{cases}\varepsilon_k\varepsilon_m\theta X & (X\in\mathfrak{g}(\mathfrak{a}_p;\ 2^{-1}(\lambda_k-\lambda_m)),\ k\neq m),\\ \theta X & (X\in\mathfrak{m}\oplus\mathfrak{a}_p).\end{cases} \tag{3.22}$$

Then we can check easily that θ_{ε} gives an involution on $\mathfrak{l}$ commuting with $\theta|\mathfrak{l}$.

Lemma 3.15. (1) *The stabilizers $\mathfrak{h}^{\varepsilon}$ and H^{ε} are θ-stable,*

(2) *$\mathfrak{h}^{\varepsilon}$ coincides with the subalgebra of the fixed points of the involution θ_{ε}.*

Proof. The assertion (2) follows from Lemma 3.14 and the definition of θ_ε. Since θ_ε commutes with $\theta|\mathfrak{l}$, $\mathfrak{h}^\varepsilon$ is θ-stable. Let us prove that H^ε is θ-stable, too. Using the bracket relation (3.5), we can get $[E_k, -\theta E_k]=H_k$ $(1\leqq k\leqq l)$. Hence, $(A(\varepsilon), H(\mathrm{o}), -\theta A(\varepsilon))$ is an $\mathfrak{sl}_2$-triplet in $\mathfrak{g}$. If $x\in H^\varepsilon=Z_L(A(\varepsilon))$, then $(A(\varepsilon), H(\mathrm{o}), -\theta\,\mathrm{Ad}(\theta x)A(\varepsilon))$ is also an $\mathfrak{sl}_2$-triplet containing $A(\varepsilon)$ and $H(\mathrm{o})$ as nilpositive and semisimple elements respectively. In view of Proposition 1.3, such a triplet is unique, hence one gets $\mathrm{Ad}(\theta x)A(\varepsilon)=A(\varepsilon)$, or $\theta x\in H^\varepsilon$. This means that $\theta H^\varepsilon=H^\varepsilon$. Q.E.D.

This lemma tells us that $(\mathfrak{l}, \mathfrak{h}^\varepsilon)$ is a "reductive symmetric pair" (cf. [2, § 1]).

At the end of this section, we now prove Proposition 3.10 which gives a Cartan decomposition of L with respect to $(K\cap L, H_0^\varepsilon)$.

Proof of Proposition 3.10. The assertion is proved just as in [12, p. 118]. Let $\mathfrak{l}=\mathfrak{h}^\varepsilon\oplus\mathfrak{q}^\varepsilon$ be the eigenspace decomposition of $\mathfrak{l}$ with respect to θ_ε, where $\mathfrak{h}^\varepsilon$ and $\mathfrak{q}^\varepsilon$ are respectively the $(+1)$- and (-1)-eigenspaces. Since θ_ε commutes with $\theta|\mathfrak{l}$, $\mathfrak{l}$ is decomposed as

$$\mathfrak{l}=\mathfrak{k}\cap\mathfrak{h}^\varepsilon\oplus\mathfrak{k}\cap\mathfrak{q}^\varepsilon\oplus\mathfrak{p}\cap\mathfrak{h}^\varepsilon\oplus\mathfrak{p}\cap\mathfrak{q}^\varepsilon. \tag{3.23}$$

Moreover, $\mathfrak{h}^\varepsilon\cap\mathfrak{p}$ is a Lie triple system contained in $\mathfrak{p}\cap\mathfrak{l}$, i.e., $[X, [Y, Z]]\in\mathfrak{h}^\varepsilon\cap\mathfrak{p}$ for any $X, Y, Z\in\mathfrak{h}^\varepsilon\cap\mathfrak{p}$. Hence, by virtue of [10, Chap. VI, Theorem 1.4], we have

$$L=(K\cap L)\exp(\mathfrak{p}\cap\mathfrak{q}^\varepsilon)\exp(\mathfrak{p}\cap\mathfrak{h}^\varepsilon). \tag{3.24}$$

Let $\mathfrak{l}(\varepsilon)$ denote the subalgebra of fixed points of the involution $\theta\theta_\varepsilon$. Then $\mathfrak{l}(\varepsilon)$ is a θ-stable, reductive subalgebra of $\mathfrak{l}$, and $\mathfrak{l}(\varepsilon)=\mathfrak{k}\cap\mathfrak{h}^\varepsilon\oplus\mathfrak{p}\cap\mathfrak{q}^\varepsilon$ gives its Cartan decomposition. Notice that $\mathfrak{a}_p\subseteqq\mathfrak{p}\cap\mathfrak{q}^\varepsilon$, hence $\mathfrak{a}_p$ is a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}^\varepsilon$. Since all the maximal abelian subspaces of $\mathfrak{p}\cap\mathfrak{q}^\varepsilon$ are conjugate under the identity component $(H^\varepsilon\cap K)_0$ of $H^\varepsilon\cap K$, we obtain

$$\exp(\mathfrak{p}\cap\mathfrak{q}^\varepsilon)\subseteqq(H^\varepsilon\cap K)_0A_p(H^\varepsilon\cap K)_0. \tag{3.25}$$

(3.24) and (3.25) prove the proposition. Q.E.D.

§ 4. (Reduced) Generalized Gelfand-Graev representations associated with the nilpotent classes ω_i

Let ω_i $(0\leqq i\leqq l)$ be the nilpotent classes of $\mathfrak{g}$ in Theorem 3.13. Consider the GGGRs (see Definition 1.11)

$$\Gamma_i\equiv\Gamma_{\omega_i}=\mathrm{Ind}_N^G(\xi_i)\quad\text{with}\quad\xi_i\equiv\xi_{A[i]}\in\hat{N}\ (N=N_{\mathrm{o}}) \tag{4.1}$$

associated with ω_i. Here o is the G_C-orbit containing ω_i. In this section, we first clarify in 4.1 the relationship between L^2-Γ_i (unitarily induced) and the left regular representation λ_G of G (Theorem 4.2). There, we find out that the direct sum $\oplus_{0\leq i\leq l}L^2$-$\Gamma_i$ is quasi-equivalent to λ_G. This shows the importance of these kinds of GGGRs.

In 4.2, we construct reduced GGGRs (=RGGGRs) associated with ω_i after Definition 2.6. But, in that definition, the following property for ξ_i is supposed:

ξ_i extends canonically to a genuine (not just projective) unitary representation $\tilde{\xi}_i$ of the semidirect product subgroup $H^i N \subseteqq P$, where $H^i \equiv H^{\varepsilon(i)}$ (see 3.3) is the stabilizer of $A[i]$ in the Levi subgroup $L=L_0$ ($H^i=H_0(A[i])$ in the notation of § 2).

So we prove the above property by applying the criterion in Proposition 2.2.

In 4.3, we give another type of realization of ξ_i, so-called Fock model, which is more convenient for our later purpose than the Kirillov model ξ_i. This new realization enables us to describe the extension $\tilde{\xi}_i$ explicitly.

4.1. Relation between GGGRs L^2-Γ_i and the regular representation λ_G.

Let $S=A_p N_m$ with $N_m=\exp\{\sum_{\lambda\in\Delta^+}\mathfrak{g}(\mathfrak{a}_p;\lambda)\}$ be an Iwasawa subgroup of G. Then, according to Duflo-Raïs [5] and Nomura [20], the left regular representation of S is decomposed into irreducibles in the following manner.

Theorem 4.1. (1) *The unitarily induced representations* $\pi_\varepsilon \equiv L^2\text{-}\mathrm{Ind}_N^S(\xi_{A(\varepsilon)})$ *are irreducible for all* $\varepsilon=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l)\in\{1, -1\}^l$, *and they are mutually inequivalent. Here* $A(\varepsilon)=\sum_{1\leq k\leq l}\varepsilon_k E_k$, *and* $\xi_{A(\varepsilon)}$ *is the irreducible unitary representation of N corresponding to the coadjoint N-orbit* $[A(\varepsilon)^*]$ *through* $A(\varepsilon)^*$ (*see* 1.3).

(2) *One can give the irreducible decomposition of the left regular representation* λ_S *of S as*

$$\lambda_S \cong \bigoplus_{\varepsilon\in\{1,-1\}^l}[\infty]\cdot\pi_\varepsilon, \tag{4.2}$$

where $[\infty]\cdot\pi_\varepsilon$ *means the infinite multiple of* π_ε.

Note. We comment here on the proof of this theorem. Let $\mathfrak{s}$ be the Lie algebra of S, then it is expressed as $\mathfrak{s}=\mathfrak{s}_0\oplus\mathfrak{n}$ with $\mathfrak{s}_0=\mathfrak{a}_p\oplus(\mathfrak{n}_m\cap\mathfrak{l})$. Extending each element of $\mathfrak{n}^*$ to a linear form on $\mathfrak{s}$ trivial on $\mathfrak{s}_0$, we identify $\mathfrak{n}^*$ with a subspace of $\mathfrak{s}^*$. T. Nomura described in [20, Proposition 1.4] the open coadjoint S-orbits in $\mathfrak{s}^*$: the map $\varepsilon\mapsto\mathrm{Ad}^*(S)A(\varepsilon)^*$ sets up a one-to-one correspondence from $\{1, -1\}^l$ onto the set of open $\mathrm{Ad}^*(S)$-orbits in $\mathfrak{s}^*$. (He gave there such a description more generally for exponential solvable Lie groups corresponding to normal j-algebras.) Moreover, the irreducible represetation π_ε corresponds to the orbit $\mathrm{Ad}^*(S)A(\varepsilon)^*$ via the Kirillov-Bernat correspondence (see [20, proof of Proposition 5.9]).

These two results of Nomura enable us to apply explicitly to our group S the general theory of Duflo-Raïs [5, p. 132] on the Plancherel theorem for exponential solvable Lie groups. As a consequence, one can get the above theorem.

In view of Theorem 3.11, for each $\varepsilon=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_l)\in\{1, -1\}^l$, $A(\varepsilon)$ is conjugate to $A[i(\varepsilon)]$ under $\mathrm{Ad}(L)$, where $i(\varepsilon)$ is the number of elements ε_k such that $\varepsilon_k=-1$. From this fact together with the stage theorem for unitarily induced representations, we obtain

(4.3) $$L^2\text{-Ind}_S^P(\pi_\varepsilon)\cong L^2\text{-Ind}_N^P(\xi_{A(\varepsilon)})\cong L^2\text{-Ind}_N^P(\xi_{i(\varepsilon)}).$$

Moreover, keeping (4.2) in mind, we find out that the left regular representation λ_P of P is decomposed as

(4.4) $$\lambda_P\cong \bigoplus_{\varepsilon\in\{1,-1\}^l}[\infty]\cdot L^2\text{-Ind}_S^P(\pi_\varepsilon)\cong \bigoplus_{0\leq i\leq l}[\infty]\cdot L^2\text{-Ind}_N^P(\xi_i).$$

Consequently, inducing up the representations in (4.4) to G, one gets

Theorem 4.2. *The left regular representation λ_G of G splits into a unitary direct sum of the infinite multiples of GGGRs L^2-Γ_i: $\lambda_G\cong\bigoplus_{0\leq i\leq l}[\infty]\cdot L^2$-$\Gamma_i$.*

Recall that our simple group G has a compact Cartan subalgebra (see 3.1). Therefore, according to Harish-Chandra, G admits discrete series representations (=irreducible subrepresentations of λ_G). So, we deduce immediately from the above theorem the following

Corollary 4.3. *Let D be a discrete series rspresentation of G. Then, there exists $0\leqq i\leqq l$ such that D occurs in the GGGR L^2-Γ_i as a subrepresentation.*

These theorem and corollary show that the GGGRs L^2-Γ_i $(0\leqq i\leqq l)$ are important in connection with the regular representation, and, especially with its discrete spectrum. Moreover, the above corollary naturally gives rise to the following

Problem EDS. Describe the embeddings of discrete series representations into GGGRs Γ_i.

In the forthcoming paper [33], we will treat this problem for holomorphic (or anti-holomorphic) discrete series representations using the method of Hashizume [9]. For our discussion there, the argument (but not the result) of Rossi-Vergne [24] plays an important role. We describe the embeddings of such type of discrete series into Γ_i's.

4.2. RGGGRs $\Gamma_i(c)$. For $0\leqq i\leqq l$, let $H^i=H^{\varepsilon(i)}$ be the centralizer of $A[i]=-\sum_{k\leq i}E_k+E_{m>i}E_m$ in the Levi subgroup L. By Lemmas 2.1 and 3.15, H^i coincides with the stabilizer of the equivalence class of ξ_i in L, denoted by $H_0(A[i])$ in the notation of § 2. As was seen in § 2, ξ_i extends generally to a projective representation of the semidirect product subgroup H^iN acting on the same Hilbert space. Moreover, in Proposition 2.2, we have given a nice sufficient condition for such an extension to be a genunine representation. Making use of this criterion, we prove here that ξ_i extends to a genuine representation $\hat{\xi}_i$ for any $0\leqq i\leqq l$. For this purpose, we have only to construct an $\mathrm{Ad}(H^i)$-invariant complex structure on $\mathfrak{g}(1)$. This is achieved in the following way.

Let r be a linear map from $\mathfrak{s}=\mathfrak{a}_p\oplus\mathfrak{n}_m$ to $\mathfrak{p}$ defined by

(4.5) $$r(X)=2^{-1}(X-\theta X) \quad \text{for} \quad X\in\mathfrak{s}.$$

Then, in view of the decompositions $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}=\mathfrak{k}\oplus\mathfrak{s}$, r gives a bijection. Therefore, the $\mathrm{Ad}(K)$-invariant complex structure J on $\mathfrak{p}$ (see (3.1)) is transferred to a complex structure J' on $\mathfrak{s}$ through r:

(4.6) $$J'=r^{-1}\circ J\circ r.$$

We show that the restriction $J''\equiv J'|\mathfrak{g}(1)$ is the desired complex structure on $\mathfrak{g}(1)$.

Lemma 4.4. (1) *For any* $1\leqq k\leqq l$, *the complexification* $\mathfrak{g}(\mathfrak{a}_p; 2^{-1}\lambda_k)_C$ *of the root space* $\mathfrak{g}(\mathfrak{a}_p; 2^{-1}\lambda_k)$ *is expressed as*

(4.7) $$\mathfrak{g}(\mathfrak{a}_p; 2^{-1}\lambda_k)_C=\mu^{-1}\Big(\sum_{\gamma\in P_k\cup C_k}\mathfrak{g}_C(\mathfrak{t}_C; \gamma)\Big),$$

where μ *is the Cayley transform in* (3.8), *and* P_k *and* C_k *are the subsets of* $\Sigma=\Sigma(\mathfrak{g}_C, \mathfrak{t}_C)$ *in Proposition* 3.1.

(2) *The subspaces* $\mathfrak{g}(\mathfrak{a}_p; 2^{-1}\lambda_k)$ $(1\leqq k\leqq l)$ *are stable under* J'.

(3) *Extend* J' *to a map on* $\mathfrak{s}_C$ *by complex linearity. Then, the* $(\pm\sqrt{-1})$-*eigenspaces* $V^{\pm}(k)$ *for* J' *in* $\mathfrak{g}(\mathfrak{a}_p; 2^{-1}\lambda_k)_C$ *are described as*

(4.8) $$V^{+}(k)=\mu^{-1}\Big\{\sum_{\gamma\in P_k}\mathfrak{g}_C(\mathfrak{t}_C; \gamma)\Big\}, \quad V^{-}(k)=\mu^{-1}\Big\{\sum_{\gamma\in C_k}\mathfrak{g}_C(\mathfrak{t}_C; \gamma)\Big\}.$$

Proof. The assertion (1) follows from Proposition 3.2 and Theorem 3.3. In order to prove (2) and (3), we calculate $J'(\mu^{-1}(X_\gamma))$ $(\gamma\in P_k\cup C_k)$. Keeping in mind Moore's restricted root theorem (Theorem 3.3) and (3.5), one has for $\gamma\in P_k\cup C_k$

$$\sum_{1\leq m\leq l}\mathrm{ad}(X_{-\gamma_m}-X_{\gamma_m})X_\gamma=[X_{-\gamma_k}, X_\gamma],$$

$$\Big\{\sum_{1\leq m\leq l}\mathrm{ad}(X_{-\gamma_m}-X_{\gamma_m})\Big\}^2X_\gamma=-[X_{\gamma_k}, [X_{-\gamma_k}, X_\gamma]]=-[H'_{\gamma_k}, X_\gamma]=-X_\gamma.$$

Using these relations we see easily

(4.9)
$$\begin{aligned}\mu^{-1}(X_\gamma)&=\sum_{j\geq 0}\frac{1}{j!}\Big(\frac{\pi}{4}\sum_{1\leq m\leq l}\mathrm{ad}(X_{-\gamma_m}-X_{\gamma_m})\Big)^jX\\&=\cos\frac{\pi}{4}X_\gamma+\sin\frac{\pi}{4}[X_{-\gamma_k}, X_\gamma]=\frac{1}{\sqrt{2}}(X_\gamma+[X_{-\gamma_k}, X_\gamma]).\end{aligned}$$

By virtue of (4.9), $J'(\mu^{-1}(X_\gamma))$ can be calculated as follows.

Case 1. If $\gamma\in P_k$, then one has $J'(\mu^{-1}(X_\gamma))=\sqrt{-1}\mu^{-1}(X_\gamma)$. Indeed,

$$\begin{aligned}J'(\mu^{-1}(X_\gamma))&=2^{-1/2}(r^{-1}\circ J\circ r)(X_\gamma+[X_{-\gamma_k}, X_\gamma]) && \text{(by (4.9))}\\&=2^{-1/2}(r^{-1}\circ J)(X_\gamma) && \text{(since } X_\gamma\in\mathfrak{p}_C \text{ and } [X_{-\gamma_k}, X_\gamma]\in\mathfrak{k}_C)\\&=2^{-1/2}\sqrt{-1}r^{-1}(X_\gamma) && \text{(by (3.1))}\\&=\sqrt{-1}\mu^{-1}(X_\gamma).\end{aligned}$$

Case 2. Let $\gamma \in C_k$. A calculation similar to that in Case 1 leads us to $J'(\mu^{-1}(X_\gamma)) = -\sqrt{-1}\mu^{-1}(X_\gamma)$.

Thus we have proved the assertions (2) and (3) at the same time.

Q. E. D.

By virtue of this lemma, it becomes clear that the restriction J'' actually gives a complex structure on $\mathfrak{g}(1)$.

Proposition 4.5. *The complex structure J'' on $\mathfrak{g}(1)$ is invariant under the adjoint action of L: $\mathrm{Ad}(x) \circ J'' \circ \mathrm{Ad}(x^{-1}) = J''$ for all $x \in L$.*

Proof. It sufficies to prove that the eigenspaces $V^{\pm} = \sum_{1 \leq k \leq l} V^{\pm}(k) \subseteqq \mathfrak{g}(1)_C$ for (the complex linear extension of) J'' are stable under $\mathrm{Ad}(L)$. We see from Lemma 4.4

$$V^{+} = \mathfrak{g}(1)_C \cap \mu^{-1}(\mathfrak{p}_{+}), \quad V^{-} = \mathfrak{g}(1)_C \cap \mu^{-1}(\mathfrak{k}_C). \tag{4.10}$$

In view of Proposition 3.1, one can get esaily that $\mu(\mathfrak{l}_C) \subseteqq \mathfrak{k}_C$. (Moreover, the equality holds if and only if $\mathfrak{g}(1) = (0)$, which happens just in the (CASE I) of Theorem 3.3.) This inclusion implies that

$$L_C \subseteqq \mu^{-1} K_C \mu, \tag{4.11}$$

where K_C and L_C denote respectively the analytic subgroups of $G_C = \mathrm{Int}(\mathfrak{g}_C)$ corresponding to $\mathfrak{k}_C$ and $\mathfrak{l}_C$. Notice that $\mathrm{Ad}(L)$ is contained in L_C and that $\mathfrak{k}_C$ and $\mathfrak{p}_{+}$ are stable under the adjoint action of K_C. Then we get from (4.10) and (4.11) the desired reselt: $\mathrm{Ad}(x) V^{\pm} = V^{\pm}$ for all $x \in L$. Q. E. D.

Thanks to this proposition, one can apply Proposition 2.2 successfully to $\xi_i = \xi_{A[i]}$, and gets the following

Theorem 4.6. *For each $0 \leqq i \leqq l$, the rpresentation ξ_i of N extends to a genuine unitary representation $\tilde{\xi}_i$ of the semidirect product group $H^i N$ acting on the same Hilbert space.*

Thus, in view of Definition 2.6, the RGGGRs $\Gamma_i(c) \equiv \Gamma_{\omega_i}(c)$ associated to ω_i are constructed as

$$\Gamma_i(c) = (L^2\text{- or } C^\infty\text{-})\,\mathrm{Ind}_{H^i N}^{G}(\tilde{c} \otimes \tilde{\xi}_i) \quad (\tilde{c} = c \otimes 1_N), \tag{4.12}$$

where c ranges over irreducible (unitary; in case of L^2-induction) representations of the reductive group H^i.

For each $0 \leqq i \leqq l$, the important type of GGGR L^2-Γ_i is decomposed into a direct integral of the RGGGRs L^2-$\Gamma_i(c)$ (see (2.8)). Especially, if $i=0$ or l, then the GGGR splits into a direct sum of the corresponding RGGGRs:

$$\Gamma_i = \bigoplus_{c \in (K \cap L)^\wedge} [\dim c] \cdot \Gamma_i(c) \quad (i = 0, l), \tag{4.13}$$

since $H^0 = H^l = K \cap L$ is a compact group.

Henceforth, we concentrate on these RGGGRs $\Gamma_i(c)$ $(0\leqq i\leqq l)$, and give in §6 finite multiplicity theorems for them by applying the results of [I]. For this purpose, we make some preparations.

4.3. The Fock model ρ_i. In the definition of the GGGRs $\Gamma_\omega=\mathrm{Ind}_{N_0}^G(\xi_X)$ $(o=G_C\cdot\omega,\ X\in\omega,\ X\in\hat{\mathfrak{g}}(2)_0$; see Definition 1.11), we constructed the irreducible unitary representation ξ_X of N_0 making use of a real polarization at X^*. But, in our specified case $\omega=\omega_i$, $X=A[i]$, it is convenient for the laetr use to adopt another type of realization of $\xi_i=\xi_{A[i]}$, so-called Fock model. Here, we explain such kind of realization mainly after [20].

We first construct a positive polarization at $A[i]^*$.

Let Ω be a linear form on the Iwasawa subalgebra $\mathfrak{s}=\mathfrak{a}_p\oplus\mathfrak{n}_m$ given as

$$\Omega(X)=B(Z_0, X)\quad (X\in\mathfrak{s}), \tag{4.14}$$

where Z_0 is the central element of $\mathfrak{k}$ in (3.1). Then, according to Rossi-Vergne, Ω has the following property:

Proposition 4.7 [24, p. 372]. *It holds that for any $X, Y\in\mathfrak{s}$,*

(1) $\Omega([J'X, J'Y])=\Omega([X, Y])$,

(2) $\Omega([J'X, X])\leqq 0$. *Moreover, the equality holds if and only if $X=0$. Here J' is the complex structure on $\mathfrak{s}$ in (4.6).*

Remark 4.8. Using this proposition, one can show that the triplet (s, Ω, J') has a structure of normal j-algebra (see Definition 1.1 of [20]). Therefore, Nomura's results in [20] for solvable Lie groups associated with normal j-algebras can be applied to our situation.

For any $0\leqq i\leqq l$, let J_i'' be the complex structure on $\mathfrak{g}(1)$ defined by

$$J_i''(X)=\begin{cases} -J''(X) & \text{if } X\in\sum_{k\leq i}\mathfrak{g}(\mathfrak{a}_p;2^{-1}\lambda_k),\\ J''(X) & \text{if } X\in\sum_{m>i}\mathfrak{g}(\mathfrak{a}_p;2^{-1}\lambda_m).\end{cases} \tag{4.15}$$

Notice that $J_0''=J''$ and $J_l''=-J''$. Extend J_i'' to a complex linear map on $\mathfrak{g}(1)_C$. Then, the $(\pm\sqrt{-1})$-eigenspaces $V_i^\pm$ for J_i'' are described as

$$V_i^\pm=(\bigoplus_{k\leq i}V^\mp(k))\oplus(\bigoplus_{m>i}V^\pm(m)),\qquad \overline{V_i^+}=V_i^-, \tag{4.16}$$

where the bar means the complex conjugation in $\mathfrak{g}_C$ with respect to the real form $\mathfrak{g}$.

Proposition 4.9. *For each $0\leqq i\leqq l$, the alternating bilinear form*

$$(X, Y)\longmapsto A[i]^*([J_i''X, Y])=B(\theta A[i], [J_i''X, Y]) \tag{4.17}$$

on $\mathfrak{g}(1)\times\mathfrak{g}(1)$ is negative definite. Moreover it holds that

$$A[i]^*([J_i''X, J_i''Y])=A[i]^*([X, Y])\quad \textit{for all}\ X, Y\in\mathfrak{g}(1). \tag{4.18}$$

Proof. The assertions for $A=A[0]$ follow from Lemma 3.5 and Proposition 4.7. For general $0\leqq i\leqq l$, we can check easily

$$A[i]^*([J_i''X, Y])=A^*([J''X, Y]) \qquad (X, Y\in\mathfrak{g}(1)),$$

which proves (4.17) for any i. To prove (4.18), express X, Y as $X=\sum_k X_k$, $Y=\sum_k Y_k$ with $X_k, Y_k\in\mathfrak{g}(\mathfrak{a}_p; 2^{-1}\lambda_k)$. Then the left hand side of (4.18) is calculated as

$$\begin{aligned}A[i]^*([J_i''X, J_i''Y])&=-\sum_{k\leq i}A^*([J''X_k, J''Y_k])+\sum_{m>i}A^*([J''X_m, J''Y_m])\\&=-\sum_{k\leq i}A^*([X_k, Y_k])+\sum_{m>i}A^*([X_m, Y_m])\\&=A[i]^*([X, Y]).\end{aligned}$$

For the second equality, we used (4.18) for $A=A[0]$ (already proved). This completes the proof. Q. E. D.

Now we put for each $0\leqq i\leqq l$,

$$\mathfrak{u}_i=V_i^-\oplus\mathfrak{g}(2)_C. \tag{4.19}$$

Then $\mathfrak{u}_i$ is a complex subalgebra of $\mathfrak{n}_C$ because $\mathfrak{g}(2)_C$ is an ideal of $\mathfrak{n}_C$.

Proposition 4.10 [20, 2.2]. *The subalgebra $\mathfrak{u}_i\subseteqq\mathfrak{n}_C$ is a totally complex, positive polarization at $A[i]^*\in\mathfrak{n}^*$. Namely, it satisfies the following three conditions:*

(1) *$\mathfrak{u}_i$ is a maximally totally isotropic subspace (=a Lagrangian subspace) with respect to the skew-symmetric form $b_i: (X, Y)\mapsto A[i]^*([X, Y])$ on $\mathfrak{n}_C\times\mathfrak{n}_C$,*

(2) *the hermitian form $h_i: (Z, W)\mapsto\sqrt{-1}A[i]^*([Z, \overline{W}])$ on $\mathfrak{u}_i\times\mathfrak{u}_i$ is positive semidefinite,*

(3) $\mathfrak{u}_i+\overline{\mathfrak{u}_i}=\mathfrak{n}_C$.

Proof. We prove here this proposition in order to clarify our arguments.

(1) Let $X, Y\in\mathfrak{u}_i$. Then, they are expressed uniquely as

$$X=X_1+\sqrt{-1}J_i''X_1+X_2, \quad Y=Y_1+\sqrt{-1}J_i''Y_1+Y_2$$

with $X_1, Y_1\in\mathfrak{g}(1)$ and $X_2, Y_2\in\mathfrak{g}(2)_C$. Then, a simple calculation leads to

$$\begin{aligned}A[i]^*([X, Y])&=A[i]^*([X_1, Y_1]-[J_i''X_1, J_i''Y_1])\\&\quad+\sqrt{-1}A[i]^*([X_1, J_i''Y_1]+[J_i''X_1, Y_1]).\end{aligned}$$

The right hand side is equal to 0 thanks to (4.18). This means that $b_i([\mathfrak{u}_i, \mathfrak{u}_i])=(0)$, i.e., $\mathfrak{u}_i$ is a totally isotropic subspace for b_i. On the other hand, we see easily that the radical of b_i coincides with $\mathfrak{g}(2)_C$. So, the dimension of any Lagrangian subspace must be equal to $2^{-1}\dim\mathfrak{g}(1)_C+\dim\mathfrak{g}(2)_C=\dim\mathfrak{u}_i$. This proves that $\mathfrak{u}_i$ itself is a Lagrangian subspace.

(2) Let $X\in\mathfrak{u}_i$ be as above. Then $h_i(X, X)$ is calculated as

$$h_i(X, X)=h_i(X_1+\sqrt{-1}J_i''X_1, X_1+\sqrt{-1}J_i''X_1)$$

$$=\sqrt{-1}A[i]^*([X_1+\sqrt{-1}J_i''X_1,\ X_1-\sqrt{-1}J_i''X_1])$$
$$=-2A[i]^*([J_i''X_1,\ X_1]).$$

By virtue of Proposition 4.9, h_i is positive semidefinite.

(3) follows from (4.16). Q. E. D.

The general theory for holomorphically induced representations (see e. g. [3, Chap. VIII]) tells us how to construct the irreducible unitary representation of N corresponding to $A[i]^*$ through the Kirillov correspondence, by making use of the positive polarization $\mathfrak{u}_i$. This is done in the following way. We identify the enveloping algebra $U(\mathfrak{n}_C)$ of $\mathfrak{n}_C$ with the algebra of left N-invariant differential operators on N via

$$R(X)f(n)=\frac{d}{dt}f(n\exp tX)|_{t=0} \qquad (n\in N)$$

for $X\in\mathfrak{n}$ and $f\in C^\infty(N)$. Denote by N_2 the center of N: $N_2=\exp\mathfrak{g}(2)$. Let $\mathcal{H}_i$ be the space of C^∞-functions f on N satisfying

$$R(Y)f=-\sqrt{-1}A[i]^*(Y)f \qquad \text{for all} \quad Y\in\mathfrak{u}_i, \tag{4.20}$$

$$\|f\|^2_{\mathcal{H}_i}=\int_{N/N_2}|f(\dot{n})|^2d\dot{n}<+\infty, \tag{4.21}$$

where $N\ni n\mapsto\dot{n}\in N/N_2$ denotes the canonical homomorphism, and $d\dot{n}$ the N-invariant measure on the factor group N/N_2.

One should note the following point. Since N_2 is simply connected, (4.20) implies that

$$f(nn_2)=\eta_i(n_2)^{-1}f(n) \qquad (n\in N,\ n_2\in N_2), \tag{4.22}$$

where η_i is the unitary character of N_2 given as

$$\eta_i(\exp Y)=\exp\{\sqrt{-1}A[i]^*(Y)\} \qquad (Y\in\mathfrak{g}(2)).$$

By virtue of (4.22), the function $\dot{n}\mapsto|f(\dot{n})|$ is well-defined on N/N_2. So, the integral (4.21) has a meaning.

In the present case, one can show that $\mathcal{H}_i$ has a structure of Hilbert space with the inner product induced from the norm $\|\cdot\|_{\mathcal{H}_i}$. (We need not take the completion.) The group N acts on $\mathcal{H}_i$ unitarily, through the left translations, so we thus get a unitary representation $(T_i, \mathcal{H}_i)$ of N by putting $T_i(n)f(n')=f(n^{-1}n')$ $(n\in N,\ f\in\mathcal{H}_i)$. Then, T_i is irreducible and it corresponds to $A[i]^*$ through the Kirillov correspondence: $T_i\cong\xi_i$.

We now realize the representation T_i in a more explicit manner. Notice that the exponential map $\exp:\mathfrak{n}\to N$ gives a diffeomorphism. Therefore, we can identify $\mathfrak{n}=\mathfrak{g}(1)\oplus\mathfrak{g}(2)$ with N through this map:

$$\mathfrak{n}=\mathfrak{g}(1)\oplus\mathfrak{g}(2)\ni(X,\ Y)\longmapsto n(X,\ Y)\equiv\exp(X,\ Y)\in N. \tag{4.23}$$

Then, in view of the Campbell-Hausdorff formula, the group law on N is rewritten as

$$(X_1, Y_1)\cdot(X_2, Y_2)=(X_1+X_2, Y_1+Y_2+2^{-1}[X_1, X_2]). \tag{4.24}$$

Moreover, we can identify canonically the factor space N/N_2 gith $\mathfrak{g}(1)=\{(X, 0)\}\subseteqq\mathfrak{n}$, and the N-invariant measure $d\dot{n}$ with the Lebesgue measure dX (suitably normalized) on the vector space $\mathfrak{g}(1)$.

Define a bilinear form $(\ ,\ )_i$ on $\mathfrak{g}(1)$ by

$$(X_1, X_2)_i=-\frac{1}{4}\{A[i]^*([J_i''X_1, X_2])+\sqrt{-1}A[i]^*([X_1, X_2])\} \tag{4.25}$$

for $X_1, X_2\in\mathfrak{g}(1)$. We denote by $(\mathfrak{g}(1), J_i'')$ the complex vector space $\mathfrak{g}(1)$ equipped with the complex structure J_i''. Then one has

Lemma 4.11. (1) *$(\ ,\ )_i$ gives a hermitian inner product on $(\mathfrak{g}(1), J_i'')$.*

(2) *The group $K\cap H^i\subseteqq L$ acts, through the adjoint action, on the complex finite-dimensional Hilbert space $(\mathfrak{g}(1), J_i'', (\ ,\ )_i)$ as a group of unitary operators. Namely, one has for any $k\in K\cap H^i$,*

$$\mathrm{Ad}(k)\circ J_i''\circ\mathrm{Ad}(k^{-1})=J_i'', \tag{4.26}$$

$$(\mathrm{Ad}(k)X, \mathrm{Ad}(k)X')_i=(X, X')_i \qquad (X, X'\in\mathfrak{g}(1)). \tag{4.27}$$

Proof. (1) It is easily checked that $(\ ,\ )_i$ gives a sesqui-linear hermitian form on $(\mathfrak{g}(1), J_i'')$. Moreover, it is positive definite thanks to Proposition 4.9.

(2) We first prove (4.26). Recall the expression (3.18) of the Lie algebra $\mathfrak{h}^i=\mathfrak{h}^{\varepsilon(i)}$ of H^i. Then, $\mathfrak{k}\cap\mathfrak{h}^i$ is described as

$$\mathfrak{k}\cap\mathfrak{h}^i=\mathfrak{x}_1\cap\mathfrak{k}\oplus\mathfrak{x}_2\cap\mathfrak{k}\oplus\mathfrak{m} \qquad \text{(as vector spaces)} \tag{4.28}$$

with

$$\mathfrak{x}_1=\sum_{1\leq m<k\leq i}\{\mathfrak{g}(\mathfrak{a}_p\,;2^{-1}(\lambda_k-\lambda_m))+\theta\mathfrak{g}(\mathfrak{a}_p\,;2^{-1}(\lambda_k-\lambda_m))\},$$

$$\mathfrak{x}_2=\sum_{i<m<k\leq l}\{\mathfrak{g}(\mathfrak{a}_p\,;2^{-1}(\lambda_k-\lambda_m))+\theta\mathfrak{g}(\mathfrak{a}_p\,;2^{-1}(\lambda_k-\lambda_m))\}.$$

Consider the decomposition $\mathfrak{g}(1)=\mathfrak{v}_-\oplus\mathfrak{v}_+$ with $\mathfrak{v}_-\equiv\sum_{k\leq i}\mathfrak{g}(\mathfrak{a}_p\,;2^{-1}\lambda_k)$ and $\mathfrak{v}_+\equiv\sum_{m>i}\mathfrak{g}(\mathfrak{a}_p\,;2^{-1}\lambda_m)$. By the definition of J_i'', we have $J_i''|\mathfrak{v}_\pm=\pm J''|\mathfrak{v}_\pm$. It follows from (4.28) that both the spaces $\mathfrak{v}_+$ and $\mathfrak{v}_-$ are stable under $\mathrm{ad}(\mathfrak{k}\cap\mathfrak{h}^i)$. Moreover, they are stable under $\mathrm{Ad}(M)$, $M=Z_K(\mathfrak{a}_p)$, too. In view of Proposition 4.5, we conclude $\mathrm{ad}(X)\circ J_i''=J_i''\circ\mathrm{ad}(X)$ and $\mathrm{Ad}(m)\circ J_i''=J_i''\circ\mathrm{Ad}(m)$ for all $X\in\mathfrak{k}\cap\mathfrak{h}^i$ and all $m\in M$. This implies that (4.26) holds for all $k\in M(K\cap H^i)_0=K\cap H^i$.

The equality (4.27) follows from (4.26) and the definition of H^i.

Q.E.D.

Let $\mathcal{F}_i$ be the space of holomorphic functions ϕ on the complex vector space $(\mathfrak{g}(1), J_i'')$ satisfying

$$\|\phi\|^2_{\mathcal{H}_i}=\int_{\mathfrak{g}(1)}|\phi(X)|^2\exp(-2\|X\|_i^2)dX<+\infty, \tag{4.29}$$

where $\|X\|_i^2=(X, X)_i$. Then, $\mathcal{F}_i$ is complete with respect to the norm $\|\cdot\|_{\mathcal{F}_i}$, hence it has a structure of Hilbert space indced from this norm. We call $\mathcal{F}_i$ the *Fock space* of $(\mathfrak{g}(1), J_i'')$. Since the function $\exp(-2\|X\|_i^2)$ is rapidly decreasing as $\|X\|_i\to\infty$, $\mathcal{F}_i$ contains the space $\mathcal{P}_i$ of polynomial functions on $(\mathfrak{g}(1), J_i'')$. Moreover, if $\mathcal{P}_i^q$ denotes, for an integer $q\geqq 0$, the subspace of homogeneous polynomials of degree q, then $\mathcal{F}_i$ is decomposed as

$$\mathcal{F}_i=\bigoplus_{q\geqq 0}\mathcal{P}_i^q \qquad \text{(orthogonal direct sum)}. \tag{4.30}$$

We can realize the representation T_i on the Fock space $\mathcal{F}_i$ as follows.

Proposition 4.12. (1) *For any $f\in\mathcal{H}_i$, define a function ιf on $\mathfrak{g}(1)$ by*

$$\iota f(X)=f(X)\exp(-\|X\|_i^2) \qquad (X\in\mathfrak{g}(1)). \tag{4.31}$$

Then, ι gives an isomorphism of Hilbert spoces from $\mathcal{H}_i$ onto $\mathcal{F}_i$.

(2) *Let ρ_i denote the unitary representation of N on $\mathcal{F}_i$ transferred from T_i through the isometry ι: $\rho_i(n)\equiv\iota\circ T_i(n)\circ\iota^{-1}$ $(n\in N)$. Then, the operators $\rho_i(n)$, $n=n(X_0, Y_0)\in N$, are described as*

$$\rho_i(n)\phi(X)=\exp\{2(X, X_0)_i-\|X_0\|_i^2+\sqrt{-1}A[i]^*(Y_0)\}\phi(-X_0+X) \tag{4.32}$$

for $X\in\mathfrak{g}(1)$ and $\phi\in\mathcal{F}_i$.

One can show this proposition by a simple calculation. For the proof, we refer to [20, 2.2].

As we saw for $\rho_i\cong\xi_i$, the representation ρ_i can be extended to a genuine unitary representation $\tilde{\rho}_i$ of the semidirect product group H^iN. The realization ρ_i enables us to write down explicitly $\tilde{\rho}_i(x)$ for $x\in(K\cap H^i)N$. We set for $k\in K\cap H^i$

$$\tilde{\rho}_i(k)\phi(X)=\phi(\mathrm{Ad}(k)^{-1}X) \qquad (\phi\in\mathcal{F}_i,\ X\in\mathfrak{g}(1)). \tag{4.33}$$

Then, thanks to Lemma 4.11(2), we have

Theorem 4.13. *The map $(K\cap H^i)N\ni kn\to\tilde{\rho}_i(kn)\equiv\tilde{\rho}_i(k)\rho_i(n)$ gives a unitary representation of $(K\cap H^i)N$ on the Fock space $\mathcal{F}_i$, which extends ρ_i.*

Remark 4.14. (1) If $i=0$ or l, then it holds that $K\cap H^i=H^i=K\cap L$. Therefore, in this case, the above theorem gives a complete description of the extension $\tilde{\xi}_i$.

(2) For each integer $q\geqq 0$, the finite-dimensional subspaces $\mathcal{P}_i^q$ $(0\leqq i\leqq l)$ are stable under $\tilde{\rho}_i(K\cap H^i)$.

4.4. The adjoint representation of M on S_i.

For the later use, we realize here the adjoint representation of M on the complex vector space $(\mathfrak{g}(1), J_i'')$ (see (4.15)) in a different manner. For this purpose, we first claim

Lemma 4.15. *The Cayley transform* $\mu \in G_C = \mathrm{Int}(\mathfrak{g}_C)$ *commutes with* $\mathrm{Ad}(k)$ *for all* $k \in K \cap L$: $\mathrm{Ad}(k) \circ \mu = \mu \circ \mathrm{Ad}(k)$.

Proof. Let $k \in K \cap L$. By virtue of Lemma 3.5(3), one has $\mathrm{Ad}(k)A = A$ and hence $\mathrm{Ad}(k)\theta A = \theta A$, where

$$A = 2^{-1}\sqrt{-1}\sum_{k=1}^{l}(H'_{\gamma_k} - X_{\gamma_k} + X_{-\gamma_k}) \quad \text{and} \quad \theta A = 2^{-1}\sqrt{-1}\sum_{k=1}^{l}(H'_{\gamma_k} + X_{\gamma_k} - X_{-\gamma_k}).$$

This implies that

$$\mathrm{Ad}(k)\left\{\sum_{k=1}^{l}(X_{\gamma_k} - X_{-\gamma_k})\right\} = \sqrt{-1}\mathrm{Ad}(k)(A - \theta A) = \sum_{k=1}^{l}(X_{\gamma_k} - X_{-\gamma_k}).$$

From the definition of μ in (3.8), we obtain the desired result. Q. E. D.

Now set $S(k) = \sum_{\gamma \in C_k} \mathfrak{g}_C(\mathfrak{t}_C; \gamma) \subseteq \mathfrak{g}_C$ $(1 \leqq k \leqq l)$. For $0 \leqq i \leqq l$, define a complex vector space S_i by

$$S_i = (\bigoplus_{k \leq i} S(k)) \oplus (\bigoplus_{m > i} S(m)^{\dagger}). \tag{4.34}$$

Here, for a vector space X over $\boldsymbol{C}$, $X^{\dagger}$ is the space X equipped with the complex structure $x \mapsto -\sqrt{-1}x$ $(x \in X)$ instead of $x \mapsto \sqrt{-1}x$. We will denote by $X \ni x \mapsto x^{\dagger} \equiv x \in X^{\dagger}$ the identical anti-linear isomorphism. Then, taking into account the definition (4.15) of J''_i and Lemma 4.15, we obtain immediately from Lemma 4.4(3) the following

Proposition 4.16. (1) *The map*

$$\tau : \mathfrak{g}(1) \ni X \longmapsto \mu(X + \sqrt{-1}J''X) \in \sum_{k=1}^{l} S(k) \tag{4.35}$$

gives rise to an isomorphism of complex vector spaces from $(\mathfrak{g}(1), J''_i)$ *onto* S_i *for any* $0 \leqq i \leqq l$. *Here,* J'' *is the restriction of the complex structure* J' *in* (4.6) *onto* $\mathfrak{g}(1)$.

(2) *The subspaces* $S(k) \subseteq \mathfrak{g}_C$ $(1 \leqq k \leqq l)$ *are stable under* $\mathrm{Ad}(M)$. *Equip* $S(k)^{\dagger}$ *with a structure of* M*-module through* $m \cdot s^{\dagger} = (\mathrm{Ad}(m)s)^{\dagger}$ $(m \in M,\ s \in S(k))$. *Then,* $(\mathfrak{g}(1), J''_i)$ *is isomorphic, as an* M*-module, to* $S_i = \sum_{k \leq i} S(k) \oplus \sum_{m > i} S(m)^{\dagger}$ *through the map* τ.

This proposition gives another realization S_i of the M-module $(\mathfrak{g}(1), J''_i)$, which is useful for our later calculation in 5. 3. 6 for type EIII.

§ 5. Finite multiplicity theorems for $(c|M) \otimes (\tilde{\rho}_i|M)$ in connection with RGGGRs $\Gamma_i(c)$

Let $(\tilde{\rho}_i, \mathcal{F}_i)$ $(0 \leqq i \leqq l)$ be the irreducible unitary representations of $(K \cap H^i)N$ constructed in 4.3. For irreducible representations c of H^i, consider the representations $\psi_i(c) \equiv (c|M) \otimes (\tilde{\rho}_i|M)$ of M. In this section, we give finite multiplicity theorems for these representations. (Our results of this section is summarized in Theorem 5.17.) These theorems combined with the results of [1] enable us to produce finite multiplicity theorems for RGGGRs $\Gamma_i(c)$, given in the succeeding section.

5.1. A central element Z_0^+ of $\mathfrak{l}=\mathrm{Lie}\ L$.

Firstly, we study the multiplicities in $\psi_i(c)$ for $i=0$ or l. For this purpose, we construct here a central element Z_0^+ of $\mathfrak{l}$ contained in the maximal abelian subalgebra $\mathfrak{t}^+$ of $\mathfrak{m}$. The element Z_0^+ reduces to 0 if and only if G/K is holomorphically equivalent to a tube domain. We describe the adjoint action of the one-parameter subgroup $C_+\equiv\exp \boldsymbol{R}Z_0^+\subseteqq M$ on the complex vector space $(\mathfrak{g}(1), J'')$, which is given in 4.2.

We keep to the notations in §3. Let $Z_0\in\mathfrak{t}$ be the central element of $\mathfrak{k}$ in (3.1). Express Z_0 as $Z_0=Z_0^++Z_0^-$ with $Z_0^+\in\mathfrak{t}^+$, $Z_0^-\in\sqrt{-1}\,\mathfrak{t}^-$, according as the decomposition $\mathfrak{t}=\mathfrak{t}^+\oplus\sqrt{-1}\,\mathfrak{t}^-$ of the compact Cartan subalgebra $\mathfrak{t}$. Then one has

Lemma 5.1. (1) $Z_0^-=2^{-1}\sqrt{-1}\sum_{1\leq k\leq l}H'_{\gamma_k}$.

(2) *For any positive root $\gamma\in\Sigma^+$, the values $\gamma(Z_0^\pm)$ are given as*

$$\gamma(Z_0^+)=\begin{cases} 2^{-1}\sqrt{-1} & \text{if } \gamma\in\bigcup_{1\leq k\leq l}P_k\,, \\ -2^{-1}\sqrt{-1} & \text{if } \gamma\in\bigcup_{1\leq k\leq l}C_k\,, \\ 0 & \text{otherwise.}\end{cases} \tag{5.1}$$

$$\gamma(Z_0^-)=\begin{cases} \sqrt{-1} & \text{if } \gamma\in\bigcup_{1\leq m<k\leq l}P_{km}\cup P_0 \\ 2^{-1}\sqrt{-1} & \text{if } \gamma\in\bigcup_{1\leq k\leq l}(C_k\cup P_k)\,, \\ 0 & \text{otherwise.}\end{cases} \tag{5.2}$$

(3) *Z_0^+ reduces to 0 if and only if $\mathfrak{g}(1)=(0)$. This happens just in the* (CASE I) *in Theorem* 3.3.

Proof. (1) Put $Z_0^-=2^{-1}\sqrt{-1}\sum_{1\leq k\leq l}a_kH'_{\gamma_k}$ with $a_k\in\boldsymbol{R}$. Then, for any $1\leqq k\leqq l$, we have $\gamma_k(Z_0)=\gamma_k(Z_0^-)=\sqrt{-1}\,a_k$ because $\gamma_k(H'_{\gamma_m})=2\delta_{km}$ (see (3.5)). On the other hand, (3.1) and (3.2) imply that $\gamma_k(Z_0)=\sqrt{-1}$, whence $a_k=1$ for $1\leqq k\leqq l$.

(2) (5.2) follows immediately from the expression of Z_0^- in (1). Keeping in mind

$$\gamma(Z_0^+)+\gamma(Z_0^-)=\gamma(Z_0)=\begin{cases}\sqrt{-1} & \text{if } \gamma\in\Sigma_p^+\,, \\ 0 & \text{if } \gamma\in\Sigma_k^+\,,\end{cases}$$

we obtain (5.1) from (5.2).

(3) The assertion (3) is a direct consequence of (5.2) because $P_k=C_k=\emptyset$ for all k if and only if $\mathfrak{g}(1)=(0)$. Q.E.D.

Thanks to this lemma, we can deduce the following propositions.

Proposition 5.2. *Z_0^+ is a central element of $\mathfrak{l}=\mathrm{Lie}\,L$. Moreover, the one parameter subgronp $C_+=\exp \boldsymbol{R}Z_0^+$ ($\subseteqq M\subseteqq L$) is contained in the center of L.*

Proof. Recall the restricted root space decomposition (3.10) of $\mathfrak{l}=\mathfrak{g}(0)$. Then, in view of Moore's restricted root theorem (see 3.1), the complexification $\mathfrak{l}_C$ is expressed as

$$\mathfrak{l}_C=\mu^{-1}\{\mathfrak{t}_C\oplus\Sigma_\gamma(\mathfrak{g}_C(\mathfrak{t}_C\,;\,\gamma)\oplus\mathfrak{g}_C(\mathfrak{t}_C\,;\,-\gamma))\}, \tag{5.3}$$

where, in the above summand, γ runs through the roots of $(\mathfrak{g}_C, \mathfrak{t}_C)$ in $C_0\cup(\bigcup_{k>m}C_{km})$. Notice that $\mu^{-1}(Z_0^+)=Z_0^+$ because the Cayley transform μ is identity on $\mathfrak{t}^+$ (see (3.9)). Then we get

$$[Z_0^+, \mathfrak{l}_C]=\mu^{-1}([Z_0^+, \mathfrak{t}_C\oplus\Sigma_\gamma(\mathfrak{g}_C(\mathfrak{t}_C\,;\,\gamma)\oplus\mathfrak{g}_C(\mathfrak{t}_C\,;\,-\gamma))])=(0)$$

by virtue of (5.1).

The latter assertion follows from the former one, since $\mathrm{Ad}(L)|\mathfrak{l}_C$ is contained in the adjoint group of $\mathfrak{l}_C$ (see the proof of Lemma 2.7 of [I]). Q.E.D.

Proposition 5.3. *The adjoint action of Z_0^+ on $\mathfrak{g}(1)$ is described as*

$$\mathrm{ad}(Z_0^+)|\mathfrak{g}(1)=\frac{1}{2}J''. \tag{5.4}$$

So, the one-parameter subgroup C_+ acts on the complex vector space $(\mathfrak{g}(1), J'')$ in such a way that

$$\mathrm{Ad}(e(s))|\mathfrak{g}(1)=\exp\sqrt{-1}\,s\cdot I \qquad \textit{for } e(s)=\exp(2sZ_0^+)\quad (s\in\boldsymbol{R}), \tag{5.5}$$

where I is the identity operator on $\mathfrak{g}(1)$.

Proof. By virtue of Lemma 4.4(3), the $(\pm\sqrt{-1})$-eigenspaces $V^\pm\subseteqq\mathfrak{g}(1)_C$ for J'' are expressed as

$$V^+=\mu^{-1}\Big\{\sum_{k=1}^{l}\sum_{\gamma\in P_k}\mathfrak{g}_C(\mathfrak{t}_C\,;\,\gamma)\Big\},\qquad V^-=\mu^{-1}\Big\{\sum_{k=1}^{l}\sum_{\gamma\in C_k}\mathfrak{g}_C(\mathfrak{t}_C\,;\,\gamma)\Big\}.$$

Then, (5.1) together with the fact $\mu^{-1}(Z_0^+)=Z_0^+$ implies (5.4). Furthermore, (5.5) is a direct censequence of (5.4). Q.E.D.

5.2. Finite multiplicity property for $\phi_i(c)$; $i=0$, l.

Thanks to Proposition 5.3, we can show that the restriction $\phi_i(c)|C_+$ ($i=0, l$), and hence $\phi_i(c)$ itself has finite multiplicity property for any irreducible unitary representation c of $H^0=H^l=K\cap L$. This is done as follows. Let $i=0$ or l. It suffides to study the case $i=0$ only, because $\tilde{\rho}_l=\tilde{\rho}_0^*$, the contragredient representation of $\tilde{\rho}_0$. By Proposition 5.3, we have for each integer $q\geqq 0$

$$\tilde{\rho}_0(e(s))|\mathcal{P}_0^q=\exp(-\sqrt{-1}\,sq)\cdot I_q, \tag{5.6}$$

where I_q is the identity map on the subspace $\mathcal{P}_0^q$ (see 4.3). In view of the decomposition (4.30), we conclude

Theorem 5.4. *The restriction of* $\tilde{\rho}_0$ *to the one-parameter subgroup* C_+ *is decomposed into irreducibles as*

$$\tilde{\rho}_0 | C_+ \cong \bigoplus [D^q] \cdot e^{-\sqrt{-1}q}, \tag{5.7}$$

where $D^q \equiv \dim \mathcal{P}_0^q$, *and* $e^{-\sqrt{-1}q}$ *denotes the unitary character of* C_+ *such that* $e(s) \mapsto \exp(-\sqrt{-1}qs)$ $(s \in \boldsymbol{R})$. *Especially, the restrictions* $\tilde{\rho}_i | C_+$ $(i=0, l)$ *are of multiplicity finite.*

From this theorem, one gets immediately the following finite multiplicity theorem.

Theorem 5.5. *If* $i=0$ *or* l, *then the representation* $\psi_i(c)=(c|M)\otimes(\tilde{\rho}_i|M)$ *of* M *has finite multiplicity for any irreducible unitary representation* c *of the compact group* $H^0=H^l=K\cap L$.

This is a complete result for $i=0, l$.

5.3. Finite multiplicity property for $\psi_i(c)$; $i\neq 0, l$.

Let us now proceed to the cases $i \neq 0, l$. Then, the reductive subgroups $H^i \subsetneqq L$ are non-compact. So, irreducible admissible representations c of H^i are infinite-dimensional in general, and it is unreasonable to expect that the representations $\psi_i(c)$ are of multiplicity finite for all c. Therefore, in the following we put some reasonable assumption on c, and give finite multiplicity theorems for $\psi_i(c)$. For this purpose, we study separately each type of simple Lie groups of hermitian type (see 3.1 for the classification of such groups).

5.3.1. (CASE I) **in 3.1.** This is just the case where G/K is holomorphically equivalent to a tube domain, or equivalently, $\mathfrak{g}(1)=(0)$. Then, $\tilde{\rho}_i$ is, by construction, a one-dimensional unitary representation of H^iN for each i. Therefore, if a representation c of H^i is finite-dimensional, then $\psi_i(c)$ is finite-dimensional, too, and in particular, of multiplicity finite.

In this case, we can not hope any more result in general. However, in the special case $G=SU(p, p)$ $(l=p)$, $\psi_i(c)$ has finite multiplicity for any irreducible admissible highest weight representation c of $H^i \cong SU(i, p-i)$. This property is proved in the next 5.3.2.

5.3.2. Case of type AIII($\mathfrak{su}(p, q)$; $p \geqq q$). For positive integers p, q such that $p \geqq q$, let $\mathfrak{g}$ be the Lie algebra of the special unitary group $SU(p, q)$ with respect to a hermitian form on $\boldsymbol{C}^n \times \boldsymbol{C}^n$ $(n=p+q)$ with signature (p, q). We realize $\mathfrak{g}$ in the following manner. Define a matrix $I'_{q,q}$ of degree n by

$$I'_{p,q}=\begin{pmatrix} 0_{qq} & 0_{qr} & \sigma_q \\ 0_{rq} & I_r & 0_{rq} \\ \sigma_q & 0_{qr} & 0_{qq} \end{pmatrix} \quad (0_{ab}=\text{the zero matrix of size } a\times b) \tag{5.8}$$

with $r=p-q$, $I_r=(r\times r)$-identity matrix, and

$$\sigma_q \equiv \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & 0 \end{pmatrix} \quad (q \times q \text{ matrix}).$$

Then, $\mathfrak{g}$ consists of complex matrices X of degree n satisfying

$$XI'_{p,q}+I'_{p,q}{}^t\bar{X}=0 \quad \text{and} \quad \operatorname{tr} X=0\,, \tag{5.9}$$

where $\operatorname{tr} X$ (resp. ${}^tX$, $\bar{X}$) denotes the trace (resp. the transpose, the complex conjugation) of a matrix X. The real linear map $X \mapsto -{}^t\bar{X}$ gives a Cartan involution of $\mathfrak{g}$ denoted by θ, and the subspace

$$\mathfrak{a}_p=\{H=\operatorname{diag}(h_q, \cdots, h_1, 0, \cdots, 0, -h_1, \cdots, -h_q);\ h_k \in \boldsymbol{R}\ (1 \leqq k \leqq q)\}$$

is a maximal abelian subspace of $\mathfrak{p}=\{X \in \mathfrak{g};\ \theta X=-X\}$. Therefore, the real rank l of $\mathfrak{g}$ coincides with q. The general element X of the centralizer $\mathfrak{m}=\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}_p)$ is expressed as

$$\mathfrak{m}:\ X=\left[\begin{array}{c|c|c} \begin{matrix}\sqrt{-1}\phi_l & & \\ & \ddots & \\ & & \sqrt{-1}\phi_1\end{matrix} & 0_{lr} & 0_{ll} \\ \hline 0_{rl} & Y & 0_{rl} \\ \hline 0_{ll} & 0_{lr} & \begin{matrix}\sqrt{-1}\phi_1 & & \\ & \ddots & \\ & & \sqrt{-1}\phi_l\end{matrix} \end{array}\right]; \quad \begin{array}{l} -{}^t\bar{Y}=Y\ (r \times r \text{ matrix}), \\ \phi_k \in \boldsymbol{R}\ (1 \leqq k \leqq l), \\ \operatorname{tr} X=0. \end{array} \tag{5.10}$$

So, one gets a canonical isomorphism of Lie algebras

$$\mathfrak{m} \cong \begin{cases} \sqrt{-1}\,\boldsymbol{R}^l \oplus \mathfrak{su}(r) & (r \neq 0)\,, \\ \sqrt{-1}\,\boldsymbol{R}^{l-1} & (r=0)\,. \end{cases} \tag{5.11}$$

We may choose elements $H(\mathrm{o})$ and $A[i]$ $(0 \leqq i \leqq l)$ respectively as

$$H(\mathrm{o})=\left[\begin{array}{c|c|c} I_l & 0 & 0 \\ \hline 0 & 0_{rr} & 0 \\ \hline 0 & 0 & -I_l \end{array}\right], \quad A[i]=\sqrt{-1}\left[\begin{array}{c|c} 0_{lp} & \begin{matrix} & \sigma_{l-i} \\ -\sigma_i & \end{matrix} \\ \hline 0_{pp} & 0_{pl} \end{array}\right]. \tag{5.12}$$

From this choice, the subalgebra $\mathfrak{l}$ and the subspaces $\mathfrak{g}(1)$ and $\mathfrak{g}(2)$ are expressed as

$$(5.13)\quad \mathfrak{l}=\left\{X=\left[\begin{array}{c|c|c} X_1 & 0 & 0 \\ \hline 0 & Y & 0 \\ \hline 0 & 0 & -{}^T\bar{X}_1 \end{array}\right];\ \begin{array}{l} X_1\in\mathfrak{gl}(l, \boldsymbol{C}) \\ -{}^t\bar{Y}=Y \\ \operatorname{tr} X=0 \end{array}\right\}\cong\begin{cases}\mathfrak{gl}(l, \boldsymbol{C})\oplus\mathfrak{su}(r) & (r\neq 0), \\ \mathfrak{sl}(l, \boldsymbol{C})\oplus\boldsymbol{R} & (r=0),\end{cases}$$

$$(5.14)\quad \mathfrak{g}(1)=\left\{\tilde{Z}=\left[\begin{array}{c|c|c} 0 & Z & 0 \\ \hline 0 & 0 & -{}^t\bar{Z}\sigma_l \\ \hline 0 & 0 & 0 \end{array}\right];\ Z\ l\times r \text{ complex matrix}\right\},$$

and

$$(5.15)\qquad \mathfrak{g}(2)=\left\{\begin{bmatrix} 0_{ll} & 0_{lr} & W \\ 0_{rl} & 0_{rr} & 0_{rl} \\ 0_{ll} & 0_{lr} & 0_{ll} \end{bmatrix};\ {}^T\bar{W}=-W\right\}.$$

Here we put, for a matrix V of degree a, ${}^TV=\sigma_a{}^tV\sigma_a$. Moreover, for each $1\leqq k\leqq l$, the root space $\mathfrak{g}(\mathfrak{a}_p;2^{-1}\lambda_k)$ is described as

$$(5.16)\qquad \mathfrak{g}(\mathfrak{a}_p;2^{-1}\lambda_k)=\{\tilde{Z}\in\mathfrak{g}(1);\ Z=(z_{ij})_{1\leq i\leq l,1\leq j\leq r} \text{ with } z_{ij}=0 \text{ if } i\neq l-k\}.$$

Now we suppose that $r\neq 0$, i.e., $\mathfrak{g}(1)\neq(0)$. The natural complex structure on the space $M_{l,r}(\boldsymbol{C})\cong\boldsymbol{C}^r\times\cdots\times\boldsymbol{C}^r$ (l-copies) of $l\times r$ complex matrices gives rise to the complex structure J'' on $\mathfrak{g}(1)$ through the map $Z\mapsto\tilde{Z}$. So we identify them. Under this identification, one has for each $0\leqq i\leqq l$

$$(5.17)\qquad (\mathfrak{g}(1), J_i'')\cong\underbrace{(\boldsymbol{C}^r)^\dagger\times\cdots\times(\boldsymbol{C}^r)^\dagger}_{i}\times\underbrace{\boldsymbol{C}^r\times\cdots\times\boldsymbol{C}^r}_{l-i}$$

where $(\boldsymbol{C}^r)^\dagger$ is the complex vector space conjugate to $\boldsymbol{C}^r$ as in 4.4. Furthermore, the adjoint action of $\mathfrak{m}$ on $(\mathfrak{g}(1), J'')$ is written as

$$(5.18)\qquad \operatorname{ad}(\sqrt{-1}\,\phi, Y)\cdot(z_1, z_2, \cdots, z_l)=(\sqrt{-1}\,\nu(\phi)_1\bar{Y}z_1, \cdots, \sqrt{-1}\,\nu(\phi)_l\bar{Y}z_l)$$

for $(\sqrt{-1}\phi, Y)\in\sqrt{-1}\,\boldsymbol{R}^l\oplus\mathfrak{su}(r)\cong\mathfrak{m}$ and $z_k\in\boldsymbol{C}^r$ $(1\leqq k\leqq l)$. Here we put

$$(5.19)\qquad \nu(\phi)_k=\phi_k+2r^{-1}\sum_{1\leq k'\leq l}\phi_{k'}\qquad \text{for } \phi=(\phi_1, \cdots, \phi_l)\in\boldsymbol{R}^l.$$

We see easily that the linear map $\phi\mapsto\nu(\phi)=(\nu(\phi)_1, \cdots, \nu(\phi)_l)$ on $\boldsymbol{R}^l$ is non-degenerate. One thus obtains the following

Proposition 5.6. *Let* $\mathfrak{g}=\mathfrak{su}(p, q)$ $(n=p+q,\ p\geqq q=l)$ *with* $r=p-q\neq 0$. *Then, for each* $0\leqq i\leqq l$, *there exists a unique* $\phi[i]\in\boldsymbol{R}^l$ *such that*

(5.20) $$\mathrm{ad}(\sqrt{-1}\phi[i], 0)|\mathfrak{g}(1)=J_i''.$$

Moreover, the element $\phi[i]\in \boldsymbol{R}^l$ *is expressed as*

(5.21) $$\phi[i]_k=\begin{cases} -(r+4l-4i)/n & (k\leqq i), \\ (r+4i)/n & (k>i). \end{cases}$$

Thanks to this proposition, we deduce, just as in the case of $l=0$ or l (see 5.1 and 5.2), the following finite multiplicity theorem.

Theorem 5.7. *Let* $\mathfrak{g}$ *be as above. Then, for any* i *and any finite dimensional representation* c *of* H^i, *the representation* $\phi_i(c)=(c|M)\otimes(\tilde{\rho}_i|M)$ *has finite multiplicity property.*

5.3.3. Interesting infinite-dimensional c. Furthermore, in case of $i\neq 0$, l, there exist very interesting examples of infinite-dimensional admissible representations c of H^i, for which $\phi_i(c)$ are still of multiplicity finite. Such c's are given in the following way as irreducible admissible highest weight representations. Suppose that $G=SU(p, q)$ $(p\geqq q=l)$ for simplicity, and consider the cases $i\neq 0$, l. Then the subgroups M, H^i $(M\subseteqq H^i)$ are described as

(5.22) $$\begin{cases} M\cong \boldsymbol{T}^l\times SU(r) \hookrightarrow U(l-i, i)\times SU(r)\cong H^i & (r\neq 0) \\ M\cong \boldsymbol{T}^{l-1} \hookrightarrow SU(l-i, i)\cong H^i & (r=0), \end{cases}$$

where $\boldsymbol{T}$ denotes the one-dimensional torus. So, in particular, H^i is a reductive group of hermitian type for each i. Take a central element Z_i^+ of $\mathfrak{k}\cap\mathfrak{h}^i$ ($\mathfrak{h}^i=\mathrm{Lie}\, H^i$) which gives an $\mathrm{Ad}(K\cap H^i)$-invariant complex structure on $\mathfrak{p}\cap\mathfrak{h}^i$. By virtue of (5.21), we can (and do) choose Z_i^+ as

(5.23) $$Z_i^+=2^{-1}(\sqrt{-1}\phi[i], 0)\in\mathfrak{t}^+\subseteqq\mathfrak{m}$$

in case of $r\neq 0$. In any case, Z_i^+ is in $\mathfrak{t}^+\subseteqq\mathfrak{m}$. Notice that $\mathfrak{t}^+$ is a compact Cartan subalgebra of $\mathfrak{h}^i$. Let $\Sigma(\mathfrak{h}_C^i)$ denote the root system of $\mathfrak{h}_C^i$ with respect to $\mathfrak{t}_C^+$. We select as in 3.1 a positive system $\Sigma^+(\mathfrak{h}_C^i)$ of $\Sigma(\mathfrak{h}_C^i)$ compatible with the above complex structure on $\mathfrak{p}\cap\mathfrak{h}^i$. Then, by definition, one has

(5.24) $$\sqrt{-1}\,\gamma(Z_i^+)<0$$

for any non-compact positive root $\gamma\in\Sigma^+(\mathfrak{h}_C^i)$.

Now let $\Phi\in\sqrt{-1}\,\mathfrak{t}^{+*}$ be a $(K\cap H^i)$-dominant, integral element:

(5.25) $(\Phi, \gamma)\geqq 0$ for all positive compact roots $\gamma\in\Sigma^+(\mathfrak{h}_C^i)$,

(5.26) $\mathfrak{t}^+\ni X\longmapsto \exp\langle\Phi, X\rangle$ gives a unitary character of compact Cartan subgroup $T^+\equiv\exp\mathfrak{t}^+$ of H^i.

Here, $(\,,\,)$ denotes the inner product on $\sqrt{-1}\,\mathfrak{t}^{+*}$ defined through the Killing form of $\mathfrak{g}_C$. Then, according to Harish-Chandra [7, IV, Theorem 2], there exists a unique (up to isomorphisms) irreducible admissible $(\mathfrak{h}_C^i, K\cap H^i)$-module

$I(\Phi)$ with highest weight $\Phi \in \sqrt{-1}\,\mathfrak{t}^{+*}$ with respect to $\Sigma^+(\mathfrak{h}_C^i)$. Furthermore, $I(\Phi)$ extends to an irreducible admissible representation $c(\Phi)$ of H^i on a Hilbert space $\mathcal{H}(\Phi)$, in such a way that $I(\Phi)$ coincides with the $(\mathfrak{h}_C^i, K\cap H^i)$-module of $(K\cap H^i)$-finite vectors for $c(\Phi)$.

The restrictions of $c(\Phi)$ and $\tilde{\rho}_i$ to the one-parameter subgroup $C_+^i = \exp \boldsymbol{R} Z_i^+ \subseteq T^+$ are described as follows. Let η_0 be the smallest positive number such that $\exp(\eta_0 Z_i^+)=1$. Then, for any integer m,

$$e(m)(\exp \nu Z_i^+) = \exp\{2\pi\sqrt{-1}\, m\nu\eta_0^{-1}\} \qquad (\nu \in \boldsymbol{R}) \tag{5.27}$$

gives a unitary character $e(m)$ of C_+^i. Moreover, the map $m \mapsto e(m)$ defines an isomorphism of abelian groups from $\boldsymbol{Z}$ to the dual group $(C_+^i)^\wedge$ of C_+^i.

Proposition 5.8. *The restrictions $c(\Phi)|C_+^i$ and $\tilde{\rho}_i|C_+^i$ are decomposed into irreducibles as follows respectively.*

$$c(\Phi)|C_+^i \cong \bigoplus_{m \leq m(\Phi)} [D_m]\cdot e(m), \tag{5.28}$$

$$\tilde{\rho}_i|C_+^i \cong \bigoplus_{m \leq 0} [D'_m]\cdot e(m). \tag{5.29}$$

Here $m(\Phi)$ is the integer such that $e(m(\Phi)) = (\exp\langle\Phi, \cdot\rangle)|C_+^i$, and the multiplicities D_m and D'_m are all finite. (If $r=0$, $\tilde{\rho}_i|C_+^i$ is the one-dimensianal trivial representation.)

Proof. The decomposition (5.28) follows from (5.24) and the admissibility of the highest weight representation $c(\Phi)$. We deduce (5.29) from Proposition 5.6, keeping in mind the decomposition (4.30) of the Fock space $\mathcal{F}_i$ (=the representation space of $\tilde{\rho}_i$). Q.E.D.

From this proposition, we obtain immediately the following finite multiplicity theorem for $\psi_i(c(\Phi))$, with irreducible highest weight representations $c(\Phi)$.

Theorem 5.9. *Suppose that $G=SU(p,q)$ $(p\geqq q=l)$. For $0\leqq i\leqq l$, let $c=c(\Phi)$ be the irreducible admissible representation of $H^i \cong U(l-i, i)\times SU(r)$ $(r=p-q\neq 0)$; $H^i \cong SU(l-i, i)$ $(r=0)$, with highest wieght $\Phi\in\sqrt{-1}\,\mathfrak{t}^{+*}$. Then, the restriction of $\psi_i(c)$ to the one-parameter subgroup $C_+^i = \exp \boldsymbol{R} Z_i^+$ is of multiplicity finite. So, in particular, the representation $\psi_i(c)$ itself has finite multiplicity property.*

5.3.4. Irreducible representations of $SU(n)$. Before proceeding to other types of simple Lie groups, we recall here theory of finite-dimensional irreducible representations for $SU(n)$ and $\mathfrak{sl}(n, \boldsymbol{C})$ $(n\geqq 1)$.

Let (π, V) be a finite-dimensional representation of $SU(n)$. Then, the differential of the group action naturally gives rise to a representation of the Lie algebra $\mathfrak{sl}(n, \boldsymbol{C}) = \mathfrak{su}(n) \oplus \sqrt{-1}\,\mathfrak{su}(n)$ on V, which is denoted again by π. Suppose that π be irreducible. Then, there exists a unique (up to constant multiples) non-zero vector $v\in V$ satisfying following two conditions:

$$\pi(X_{ij})v = 0 \qquad \text{for } 1\leqq i<j\leqq n, \tag{5.30}$$

(5.31) $\pi(X_{kk}-X_{k+1,k+1})v=\mu_k\cdot v$ for some non-negative integers μ_k $(1\leqq k\leqq n-1)$.

Here, for $1\leqq i,\ j\leqq n$, we put $X_{ij}=(x_{km}^{ij})_{1\leq k,m\leq n}$ with $x_{km}^{ij}=1$ if $(i,j)=(k,m)$; $x_{km}^{ij}=0$ otherwise. The sequence $\boldsymbol{\mu}=(\mu_k)_{1\leq k\leq n-1}$ is said to be the highest weight for π, and v the highest weight vector.

We can thus attach, to an irreducible finite-dimensional representation π of $SU(n)$, its highest weight $\boldsymbol{\mu}$. Moreover, this assignment $\pi\mapsto\boldsymbol{\mu}$ sets up a one-to-one correspondence from the set of equivalence classes of irreducible finite-dimensional representations of $SU(n)$ onto $(\boldsymbol{Z}_+)^{n-1}$, where $\boldsymbol{Z}_+=\{0,1,\cdots\}$. For $\boldsymbol{\mu}=(\mu_k)\in(\boldsymbol{Z}_+)^{n-1}$, we will denote by $R[\boldsymbol{\mu}]=R[\mu_1,\mu_2,\cdots,\mu_{n-1}]$ the (equivalence class of) irreducible representation of $SU(n)$ (or $\mathfrak{sl}(n,\boldsymbol{C})$) with highest weight $\boldsymbol{\mu}$.

For an integer $q\geqq 0$, let $\mathcal{P}^q(n)$ denote the space of homogeneous polynomials of degree q on $\boldsymbol{C}^n$. $\mathcal{P}^q(n)$ has a structure of $SU(n)$-module, denoted by $d_q(n)$, through

$$d_q(n)(g)\phi(z)=\phi(g^{-1}z)\qquad(g\in SU(n),\ \phi\in\mathcal{P}^q(n)).\tag{5.32}$$

One can check easily that $d_q(n)$ is an irreducible representation with highest weight $(0,\cdots,0,q)$.

For the later use, we now give the irreducible decompositions of the tensor product representations $d_p(n)\otimes d_r(n)$ $(q,r\in\boldsymbol{Z}_+)$.

Proposition 5.10. *For integers q, r such that $q\geqq r\geqq 0$, the representation $d_q(n)\otimes d_r(n)$ is decomposed into irreducibles as*

$$d_q(n)\otimes d_r(n)\cong\bigoplus_{j=0}^{r}R[0,\cdots,0,j,q+r-2j].\tag{5.33}$$

Proof. For a sequence $\sigma=(\sigma_k)_{1\leq k\leq n}$ of non-negative integers σ_k, define a homogeneous polynomial $\phi(\sigma)=\phi(\sigma_1,\sigma_2,\cdots,\sigma_n)\in\mathcal{P}^{|\sigma|}(n)$ with $|\sigma|=\sum_{1\leq k\leq n}\sigma_k$ by $\phi(\sigma)(z_1,z_2,\cdots,z_n)=z_1^{\sigma_1}z_2^{\sigma_2}\cdots z_n^{\sigma_n}$. Then, $\{\phi(\sigma);\ |\sigma|=q\}$ forms a basis of $\mathcal{P}^q(n)$ for any $q\geqq 0$. Moreover, making use of this basis, the representation operators $d_q(n)(X_{ij})$ $(1\leqq i\neq j\leqq n)$ are described explicitly as

$$d_q(n)(X_{ij})\phi(\sigma)=-\sigma_i\phi(\sigma-\delta_i+\delta_j),\tag{5.34}$$

where $\delta_i=(\delta_{ik})$ with $\delta_{ik}=0$ $(k\neq i)$; $\delta_{ik}=1$ $(k=i)$. Here we understand that the right hand side is equal to 0 if $\sigma_i=0$.

Using this description, one can determine easily all the highest weight vectors in $\mathcal{P}^q(n)\otimes\mathcal{P}^r(n)$ as follows. For an integer $0\leqq j\leqq r$, set

$$\Psi^{(j)}=\sum_{\beta=0}^{j}(-1)^\beta\binom{j}{\beta}\phi(0,\cdots,\beta,q-\beta)\otimes\phi(0,\cdots,0,j-\beta,r-j+\beta),$$

where $\binom{j}{\beta}=j!/\beta!((j-\beta)!$. Then $\Psi^{(j)}\in\mathcal{P}^q(n)\otimes\mathcal{P}^r(n)$ is a highest weight vectors with highest weight $(0,\cdots,0,j,q+r-2j)$. Conversely, any highest weight vector in $\mathcal{P}^q(n)\otimes\mathcal{P}^r(n)$ is a scalar multiple of some $\Psi^{(j)}$. This proves the assertion. Q. E. D.

5.3.5. Case of type DIII ($\mathfrak{so}^*(4n+2)$). Let $\mathfrak{g}=\mathfrak{so}^*(4n+2)=\mathfrak{so}(2n+1, \boldsymbol{H})$ be the Lie algebra of special orthogonal group of degree $2n+1$ over the quaternion field $\boldsymbol{H}$. In this case, the real rank $l=\dim\mathfrak{a}_p$ is equal to n. By realizing $\mathfrak{g}$, as in the case of type AIII, as a Lie algebra of matrices explicitly, one can describe the adjoint action of $\mathfrak{m}$ on $(\mathfrak{g}(1), J'')$ as follows. For each $1\leqq k\leqq l$, the complex vector space $(\mathfrak{g}(\mathfrak{a}_p; 2^{-1}\lambda_k), J'')$ is two-dimensional. So, we may identify this space with $\boldsymbol{C}^2$. The Lie subalgebra $\mathfrak{m}$ is isomorphic to (hence it can be identified with)

$$\underbrace{\mathfrak{su}(2)\times\cdots\times\mathfrak{su}(2)}_{l}\times\sqrt{-1}\boldsymbol{R}.$$

Under this identification, the adjoint action of $\mathfrak{m}$ on $(\mathfrak{g}(1), J'')$ is described as

$$\mathrm{ad}(Y_1, \cdots, Y_l, \sqrt{-1}\phi)\cdot(z_1, \cdots, z_l)=(\sqrt{-1}\phi z_1+Y_1z_1, \cdots, \sqrt{-1}\phi z_l+Y_lz_l), \tag{5.35}$$

where $(Y_1, \cdots, Y_l, \sqrt{-1}\phi)\in\mathfrak{m}$ with $Y_k\in\mathfrak{su}(2)$ $(1\leqq k\leqq l)$, $\phi\in\boldsymbol{R}$, and $(z_1, \cdots, z_l)\in\mathfrak{g}(1)$ with $z_k\in\boldsymbol{C}^2$.

Let M' denote the analytic subgroup of M corresponding to $\mathfrak{su}(2)\times\cdots\times\mathfrak{su}(2)$ (l-copies). Then, M' is identified canonically with $SU(2)\times\cdots\times SU(2)$, and the adjoint action of M' on $(\mathfrak{g}(1), J'')$ coincides with its natural action on $\boldsymbol{C}^2\times\cdots\times\boldsymbol{C}^2$. The restriction $\tilde{\rho}_i|M'$ is decomposed into irreducibles as follows.

Proposition 5.11. *For any $0\leqq i\leqq l$, one has an isomorphism of unitary representations of $M'=SU(2)\times\cdots\times SU(2)$ (l-copies) as*

$$\tilde{\rho}_i|M'\cong\sum_{q_1,q_2,\cdots,q_l\geqq 0}^{\oplus} d_{q_1}(2)\hat{\otimes}d_{q_2}(2)\hat{\otimes}\cdots\hat{\otimes}d_{q_l}(2), \tag{5.36}$$

where $d_q(2)$ is the irreducible representation of $SU(2)$ given as in (5.32). *Especially, $\tilde{\rho}_i|M'$ is of multiplicity free, and its unitary equivalence class does not depend on i.*

Proof. In view of (4.30), the Fock space $\mathcal{F}_i$ of $(\mathfrak{g}(1), J''_i)$ is decomposed canonically into an orthogonal direct sum of M'-modules as

$$\mathcal{F}_i=\sum_{q_1,q_2,\cdots,q_l\geqq 0}^{\oplus}(\mathcal{P}^{q_1}(2))^{\dagger}\hat{\otimes}\cdots\hat{\otimes}(\mathcal{P}^{q_i}(2))^{\dagger}\hat{\otimes}\mathcal{P}^{q_{i+1}}(2)\hat{\otimes}\cdots\hat{\otimes}\mathcal{P}^{q_l}(2).$$

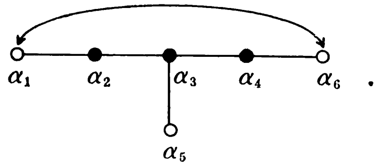
Here $(\mathcal{P}^q(2))^{\dagger}$ is the $SU(2)$-module contragredient to $\mathcal{P}^q(2)$. But, in this case, $(\mathcal{P}^q(2))^{\dagger}$ is equivalent to $\mathcal{P}^q(2)$ as an $SU(2)$-module, which proves (5.36). Q.E.D.

From this proposition, we can deduce immediately the following

Theorem 5.12. *Suppose that $\mathfrak{g}=\mathfrak{so}^*(4l+2)$. Then, for any $0\leqq i\leqq l$ and any finite-dimensional representation c of H^i, the restriction $(c|M')\otimes(\tilde{\rho}_i|M')$ and so $\phi_i(c)$ have finite multiplicity property.*

5.3.6. Case of type EIII. We now proceed to the last case of type EIII. Then, the Satake diagram of $\mathfrak{g}$ (see [11, p. 534]) is given as

(5.37)



This diagram should be understood as follows. Let Π' denote the totality of simple roots in $\Sigma^+=\Sigma^+(\mathfrak{g}_C, \mathfrak{t}_C)$. First, consider the Dynkin diagram of the root system Σ of type E_6. Each node of this graph expresses an element of Π': $\Pi'=\{\alpha_k\,;\, 1\leqq k\leqq 6\}$. The elements in $\Pi'_0\equiv\Pi'\cap C_0$ (i.e., α_2, α_3 and α_4) are denoted by black circles, and the others (i.e., α_1, α_5 and α_6) by white circles. If $\alpha, \beta\in\Pi'\setminus\Pi'_0$ are such that $\alpha|\mathfrak{t}_{\bar{C}}=\beta|\mathfrak{t}_{\bar{C}}$, then α and β are jointed by a curved arrow $\curvearrowleft$. (This is the case of α_1 and α_6.)

From the diagram (5.37), one sees that $\mathfrak{g}$ is of real rank 2: $l=2$. In view of Theorem 5.5, we have only to study the cases $i\neq 0$, l, so the case $i=1$ here. In the following, we give the irreducible decomposition of $\tilde{\rho}_1|M'$ on the Fock space $\mathcal{F}_1$, where $M'\subseteqq M$ is the analytic subgroup of M with Lie algebra $\mathfrak{m}'\equiv[\mathfrak{m}, \mathfrak{m}]$. As a consequence, we will find out that $\psi_1(c)=(c|M)\otimes(\tilde{\rho}_1|M)$ is of multiplicity finite for any finite-dimensional representation c of H^1.

First, let us describe the adjoint action of $\mathfrak{m}'$ on the complex vector space $(\mathfrak{g}(1), J''_i)$, which is equivalent to such action on the space S_1 constructed in 4.4. For this purpose, we realize the complex simple Lie algebra $\mathfrak{k}'_C=\mathfrak{k}'\otimes_{R}C$ with $\mathfrak{k}'\equiv[\mathfrak{k}, \mathfrak{k}]$ explicitly as a Lie algebra of matrices.

It follows from Theorem 3.3 that Π' splits into a disjoint union of subsets as

(5.38) $$\Pi'=\Pi'_0\cup(\Pi'\cap C_{21})\cup(\Pi'\cap C_1)\cup(\Pi'\cap P_1).$$

In the Satake diagram, the unique element of $\Pi'\cap C_1$ is jointed with the element of $\Pi'\cap P_1$ by an arrow. So we may assume without loss of generality

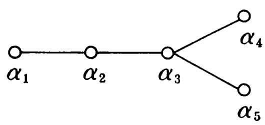
(5.39) $$\Pi'\cap C_1=\{\alpha_1\} \quad\text{and}\quad \Pi'\cap P_1=\{\alpha_6\}.$$

Moreover, we get

(5.40) $$\Pi'\cap C_{21}=\{\alpha_5\}.$$

This implies that the Dynkin diagram of the root system $\Sigma_{\mathfrak{t}}$ of $(\mathfrak{k}'_C, \mathfrak{t}'_C)$ with $\mathfrak{t}'=\mathfrak{t}\cap\mathfrak{k}'$ is of type D_5:

(5.41)



Therefore, $\mathfrak{k}'_C$ is isomorphic to (and hence we identify it with) the Lie algebra $\mathfrak{o}(10, C)$, consisting of complex matrices X of degree 10 such that $X=-{}^TX$.

Under this identification, we may assume that

$$\mathfrak{t}'_C=\{h=\mathrm{diag}(h_1,\cdots,h_5,-h_5,\cdots,-h_1);\ h_k\in \boldsymbol{C}\ (1\leqq k\leqq 5)\} \tag{5.42}$$

and that

$$\alpha_k=e_k-e_{k+1}\ (1\leqq k\leqq 4),\qquad \alpha_5=e_4+e_5. \tag{5.43}$$

Here, for $1\leqq k\leqq 5$, e_k denotes the linear form on $\mathfrak{t}'_C$ given as $e_k(h)=h_k$ ($h\in\mathfrak{t}'_C$). Then the subsets C_0, C_{21}, C_1 and C_2 of Σ_k^+ are expressed respectively as follows:

$$\begin{cases} C_0=\{e_r-e_s;\ 2\leqq r<s\leqq 5\}, & C_{21}=\{e_r+e_s;\ 2\leqq r<s\leqq 5\}, \\ C_1=\{e_1-e_s;\ 2\leqq s\leqq 5\}, & C_2=\{e_1+e_s;\ 2\leqq s\leqq 5\}. \end{cases} \tag{5.44}$$

Moreover, the subalgebra $\mathfrak{m}'_C\subseteqq\mathfrak{k}'_C$ is isomorphic to $\mathfrak{sl}(4,\boldsymbol{C})$ and it is described as

$$\mathfrak{m}'_C=\left\{\tilde{Y}=\left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & -{}^TY & 0 & 0 \\ \hline 0 & 0 & Y & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right];\ \begin{array}{l} Y\ \ 4\times 4\ \ \text{matrix} \\ \mathrm{tr}\,Y=0 \end{array}\right\}. \tag{5.45}$$

Therefore, we can take a compact real form $\mathfrak{m}'$ of $\mathfrak{m}'_C$ as

$$\mathfrak{m}'=\{\tilde{Y}\in\mathfrak{m}'_C;\ {}^tY=-\bar{Y}\}\cong\mathfrak{su}(4). \tag{5.46}$$

Let $S_1=S(1)\oplus S(2)^\dagger$ with $S(k)=\sum_{\gamma\in C_k}\mathfrak{g}_C(\mathfrak{t}_C;\gamma)\subseteqq\mathfrak{k}'_C$ be the complex vector space in 4.4. The adjoint $\mathfrak{m}'$-action on S_1 is described as follows. The subspaces $S(1)$ and $S(2)^\dagger$ are identified with $\boldsymbol{C}^4$ respectively through the maps

$$\boldsymbol{C}^4\ni z=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}\longmapsto \tilde{z}=\left[\begin{array}{c|c|c|c} 0 & {}^tz\sigma_4 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & z \\ \hline 0 & 0 & 0 & 0 \end{array}\right]\in S(1),$$

$$C^4 \ni w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \longmapsto \tilde{\tilde{w}} = \left[\begin{array}{c|c|c|c} 0 & 0 & -{}^t\bar{w} & 0 \\ \hline 0 & 0 & 0 & \sigma_4\bar{w} \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \in S(2)^{\dagger}.$$

Under this identification, the adjoint action of $\mathfrak{m}'$ on $S_1 = C^4 \oplus C^4$ is expressed as

(5.47) $\quad \operatorname{ad}(\tilde{Y})\cdot(z, w) = (Yz, Yw) \quad$ for $Y \in \mathfrak{su}(4)$ and $z, w \in C^4$.

In view of Proposition 4.16, we deduce immediately from this description the following

Lemma 5.13. *The analytic subgroup $M' \subseteqq M$ with Lie algebra $\mathfrak{m}' = [\mathfrak{m}, \mathfrak{m}]$ is isomorphic to $SU(4)$: $M' \cong SU(4)$, because of the action (5.47) of $\mathfrak{m}'$ on S_1. Through this isomorphism, the adjoint representation of M' on $(\mathfrak{g}(1), J_1'')$ is equivalent to $(\alpha, C^4) \oplus (\alpha, C^4)$, where α is the natural representation of $SU(4)$ on C^4.*

Let us now decompose the representation $\tilde{\rho}_1 | M'$ on the Fock space $\mathcal{F}_1$ into irreducibles. Identify M' with $SU(4)$ as in the above lemma. Then, keeping (4.30) in mind, one gets immediately from Lemma 5.13 a decomposition of $\tilde{\rho}_1 | M'$ as

(5.48) $$\tilde{\rho}_1 | M' \cong \bigoplus_{q, r \geq 0} d_q(4) \otimes d_r(4).$$

This equivalence together with Proposition 5.10 yields the following

Theorem 5.14. *Keep to the notation in* 5.3.4 *for representations of $M' = SU(4)$. Then, the representation $\tilde{\rho}_1 | M'$ is decomposed into irreducibles as*

(5.49) $$\tilde{\rho}_1 | M' \cong \bigoplus_{m, k \geq 0} [k+1] \cdot R[0, m, k].$$

In particular, $\tilde{\rho}_1 | M'$ has finite multiplicity property.

From this theorem and Theorem 5.5, one gets immediately a finite multiplicity theorem for $\phi_i(c)$ as follows.

Theorem 5.15. *Suppose that $\mathfrak{g}$ is of type EIII. Then the representation $\phi_i(c) = (c | M) \otimes (\tilde{\rho}_i | M)$ is of multiplicity finite for any i: $0 \leqq i \leqq 2$, and any finite-pimensional c.*

Remark 5.16. In any case of type AIII, DIII and EIII, we have proved that the representation $\phi_i(c)$ has finite multiplicity property for any i and any finite-dimensional c (Theorems 5.7, 5.12 and 5.15). But, the proofs for theese three cases are quite different from each other. In case of type AIII, we can construct a one-parameter subgroup $C_+^i \subseteq T^+ = \exp \mathfrak{t}^+$ such that $\phi_i(c)$, restricted to C_+^i, is already of multiplicity finite. On the other hand, in the latter two cases, the restriction $\phi_i(c)|T^+$ is no longer of multiplicity finite if $i \neq 0, l$. Actually, it contains the trivial character with infinite multiplicity. In these cases, we can not prove finite multiplicity property for $\phi_i(c)$ until we study $\phi_i(c)$ itself detailedly.

5.4. Finite mutipllicity theorems for $\phi_i(c) = (c|M) \otimes (\tilde{\rho}_i|M)$.

We summarize the results of this section as follows.

Theorem 5.17. *For $0 \leqq i \leqq l = \dim \mathfrak{a}_p$ and an irreducible admissible representation c of the reductive subgroup $H^i \subseteq L$, consider the representation $\phi_i(c) = (c|M) \otimes (\tilde{\rho}_i|M)$ of $M = Z_K(\mathfrak{a}_p)$. Then one has*

(1) *$\phi_i(c)$ has finite multiplicity property for any i and any finite-dimensional c.*

(2) *Suppose that $G = SU(p, l)$ with $p \geqq l$. For $0 < i < l$, let $c = c(\Phi)$ be the irreducible admissible representation of H^i with highest weight $\Phi \in \sqrt{-1}\,\mathfrak{t}^{+*}$ (see 5.3.3). Then the representation $\phi_i(c)$ is of multiplicity finite*

Remark 5.18. Notice that (L, H^i) is a reductive symmetric part at least on the level of Lie algebras (see Lemma 3.15). So, we can apply the results of [I] to RGGGRs $\Gamma_i(c) = \mathrm{Ind}_{H^i N}^G(\check{c} \otimes \tilde{\rho}_i)$. The above theorem enables us to prove finite multiplicity property for RGGGRs, which will be discussed in the succeeding section.

Remark 5.19. One can describe, developing the argument in this section, the multiplicities in $\phi_i(c)$ explicitly, although we do not carry out it here. Using such description, we can give a nice upper bound for multiplicities in RGGGRs $\Gamma_i(c)$.

§ 6. Finite multiplicity theorems for RGGGRs $\Gamma_i(c)$

We now come to the climax of the present article. In this final section, we give finite multiplicity theorems for RGGGRs $\Gamma_i(c)$ associated with the nilpotent classes $\omega_i = \mathrm{Ad}(G)A[i]$ (see Theorem 3.13), using the results of [I], summarized in 6.1, and those in the previous sections. Our results for $i=0$ or l are complete: all the RGGGRs associated with the nilpotent classes ω_i ($i=0, l$) have finite multiplicity property. Also in other cases: $i \neq 0, l$, we can prove the finiteness of multiplicities in differentiably induced RGGGRs C^∞-$\Gamma_i(c)$, if c is finite-dimensional.

Moreover, even for infinite-dimensional c's, there are interesting examples for $G = SU(p, q)$ ($p \geqq q$) such that the RGGGRs C^∞-$\Gamma_i(c)$ are of multiplicity finite. Such c's are given as highest weight representations.

Theorems 6.5 and 6.6 are the main results of this paper. We treat in this paper finite multiplicity property only. Nevertheless, the method of this series of papers is applicable also for estimating multiplicities in RGGGRs.

At the end of this section, we will announce a multiplicity one theorem (Theorem 6.9), proved in the subsequent paper [33]. This theorem asserts that, under some reasonable assumptions on G and c, the reduced generalized Gelfand-Graev representations $\Gamma_i(c)$ have multiplicity free property for $i=0$ or l.

6.1. A summary of the results of [I]. In this subsection, let G be a connected semisimple Lie group with finite center. We sum up here the results of [I] on (finiteness of) multiplicities in induced representations of G.

Let K denote a maximal compact subgroup of G, and θ a Cartan involution of G such that $K=\{g\in G;\ \theta(g)=g\}$. Denote by $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}=\mathrm{Lie}\,G$ determined by θ. Let $P'=LN$ with $L=\theta P'\cap P'$ be a Levi decomposition of an arbitrary parabolic subgroup P' of G. Then $\mathfrak{a}=\mathfrak{z}_{\mathfrak{l}}\cap\mathfrak{p}$ is the split component of the Lie algebra $\mathfrak{l}$ of L, where $\mathfrak{z}_{\mathfrak{l}}$ expresses the center of $\mathfrak{l}$. Consider an involutive automorphism σ of $\mathfrak{l}$ with following two properties:

(i) σ commutes with θ restricted to $\mathfrak{l}$,

(ii) $\sigma|\mathfrak{a}=-I_{\mathfrak{a}}$, $I_{\mathfrak{a}}$=the identity map on $\mathfrak{a}$.

Let $\mathfrak{l}=\mathfrak{h}\oplus\mathfrak{q}$ be the eigenspace decomposition of $\mathfrak{l}$ with respect to σ, where $\mathfrak{h}$ and $\mathfrak{q}$ are respectively $(+1)$- and (-1)-eigenspaces. Then $(\mathfrak{l}, \mathfrak{h})$ has a structure of so-called reductive symmetric pair. Take a closed subgroup H of L with Lie algebra $\mathfrak{h}$, and consider the semidirect product subgroup $HN=H\ltimes N$ of G.

Extend $\mathfrak{a}$ to a maximal abelian subspace $\mathfrak{a}_{pq}\supseteqq\mathfrak{a}$ of $\mathfrak{p}\cap\mathfrak{q}$. We set $M_{kh}=Z_{K\cap H}(\mathfrak{a}_{pq})$, the centralizer of $\mathfrak{a}_{pq}$ in $K\cap H$.

In the previous paper [I], we studied multiplicities in C^∞- or L^2-induced representations $\mathrm{Ind}_{HN}^G(\zeta)$ of G, generalizing the theory of spherical functions developed by Harish-Chandra [8] and van den Ban [1]. We gave nice sufficient conditions for a representation ζ of HN that $\mathrm{Ind}_{HN}^G(\zeta)$ has finite multiplicity property.

Remark 6.1. In [I], an assumption stronger than the above one is imposed on σ and H. Namely, we supposed there that (1) σ extends to an involution of the Levi subgroup L (denoted again by σ), and that (2) H satisfies the inclusion relation $(L_\sigma)_0\subseteqq H\subseteqq L_\sigma$. Here L_σ is the fixed subgroup of σ, and $(L_\sigma)_0$ its identity component. But these additional assumptions are not used. Therefore, all the results of [I] remain true in the present setting.

6.1.1. Case of C^∞-induced representations.

First, let us consider the representation $C^\infty\text{-}\mathrm{Ind}_{HN}^G(\zeta)=(\pi_\zeta, C^\infty(G;\zeta))$ induced in C^∞-context (see [I, 2.1] for the precise definition of C^∞-induced representations). Here ζ is a continuous representation of HN on a Fréchet space. The differential of the G-action gives on the representation space $C^\infty(G;\zeta)$ a compatible $(\mathfrak{g}_C, K)$-module structure, denoted again by π_ζ. Then we obtain a criterion for the finiteness of multiplicities in π_ζ as follows.

Theorem 6.2 [I, Theorem 2.12]. *The induced representation* $\pi_\zeta = C^\infty\text{-Ind}_{HN}^G(\zeta)$ *has finite multiplicity property, if so does the restriction of* ζ *to the compact subgroup* M_{kh}. *Namely, if* $\dim \mathrm{Hom}_{M_{kh}}(\nu, \zeta) < +\infty$ *for any finite-dimensional irreducible representation* ν *of* M_{kh}, *then every irreducible admissible* $(\mathfrak{g}_C, K)$*-module* $\mathcal{H}_K$ *(see* [I, 2.1]*) occurs as a submodule of* $C^\infty(G; \zeta)$ *with at most finite multiplicity, i.e.,* $\dim \mathrm{Hom}_{\mathfrak{g}_C - K}(\mathcal{H}_K, C^\infty(G; \zeta)) < +\infty$.

6.1.2. Case of unitarily induced representations.

We can relate the multiplicities in unitarily induced representations to those in the corresponding C^∞-induced representations. To be more precise, for a unitary representation ζ of a closed subgroup Q of G, let $L^2\text{-Ind}_Q^G(\zeta) = (\mathcal{U}_\zeta, L^2(G; \zeta))$ denote the representation of G induced unitarily from ζ. According to Harich-Chandra, the connected semisimple Lie group G is of type I. Therefore, the unitary representation $\mathcal{U}_\zeta$ admits a unique factor decomposition on the unitary dual $\hat{G}$ of G, equipped with the usual Borel structure (cf. [I, Lemma 3.5]):

$$\mathcal{U}_\zeta \cong \int_{\hat{G}}^{\oplus} \mathcal{U}_\zeta(\pi) d\mu_\zeta(\pi), \qquad L^2(G; \zeta) \cong \int_{\hat{G}}^{\oplus} \mathcal{H}(\zeta, \pi) d\mu_\zeta(\pi). \tag{6.1}$$

Here μ_ζ is a Borel measure on $\hat{G}$, and the representations $(\mathcal{U}_\zeta(\pi), \mathcal{H}(\zeta, \pi))$ are, for almost evey (=a.e.) $\pi \in \hat{G}$ with respect to μ_ζ, factor representations of type π: $\mathcal{U}_\zeta(\pi) \cong [m_\zeta(\pi)] \cdot \pi$. The function $\pi \mapsto m_\zeta(\pi)$ is said to be the multiplicity function for $\mathcal{U}_\zeta$. (This is a Borel function on $\hat{G}$.) Then, using the results of R. Penney on desintegrations of C^∞-vectors for unitary representations, we get

Theorem 6.3 [I, Theorem 3.12]. *Suppose that* ζ *be finite-dimensional. Then, the multiplicity function* m_ζ *for* $\mathcal{U}_\zeta$ *admits an upper bound as*

$$m_\zeta(\pi) \leqq \dim \mathrm{Hom}_{\mathfrak{g}_C - K}(\pi_K, \pi_\zeta) \qquad \textit{with} \quad \pi_\zeta = C^\infty\text{-Ind}_Q^G(\zeta) \tag{6.2}$$

for a.e. $\pi \in \hat{G}$. *Here, for an irreducible unitary representation* π *of* G, π_K *denotes the irreducible admissible representation of* $(\mathfrak{g}_C, K)$ *acting on the space of* K*-finite vectors for* π.

From Theorems 6.2 and 6.3, one obtains

Theorem 6.4 [I, Theorem 3.13]. *Let* ζ *be a finite-dimensional unitary representation of HN. Then, the unitarily induced representation* $\mathcal{U}_\zeta = L^2\text{-Ind}_{HN}^G(\zeta)$ *has finite multiplicity property: the multiplicity function* $\pi \mapsto m_\zeta(\pi)$ *takes finite values for a.e.* $\pi \in \hat{G}$.

6.2. Application to RGGGRs $\Gamma_i(c)$. Return to our original objects in §§3-5 and keep to the notations there. Put $\sigma = \theta_{\varepsilon(i)}$ and $H = H^i$ ($0 \leqq i \leqq l$; see 3.2 and 3.3). Thanks to Lemma 3.15, the pair $(\theta_{\varepsilon(i)}, H^i)$ satisfies the assumption for (σ, H) in 6.1. In this case, the subspace $\mathfrak{a}_{pq}$ coincides with the maximal abelian subspace $\mathfrak{a}_p = \sum_{1 \leq k \leq l} \boldsymbol{R} H_k$ of $\mathfrak{p}$ in 3.1. Moreover, each H^i contains $M = Z_K(\mathfrak{a}_p)$. Therefore we have $M_{kh} = M$. We can thus apply the results of [I], quoted as

Theorems 6.2-6.4, to RGGGRs $\Gamma_i(c)=\mathrm{Ind}_{H^iN}^G(\tilde{c}\otimes\tilde{\xi}_i)$ associated with the nilpotent classes ω_i. In the succeeding subsections, we present the results and prove them.

6.3. Finite multiplicity theorems for C^∞-$\Gamma_i(c)$.

First, let us consider the RGGGRs C^∞-$\Gamma_i(c)$ induced in C^∞-context. Thanks to Theorems 5.17 and 6.2, we immediately obtain our first main result, on finiteness of multiplicities in C^∞-$\Gamma_i(c)$, as follows.

Theorem 6.5. *Let G be a connected simple Lie group of hermitian type, with finite center. Denote by ω_i ($0\leqq i\leqq l=$the real rank of G) the nilpotent* $\mathrm{Ad}(G)$-*orbits in $\mathfrak{g}=\mathrm{Lie}\,G$ defined in Theorem* 3.13. *Consider the RGGGRs C^∞-$\Gamma_i(c)=C^\infty$-$\mathrm{Ind}_{H^iN}^G(\tilde{c}\otimes\tilde{\xi}_i)$ (induced in C^∞-context) associated with ω_i. Then one has*

(1) *C^∞-$\Gamma_i(c)$ has finite multiplicity property in the sense of Theorem* 6.2 *for any $0\leqq i\leqq l$ and any finite-dimensional representation c of H^i.*

(2) *Suppose that $G=SU(p, l)$ $(p\geqq l)$. Let $c=c(\Phi)$ be the irreducible admissible representation of $H^i\cong U(l-i, i)\times SU(p-l)$ $(p\neq l)$; $H^i\cong SU(l-i, i)$ $(p=l)$ with highest weight $\Phi\in\sqrt{-1}\mathfrak{t}^{+*}$, constructed in* 5.3.3. *Then, the representation C^∞-$\Gamma_i(c)$ is of multiplicity finite.*

6.4. Finite multiplicity theorems for L^2-$\Gamma_i(c)$.

Secondly, we study the multiplicities in the unitarily induced RGGGRs. Our result for these representations is stated as follows, which is our second main result of this article.

Theorem 6.6. *Let G be a simple Lie group as in Theorem* 6.5. *For an irreducible unitary representation c of tbe reductive subgroup H^i, consider the unitary RGGGR L^2-$\Gamma_i(c)=L^2$-$\mathrm{Ind}_{H^iN}^G(\tilde{c}\otimes\tilde{\xi}_i)$. Then,*

(1) *if $i=0$ or l, then all the RGGGRs L^2-$\Gamma_i(c)$ $(c\in(K\cap L)^\wedge)$ associated with ω_i have finite multiplicity property in the sense of Theorem* 6.4. *Here $(K\cap L)^\wedge$ denotes the unitary dual of the compact group $K\cap L=H^0=H^l$.*

(2) *Suppose that G/K reduces to a tube domain. Then, the RGGGR L^2-$\Gamma_i(c)$ is of multiplicity finite for any i and any finite-dimensional c.*

6.5. Proof of Theorem 6.6. First, notice that the representations $\tilde{\xi}_i$ $(0\leqq i\leqq l)$ are all one-dimensional if G/K reduces to a tube domain. So, the assertion (2) follows from Theorem 6.4.

In the following, we show the assertion (1) by reducing the proof to the case of C^∞-induced representations. (Since $\tilde{\xi}_i$ is infinite-dimensional in general, one can not apply Theorem 6.4 directly. In order to prove (1), we need get back to the argument in [I, §3], which proves Theorems 6.3 and 6.4.) Our proof consists of four steps.

STEP I. Throughout this proof, we always assume that $i=0$ or l. Let $(\rho_i, \mathcal{F}_i)$ and $(T_i, \mathcal{H}_i)$ be two kinds of realizations of $[\xi_i]\in\hat{N}$ constructed in 4.3, where $\mathcal{F}_i$ is the Fock space of $(\mathfrak{g}(1), J_i'')$. Denote by ι the intertwining operator from $\mathcal{H}_i$ onto $\mathcal{F}_i$ in Proposition 4.12. Extend $(\rho_i, \mathcal{F}_i)$ to a representation $(\tilde{\rho}_i, \mathcal{F}_i)$ of $H^i N$ as in Theorem 4.13, and define a representation $\tilde{T}_i$ of $H^i N$ by $\tilde{T}_i = \iota^{-1} \circ \tilde{\rho}_i \circ \iota$. By making use of these realizations $\tilde{\rho}_i$ and $\tilde{T}_i$, the unitary RGGGRs associated with ω_i are constructed as

$$L^2\text{-}\Gamma_i(c) = L^2\text{-}\mathrm{Ind}_{H^i N}^{G}(\tilde{c} \otimes \tilde{\rho}_i) \cong L^2\text{-}\mathrm{Ind}_{H^i N}^{G}(\tilde{c} \otimes \tilde{T}_i) \tag{6.3}$$

for irreducible unitary representations (c, E_c) of $H^i = K \cap L$.

We identify the tensor product Hilbert space $E_c \otimes \mathcal{F}_i$ canonically with the Hilbert space consisting of E_c-valued holomorphic functions ϕ on $(\mathfrak{g}(1), J_i'')$ such that

$$\|\phi\|^2 = \int_{\mathfrak{g}(1)} \|\phi(X)\|_c^2 \exp(-2\|X\|_i^2) dX < +\infty, \tag{6.4}$$

where $\|\cdot\|_c$ denotes the norm on E_c. (Compare with (4.29).) Then, the representation $\zeta \equiv \tilde{c} \otimes \tilde{\rho}_i$ of $H^i N$ on $E_c \otimes \mathcal{F}_i$ is described as

$$\begin{cases} \zeta(k)\phi(X) = c(k) \circ \phi(\mathrm{Ad}(k)^{-1} X) & (k \in H^i) \\ \zeta(n) = I_c \otimes \rho_i(n) & (n \in N;\ I_c = \text{the identity map on } E_c) \end{cases} \tag{6.5}$$

for $\phi \in E_c \otimes \mathcal{F}_i$.

We adopt the same kind of identification of $E_c \otimes \mathcal{H}_i$ with the space of E_c-valued C^∞-functions on N satisfying (4.20) and (4.21; replace the absolute value $|\cdot|$ in the integrand by $\|\cdot\|_c$). Put $\zeta' = \tilde{c} \otimes \tilde{T}_i$, then an element $k \in H^i$ is represented by ζ' as

$$\zeta'(k) f(n) = c(k) f(k^{-1} n k) \qquad (n \in N,\ f \in E_c \otimes \mathcal{H}_i). \tag{6.6}$$

STEP II. We put $(\mathcal{U}_\zeta, L^2(G;\zeta)) \equiv L^2\text{-}\Gamma_i(c) = L^2\text{-}\mathrm{Ind}_{H^i N}^{G}(\zeta)$ with $\zeta = \tilde{c} \otimes \tilde{\rho}_i$. Then, in view of Lemma 3.9 of [I], the subspace $L^2(G;\zeta)^\infty$ is contained in the space $C^\infty(G;\zeta)$ consisting of $\mathcal{H}(\zeta)$ $(\equiv E_c \otimes \mathcal{F}_i)$-valued C^∞-fuctions F on G such that $F(gz) = \zeta(z)^{-1} F(g)$ $(g \in G, z \in H^i N)$. Here, for a unitary representation $(\mathcal{U}, \mathcal{H})$ of G, $\mathcal{U}_\infty$ denotes the corresponding smooth representation of G and $\mathfrak{g}_C$ on the space $\mathcal{H}^\infty$ of C^∞-vectors for $\mathcal{U}$, equipped with the usual Fréchet space topology. Moreover, the linear map

$$\delta^{(1)}: L^2(G;\zeta)^\infty \longrightarrow \mathcal{H}(\zeta), \qquad \delta^{(1)}(F) = F(1)$$

is continuous, where 1 expresses the unit element of G.

We set $\delta^{(2)}(\phi) = \phi(0) \in E_c$ for $\phi \in \mathcal{H}(\zeta)$. Then, $\delta^{(2)}$ gives a continuous linear map from $\mathcal{H}(\zeta)$ to E_c. To verify this, we may assume without loss of generality that $c = 1_{H^i}$, the trivial character of H^i. Then it holds that $\mathcal{H}(\zeta) = \mathcal{F}_i$. Take an orthogonal basis $X_1, X_2, \cdots, X_r$ of the complex Hilbert space $(\mathfrak{g}(1), J_i'', (\ ,\ )_i)$ (see Lemma 4.11), and express $X \in \mathfrak{g}(1)$ as $X = \sum_{1 \le j \le r} z_j X_j$ with $z_j \in \boldsymbol{C}$. The Lebesgue measure dX on $\mathfrak{g}(1)$ in (6.4) is described as

$$dX = b \prod_{j=1}^{r} dz_j d\bar{z}_j \qquad \text{for some positive constant } b.$$

For $\phi \in \mathcal{F}_i$, $\phi(X) = \sum_N a_N z^N$ be the Taylor expansion of the entire function ϕ, where $z^N = z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r}$ for a multi-index $N = (n_1, n_2, \cdots, n_r)$ of non-negative integers n_j. Then, a simple calculation yields

$$\|\phi\|_{\mathcal{F}_i}^2 = b \sum_N |a_N|^2 (2\pi)^{|N|} \left(\prod_{j=1}^{r} \int_0^\infty t^{2n_j+1} e^{-t^2} dt \right), \tag{6.7}$$

where $|N| = n_1 + n_2 + \cdots + n_r$. This shows the continuity of $\delta^{(2)}$ because $a_{(0,0,\ldots,0)} = \phi(0) = \delta^{(2)}(\phi)$.

Thus, we find out that the composition $\delta = \delta^{(2)} \circ \delta^{(1)}$ gives a continuous linear map from $L^2(G;\zeta)^\infty$ to E_c. Furthermore, δ satisfies, by construction, following three properties: for $F \in L^2(G;\zeta)^\infty$, one has

$$\delta((\mathcal{U}_\zeta)_\infty(Y)F) = \sqrt{-1}\, A[i]^*(Y)\delta(F) \qquad (Y \in \mathfrak{u}_i), \tag{6.8}$$

$$\delta((\mathcal{U}_\zeta)_\infty(k)F) = c(k) \circ \delta(F) \qquad (k \in H^i), \tag{6.9}$$

$$\delta((\mathcal{U}_\zeta)_\infty(G)F) = (0) \quad \text{implies that } F = 0. \tag{6.10}$$

Here, $\mathfrak{u}_i = V_i^- \oplus \mathfrak{g}(2)_C$ is the positive polarization at $A[i]^* \in \mathfrak{n}^*$ in 4.3. The third property means that $\delta: L^2(G;\zeta)^\infty \to E_c$ is a generalized cyclic map (see [I, 3.5]).

STEP III. Consider the factor decomposition of $\mathcal{U}_\zeta$ in 6.1. Then, by virtue of Propositions 3.7 and 3.8 of [I], there exist, for a.e. $\pi \in \hat{G}$, linear maps $\delta_\pi: \mathcal{H}(\zeta, \pi)^\infty \mapsto E_c$ with properties (6.8), (6.9) and (6.10). (There, replace $(\mathcal{U}(t), \mathcal{H}(t))$ by $(\mathcal{U}_\zeta(\pi), \mathcal{H}(\zeta, \pi)$.) And δ is decomposed into a "direct integral" of δ_π's.

For $F \in \mathcal{H}(\zeta, \pi)^\infty$, define an E_c-valued C^∞-function $A_\pi(F)$ on G via

$$A_\pi(F)(g) = \delta_\pi(\mathcal{U}_\zeta(\pi)(g^{-1})F) \qquad (g \in G). \tag{6.11}$$

This map A_π naturally gives rise to an injective, continuous intertwining operator from $\mathcal{U}_\zeta(\pi)_\infty$ to the representation $C^\infty\text{-Ind}_{H^iN}^G(\zeta'^+)$. Here ζ'^+ is a representation of H^iN defined as follows. $\mathcal{H}_i^+$ ($\supseteqq \mathcal{H}_i$) denotes the space of C^∞-functions f on N satisfying (4.20) only. The representation $\tilde{T}_i$ on $\mathcal{H}_i$ extends naturally to a representation $\tilde{T}_i^+$ acting on $\mathcal{H}_i^+$. We set $\zeta'^+ = \tilde{c} \otimes \tilde{T}_i^+$.

In the same way, we can construct a representation $\zeta^+ = \tilde{c} \otimes \tilde{\rho}_i^+$ of H^iN acting on $E_c \otimes \mathcal{F}_i^+$. Then $\iota: \mathcal{H}_i \xrightarrow{\sim} \mathcal{F}_i$ is extended naturally to an isomorphism ι^+ of H^iN-modules from $\mathcal{H}_i^+$ onto $\mathcal{F}_i^+$. One thus gets a commutative diagram of H^iN-modules:

$$\begin{tikzcd} E_c \otimes \mathcal{H}_i^+ \arrow[r, "I_c \otimes \iota^+", "\sim"'] & E_c \otimes \mathcal{F}_i^+ \\ E_c \otimes \mathcal{H}_i \arrow[u, hook] \arrow[r, "\sim", "I_c \otimes \iota"'] & E_c \otimes \mathcal{F}_i . \arrow[u, hook] \end{tikzcd} \tag{6.12}$$

From the above discussion, we see that the multiplicity function m_ζ for the RGGGR $\mathcal{U}_\zeta = L^2\text{-}\Gamma_i(c)$ is bounded as

(6.13) $m_\zeta(\pi) \leqq \dim \mathrm{Hom}_G(\pi_\infty, C^\infty\text{-}\mathrm{Ind}_{HiN}^G(\zeta^+)) \leqq \dim \mathrm{Hom}_{\mathfrak{g}_C-K}(\pi_K, C^\infty\text{-}\mathrm{Ind}_{HiN}^G(\zeta^+))$

for a.e. $\pi \in \hat{G}$.

STEP IV. Thus, in order to complete the proof, it suffices to show that the representation $C^\infty\text{-}\mathrm{Ind}_{HiN}^G(\zeta^+)$ has finite multiplicity property. This is achieved as follows. Thanks to Theorem 6.2, it is enough to prove that the restriction $\zeta^+|M$ is of multiplicity finite. Let (ν, V) be an irreducible unitary representation of M, and T a non-zero intertwining operator from ν to $\zeta^+|M$. Then, the central one-parameter subgroup C_+ of M in 5.1 acts on the subspace $T(V) \subseteqq E_c \otimes \mathcal{F}_i^+$ by scalars, which implies that all the elements in $T(V)$ are polynomials on $(\mathfrak{g}(1), J_i'')$. Hence one gets $T(V) \subseteqq E_c \otimes \mathcal{F}_i$. This means that $\mathrm{Hom}_M(\nu, \zeta^+) = \mathrm{Hom}_M(\nu, \zeta)$. Consequently, $\zeta^+|M$ has finite multiplicity property thanks to Theorem 5.5. Q.E.D.

Remark 6.7. In the proof of the assertion (1), we have made use of the Fock model realization of $[\tilde{\xi}_i] \in (H^i N)^\wedge$ $(i=0, l)$. But, if $i \neq 0, l$, then $\tilde{\rho}_i$ constructed in 4.3 can not give a complete description of a genuine extension $\tilde{\xi}_i$ of ξ_i. This is because, in such a case, the positive polarization $\mathfrak{n}_i = V_i^- \oplus \mathfrak{g}(2)_C$ at $A[i]^* \in \mathfrak{n}^*$ is no longer stable under the whole $\mathrm{Ad}(H^i)$. For this reason, one can not generalize the proof of (1) directly to the case of arbitrary i. At the present time, we do not know whether or not the unitary RGGGRs $L^2\text{-}\Gamma_i(c)$ have finite multiplicity property for $i \neq 0, l$.

6.6. Case of $i=0, l$. Our results for this case are quite satisfactory. We restate here them and comment on them in connection with Problem EDS (see 4.1).

Theorem 6.8. *All the RGGGRs (L^2- or C^∞-)$\Gamma_i(c)$ $(c \in (K \cap L)^\wedge)$ associated with the nilpotent class ω_i have finite multiplicity property for $i=0$ or l.*

The original non-reduced GGGR Γ_i is far from being of multiplicity finite in general, and its study is reduced to that of the corresponding RGGGRs $\Gamma_i(c)$. Therefore, thanks to our results, one can avoid the difficulties in the study of GGGRs Γ_i $(i=0, l)$, coming from the infiniteness of multiplicities in Γ_i.

For instance, the discrete series representations of G occur in Γ_i generally with infinite multiplicities (cf. [23]). We believe that, in such a case, our RGGGRs are useful to settle Problem EDS brought up in 4.1. To be more precise, in view of (4.13) this problem for Γ_i $(i=0, l)$ is reduced to the following

Problem EDSbis. For $i=0, l$, describe the embeddings of discrete series representations of G into RGGGRs $\Gamma_i(c)$ $(c \in (K \cap L)^\wedge)$.

We think that this problem of reduced type is, thanks to our results, much more manageble than the original one.

6.7. A multiplicity one theorem and embeddings of discrete series.

At the end of this paper, we refer to the contents of the subsequent paper [33].

Under some additional assumptions on G and c, we can show multiplicity free property for RGGGRs $\Gamma_i(c)$ $(i=0, l)$.

Theorem 6.9 [33]. *Suppose that G/K is holomorphically equivalent to a tube domain. Then, for any real valued character c of the compact subgroup $K\cap L=H^0=H^l$, the unitary RGGGR $L^2\text{-}\Gamma_i(c)=L^2\text{-}\mathrm{Ind}_{(K\cap L)N}^G(\tilde{c}\otimes\tilde{\xi}_i)$ $(i=0$ or $l)$ is of multiplicity one: the multiplicity function for $L^2\text{-}\Gamma_i(c)$ takes values 0 or 1 almost everywhere on $\hat{G}$ with respect to the Borel measure giving its factor decomposition (see 6.1). Here, one chooses an extension $\tilde{\xi}_i$ of the unitary character ξ_i of N in such a way $\tilde{\xi}_i|(K\cap L)\equiv 1$.*

This theorem generalizes, in a certain sense, the result of Piatetskii-Shapiro and Novodvorskii ([21], [22]) on the uniqueness of generalized Bessel models for $Sp(4)$.

In [33], we study also the embeddings of discrete series into (reduced) GGGRs L^2- or $C^\infty\text{-}\Gamma_i$ $(\Gamma_i(c))$ (Problems EDS and EDSbis). We remark here that there is a difference between the embeddings into unitary GGGRs $L^2\text{-}\Gamma_i$ and those into $C^\infty\text{-}\Gamma_i(c)$ (in C^∞-context). In other words, even if a discrete series representation D can be embedded into $C^\infty\text{-}\Gamma_i$ as a $(\mathfrak{g}_C, K)$-module, D does not necessarily occur in $L^2\text{-}\Gamma_i$.

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