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Plurianonical divisors of elliptic fiber spaces

By

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Introduction.

By an elliptic fiber space $f: V \to W$, we mean that f is a proper surjective morphism of a compact complex manifold V to a compact complex manifold W, where each fiber is connected and the general fibers are smooth elliptic curves. In particular, when V is a surface and W is a curve, we say that V is an elliptic surface over W.

By an n-dimensional elliptic fiber space $V \to W$ with $\kappa(V)=n-1$, we mean that the image of a rational map $\Phi_{|mK_V|}$ is $(n-1)$-dimensional for sufficiently large m. In this case if an m-th pluricanonical mapping $\Phi_{|mK_V|}: V \to \Phi_{|mK_V|}(V)$ is bimeromorphic to the original elliptic fiber space, we say that $\Phi_{|mK_V|}$ gives the Iitaka fibration.

Iitaka [6] showed that for any elliptic surface $f: S \to C$ with $\kappa(S)=1$, the m-th pluricanonical mapping $\Phi_{|mK_S|}$ gives the unique structure of the elliptic surface $f: S \to C$ if $m \geq 86$. Moreover he showed that 86 is the best possible number. On the other hand, Katsura and Ueno [7] showed that if S is an *algebraic* elliptic surface defined over an algebraically closed field k of characteristic $p \geq 0$ with $\kappa(S)=1$, then $\Phi_{|mK_S|}$ gives the unique structure of the elliptic surface for every $m \geq 14$.

One of the main purpose of this paper is to obtain the bound of the Iitaka fibration of an elliptic threefold when the Kodaira dimension of the base space is greater than or equal to 1.

We prove the following.

Main theorem A. *If $f: X \to S$ is an elliptic threefold with $\kappa(X)=2$ and $\kappa(S) \geq 1$, then $\Phi_{|mK_X|}$ gives the Iitaka fibration for all even integer $m \geq 16$.*

The main difficulty is that if $f: X \to Y$ is an elliptic fiber space, $f_*(mK_{X/Y})$ is *not necessarily invertible* for a positive integer m, as was remarked by Fujita [5]. So we take a suitable bimeromorphic model $\tilde{f}: \tilde{X} \to \tilde{Y}$ of f and express an holomorphic section of $mK_{\tilde{X}/\tilde{Y}}$ by means of the modular form of weight m on the upper half plane. (cf. [5] [14]) Then if the Kodaira dimension of the base space is equal to or more than one, we can apply the results about pluricanonical mappings of surfaces.

Though we have not completely proved the counterpart of Iitaka's theorem for

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elliptic threefolds, we conjecture that such a theorem holds and 5420 is the best best possibl number. In [4], the author constructed series of examples of elliptic fiber spaces, which gives an evidence for the existence of such bounds. Our result is the following.

Main theorem B. *Let $\{a_n\}_{n=1,2,\cdots}$ be a sequence of natural numbers defined by*

$$a_1 = 2\,, \quad a_{n+1} = a_1 a_2 \cdots a_n + 1\,.$$

And let $\{b_n\}_{n=1,2,\cdots}$ be a sequence of natural numbers defined by

$$b_n = (n+1)\,(a_{n+3} - 1) + 2\,.$$

Then for every positive integer n, there exists an elliptic fiber space $X^{(n+1)} \to \boldsymbol{P}^n$ over $\boldsymbol{P}^n$ which satisfies the following conditions.

(1) $\kappa(X^{(n+1)}) = n$.

(2) b_n *is the best possible number of the Iitaka fibering of* $X^{(n+1)}$, *that is,* $\dim|mK_X| = 0$ *if* $m = b_n - 1$, *and the m-th pluricanonical mapping* $\Phi_{|mK_X|}$ *gives the Iitaka fibering for all* $m \geq b_n$.

Moreover, $X^{(n+1)}$ *is not in the class* $\mathcal{C}$ *in the sense of Fujiki* [1]. (*That is,* $X^{(n+1)}$ *cannot be bimeromorphic to any compact Kähler manifold.*)

Examples. Now, we write down the first few terms of $\{a_n\}$ and $\{b_n\}$.

n	1	2	3	4	5	6
a_n	2	3	7	43	1807	3263443
b_n	86	5420	13053770	$\sim 10^{13}$	$\sim 10^{26}$	$\sim 10^{52}$

(1) $b_1 = 86$. This is the well-known result of the elliptic surface. An elliptic surface $f\colon S \to \boldsymbol{P}^1$ over $\boldsymbol{P}^1$ with three multiple fibers of multiplicity 2, 3, 7 and with constant moduli has the property that $\dim|85K_S| = 0$.

(2) $b_2 = 5420$. There exists an elliptic threefold $f\colon X \to \boldsymbol{P}^2$ over $\boldsymbol{P}^2$ with constant moduli which has multiple fibers of multiplicity 2, 3, 7, 43 along the four lines on $\boldsymbol{P}^2$ in a general position. X has the property that $\dim|5419K_X| = 0$.

To prove Theorem B, we need to study multiple fibers of elliptic fiber spaces and generalize the notion of a logarithmic transformation defined by Kodaira. Our construction is as follows.

Let $H_i\,(1 \leq i \leq n+2)$ be $(n+2)$ hyperplanes on $\boldsymbol{P}^n$ which are in general position. Let $(a_1, a_2, \cdots, a_{n+2})$ be $(n+2)$-tuple of positive integers defined as in theorem B. Then $X^{(n+1)} \to \boldsymbol{P}^n$ is an elliptic fiber space over $\boldsymbol{P}^n$ which has multiple fibers of multiplicity a_i along each $H_i\,(1 \leq i \leq n+2)$. Note that there exists *no* finite *abelian* covering of $\boldsymbol{P}^n$ which branches along H_i's $(1 \leq i \leq n+2)$ with the ramification index a_i respectively.

We prove Theorem B in two different methods. One way is to use generalized logarithmic transformations along the divisors which have only normal crossings and another way is to construct $X^{(n+1)}$ as a submanifold of a Hopf manifold, which was suggested by M. Kato. The latter proof is much simpler than the former, while the former is applicable to many other situations. (cf. §5).

On the other hand, if we consider only algebraic elliptic fiber spaces, the best possible number of the Iitaka fibration seems to be much smaller than that of the analytic case. One of the main reason is that the multiplicities of the multiple fibers of an algebraic elliptic fiber space with constant moduli should satisfy certain numerical conditions, as was shown by Katsura and Ueno [7]. Moreover, there is a deep connection with the theory of branched coverings of complex manifolds which was developed by Namba.

In [13], Namba obtained the necessary and sufficient conditions for the existence of finite abelian coverings of $\boldsymbol{P}^n$. It is almost equivalent to the one obtained by Katsura and Ueno [7]. Combining these two results, we see that an algebraic elliptic fiber space over $\boldsymbol{P}^n$ with constant moduli which has multiple fibers along hyperplanes can be constructed globally by taking finite abelian coverings of $\boldsymbol{P}^n$.

Our result is the following.

Theorem C. *Let* $\{d_n\}_{n=1,2,\dots}$ *be a sequence of natural numbers defined as follows*: $d_n=2(n^2+3n+3)$.
Then for every positive integer n, there exists an algebraic elliptic fiber space $Z^{(n+1)}\to\boldsymbol{P}^n$ *over* $\boldsymbol{P}^n$ *which satisfies the following conditions.*
(1) $\kappa(Z^{(n+1)})=n$.
(2) d_n *is the best possible number of the Iitaka fibration of* $Z^{(n+1)}$, *that is,* $\dim|mK_Z|=0$ *if* $m=d_n-1$, *and the m-th pluricanonical mapping* $\Phi|_{mK_Z}|$ *gives the Iitaka fibration for all* $m\geq d_n$.

Examples. We write down the first few terms of $\{d_n\}$.

n	1	2	3	4	5	6
d_n	14	26	42	62	86	114

Finally, let us explain briefly the contents of our paper.

In §1, we shall review the canonical bundle formula of elliptic fiber spaces due to T. Fujita [5]. In §2, we shall consider pluricanonical mappings of elliptic threefolds when the Kodaira dimension of the base space is greater than or equal to 0. In §3, we shall consider the structure of algebraic elliptic fiber spaces with constant moduli. In §4, we shall prove Main theorem B. In §5, we shall consider generalized logarithmic transformations along the divisors which have only normal crossings and reprove Theorem B in a different way. In §6, as an application of Theorem 5.1, we shall construct examples of elliptic fiber spaces with $\kappa=0$.

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advices and to Professor M. Kato, who suggested to me another proof of Theorem B.

Notation and convention. If X is a compact complex manifold, we use the following notation.
$\kappa(X)$: the Kodaira dimension of X
K_X: the canonical bundle of X
$P_m(X)=\dim_C H^0(X, \mathcal{O}(mK_X))$
$h^{p,q}(X)=\dim_C H^q(X, \Omega_X^p)$
$q(X)=\dim_C H^1(X, \mathcal{O}_X)$
$e_m=\exp(2\pi\sqrt{-1}/m)$
$d\mathcal{O}_X$: the subsheaf of Ω_X^1 whose elements are d-closed.
For an integer n, $[n]$ denotes the greatest integer that does not exceed n.

§1. Preliminaries.

By an elliptic fiber space $f: V\to W$, we mean that f is a proper surjective morphism of a complex manifold V to a complex manifold W, where each fiber is connected and the general fibers are non-singular elliptic curves.

Put $\Sigma:=\{w\in W \mid f$ is not smooth over $f^{-1}(w)\}$ and let F be an irreducible component of Σ with $\dim F=\dim W-1$. For a general point x of F, there exists a curve Z in W passing through x such that Z meets F transversally and $f^{-1}(Z)$ is a non-singular elliptic surface over Z.

Furthermore we assume that $f^{-1}(Z)\to Z$ is relatively minimal. Then $f^{-1}(Z)$ has a singular fiber at x and the type of the singular fiber in Kodaira [12] is independent of the choice of Z and x. Hence we can define it to be the type of the singular fibers of f along F. In particular, if $f^{-1}(Z)$ has multiple fibers of multiplicity m at x, we say that *V has multiple fibers of multiplicity m along F.*

Now, for each type of singular fibers we can define a number α_i as follows.

Type	$m\mathrm{I}_b$	I_b^*	II	II*	III	III*	IV	IV*
α	$1-m^{-1}$	1/2	1/6	5/6	1/4	3/4	1/3	2/3

For an elliptic threefold, the following theorem is fundamental.

Theorem 1.1. (*Ueno* [14], *Corollary* (1.10)). *Let $f: V\to W$ be an elliptic threefold. Then there exists a bimeromorphically equivalent model $\hat{f}: \hat{V}\to\hat{W}$ of f which satisfies the following conditions.*

(*) *Let F be an irreducible component of the discriminant locus Σ of $\hat{f}$ with $\dim F=\dim\hat{W}-1$. For a general point x of F, there exists an analytic arc Z in $\hat{W}$ meeting F transversally and passing through x such that the elliptic surface $\hat{f}^{-1}(Z)\to Z$ is relatively minimal.*

Thanks to Theorem 1.1, our definition of the type of the singular fibers are well-defined.

Now, we recall the canonical bundle formula due to T. Fujita [5].

Theorem 1.2. (*T. Fujita* [5]). *Let f: $V\to W$ be an elliptic threefold such that the J-invariant J: $W\to \boldsymbol{P}^1$ is holomorphic. Let m be a positive integer such that $k=12m$ is divisible by the multiplicities of all the components D_i of the discriminant locus $\sum$ of f. Then*

$$\omega_V^{\otimes k}\simeq f^*(\omega_W^{\otimes k}\otimes J^*\mathcal{O}_{\boldsymbol{P}^1}(m)\otimes\mathcal{O}_W[\sum_i k\alpha_i D_i])\otimes\mathcal{O}_V[E-X]$$

for some effective divisor E, X on V such that
1) $\operatorname{codim} f(X)\geq 2$.
2) $f_*\mathcal{O}_V\simeq f_*\mathcal{O}_V(mE)$ *for any positive integer m.*

§2. Pluricanonical mappings of elliptic threefolds.

In this section, we consider pluricanonical mappings of elliptic threefolds only when the Kodaira dimension of the base space is greater than or equal to 1.

Proposition 2.1. (cf. Fujita [5], Ueno [14]). *Let f: $V\to W$ be an elliptic threefold. Assume that the discriminant locus $\sum$ of f are divisors with only normal crossings and the condition* (*) *in Theorem* 1.1 *is satisfied. Then for an arbitrary even positive integer $m>2$, we have: $mK_V\simeq f^*(mK_W+\Gamma)+E-G$ for some effective divisor Γ on W and E, G on V such that*
(1) $f_*\mathcal{O}_V(E)\simeq\mathcal{O}_W$
(2) $\operatorname{codim} f(G)\geq 2$.
(3) *Let D_i be an irreducible component of $\sum$ with* $\dim D_i=\dim W-1$. *Then we have $\Gamma=\sum_i [m\alpha_i]D_i+(p\nabla_1+q\nabla_2)$, where ∇_i's are effective divisors on W such that $3\nabla_1\sim J^*\mathcal{O}_{\boldsymbol{P}^1}(1)$ and $2\nabla_2\sim J^*\mathcal{O}_{\boldsymbol{P}^1}(1)$ and p, q are positive integers such that $m=4p+6q$.*

Remark 2.2. If $V\to W$ is an elliptic bundle over a Zariski open set of W and has only multiple singular fibers, the above result holds for *all* positive integer m.

Proof. We follow the idea of Fujita [5] and Ueno [14]. Let T: $W^\circ=W\backslash\sum\to H=\{\tau\in\boldsymbol{C}\,|\,\operatorname{Im}(\tau)>0\}$ be the period mapping associated to a holomorphic 1-form. T gives a single-valued holomorphic mapping T: $\tilde{W}^\circ\to H$ on the universal covering $\tilde{W}^\circ$ of W°. Let Φ: $\pi_1(W^\circ)\to SL(2,\boldsymbol{Z})$ be a monodromy representation. $\pi_1(W^\circ)$ can be considered as a covering transformation group of $\tilde{W}^\circ$ and we have

$$T(\gamma x)=\Phi(\gamma)T(x)\quad\text{for every}\quad \gamma\in\pi_1(W^\circ)\,.$$

The semi-direct product $G=\pi_1(W^\circ)\ltimes\boldsymbol{Z}^2$ acts on $\tilde{W}^\circ\times\boldsymbol{C}$ in a canonical way such that the quotient space $M=\tilde{W}^\circ\times\boldsymbol{C}\,|_G$ is non-singular and $f^\circ=f\,|_{W^\circ}$: $f^{-1}(W^\circ)\to W^\circ$ can be obtained from $M\to W^\circ$ by repatching a fiber coordinate. (cf. [15])

Now, let $G_k(z)=\sum_{m,n}'\dfrac{1}{(mz+n)^{2k}}$ be the Eisenstein series of index k. Then $G_2(z)$ (resp. $G_3(z)$) is the modular form of weight 4 (resp. 6) on the upper half plane with respect to $SL(2, \boldsymbol{Z})$ and has a zero of order 1 at $z=\exp(2\pi\sqrt{-1}/3)$. (resp. $z=\sqrt{-1}$.) And the elliptic modular function $j(z)$ can be written as

$$j(z)=1728 g_2^3/\Delta\,,\quad \Delta=g_2^3-27g_3^2\,,\quad g_2=60G_2 \quad\text{and}\quad g_3=140G_3\,.$$

Given an arbitrary even positive integer $m>2$, there exist positive integers p, q such that $m=4p+6q$ and $F(z)=G_2^p(z)G_3^q(z)$ is the modular form of weight m on the upper half plane with respect to $SL(2, \boldsymbol{Z})$.

Then for any $\omega\in H^0(W, K_W^{\otimes m})$, put $\Xi=F(T(w))f^*\omega\otimes(d\zeta)^m$, where ζ is the fiber coordinate of $f^\circ\colon V^\circ\to W^\circ$. Ξ is $\pi_1(W^0)$-invariant and gives an element of $H^0(V^\circ, K_{V^\circ}^{\otimes m})$. Set $\sum^\circ=\{x\in\sum\mid\sum$ is non-singular at x and there exists a curve Z in W passing through x and meeting $\sum$ transversally such that an elliptic surface $f^{-1}(Z)\to Z$ is non-singular.$\}$. Put $W'=W^\circ\cup\sum^\circ$. Clearly we have codim $(W\backslash W')\geq 2$.

By Ueno [14], Theorem (2.3), Ξ can be extended holomorphically to an element of $\Gamma(f^{-1}(W'), K_V^{\otimes m})$. And by writing down the zeros of Ξ explicitly, we have the following isomorphism on W':

$$f_*(K_V^{\otimes m})\to K_W^{\otimes m}\otimes\mathcal{O}(\Gamma)\,,\quad \Gamma=\sum_{i=1}[m\alpha_i]D_i+(pV_1+qV_2)\,, \qquad \text{——}(**)$$

where $V_i (i=1, 2)$ are effective divisors on W such that $3V_1\sim J^*\mathcal{O}_{P^1}(1)$ and $2V_2\sim J^*\mathcal{O}_{P^1}(1)$ and p and q are positive integers such that $m=4p+6q$.

Since codim $(W\backslash W')\geq 2$, the above isomorphism can be extended to a homomorphism on W.

Let E be an effective divisor on V such that

$$K_V^{\otimes m}\otimes\mathcal{O}_V(-E)=\{\mathrm{Image}\,(f^*f_*K_V^{\otimes m}\to K_V^{\otimes m})\}^{\vee\vee}\,.$$

Then $f^*f_*K_V^{\otimes m}\to f^*(K_W^{\otimes m}\otimes\mathcal{O}_W(\Gamma))$ induces an injective homomorphism $K_V^{\otimes m}\otimes\mathcal{O}_V(-E)\to f^*(K_W^{\otimes m}\otimes\mathcal{O}_W(\Gamma))$. Therefore we have $K_V^{\otimes m}\otimes\mathcal{O}_V(E-G)\simeq f^*(K_W^{\otimes m}\otimes\mathcal{O}_W(\Gamma))$ for an effective divisor G on V and this implies the claim. And (1) and (2) is clear from our construction. q.e.d.

Proof of remark 2.2. In this case, we can show (**) directly without using the modular forms, so our proof works for all positive integer m.

Proposition 2.3. *Let $f\colon X\to Y$ be an elliptic threefold. Then we have $P_m(X)\geq P_m(Y)$ for an arbitrary positive even positive integer $m>2$, except the case where X is bimeromorphic to an elliptic threefold which is a fiber bundle with the structure group* Z/2, *$\boldsymbol{Z}/3$, $\boldsymbol{Z}/4$ or $\boldsymbol{Z}/6$ over a Zariski open set of the base space.*

Proof. By Hironaka's flattening theorem and resolution of singularities, we have the following commutative diagram.

$$\begin{array}{ccccc} V & \xrightarrow{\mu} & M & \xrightarrow{\nu} & X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\phi} & S & \xrightarrow{\pi} & Y \end{array}$$

1) V, T and S are non-singular.
2) μ, ν, ϕ and π are bimeromorphic morphism and g is flat.
3) The J-invariant J: $T \to \boldsymbol{P}^1$ is a morphism.
4) The discriminant locus $\sum = \sum_i D_i$ of h are divisors with normal crossings.

Furthermore we may assume that the condition (*) in Theorem 1.1 is satisfied for h: $V \to T$.

Then it follows from Proposition 2.1 that for an arbitrary *even* positive integer m, we have $mK_V \sim h^*(mK_T + \Gamma) + E - G$ for some effective divisor Γ on T and E, G on V. Since g is flat, G is $(\nu \circ \mu)$-exceptional. Therefore we have an isomorphism

$$\begin{aligned} H^\circ(V, \mathcal{O}(mK_V)) &\simeq H^\circ(V, \mathcal{O}(mK_V + G)) \\ &\simeq H^\circ(T, \mathcal{O}(mK_T + \Gamma)). \end{aligned}$$

This implie sthat $P_m(X) \geq P_m(Y)$ for any even positive integer $m > 2$.

Remark 2.4. If f: $X \to Y$ is an elliptic threefold with constant moduli which has only multiple singular fibers, we have $P_m(X) \geq P_m(Y)$ for every positive integer m.

Proposition 2.5 (*The canonical bundle formula of elliptic threefolds*). *Let f: $V \to W$ be an elliptic threefold over an algebraic surface W. Assume that the discriminant locus $\sum$ of f are divisors with only normal crossings and the condition* (*) *in theorem* 1.1 *is satisfied. Then the canonical bundle of V can be written as follows*:

$$K_V \simeq f^*(K_W + L) + M - G$$

where 1) *L is a line bundle on W.*

2) *M is an effective divisor on V such that*

$$M \underset{Q}{=} f^*\Big(\sum_i \frac{m_i - 1}{m_i} D_i\Big) + E_1 - E_2\,,$$

where V has multiple fibers of multiplicity m_i along the irreducible component D_i of $\sum$ and E_i is an effective divisor on V such that $f(\operatorname{supp}(E_i))$ is a point.

3) *G is an effective divisor on V such that $f(G)$ is a point.*

Proof. Since $f_* K_{V/W}$ is coherent, it follows from Serre's theorem that there exists a very ample divisor H on W such that $H^\circ(W, f_* K_{V/W}(H)) \neq 0$. Hence if we put $\bar{H} = f^* H$, we have $H^\circ(V, K_{V/W}(\bar{H})) \neq 0$ and the complete linear system $|K_{V/W}(\bar{H})|$ contains an effective divisor $F = \sum_j F_j$.

Let C be ageneral hyperplane section of W and put $V(C) := f^{-1}(C)$. Then $V(C)$ is a non-singular elliptic surface over C and is *relatively minimal.* Clearly we have $K_{V/W}|_{V(C)} = K_{V(C)/C}$. Hence by the canonical bundle formula of elliptic surfaces (cf. [12]), each $f(F_j)$ is a curve or a point. The same argument as in Kodaira [12], Theorem (12.1) can be applied to our situation and we can easily see that there exists a line bundle L on W such that $f_*K_{V/W} \simeq \mathcal{O}(L)$ except a finite number of points on W.

By Krull's theorem, we can extend this to a homomorphism $f_*K_{V/W} \to \mathcal{O}(L)$ on W. Let M be an effective divisor on V such that

$$K_{V/W} \otimes \mathcal{O}_V(-M) = \{\text{Image}\,(f^*f_*K_{V/W} \to K_{V/W})\}^{\vee\vee}.$$

Then the homomorphism $f^*f_*K_{V/W} \to f^*\mathcal{O}(L)$ induces an injective homomorphism $K_{V/W} \otimes \mathcal{O}_V(-M) \to f^*\mathcal{O}(L)$. Hence there exists an effective divisor G on V such that

$$K_V \simeq f^*(K_W + L) + M - G\,.$$

Again by the canonical bundle formula of Kodaira, M can be expressed as

$$M \underset{Q}{\sim} f^*\Big(\sum_i \frac{m_i-1}{m_i} D_i\Big)\,, \quad \text{in codimension one on } W.$$

Therefore by applying the same argument as above, we obtain (2). q.e.d.

Proposition 2.6. *Let $f\colon X \to Y$ be an elliptic threefold over an algebraic surface Y and assume that $\kappa(X) = 2$. Then there exists a positive integer m_0 (which may depend on X) such that the pluricanonical mapping $\Phi_{|mK_X|}$ gives the Iitaka fibration for all $m \geq m_0$.*

Proof. By Hironaka's flattening theorem and Theorem 1.1, we may assume that $f\colon X \to Y$ satisfies the same conditions as in the proof of proposition 2.3. Since $\kappa(X) = 2$, there exist positive numbers α, β and positive integers p_0, d such that the following inequalities hold for any integer $p \geq p_0$: $\alpha p^2 \leq h^\circ(X, pdK_X) \leq \beta p^2$ and $\Phi_{|pdK_X|}$ gives the Iitaka fibration for all $p \geq p_\circ$. By Proposition 2.5, we have $K_X \sim f^*(K_Y + L) + M - G$,

$$M \underset{Q}{\sim} f^*\Big(\sum_i \frac{m_i-1}{m_i} D_i\Big) + E_1 - E_2, \text{ and codim } f(\text{supp}\,(E_i)) \geq 2, \text{ codim } f(\text{supp}\,(G)) \geq 2.$$

Fix a positive integer r such that $1 \leq r < d$ and for any positive integer p, put $a := pd - r$. Then we have

$$\begin{aligned} H^\circ(X, aK_X) &\simeq H^\circ\Big(X, f^*\Big(a(K_Y+L) + \sum_i \Big[\frac{a(m_i-1)}{m_i}\Big] D_i\Big)\Big) \\ &\simeq H^\circ\Big(Y, a(K_Y+L) + \sum_i \Big[\frac{a(m_i-1)}{m_i}\Big] D_i\Big) \end{aligned}$$

$$H^\circ(X, (a+r)K_X) \simeq H^\circ\Big(Y, (a+r)(K_Y+L) + \sum_i \Big[\frac{(a+r)(m_i-1)}{m_i}\Big] D_i\Big)\,.$$

$$\text{Since } \Big[\frac{(a+r)(m_i-1)}{m_i}\Big] - \Big[\frac{a(m_i-1)}{m_i}\Big] \leq \frac{(a+r)(m_i-1)}{m_i} - \Big(\frac{a(m_i-1)}{m_i} - 1\Big) < r+1\,,$$

we have the following inclusions.

$$H^\circ(X, aK_X) \supsetneq H^\circ(Y, (a+r)(K_Y+L)+ \sum_i \left[\frac{(a+r)(m_i-1)}{m_i}\right] D_i \\ -r(K_Y+L)-r\sum_i D_i) .$$

Let H be a very ample line bundle on Y such that $\bar{H}:=r(K_Y+L)+r\sum_i D_i+H$ is also very ample. Then we have $H^\circ(Y, \mathcal{O}(\Gamma-\bar{H}))\hookrightarrow H^\circ(X, aK_X)$, where we put $\Gamma=(a+r)(K_Y+L)+\sum_i \left[\frac{(a+r)(m_i-1)}{m_i}\right]D_i$.

There is an exact sequence

$$0\to H^\circ(Y, \mathcal{O}(\Gamma-\bar{H}))\to H^\circ(Y, \mathcal{O}(\Gamma))\to H^\circ(\bar{H}, \mathcal{O}(\Gamma)\otimes\mathcal{O}_{\bar{H}})\to ,$$

where $\bar{H}$ also denotes a general member of the complete linear system $|\bar{H}|$. Since $H^\circ(X, \mathcal{O}(\Gamma))\supsetneq H^\circ(X, (a+r)K_X)\supsetneq H^\circ(X, pdK_X)$, we have $\alpha p^2\leq \dim H^\circ(Y, \mathcal{O}(\Gamma))\leq \beta p^2$ for all $p\geq p_0$.

On the other hand, there is a positive integer γ such that $\dim H^\circ(\bar{H}, \mathcal{O}(\Gamma)\otimes O_{\bar{H}})\leq\gamma p$ by the consideration of the dimension.

Therefore there exists a positive integer $k(r)$ such that for all $p\geq k(r)$, we have $H^\circ(Y, \mathcal{O}(\Gamma-\bar{H}))\neq 0$ and hence $H^\circ(X, aK_X)\neq 0$, where $a=pd-r$. Since $H^\circ(X, (k(r)d-r)K_X)\neq 0$ for $0<r<d$, we have

$$H^\circ(X, pdK_X)\hookrightarrow H^\circ(X, ((p+k(r))d-r))K_X) .$$

So if we put $m_0:=\operatorname*{Max}_{0<r<d}\{(p_0+k(r))d-r\}$, $\Phi_{|mK_X|}$gives the Iitaka fibration for all $m\geq m_0$. q.e.d

Theorem 2.7. *Let f: $X\to S$ be an elliptic threefold over a surface S. Assume that $\kappa(X)=2$ and S is a surface of general type. Then the pluricanonical mapping $\Phi_{|mK_X|}$ associated to the complete linear system $|mK_X|$ gives the Iitaka fibration for all even positive integer $m\geq 6$.*

Remark 2.8. If X is an elliptic threefold with constant moduli which has only multiple singular fibers, the theorem holds for all $m\geq 5$.

Proof of 2.7. Let $M\to S$ be a flattening of f and let $V\to T$ be a non-singular model of M. We may assume that $V\to T$ satisfies the same conditions as in the proof of Proposition 2.3.

Then it follows from Proposition 2.1 that for an *even* positive integer $m>2$, we have $mK_V\sim f^*(mK_T+\Gamma)+E-G$ for some effective divisor Γ on T and E, G on V. By the same reason as in the proof of Proposition 2.3, we have an isomorphism $H^\circ(V, \mathcal{O}(mK_V))\simeq H^\circ(T, \mathcal{O}(mK_T+\Gamma))$.

If Y is a minimal surface of general type, then the pluricanoical mapping $\Phi_{|mK_Y|}$ gives a birational morphism for all $m\geq 5$.

Therefore $\Phi_{|mK_X|}$ gives the Iitaka fibration for all *even* integer $m\geq 6$.

Theorem 2.9. *Let T be a simple abelian surface and let $f: X \to T$ be an elliptic threefold over T with constant moduli which has only multiple singular fibers. Assume that $\kappa(X)=2$. Then we have $P_m(X)>1$ for all $m \geq 10$.*

Proof. From Remark 2.4, we have $P_g(X)=1$ and $P_m(X) \geq P_m(Y)$ for all $m \geq 2$. So we may assume that K_X is effective. Take a sufficiently fine open covering $\{U_\lambda\}_\lambda$ of X and K_X is locally defined by $\psi_\lambda=0$. Then we can take a double covering $\tilde{X}$ of X defined by $\tilde{X}=\bigcup_\lambda \{\zeta_\lambda^2=\psi_\lambda^2\}$, where $\{\zeta_\lambda\}$ is a fiber coordinate of the canonical bundle K_X. Take the normalization $\tilde{X}^*$ of $\tilde{X}$. Clearly $\tilde{X}^* \xrightarrow{\pi} X$ is a two-sheeted unramified covering of X. Take the Stein factorization of $\tilde{X}^* \to T$.

$$\begin{array}{ccc} \tilde{X}^* & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow f \\ \tilde{T} & \longrightarrow & T \end{array}$$

From our construction, it is clear that $\tilde{T} \to T$ is a double covering of T ramified only along $f(\operatorname{supp}(K_X))$ and $\tilde{T}$ is irreducible. By taking a suitable bimeromorphic model of $\tilde{X}^* \to \tilde{T}$, we may assume that $\tilde{T}$ is non-singular. Because T is a simple abelian surface, $\tilde{T}$ is a surface of general type. Clearly we have $\kappa(\tilde{X}^*) \geq \kappa(X)=2$. Therefore it follows from Proposition 2.3 that the pluricanonical mapping $\Phi_{|mK_{\tilde{X}^*}|}$ gives the Iitaka fibration for all $m \geq 5$. In particular we have $P_m(\tilde{X}^*) \geq 4$ for all $m \geq 5$.

Now, $\pi: \tilde{X}^* \to X$ is an unramified double covering of X with the Galois group G. Let L be the line bundle associated to the non-trivial character on G. Then we have $L^{\otimes 2} \simeq \mathcal{O}_X$ and $\pi_* \mathcal{O}_{\tilde{X}^*} \simeq \mathcal{O}_X \oplus \mathcal{O}_X(L)$.

Note that $\pi_*(\mathcal{O}(mK_{\tilde{X}^*})) \simeq \mathcal{O}(mK_X) \oplus \mathcal{O}(mK_X+L)$.

By considering the Leray's spectral sequence, we have

$$H^\circ(\tilde{X}^*, \mathcal{O}(mK_{\tilde{X}^*})) \xrightarrow{\sim} H^\circ(X, \mathcal{O}(mK_X)) \oplus H^\circ(X, \mathcal{O}(mK_X+L)) .$$

Thus for all $m \geq 5$, we have $h^\circ(X, \mathcal{O}(mK_X)) \geq 2$ or $h^\circ(X, \mathcal{O}(mK_X+L)) \geq 2$. Noting that $L^2 \simeq \mathcal{O}_X$ and K_X is effective, we have $P_m(X) \geq 2$ for all $m \geq 10$. q.e.d.

Proposition 2.10. *Let $f: X \to S$ be an elliptic threefold over an elliptic surface $\phi: S \to C$ such that $\kappa(X)=2$ and $\kappa(S)=1$. Let $\Sigma \subset S$ be a discriminant locus of f. Then S is algebraic and there exists an irreducible component D_0 of Σ with $\phi(D_0)=C$.*

Proof. By taking a suitable bimeromorphic model of $f: X \to S$, we may assume that

1) Σ have only normal crossings.
2) The J-invariant $J: S \to \boldsymbol{P}^1$ is holomorphic.

By the canonical bundle formula of Fujita [5], we have

*) $$K_X^{\otimes k} \simeq f^*(K_S^{\otimes k} \otimes J^* \mathcal{O}_{\boldsymbol{P}^1}(m)) \otimes \mathcal{O}_S(\sum_Y k\mu_Y Y) \otimes \mathcal{O}_X(V-W) \quad \text{where}$$

1) codim $f(W)\geq 2$.
2) $f_*\mathcal{O}_X(pV)\simeq\mathcal{O}_S$ for all $p\geq 1$,
and $k=12\mathrm{m}$ is a positive integer which is a multiple of all the multiplicities of the irreducible component of $\sum$.

Let $\mu\colon S\to S'$ be a contraction of exceptional curves of the first kind in fibers. Then $\phi'\colon S'\to C$ is relatively minimal and we have

$$K_{S'}\simeq\phi'^*(K_C-f)+\sum_i (m_i-1)[E_i]\,,\quad \phi'^*[p_i]=[m_iE_i]$$

and

$$K_S\simeq\mu^*K_{S'}+\sum_j e_j\,,$$

where e_j is an exceptional curve.

Moreover we may assume that $k=12\,\mathrm{m}$ is divisible by all m_i's. Then we have

$$K_S^{\otimes k}\simeq\phi^*(k(K_C-f)+\sum_i\frac{k}{m_i}(m_i-1)P_i)+k\sum_j e_j\,. \tag{**}$$

From (*) and (**), if we put $g=\phi\circ f$, we have

$$K_X^{\otimes k}\simeq g^*(k(K_C-f)+\sum_i\frac{k}{m_i}(m_i-1)P_i)\otimes f^*(J^*\mathcal{O}_{\boldsymbol{P}^1}(m)\otimes\mathcal{O}[k\sum_j e_j]$$
$$\otimes\mathcal{O}_S[\sum_Y k\mu_Y Y])\otimes\mathcal{O}_X(V-W)\,.$$

Hence, if there exists no irreducible component of $\sum$ which is mapped surjectively onto C by ϕ, we have $\kappa(X)\leq 1$ and this is a contradiction. q.e.d.

Theorem 2.11. *Let $f\colon X\to Y$ be an elliptic threefold over Y. Assume that $\kappa(X)=2$ and $\kappa(Y)=1$. Then the pluricanonical mapping $\Phi_{|mK_X|}$ gives the Iitaka fibration for all even integer $m\geq 16$.*

Remark 2.12. If X is an elliptic threefold with constant moduli which has only multiple singular fibers, the theorem holds for all $m\geq 15$.

Proof. We use the same notation as in Proposition 2.1. By Hironaka's flattening theorem and resolution of singularities, we may assume that
1) The discriminant locus $D=\sum_i D_i$ are divisors which have only normal crossings.
2) The J-invariant $J\colon Y\to\boldsymbol{P}^1$ is a morphism.
3) The condition (*) in Theorem 1.1 is satisfied.
4) There is an isomorphism $H^0(X,\mathcal{O}(mK_X))\simeq H^0(Y,\mathcal{O}(mK_Y+\Gamma))$ for an even integer $m>2$, where $\Gamma=\sum_i[m\alpha_i]D_i+(p\nabla_1+q\nabla_2)$.

Y has the structure of an elliptic surface $\phi\colon Y\to C$ and let $\mu\colon Y\to Y'$ be the contraction of the exceptional curves of the first kind in fibers. Then $\phi'\colon Y'\to C$ is relatively minimal and we have

$$K_{Y'} \simeq \phi'^*(K_C - f) + \sum_i (q_i - 1)[E_i], \quad \phi'^*[p_i] = [q_i E_i]$$

and

$$K_Y \simeq \mu^* K_{Y'} + \sum_j e_j,$$

where e_j is an exceptional curve. Hence we have

$$K_Y \simeq \phi^*(K_C - f) + \sum_i (q_i - 1)[\mu^* E_i] + \sum_j e_j,$$

where

$$\phi^*[p_i] = [q_i \mu^* E_i].$$

By Proposition 2.10, ϕ: $Y \to C$ is an *algebraic* elliptic surface and there exists some irreducible component of the discriminant locus which is mapped surjectively onto C by ϕ. Let $D_1, D_2, \cdots, D_p$ (resp. $D_{p+1}, \cdots, D_\lambda$) be horizontal with respect to ϕ. (resp. be contained in fibers of ϕ.)

By Katsura and Ueno [7], the pluricanonical mapping $\Phi_{|mK_Y|}$ gives the unique structure of the elliptic surface for every $m \geq 14$. If we put $\Gamma = \mathcal{O}(mK_Y + \sum_{i=1}^{p} [m\alpha_i] D_i)$ there is an injection $H^0(Y, \mathcal{O}(\Gamma)) \hookrightarrow H^0(X, \mathcal{O}(mK_X))$. Hence it suffices to show that Φ_Γ seperates points on the general fiber f of Φ: $Y \to C$. Since $\Gamma \cdot f > 0$, the restriction $\mathcal{O}(\Gamma) \otimes \mathcal{O}_f$ is very ample for every $m \geq 14$.

So it suffices to show that the restriction map

R: $H^0(Y, \mathcal{O}(\Gamma)) \to H^0(f, \mathcal{O}(\Gamma) \otimes O_f)$ is surjective for every $m \geq 14$.

We have the following exact sequence:

$$\begin{aligned} 0 &\to H^0(Y, \mathcal{O}(\Gamma - f)) \to H^0(Y, \mathcal{O}(\Gamma)) \to H^0(f, \mathcal{O}(\Gamma) \otimes O_f) \\ &\to H^1(Y, \mathcal{O}(\Gamma - f)) \to H^0(Y, \mathcal{O}(\Gamma)) \to 0. \end{aligned}$$

We will show that $H^1(Y, \mathcal{O}(\Gamma - f)) = 0$. We need the following lemma.

Lemma 2.13 (Kodaira). *Let V be a Kähler surface and let C be a curve composed of m connected components on V. Then the integer $k = h^1(K_V + C) - m + 1$ is equal to the number of lineary independent holomorphic 1-forms on V which vanishes on C.*

We have $H^1(Y, \mathcal{O}(\Gamma - f)) \simeq H^1(Y, K_Y + ((m-1)K_Y - f) + \sum_{i=1}^{p} [m\alpha_i] D_i))$.

From Katsura and Ueno [7], we have $\dim |(m-1)K_Y - f| \geq 0$ for all $m \geq 15$. And there is *no* holomorphic 1-form on Y which vanishes on some $D_i (1 \leq i \leq p)$, since D_i is a horizontal component of ϕ: $Y \to C$. Hence Lemma 2.13 implies the claim.

Hence for every *even* integer $m \geq 16$, $\Phi_{|mK_X|}$ gives the Iitaka fibration. q.e.d.

§1. The structure of algebraic elliptic fiber spaces.

In this section, we shall consider an *algebraic* elliptic fiber space with constant moduli which has only multiple singular fibers. There is great difference between algebraic elliptic fiber spaces and analytic elliptic fiber spaces. Katsura and Ueno [7] showed that the multiplicities of the multiple fibers of an *algebraic* elliptic surface satisfy certain numerical conditions.

On the other hand, there is a deep connection between *algebraic* elliptic fiber spaces and the theory of branched coverings of complex manifolds developed by Namba [13].

First, we quote the following important theorems.

Theorem 3.1 (Katsura and Ueno [7]). *Let $f: S \to \boldsymbol{P}^1$ be an algebraic elliptic surface of type $(m_1, m_2, \cdots, m_\lambda)$.*

(*) *Let m be the least common multiple of $m_1, m_2, \cdots, m_\lambda$.*

For a prime number q, let α be the maximal integer such that q^α divides m. Then there exists at least two indices i and j such that q^α divides both m_i and m_j. (We call (*) condition (U).)

We need the following definition.

Definition 3.2 (c.f. Namba [13]). Let M (resp. X) be an n-dimensional complex manifold. (resp. n-dimensional normal complex space.) A Galois covering $f: X \to M$ which branches at the divisor D is said to be *maximal* if for any covering $f': X' \to M$ which branches at at most D, there is a morphism g of X onto X' such that $f = f' \circ g$.

Theorem 3.3 (Namba [13]). *Let $\bar{D}_j$ $(1 \leq j \leq \lambda)$ be distinct irreducible hypersurfaces of degree d_j of $\boldsymbol{P}^n$. Let m_j $(1 \leq j \leq \lambda)$ be positive integers and put $\bar{D} = m_1\bar{D}_1 + m_2\bar{D}_2 + \cdots + m_\lambda\bar{D}_\lambda$. Then there is a finite abelian covering of $\boldsymbol{P}^n$ which branches at $\bar{D}$ if and only if the following condition is satisfied.*

Condition (N)

$$\frac{m_j}{(d_j, m_j)} \quad \textit{divides} \quad \left\langle \frac{m_1}{(d_1, m_1)}, \cdots, \overset{\vee}{\frac{m_j}{(d_j, m_j)}}, \cdots, \frac{m_\lambda}{(d_\lambda, m_\lambda)} \right\rangle$$

for $1 \leq j \leq \lambda$, where (d_j, m_j) denotes the greatest common divisor of d_j and m_j and $\langle a_1, a_2, \cdots, \overset{\vee}{a_j}, \cdots, a_s \rangle$ denotes the least common multiple of $a_1, a_2, \cdots, a_s$ except a_j. (If $n=1$, put $d_j = 1$ for all $1 \leq j \leq \lambda$.)

Moreover, if $n \geq 2$ and $\bar{D}_j$'s $(1 \leq j \leq \lambda)$ are smooth and crossing normally, such a finite abelian covering $\pi: \tilde{\boldsymbol{P}}^n \to \boldsymbol{P}^n$ is maximal and the Galois group G_π is isomorphic to $\boldsymbol{Z}\gamma_1 + \boldsymbol{Z}\gamma_2 + \cdots + \boldsymbol{Z}\gamma_\lambda$, where $d_1\gamma_1 + d_2\gamma_2 + \cdots + d_\lambda\gamma_\lambda = 0$ and $m_j\gamma_j = 0$ for $1 \leq j \leq \lambda$.

Combining these two results, we obtain the following proposition.

Proposition 3.4. *Let $f: S \to \boldsymbol{P}^1$ be an elliptic surface with constant moduli which has only multiple singular fibers. Put*

$$S = L_{p_1}(m_1, a_1)L_{p_2}(m_2, a_2) \cdots L_{p_\lambda}(m_\lambda, a_\lambda)(\boldsymbol{P}^1 \times E).$$

(Here we use the notation of Kodaira [11].) *Then the following conditions are equivalent.*

(1) *S is projective.*

(2) *S is Kähler.*

(3) $\sum_{i=1}^{\lambda} a_i = 0$.

(4) *There exists a finite abelian covering $\pi: C \to \boldsymbol{P}^1$ of $\boldsymbol{P}^1$ which branches at $D = \sum_{i=1}^{\lambda} m_i p_i$. Let G be the Galois group of π and let X be the normalization of the pull-back $S \times_{\boldsymbol{P}^1} C \to C$. Then we have $X = (\boldsymbol{P}^1 \times E)^{\eta}$, where $\eta \in H^1(C, \mathcal{O}(E))$ is of finite order, and the quotient space X/G is isomorphic to S. (Here E denotes a smooth elliptic curve.)*

Proof. (1)$\leftrightarrow$(2) follows from Kodaira [11].
(1)$\leftrightarrow$(3) follows from Katsura and Ueno [7]; appendix.
(4)$\to$(1) is clear, so we prove that (3) implies (4).
As is shown in Katsura and Ueno [7], (3) implies condition (U) in Theorem 3.1.

On the other hand, one can see easily that condition (U) is equivalent to the following condition.

$$(*) \quad m_j \text{ divides } \langle m_1, \cdots, \check{m}_j, \cdots, m_\lambda \rangle \quad \text{for} \quad 1 \leq j \leq \lambda.$$

Therefore if we put $d_j = 1$ for all j in Theorem 3.3, there exists a finite *abelian* covering which branches at $D = \sum_{i=1}^{\lambda} m_i p_i$ and the claim follows. q.e.d.

Remark 3.5. Theorem 3.1 and Proposition 3.4 are still true if we replace $\boldsymbol{P}^1$ by any compact smooth curve C. Here we give another proof.

Proof. Put $S = L_{p_1}(m_1, a_1) \cdots L_{p_\lambda}(m_\lambda, a_\lambda)(C \times E)$ and let m be the least common multiple of m_i's. ($1 \leq i \leq \lambda$). The multiplication map $m: E \to E$ induces a finite surjective morphism $\phi: S \to Y$, where $Y \to C$ is an elliptic bundle over C. Hence S is Kähler if and only if Y is Kähler. If we express Y as $Y = (C \times E)^{\eta}$, $\eta \in H^1(C, \mathcal{O}(E))$, we can easily see that the Chern class of η is $c(\eta) = m \sum_{i=1}^{\lambda} a_i$. By Kodaira [11], Y is Kähler if and only if $c(\eta) = 0$. Hence the claim follows.

Now, we give a generalization of Proposition 3.4.

Theorem 3.6. *Let H_j ($1 \leq j \leq \lambda$) be distinct hyperplanes of $\boldsymbol{P}^n$ ($n \geq 2$) which are crossing normally and let $f: X \to \boldsymbol{P}^n$ ($n \geq 2$) be an elliptic fiber space over $\boldsymbol{P}^n$ with constant moduli which has multiple fibers of multiplicity m_j along each H_j ($1 \leq j \leq \lambda$) and is a principal fiber bundle over $\boldsymbol{P}^n \setminus \bigcup_{j=1}^{\lambda} H_j$ with the structure group E, where E is a smooth elliptic curve. Then the following conditions are equivalent.*

(1) *X is Moishezon.*

(2) *X is in the class $\mathcal{C}$.*

(3) *$t(X)=1$, where $t(X)$ denotes the Albanese dimension of X.*

(4) *There exists a finite abelian covering $\pi\colon Y\to \boldsymbol{P}^n$ of $\boldsymbol{P}^n$ which branches at $D=\sum_{j=1}^{\lambda} m_j H_j$. Let G be the Galois group of $\pi\colon Y\to\boldsymbol{P}^n$. Then the pull-back $X\times_{\boldsymbol{P}^n} Y\to Y$ of π is bimeromorphic to $Y\times E$ and the quotient space $Y\times E/G$ is bimeromorphic to X.*

Proof. (1)$\leftrightarrow$(2) is well-known. (c.f. Fujiki [1])

(2)$\to$(4)

Take $(n-1)$ general hyperplane sections of $\boldsymbol{P}^n$ and restrict the elliptic fiber space $f\colon X\to\boldsymbol{P}^n$ over the intersection of them. Then we get an elliptic surface S over $\boldsymbol{P}^1$ of type $(m_1, m_2, \cdots, m_\lambda)$. Because X is in the class $\mathcal{C}$, S is also in the class $\mathcal{C}$, so is Kähler. Therefore by Theorem 3.1, $(m_1, m_2, \cdots, m_\lambda)$ satisfies condition (U).

The same arguments as in Proposition 3.4 can be applied to our situation, so there exists a finite *abelian* covering $\pi\colon Y\to\boldsymbol{P}^n$ of $\boldsymbol{P}^n$ which branches at $D=\sum_{j=1}^{\lambda} m_j H_j$ and the pull-back $X\times_{\boldsymbol{P}^n} Y\to Y$ is bimeromorphic to $(Y\times E)^\eta\to Y$, where $\eta\in H^1(Y, \mathcal{O}(E))$ is of finite order. There exists an ètale cover $Z\to Y$ such that $(Y\times E)^\eta\times_Y Z$ is isomorphic to $Z\times_Y E$. Since $\pi\colon Y\to\boldsymbol{P}^n$ $(n\geq 2)$ is a *maximal* covering, we have $Z\simeq Y$ and $\eta=0$ in $H^1(Y, \mathcal{O}(E))$. So the claim follows.

(4)$\to$(3) is trivial.

(3)$\to$(1) Since $t(X)=1$, $\mathrm{Alb}(X)$ is a smooth elliptic curve and $\alpha_X\colon X\to\mathrm{Alb}(X)$ (the Albanese map of X) is surjective and has connected fibers. Then the morphism $\Phi=(f, \alpha_X)\colon X\to\boldsymbol{P}^2\times\mathrm{Alb}(X)$ is surjective, hence X is Moishezon. q.e.d.

§4. Proof of main theorem B.

The following proposition is due to M. Kato.

Proposition 4.1 (M. Kato). *For an arbitrary integer $\lambda\geq 2$, let $(m_1, m_2, \cdots, m_\lambda)$ be a λ-tuple of positive integers with $m_i\geq 2$ for all i, and assume that any two of them are relatively prime.—(*) Then there exists an elliptic fiber space $f\colon X\to\boldsymbol{P}^{\lambda-1}$ over $\boldsymbol{P}^{\lambda-1}$ with constant moduli which satisfies the following conditions.*

(1) *X has multiple fibers of multiplicity m_i along $\{\zeta_i=0\}$ for each i, and is trivial over $\boldsymbol{P}^{\lambda-1}\setminus\bigcup_{i=1}^{\lambda}\{\zeta_i=0\}$, where $(\zeta_1\colon\zeta_2\colon\cdots\colon\zeta_\lambda)$ is the homogeneous coordinate of $\boldsymbol{P}^{\lambda-1}$.*

(2) *$F\colon X\to\boldsymbol{P}^{\lambda-1}$ is flat.*

Remark 4.2. X is not in the class $\mathcal{C}$ in the sense of Fujiki [1]. That is, X cannot be bimeromorphic to any compact Kähler manifold. (c.f. §3).

Remark 4.1. X is a Hopf manifold. Conversely, any elliptic fiber space which satisfies (1) and (2) is a submanifold of a Hopf manifold.

Proof. Let us consider an analytic automorphism of $C^\lambda\setminus\{0\}$ defined by

$$\begin{array}{cccc} g: & C^\lambda\setminus\{0\} & \longrightarrow & C^\lambda\setminus\{0\} \\ & \Psi & & \Psi \\ & (z_1, \cdots, z_i, \cdots, z_\lambda) & \mapsto & (\rho^{m/m_1} z_1, \cdots, \rho^{m/m_i} z_i, \cdots, \rho^{m/m_\lambda} z_\lambda)\,, \end{array}$$

where we fix a constant $\rho\in C\,(0<|\rho|<1)$ and put $m=m_1m_2\cdots m_\lambda$. Put $X=C^\lambda\setminus\{0\}/\langle g\rangle$. The automorphism g acts on $C^\lambda\setminus\{0\}$ freely and properly discontinuously, hence X is smooth. There is a natural holomorphic map

$$\begin{array}{cccc} f: & X & \longrightarrow & P^{\lambda-1} \\ & \Psi & & \Psi \\ & \overline{(z_1, z_2, \cdots, z_\lambda)} & \to & (z_1^{m_1}: z_2^{m_2}: \cdots : z_\lambda^{m_\lambda})\,, \end{array}$$

where by $\overline{(z_1, z_2, \cdots, z_\lambda)}$ we denote the point of X corresponding to a point $(z_1, z_2, \cdots, z_\lambda)\in C^\lambda\setminus\{0\}$.

Because any two of m_i's are relatively prime, the morphism f gives an algebraic reduction of X and *each fiber of f is connected.* Thus X is an elliptic fiber space over $P^{\lambda-1}$ which satisfies the desired properties. q.e.d.

Proof of remark 4.3. Assume that an elliptic fiber space $f: X\to P^{\lambda-1}$ satisfies the conditions (1) and (2). Because any two of m_i's are relatively prime, it follows from Katsura and Ueno [7]; appendix that X is not in the class $\mathcal{C}$ and $h^0(X, d\mathcal{O}_X)=0$. (cf. theorem 3.5) First, we show that $H^1(X, C)\xrightarrow{\sim}H^1(X, \mathcal{O}_X)\xrightarrow{\sim}C$. Since $R^1f_*\mathcal{O}_X\xrightarrow{\sim}\mathcal{O}_{P^{\lambda-1}}$, it follows easily from the Leray spectral sequence that $q(X)=1$. By the exact sequence $0\to C\to\mathcal{O}_X\to d\mathcal{O}_X\to 0$, we have $0\to H^0(X, d\mathcal{O}_X)\to H^1(X, C)\to H^1(X, \mathcal{O}_X)$, and $b_1(X)\le 1$. And by Leray's spectral sequence

$$E_2^{p,q} = H^p(P^{\lambda-1}, R^qf_*C) \Rightarrow H^{p+q}(X, C)\,,$$

we have an exact sequence

$$0\to H^1(X, C)\to H^0(X, R^1f_*C)\to H^2(P^{\lambda-1}, C)\,.$$

Since $R^1f_*C\xrightarrow{\sim}C^2$ and $b_2(P^{\lambda-1})=1$, we have $b_1(X)\ge 1$. Therefore we have $b_1(X)=1$, and $H^1(X, C)\xrightarrow{\sim}H^1(X, \mathcal{O}_X)\xrightarrow{\sim}C$.

Now, we follow the arguments of Kato [9]. By the same method as in [9], Lemma 19, 20, we can show that $f^*: H^2(P^{\lambda-1}, C)\to H^2(X, C)$ is a zero mapping and $f^*O_{P^{\lambda-1}}(1)\in \mathrm{Pic}(X)$ is a *flat* line bundle. Then from Kato's theorem [9], X is a submanifold of a Hopf manifold. q.e.d.

Remark 4.4. If m_i's do not satisfy $(*)$, the fiber of f is not connected. To prove Theorem B, we need the following lemma.

Lemma 4.5. *Let $\{a\}_{n=1,2,\dots}$ be a sequence of natural numbers defined as follows.*

$$a_1 = 2,\ a_{n+1} = a_1 a_2 \cdots a_n + 1 .$$

And let $\{b_n\}_{n=1,2,\dots}$ be a sequence of natural numbers defined as follows.

$$b_n = (n+1)(a_{n+3}-1)+2 .$$

Then for every positive integer n, we have

$$-k(n+1)+\sum_{i=1}^{n+2}\left[\frac{k(a_i-1)}{a_i}\right]>0 \qquad \text{for all } k\geq b_n ,$$

and

$$-k(n+1)+\sum_{i=1}^{n+2}\left[\frac{k(a_i-1)}{a_i}\right]=0 \qquad \text{if } k=b_n-1 .$$

(Here [ ] denotes the Gauss symbol.)

Proof. First, we need the following sublemma.

Sublemma 4.6.

$$\sum_{i=1}^{p}\frac{1}{a_i} = 1-\frac{1}{a_1a_2\cdots a_p}$$

(It is easy to prove by induction on p, so we omit the proof.) Now, we follow the method of Iitaka [6]. Because

$$\left[\frac{k(a_i-1)}{a_i}\right]\geq\frac{k(a_i-1)}{a_i}-\frac{a_i-1}{a_i}=(k-1)\frac{a_i-1}{a_i} ,$$

we have

$$\sum_{i=1}^{n+2}\left[\frac{k(a_i-1)}{a_i}\right]\geq(k-1)\sum_{i=1}^{n+2}\frac{a_i-1}{a_i}=(k-1)\left(n+2-\sum_{i=1}^{n+2}\frac{1}{a_i}\right).$$

By Sublemma 4.6, we have the following inequality.

$$(*) \qquad \sum_{i=1}^{n+2}\left[\frac{k(a_i-1)}{a_i}\right]\geq(k-1)\left(n+1+\frac{1}{a_1a_2\cdots a_{n+2}}\right).$$

So it suffices to determine the smallest integer m_0 such that we have

$$(**) \qquad (k-1)\left(n+1+\frac{1}{a_1a_2\cdots a_{n+2}}\right)>k(n+1) \qquad \text{for all } k\geq m_0 .$$

The inequality (**) holds if and only if

$$k>(n+1)(a_1a_2\cdots a_{n+2})+1=(n+1)(a_{n+3}-1)+1=b_n-1 .$$

Therefore if $k\geq b_n$, we have $-k(n+1)+\sum_{i=1}^{n+2}\left[\frac{k(a_i-1)}{a_i}\right]>0$.

Next, we consider the case when $k=b_n-1$. If we put $A=a_1a_2\cdots a_{n+2}$, we have

$k=(n+1)A+1$. Then for $1\leq i\leq n+2$, we have

$$\left[\frac{k(a_i-1)}{a_i}\right]=\left[\frac{((n+1)A+1)(a_i-1)}{a_i}\right]$$
$$=\left[(n+1)\frac{A}{a_i}(a_i-1)+\frac{a_i-1}{a_i}\right]$$
$$=(n+1)\frac{A}{a_i}(a_i-1),$$

since $\frac{A}{a_i}$ $(1\leq i\leq n+2)$ is a positive integer. So we have the following equality.

$$\sum_{i=1}^{n+2}\left[\frac{k(a_i-1)}{a_i}\right]=(n+1)A\sum_{i=1}^{n+2}\frac{a_i-1}{a_i}$$
$$=(n+1)A\Big(n+2-\sum_{i=1}^{n+2}\frac{1}{a_i}\Big)$$
$$=(n+1)A\Big((n+1+\frac{1}{A}\Big)$$
$$=(n+1)(A(n+1)+1)$$
$$=(n+1)k\,.$$

q.e.d.

Now, we are ready to prove Theorem B.

Proof of Theorem B. Let $\{a_n\}_{n=1,2,\cdots}$ and $\{b_n\}_{n=1,2,\cdots}$ be sequences of natural numbers defined as in Lemma 4.5. Take $(a_1, a_2, \cdots, a_{n+2})$, a $(n+2)$—tuple of positive integers. From the construction of $\{a_n\}_{n=1,2,\cdots}$, any two of a_n's are relatively prime. Therefore it follows from proposition (4.1) that there exists an elliptic fiber space $Z\to\boldsymbol{P}^{n+1}$ over $\boldsymbol{P}^{n+1}$ which satisfies the following conditions.

(1) Z has multiple fibers of multiplicity a_i along $\{\zeta_i=0\}$ for each i $(1\leq i\leq n+2)$. (Here $(\zeta_1:\zeta_2:\cdots:\zeta_{n+2})$ denotes the homogeneous coordinate of $\boldsymbol{P}^{n+1}$.)

(2) Z is trivial over $\boldsymbol{P}^{n+1}\setminus\bigcup_{i=1}^{n+2}\{\zeta_i=0\}$.

(3) $Z\to\boldsymbol{P}^{n+1}$ is flat.

Now, take a generic hyperplane section H of $\boldsymbol{P}^{n+1}$ and restrict the elliptic fiber space $Z\to\boldsymbol{P}^{n+1}$ over H. Put $X^{(n+1)}:=Z\,|_H$. By Bertini's theorem $X^{(n+1)}$ is smooth and $X^{(n+1)}\to H(\simeq\boldsymbol{P}^n)$ is an elliptic fiber space over $\boldsymbol{P}^n$ which satisfies the following conditions.

(1) $X^{(n+1)}$ has multiple fibers of multiplicity a_i along each H_i $(1\leq i\leq n+2)$, where H_i's are $(n+2)$ hyperplanes on $\boldsymbol{P}^n$ which are in a general position.

(2) $f\colon X^{(n+1)}\to\boldsymbol{P}^n$ is trivial over $\boldsymbol{P}^n\setminus\bigcup_{i=1}^{n+2}H_i$.

(3) $f\colon X^{(n+1)}\to\boldsymbol{P}^n$ is flat.

The canonical bundle of $X^{(n+1)}$ is as follows.

$$K_{X^{(n+1)}}\simeq f^*\Big(\mathcal{O}_{P^n}(-n-1)+\sum_{i=1}^{n+2}\frac{a_i-1}{a_i}H_i\Big)$$

Because

$$-(n+1)+\sum_{i=1}^{n+2}\frac{a_i-1}{a_i}=\frac{1}{a_1a_2\cdots a_{n+2}}>0$$

from Sublemma 4.6, we have $\kappa(X^{(n+1)})=n$.

And for any positive integer m, we have

$$|mK_{X^{(n+1)}}|=f^*(\mathcal{O}_{P^n}(-m(k+1))+\sum_{i=1}^{n+2}\left[\frac{m(a_i-1)}{a_i}\right]H_i)+\text{(fixed components)}\,,$$

where [ ] denotes the Gauss symbol.

So we have $\dim|mK_{X^{(n+1)}}|>0$ (resp. $=0$) if and only if m satisfies the the following inequality. (resp. equality.)

$$(*)\qquad -m(n+1)+\sum_{i=1}^{n+2}\left[\frac{m(a_i-1)}{a_i}\right]>0\qquad(\text{resp.}=0.)$$

Therefore from Lemma 4.5, we have

$$\dim|mK_{X^{(n+1)}}|=0\qquad \text{if } m=b_n-1$$

and

$$\dim|mK_{X^{(n+1)}}|>0\qquad \text{for all } m\geq b_n\,.$$

Thus $f\colon X^{(n+1)}\to P^n$ has the desired properties. q.e.d.

Remark 4.7. There exists *no* finite *abelian* covering of P^n which branches along each H_i $(1\leq i\leq n+2)$ with the ramification index a_i. This follows from a theorem of Namba [13]. (See §3. Theorem 3.3.)

Remark 4.8. For any positive integer n, $X^{(n+1)}$ is not in the class $\mathcal{C}$ in the sense of Fujiki [1].

Proof. Take $(n-1)$ general hyperplane sections of P^n and let C be the intersection of them. Then $f^{-1}(C)\to C(\cong P^1)$ is an elliptic surface by Bertini's theorem and is of type $(a_1, a_2, \cdots, a_{n+2})$. Because any two of a_n's are relatively prime, it follows from Katsura and Ueno [7]; appendix that $f^{-1}(C)$ is non-Kähler. There fore, $X^{(n+1)}$ is not in the class $\mathcal{C}$. (See §3, Theorem (3.1).)

Remark 4.9. In the above examples, $X^{(n+1)}$ is a submanifold of a Hopf manifold. *However, using the result of* §5, *we can construct another example of* $X^{(n+1)}$, *which cannot be bimeromorphic to any subvariety of a Hopf manifold.* (c.f. [10])

Remark 4.10. Let $f\colon X\to P^n$ be an elliptic fiber space over P^n and assume that f *is flat.* Then b_n is the best possible number of the Iitaka fibration for all such X.

Now we prove theorem C.

Proof of theorem C. In $n=1$, theorem 2 follows from Katsura and Ueno [7].

So we may assume that $n>1$.

Let H_i's $(1\leq i\leq n+2)$ be $(n+2)$ hyperplanes on $\boldsymbol{P}^n$ which are in a general position. Next, put $(a_1, a_2, \cdots, a_{n+1}, a_{n+2})=(2, \underbrace{2(n+1), \cdots, 2(n+1)}_{n+1})$.

Clearly it satisfies condition (N) in Theorem 3.3. Therefore, there exists a finite *abelian* covering $\pi\colon \tilde{\boldsymbol{P}}^n\to\boldsymbol{P}^n$ of $\boldsymbol{P}^n$ which branches at $D=2H_1+2(n+1)H_2+\cdots+2(n+1)H_{n+2}$. And the Galois group G is isomorphic to $\boldsymbol{Z}\gamma_1+\boldsymbol{Z}\gamma_2+\cdots+\boldsymbol{Z}\gamma_{n+2}$, where

$$\begin{cases} \gamma_1+\gamma_2+\cdots+\gamma_{n+1}+\gamma_{n+2}=0 \\ 2\gamma_1=0 \\ 2(n+2)\,\gamma_2=0 \\ \cdots\cdots\cdots\cdots \\ 2(n+2)\gamma_{n+2}=0\,. \end{cases}$$

Clearly G is isomorphic to $\boldsymbol{Z}/2\oplus\underbrace{\boldsymbol{Z}/2(n+2)\oplus\cdots\oplus\boldsymbol{Z}/2}_{n}(n+2)$.

Fix an elliptic curve E with the period $(1, \tau)$, $\operatorname{Im}(\tau)>0$, a torsion point $a\in E$ of order 2 and a torsion point $b\in E$ of order $2(n+2)$ such that $a\neq(n+2)b$. The group G acts on $\tilde{\boldsymbol{P}}^n\times E$ as follows.

$$\gamma_1\colon\ (z, [\zeta])\mapsto(\gamma_1 z, [\zeta+a])$$
$$\gamma_i\colon\ (z, [\zeta])\mapsto(\gamma_i z, [\zeta+b])\qquad(2\leq i\leq n+1)\,.$$

Note that $\tilde{\boldsymbol{P}}^n$ is smooth.

The action is properly discontinuous but not free, so the quotient space $\tilde{\boldsymbol{P}}^n\times E/G$ has singularities. Take a G-equivariant resolution $Z^{(n+1)}$ of it. Then by a natural holomorphic mapping $f\colon Z^{(n+1)}\to\boldsymbol{P}^n$, $Z^{(n+1)}$ is an *algebraic* elliptic fiber space over $\boldsymbol{P}^n$ which has multiple fibers of multiplicity a_i along each H_i. $(1\leq i\leq n+2)$.

The canonical bundle of $Z^{(n+1)}$ is as follows.

$$Z^{(n+1)}\simeq f^*\Big(\mathcal{O}_{\boldsymbol{P}^n}(-n-1)+\sum_{i=1}^{n+2}\frac{a_i-1}{a_i}H_i\Big)\,.$$

Because $-(n+1)+\sum_{i=1}^{n+2}\left[\frac{k(a_i-1)}{a_i}\right]=\frac{1}{2(n+2)}>0$, we have $\kappa(Z^{(n+1)})=n$. And for positive integer k, we have

$$|kK_{Z^{(n+1)}}|=f^*\Big(\mathcal{O}_{\boldsymbol{P}^n}(-k(n+1))+\sum_{i=1}^{n+2}\left[\frac{k(a_i-1)}{a_i}\right]H_i\Big)+(\text{fixed components})\,,$$

where [ ] denotes the Gauss symbol. Therefore we have $\dim|kK_{Z^{(n+1)}}|>0$ (resp. $=0$) if and only if k satisfies the following inequality. (resp. equality.)

$$(*)\qquad -k(n+1)+\sum_{i=1}^{n+2}\frac{k(a_i-1)}{a_i}>0\qquad(\text{resp. }=0)$$

By the same method as in Lemma 4.5, we have

$$\sum_{i=1}^{n+2}\left[\frac{k(a_i-1)}{a_i}\right]\geq(k-1)\left(n+2-\sum_{i=1}^{n+2}\frac{1}{a_i}\right)=(k-1)\left(n+1+\frac{1}{2(n+2)}\right).$$

So it suffices to estimate m_0 such that

$$(k-1)\left(n+1+\frac{1}{2(n+2)}\right)>k(n+1)\qquad\text{for all } k\geq m_0. \qquad \text{——}(**)$$

The inequality (**) holds if and only if $k\geq 2(n^2+3n+3)$. By a direct calculation, we see that if $k=2n^2+6n+5$, we have

$$\begin{aligned}
&-k(n+1)+\sum_{i=1}^{n+2}\left[\frac{k(a_i-1)}{a_i}\right]\\
&=-k(n+1)+(n+1)\left[\frac{k(2n+3)}{2(n+2)}\right]+\left[\frac{k}{2}\right]\\
&=-(2n^2+6n+5)\,(n+1)+(n+1)\left[2n^2+5n+4-\frac{1}{2(n+2)}\right]+\left[\frac{2n^2+6n+5}{2}\right]\\
&=0.
\end{aligned}$$

Therefore $d_n=2(n^2+3n+3)$ is the best possible number of the Iitaka fibration of $Z^{(n+1)}$. q.e.d.

Remark 4.11. Note that in this case, the exceptional divisors disappear in $K_{Z^{(n+1)}}$. (c.f. [3]).

§5. Generalized logarithmic transformations.

In this section, we shall study generalized logarithmic transformations along the divisors which have only normal crossings. First, we state our main theorem in this section.

Theorem 5.1. For an arbitrary integer $\lambda\geq 2$, let $(m_1, m_2, \cdots, m_\lambda)$ be λ-tuple of positive integers with $m_i\geq 2$ for all i, and assume that any two of them are relatively prime. Let Y be an n-dimensional compact complex manifold and let D_i's $(1\leq i\leq\lambda)$ be smooth divisors on Y which have only normal crossings. Assume that $|D_i|$ is fixed component free and base point free and $\dim|D_i|>0$ for all i. Then there exists an elliptic fiber space $f\colon X\to Y$ over Y with constant moduli which satisfies the following conditions.

(1) X has multiple fibers of multiplicity m_i along D_i for each i.

(2) $X|_{Y\setminus\bigcup_{i=1}^{\lambda}D_i}\xrightarrow{\sim}(Y\setminus\bigcup_{i=1}^{\lambda}D_i)\times E$, where E is a smooth elliptic curve with the period $(1,\tau)$, $\mathrm{Im}(\tau)>0$.

(3) For an arbitrary integer m, $f_*(\omega_{X/Y}^{\otimes m})$ is *invertible*.

Remark 5.2. If Y is isomorphic to $\boldsymbol{P}^n$, the above theorem holds automatically. Hence, thanks to (3), we can construct another example of $X^{(n+1)}$ in theorem B.

To prove Theorem 5.1, we need the following propositions.

Proposition 5.3. *For an arbitrary integer $\lambda \geq 2$, let $(m_1, m_2, \cdots, m_\lambda)$ be a λ-tuple of positive integers with $m_i \geq 2$ for all i, and assume that any two of them are relatively prime. Let $D_i = \{z_i \in \boldsymbol{C};\ |z_i| < \varepsilon\}$ $(1 \leq i \leq \lambda)$ be λ discs. Then there exists an elliptic fiber space X_0 over $D_1 \times D_2 \times \cdots \times D_\lambda$ which satisfies the following conditions.*

(1) *$X_0|_{D_1^* \times D_2^* \times \cdots \times D_\lambda^*} \xrightarrow{\sim} D_1^* \times D_2^* \times \cdots \times D_\lambda^* \times E$, where E is a smooth elliptic curve with the period $(1, \tau)$, $\operatorname{Im}(\tau) > 0$.*

(2) *X_0 has multiple fibers of multiplicity m_i along $\{z_i = 0\}$ for each i.*

Moreover, $f\colon X_0 \to D_1 \times D_2 \times \cdots \times D_\lambda$ is flat.

Proof. Let
$$\begin{array}{ccc} \tilde{D}_i = \{t_i \in \boldsymbol{C};\ |t_i| < \varepsilon^{1/m_i}\} & \rightarrow & D_i = \{z_i \in \boldsymbol{C};\ |z_i| < \varepsilon\} \\ \Uparrow & & \Uparrow \\ t_i & \longmapsto & t_i^{m_i} \end{array}$$

be an m_i-sheedted cyclic covering of D_i. $(1 \leq i \leq \lambda)$ Then by the assumption,
$$\begin{array}{ccc} \tilde{D} := \tilde{D}_1 \times \tilde{D}_2 \times \cdots \times \tilde{D}_\lambda & \rightarrow & D_1 \times D_2 \times \cdots \times D_\lambda =: D \\ \Uparrow & & \Uparrow \\ (t_1, t_2, \cdots, t_\lambda) & \mapsto & (t_1^{m_1}, t_2^{m_2}, \cdots, t_\lambda^{m_\lambda}) \end{array}$$

is an $m_1 m_2 \cdots m_\lambda$-sheeted cyclic covering of D, and we have
$$\operatorname{Gal}(\tilde{D}/D) \xrightarrow{\sim} \boldsymbol{Z}/m_1 \oplus \boldsymbol{Z}/m_2 \oplus \cdots \oplus \boldsymbol{Z}/m_\lambda\,.$$

Now, let us consider an analytic automorphism of $\tilde{D} \times E$ defined by
$$\begin{array}{ccc} g\colon \tilde{D}_1 \times \tilde{D}_2 \times \cdots \times \tilde{D}_\lambda \times E & \rightarrow & \tilde{D}_1 \times \tilde{D}_2 \times \cdots \times \tilde{D}_\lambda \times E \\ \Uparrow & & \Uparrow \\ (t_1, t_2, \cdots, t_\lambda, [\zeta]) & \mapsto & (e_{m_1} t_1, e_{m_2} t_2, \cdots, e_{m_\lambda} t_\lambda, \\ & & \left[\zeta + \dfrac{1}{m_1 m_2 \cdots m_\lambda}\right]), \end{array}$$

where e_{m_i} is a primitive m_i-th root of unity. Put $X_0 := \tilde{D} \times E / \langle g \rangle$. The automorphism g acts on $\tilde{D} \times E$ freely and properly discontinuously, hence X_0 is smooth. There is a natural holomorphic map
$$\begin{array}{ccc} f\colon X_0 & \longrightarrow & D_1 \times D_2 \times \cdots \times D_\lambda \\ \Uparrow & & \Uparrow \\ \overline{(t_2, t_1, \cdots, t_\lambda, [\zeta])} & \mapsto & (t_1^{m_1}, t_2^{m_2}, \cdots, t_\lambda^{m_\lambda}), \end{array}$$

where by $\overline{(t_1, t_2, \cdots, t_\lambda, [\zeta])}$ we denote the point of X_0 corresponding to a point $(t_1, t_2, \cdots, t_\lambda, [\zeta]) \in \tilde{D} \times E$. By this morphism, X_0 is an elliptic fiber space over D. Clearly X_0 has multiple fibers of multiplicity m_i along each $\{z_i = 0\}$. There is an isomorphism
$$\begin{array}{ccc} \Lambda\colon X_0|_{D_1^* \times D_2^* \times \cdots \times D_\lambda^*} & \xrightarrow{\sim} & D_1^* \times D_2^* \times \cdots \times D_\lambda^* \times E \\ \Uparrow & & \Uparrow \\ \overline{(t_1, t_2, \cdots, t_\lambda, [\zeta])]} & \mapsto & (t_1^{m}, t_2^{m_2}, \cdots, t_\lambda^{m_\lambda}), \\ & & [\zeta - \sum_{i=1}^{\lambda} \dfrac{\alpha_i}{2\pi\sqrt{-1}} \log(t_i)]), \end{array}$$

where α_i $(1 \leq i \leq \lambda) \in \boldsymbol{Z}$ are defined as follows. By the assumption, there exists $\alpha_i \in \boldsymbol{Z}$ such that

$$(*) \quad \alpha_i m_1 m_2 \cdots \overset{\vee}{m_i} \cdots m_1 \equiv 1 \bmod m_i \qquad \text{for each } i.$$

Take such α_i's and fix them. q.e.d.

Proposition 5.4. *Let $C_i (1 \leq i \leq \lambda)$ be a smooth curve and take one point P_i on C_i for each i. For an arbitrary integer $\lambda \geq 2$, let $(m_1, m_2, \cdots, m_\lambda)$ be a λ-tuple of integers with $m_i \geq 2$ for all i, and assume that any two of them are relatively prime. Then there exists an elliptic fiber space X over $C_1 \times C_2 \times \cdots \times C_\lambda$ which satisfies the following conditions.*

(1) *X has multiple fibers of multiplicity m_i along $C_1 \times C_2 \times \cdots \times C_{i-1} \times \{P_i\} \times C_{i+1} \times \cdots \times C_\lambda$ for each i.*

(2) *$X\Big|_{\prod_{i=1}^{\lambda} C_i^*} \simeq \prod_{i=1}^{\lambda} C_i^* \times E$, ehere $C_i^* = C_i \backslash \{P_i\}$ and E is a smooth elliptic curve with the period $(1, \tau)$, $\mathrm{Im}(\tau) > 0$.*

(3) *$f\colon X \rightarrow \prod_{i=1}^{\lambda} C_i$ is flat.*

Moreover, X is not in the class $\mathcal{C}$.

Proof. Let D_i be a small neighborhood of P_i in C_i with a coordinate z_i and put $C_i^* = C_i \backslash \{p_i\}$. Then $\prod_{i=1}^{\lambda} C_i$ can be covered by the following 2^λ open sets.

$$\begin{aligned}
&U_{00\cdots 0} := D_1 \times D_2 \times \cdots \times D_\lambda \\
&U_{00\cdots 1} := D_1 \times D_2 \times \cdots \times D_{\lambda-1} \times C_\lambda^* \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&U_{11\cdots 1} := C_1^* \times C_2^* \times \cdots \times C_\lambda^*
\end{aligned}$$

(There is a one to one correspondence between D_i (resp. C_i^*) and 0 (resp. 1) and by U_a, we denote the open set corresponding to $a = (a_1, a_2, \cdots, a_\lambda) \in \boldsymbol{Z}_2^{\oplus\lambda}$.

Step 1. By Proposition 5.3, there exists an elliptic fiber space X_0 over $U_0 = D_1 \times D_2 \times \cdots \times D_\lambda$ which satisfies the following conditions.

(1) X_0 has multiple fibers of multiplicity m_i along $\{z_i = 0\}$. $(1 \leq i \leq \lambda)$

(2) $X_0\Big|_{D_1^* \times D_2^* \times \cdots \times D_\lambda^*} \simeq D_1^* \times D_2^* \times \cdots \times D_\lambda^* \times E$, where E is a smooth elliptic curve with the period $(1, \tau)$.

Step 2. Now, we shall express the elliptic fiber space $X_0 \rightarrow U_{00\cdots 0}$ in another form. (Here we use the same notation as in proposition (3.1).)

For any $a \ni \boldsymbol{Z}_2^{\oplus\lambda}$, we denote it by $a = (0, 0, \cdots, 0, \underset{i_1}{\underset{\smile}{1}}, 0, \cdots, \underset{i_2}{\underset{\smile}{1}}, 0, \cdots, \underset{i_p}{\underset{\smile}{1}}, 0, \cdots, 0)$, $i_1 < i_2 \cdots < i_p$

that is the i_k-th component is 1 for $k = 1, 2, \cdots, p$ and all the rest are 0. Put $\{j_1, j_2, \cdots, j_{\lambda-p}\} = \{1, 2, \cdots, \lambda\} \backslash \{i_1, i_2, \cdots, i_p\}$, where $1 \leq j_1 < j_2 < \cdots < j_{\lambda-p} \leq \lambda$. Then we have $U_a = \prod_{\mu=1}^{\lambda-p} D_{j_\mu} \times \prod_{\nu=1}^{p} C_{i_\nu}^*$. Noting that Gal $(\tilde{D}/D) \simeq \boldsymbol{Z}/m_1 \oplus \boldsymbol{Z}/m_2 \oplus \cdots \oplus \boldsymbol{Z}/m_\lambda$, we have $X_0=\dfrac{\prod_{i=1}^{\lambda}\tilde{D}_i\times E/\langle g^{m_{j_1}m_{j_2}\cdots m_{j_{\lambda-p}}}\rangle}{\langle g^{m_{i_1}m_{i_2}\cdots m_{i_p}}\rangle}$. Put $\tilde{X}_0^{(a)}:=\prod_{i=1}^{\lambda}\tilde{D}_i\times E/\langle g^{m_{j_1}m_{j_2}\cdots m_{j_{\lambda-p}}}\rangle$.

Then $\tilde{X}_0^{(a)}$ is an elliptic fiber space over $\prod_{\mu=1}^{\lambda-p}\tilde{D}_{j_\mu}\times\prod_{\nu=1}^{p}D_{i_\nu}$, which satisfies the following conditions.

(1) $\tilde{X}_0^{(a)}$ has multiple fibers of multiplicity m_{i_ν} $(1\leq\nu\leq p)$ along $\{z_{i_\nu}=0\}$ $(1\leq\nu\leq p)$ respectively.

(2)
$$\begin{array}{ccl} X_0^{(a)}\Big|_{\prod_{\mu=1}^{\lambda-p}\tilde{D}_{j_\mu}\times\prod_{\nu=1}^{p}D_{i_\nu}^*} & \xrightarrow{\sim} & \prod_{\mu=1}^{\lambda-p}\tilde{D}_{j_\mu}\times\prod_{\nu=1}^{p}D_{i_\nu}^*\times E \\ \Psi & & \Psi \\ \overline{(t_1, t_2, \cdots, t_\lambda, [\zeta])} & & (t_{j_1}, t_{j_2}, \cdots, t_{j_{\lambda-p}}, t_{i_1}^{m_{i_1}}, t_{i_2}^{m_{i_2}}, \cdots, t_{i_p}^{m_{i_p}}, \\ & & [\zeta-\sum_{\nu=1}^{p}\frac{\alpha_{i_\nu}}{2\pi\sqrt{-1}}\log(t_{i_\nu})]), \end{array}$$

where by $\overline{(t_1, t_2, \cdots, t_\lambda, [\zeta])}$ we denote the point of $\tilde{X}_0^{(a)}$ corresponding to a point $(t_1, t_2, \cdots, t_\lambda, [\zeta])\in\tilde{D}\times E$.

$$\begin{array}{rccc} \text{Let } h_a\colon & \prod_{\mu=1}^{\lambda-p}\tilde{D}_{j_\mu}\times\prod_{\nu=1}^{p}D_{i_\nu} & \rightarrow & \prod_{\mu=1}^{\lambda-p}D_{j_\mu}\times\prod_{\nu=1}^{p}D_{i_\nu} \\ & \Psi & & \Psi \\ & (t_{j_1}, t_{j_2}, \cdots, t_{j_{\lambda-p}}, z_{i_1}, \cdots, z_{i_p}) & \mapsto & (t_{j_1}^{m_{j_1}}, t_{j_2}^{m_{j_2}}, \cdots, t_{j_{\lambda-p}}^{m_{j_{\lambda-p}}}, z_{i_1}, z_{i_2}, \cdots, z_{i_p}) \end{array}$$

be an $m_{j_1}m_{j_2}\cdots m_{j_{\lambda-p}}$-sheeted cyclic covering of U_0. Then we have

$$\begin{array}{ccl} X_0^{(a)}|_{h_a^{-1}(U_a\cap U_0)} & \xrightarrow{\sim} & h_a^{-1}(U_a\cap U_0)\times E \\ \Psi & & \Psi \\ \overline{(t_1, t_2, \cdots, t_\lambda, [\zeta])} & \mapsto & (t_{j_1}, t_{j_2}, \cdots, t_{j_{\lambda-p}}, t_{i_1}^{m_{i_1}}, \cdots, t_{i_p}^{m_{i_p}}, \\ & & [\zeta-\sum_{\nu=1}^{p}\frac{\alpha_{i_\nu}}{2\pi\sqrt{-1}}\log(t_{i_\nu})]), \end{array}$$

since $z_{i_\nu}\neq 0$ $(1\leq\nu\leq p)$ on $U_a\cap U_0$.

By the above isomorphism, we have

$$\begin{aligned} g^{m_{i_1}m_{i_2}\cdots m_{i_p}}\colon\ & (t_{j_1}, t_{j_2}, \cdots, t_{j_{\lambda-p}}, z_{i_1}, z_{i_2}, \cdots, z_{i_p}, [\eta_a]) \\ \mapsto & \left(e_{m_{j_1}}^{m_{i_1}m_{i_2}\cdots m_{i_p}}t_{j_1}, \cdots, e_{m_{j_{\lambda-p}}}^{m_{i_1}m_{i_2}\cdots m_{i_p}}t_{j_{\lambda-p}}, z_{i_1}, \cdots, z_{i_p}, \left[\eta_a+\frac{1}{m_{j_1}m_{j_2}\cdots m_{j_{\lambda-p}}}\right]\right), \end{aligned}$$

where we put $[\eta_a]=\left[\zeta-\sum_{\nu=1}^{p}\frac{\alpha_{i_\nu}}{2\pi\sqrt{-1}}\log(t_{i_\nu})\right]$. Therefore, there is an isomorphism $X_0|_{U_a\cap U_0}\xrightarrow{\sim}\dfrac{h_a^{-1}(U_a\cap U_0)\times E}{\langle g^{m_{i_1}m_{i_2}\cdots m_{i_p}}\rangle}$. h_a can be naturally extended to a $\prod_{\mu=1}^{\lambda-p}m_{j_\mu}$-sheeted cyclic covering $\tilde{h}_a$ of $\prod_{\mu=1}^{\lambda-p}D_{j_\mu}\times\prod_{\nu=1}^{p}C_{i_\nu}$, and the group $\langle g^{m_{i_1}m_{i_2}\cdots m_{i_p}}\rangle$ acts on $\tilde{h}_a^{-1}(U_a)\times E$ as in the same way as above.

Its action is free and properly discontinuous, so the quotient space

$$X_a := \tilde{h}_a^{-1}(U_a)\times E/\langle g^{m_{i_1} m_{i_2} \cdots m_{i_p}}\rangle \text{ is smooth}.$$

By a natural holomorphic mapping

$$\begin{array}{ccc} \Phi_a\colon X_a & \longrightarrow & U_a \\ \Cup & & \Cup \\ \overline{(t_{j_1}, t_{j_2}, \cdots, t_{j_{\lambda-p}}, z_{i_1}, \cdots, z_{i_p}, [\eta_a])} & \rightarrow & (t_{j_1}^{m_{j_1}}, \cdots, t_{j_{\lambda-p}}^{m_{j_{\lambda-p}}}, z_{i_1}, \cdots, z_{i_p}), \end{array}$$

X_a is an elliptic fiber space over U_a. By our construction, we have the following commutative diagram.

$$\begin{array}{ccc} X_0|_{U_a\cap U_0} & \hookrightarrow & X_a \\ \downarrow & & \downarrow \\ U_a\cap U_0 & \hookrightarrow & U_a. \end{array}$$

That is, $X_a \rightarrow U_a$ is a natural compactification of $X_0|_{U_a\cap U_0} \rightarrow U_a\cap U_0$.

Step 3. We can show that the elliptic fiber spaces $X_a \rightarrow U_a$ constructed in step 2 ($a \in \boldsymbol{Z}_2^{\oplus\lambda}$) can be glued together. *For that purpose, it is sufficient to show the following claims.*

Claim A. For any a, $b \in \boldsymbol{Z}_2^{\oplus\lambda}$ ($a \neq b$), there is a following commutative diagram:

$$\begin{array}{ccc} & X_c & \\ \nearrow & & \nwarrow \\ X_a|_{U_a\cap U_b} & \xrightarrow{\sim} & X_b|_{U_a\cap U_b} \end{array},$$

where $c = \max(a, b)$ and $\hookrightarrow$ (resp. $\xrightarrow{\sim}$) denotes an open immersion. (resp. an isomorphism.) (Here, for $a = (a_1, a_2, \cdots, a_\lambda)$, $b = (b_1, b_2, \cdots, b_\lambda) \in \boldsymbol{Z}_2^{\oplus\lambda}$, $a > b$ means that $a_i \leq b_i$ for all i. and $C = \max(a, b)$ means that $c_i = \max(a_i, b_i)$ for all i, where $C = (c_1, c_2, \cdots, c_\lambda)$.)

Claim B. For any a, $b \in \boldsymbol{Z}_2^{\oplus\lambda}$ such that $a < b$, there is a following commutative diagram. That is, $X_b \rightarrow U_b$ is the natural compactification of $X_a|_{U_a\cap U_b} \rightarrow U_a\cap U_b$.

$$\begin{array}{ccc} X_a|_{U_a\cap U_b} & \hookrightarrow & X_b \\ \downarrow & & \downarrow \\ U_a\cap U_b & \hookrightarrow & U_b \end{array}$$

It is easy to show that claim B implies claim A, so we shall prove claim B.

For any a, $b \in \boldsymbol{Z}_2^{\oplus\lambda}$ such that $a < b$, put

$$a = (0, \cdots, 0, \underbrace{1}_{i_1}, 0, \cdots, \underbrace{1}_{i_2}, 0, \cdots, \underbrace{1}_{i_p}, 0, \cdots, 0),\ 1 \leq i_1 < i_2 < \cdots < i_p \leq \lambda$$

$$b - a = (0, \cdots, 0, \underbrace{1}_{j_1}, 0, \cdots, \underbrace{0}_{j_2}, 1, 0, \cdots, \underbrace{0}_{j_s}, 1, 0, \cdots, 0),\ 1 \leq j_1 < j_2 < \cdots < j_s \leq \lambda.$$

Put $\{k_1 < k_2 < \cdots < k_{\lambda-p-s}\} = \{1, 2, \cdots, \lambda\} \setminus \{i_1, i_2, \cdots, i_p\} \setminus \{j_1, j, \cdots, j_s\}$. Then we have $U_a \cap U_b = \prod_{\mu=1}^{\lambda-p-s} D_{k_\mu} \times \prod_{\nu=1}^{s} D_{j_\nu}^* \times \prod_{k=1}^{p} C_{i_k}^*$.

Now, let us recall the construction of X_a. There is a following commutative diagram:

$$\begin{array}{ccc} X_0|_{U_a\cap U_0} & = & h_a^{-1}(U_a\cap U_0)\times E/\langle g^{m_{i_1}m_{i_2}\cdots m_{i_p}}\rangle \\ \cup & & \cup \\ X_a\colon & =\!=\!=\!= & \tilde{h}_a^{-1}(U_a)\times E/\langle g^{m_{i_1}m_{i_2}\cdots m_{i_p}}\rangle\,, \end{array}$$

where

$$\begin{array}{ccc} h_a\colon \prod_{\mu=1}^{\lambda-p-s}\tilde{D}_{k_\mu}\times\prod_{\nu=1}^{s}\tilde{D}_{j_\nu}\times\prod_{k=1}^{p}D_{i_k} & \to & \prod_{i=1}^{\lambda}D_i \\ \Psi & & \Psi \\ (t_{k_1},\cdots,t_{k_{\lambda-p-s}},t_{j_1},\cdots,t_{j_s},z_{i_1},\cdots,z_{i_p}) & \mapsto & (t_{k_1}^{m_{k_1}},\cdots,t_{k_{\lambda-p-s}}^{m_{k_{\lambda-p-s}}}, \\ & & t_{j_1}^{m_{j_1}},\cdots,t_{j_s}^{m_{j_s}},z_{i_1},\cdots,z_{i_p}) \end{array}$$

is a $\prod_{\mu=1}^{\lambda-p-s}m_{k_\mu}\cdot\prod_{\nu=1}^{s}m_{j_\nu}$-sheeted cyclic covering of $U_0=D_1\times D_2\times\cdots\times D_\lambda$.

Now, we shall express the elliptic fiber space $X_a\to U_a$ in another form. Because any two of m_i's are relatively prime, there is an isomorphism

$$X_a\xrightarrow{\sim}\frac{\tilde{h}_a^{-1}(U_a)\times E/\langle g^{m_{i_1}\cdots m_{i_p}m_{k_1}\cdots m_{k_{\lambda-p-s}}}\rangle}{\langle g^{m_{i_1}\cdots m_{i_p}m_{j_1}\cdots m_{j_s}}\rangle}\,.$$

On the other hand, the quotient space $\tilde{h}_a^{-1}(U_a)\times E/\langle g^{m_{i_1}\cdots m_{i_p}m_{k_1}\cdots m_{k_{\lambda-p-s}}}\rangle$ is an elliptic fiber space over $\prod_{\mu=1}^{\lambda-p-s}\tilde{D}_{k_\mu}\times\prod_{\nu=1}^{s}D_{j_\nu}\times\prod_{k=1}^{p}C^*_{i_k}$, which satisfies the following conditions.

(1) It has multiple fibers of multiplicity m_{i_ν} along $\{z_{j_\nu}=0\}$ $(1\leq\nu\leq s)$.

(2) It is trivial over $\prod_{\mu=1}^{\lambda-p-s}\tilde{D}_{k_\mu}\times\prod_{\nu=1}^{s}D^*_{j_\nu}\times\prod_{k=1}^{p}C^*_{i_k}$ and the trivialization is given as follows.

$$\begin{aligned}&\overline{(t_{k_1},\cdots,t_{k_{\lambda-p-s}},t_{j_1},\cdots,t_{j_s},z_{i_1},\cdots,z_{i_p},[\eta_a])}\\ &\xrightarrow{\sim}\left(t_{k_1},\cdots,t_{k_{\lambda-p-s}},t_{j_1}^{m_j},\cdots,t_{j_s}^{m_{j_s}},z_{i_1},\cdots,z_{i_p},\left[\eta_a-\sum_{\nu=1}^{s}\frac{\alpha_{j_\nu}}{2\pi\sqrt{-1}}\log(t_{j_\nu})\right]\right)\end{aligned}$$

Therefore, there is an ispmorphism

$$X_a|_{U_a\cap U_b}\xrightarrow{\sim}\frac{\prod_{\mu=1}^{\lambda-p-s}\tilde{D}_{k_\mu}\times\prod_{\nu=1}^{s}D^*_{j_\nu}\times\prod_{k=1}^{p}C^*_{i_k}\times E}{\langle g^{m_{i_1}m_{i_2}\cdots m_{i_p}m_{j_1}m_{j_2}\cdots m_{j_s}}\rangle}\,,$$

since $t_{j_\nu}\neq 0$ $(1\leq\nu\leq s)$ on $\tilde{h}_a^{-1}(U_a\cap U_b)$.

Here the group $\langle g^{m_{i_1}m_{i_2}\cdots m_{i_p}m_{j_1}m_{j_2}\cdots m_{j_s}}\rangle$ acts as follows.

$$\begin{aligned}&g^{m_{i_1}\cdots m_{i_p}m_{j_1}\cdots m_{j_s}}\colon(t_{k_1},\cdots,t_{k_{\lambda-p-s}},z_{j_1},\cdots,z_{j_s},z_{i_1},\cdots,z_{i_p},[\xi_a])\\ &\to\Bigg(e_{m_{k_1}}^{m_{i_1}\cdots m_{i_p}m_{j_1}\cdots m_{j_s}}t_{k_1},\cdots,e_{m_{k_{\lambda-p-s}}}^{m_{i_1}\cdots m_{i_p}m_{j_1}\cdots m_{j_s}}t_{k_{\lambda-p-s}},\\ &\qquad z_{j_1},\cdots,z_{j_s},z_{i_1},\cdots,z_{i_p},\left[\xi_a+\frac{1}{m_{k_1}m_{k_2}\cdots m_{k_{\lambda-p-s}}}\Bigg)\right],\end{aligned}$$

where we put $[\xi_a]=\left[\eta_a-\sum_{\nu=1}^{s}\frac{\alpha_{j_\nu}}{2\pi\sqrt{-1}}\log(t_{j_\nu})\right]$.

If we restrict the elliptic fiber space $X_a|_{U_a\cap U_b}\to U_a\cap U_b$ over $U_a\cap U_b\cap U_0$, we have

$$X_a|_{U_a\cap U_b\cap U_0}\simeq\frac{\prod_{\mu=1}^{\lambda-p-s}\tilde{D}_{k_u}\times\prod_{\nu=1}^{s}D^*_{j_\nu}\times\prod_{k=1}^{p}D^*_{i_k}\times E}{\langle g^{m_{i_1}\cdots m_{i_p}m_{j_1}\cdots m_{j_s}}\rangle}.$$

Here we have $[\xi_a]=\left[\zeta-\sum_{k=1}^{p}\frac{\alpha_{i_k}}{2\pi\sqrt{-1}}\log(t_{i_k})-\sum_{\nu=1}^{s}\frac{\alpha_{j_\nu}}{2\pi\sqrt{-1}}\log(t_{j_\nu})\right]$, since we have $[\eta_a]=\left[\zeta-\sum_{k=1}^{p}\frac{\alpha_{i_k}}{2\pi\sqrt{-1}}\log(t_{i_k})\right]$.

Therefore, we have the following commutative diagram.

$$\begin{array}{ccc} X_0|_{U_0\cap U_b} & \hookrightarrow & X_a|_{U_a\cap U_b} \\ \downarrow & & \downarrow \\ U_0\cap U_b & \hookrightarrow & U_a\cap U_0\,. \end{array}$$

Thus from the construction of $X_b\to U_b$ in step 2, there is an open immersion

$$\begin{array}{ccc} X_a|_{U_a\cap U_b} & \hookrightarrow & X_b \\ \downarrow & & \downarrow \\ U_b\cap U_b & \hookrightarrow & U_b \end{array}\qquad \text{for}\quad a<b\,,$$

and the claim B has been proved.

Step 4. By step 3, we can glue the elliptic fiber spaces $X_a\to U_a$ $(a\in\boldsymbol{Z}_2^{\oplus\lambda})$ to obtain the elliptic fiber space X over $C_1\times\cdots\times C_\lambda$. Clearly X satisfies the desired conditions (1) and (2). And X is not in the class $\mathcal{C}$ by the same reason as in Remark 4.8. q.e.d.

Remark 5.5. The elliptic fiber space X, which we have just constructed, depends on the choice of $\alpha_i\in\boldsymbol{Z}$ $(1\leq i\leq\lambda)$ in step 1. Hence we write X as $X(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$. (cf. Prop 5.6.)

Now, we are ready to prove our Main theorem 5.1.

Proof of Theorem 5.1.

Step 1. Take a linear pencil L_i from $|D_i|$ for each i and consider a meromorphic map Φ_{L_i}: $Y\to\boldsymbol{P}^1_{(i)}$ associated with L_i, where $\boldsymbol{P}^1_{(i)}$'s $(1\leq i\leq\lambda)$ are λ copies of $\boldsymbol{P}^1$. Let Δ_i be a smooth divisor on D_i in a sufficiently general position such that the following conditions are satisfied.

(1) $\Delta_i\not\subset D_j$ for all $j\,(\neq i)$ and $\Delta_{i_1i_2\cdots i_b}=\Delta_{i_1}\cap\Delta_{i_2}\cap\cdots\cap\Delta_{i_p}$ are $(n-2p)$-dimensional compact complex manifold.

(2) Let α_i: $Y_i\to Y$ be the blowing-up of Y with Δ_i center for each i. Then the composite map $Y_i\overset{\alpha_i}{\to}Y\to\boldsymbol{P}^1_{(i)}$ is a morphism.

Next, consider the fiber product

$$\tau\colon W = Y_1 \underset{Y}{\times} Y_2 \underset{Y}{\times} \cdots \underset{Y}{\times} Y_\lambda \to Y.$$

Because the arrangements of Δ_i's are sufficiently general, W is smooth and τ is a bimeromorphic morphism. Let $\overline{D}_i$ (resp. E_i) be the strict transform of D_i (resp. the total inverse image of Δ_i) for each i. Then W can be considered as a fiber space $f_i\colon W \to \boldsymbol{P}^1_{(i)}$ over $\boldsymbol{P}^1$, where E_i is a section and $\overline{D}_i$ is a fiber of the fiber space $(1 \leq i \leq \lambda)$. By a suitable change of coordinates, we may assume that $f_i(\overline{D}_i) = O$, where O is the origin of each $\boldsymbol{P}^1_{(i)}$. Let $\Phi\colon W \to \prod_{i=1}^{\lambda} \boldsymbol{P}^1_{(i)}$ be a holomorphic map defined by $\Phi = (f_1, f_2, \cdots, f_\lambda)$. By the construction of W, we have $\Phi^{-1}(\Phi(W) \cap \{z_i = 0\}) = \overline{D}_i$ for each i, where z_i is the inhomogeneous coordinate of $\boldsymbol{P}^1_{(i)}$.

Step 2. Because any two of m_i's are relatively prime, it follows from Proposition 5.4 that there exists an elliptic fiber space $h\colon Z \to \prod_{i=1}^{\lambda} \boldsymbol{P}^1_{(i)}$ over $\prod_{i=1}^{\lambda} \boldsymbol{P}^1_{(i)}$ which satisfies the following conditions.

(1) Z has multiple fibers of multiplicity m_i along each $\{z_i = 0\}$.

(2) Z is trivial over $\prod_{i=1}^{\lambda} (\boldsymbol{P}^1_{(i)} \setminus \{O\})$, where O denotes the origin of $\boldsymbol{P}^1_{(i)}$.

(3) h is flat.

Step 3. Next, consider the pull-back

$$g\colon X = Z \underset{\prod_{i=1}^{\lambda} \boldsymbol{P}^1_{(i)}}{\times} W \to W.$$

Because D_i's have only normal crossings, X is smooth. By the composition of the morphisms $X \xrightarrow{g} W \xrightarrow{\tau} Y$, $f\colon = \tau \circ g$ is an elliptic fiber space over Y which satisfies the following conditions.

(1) X has multiple fibers of multiplicity m_i along each D_i $(1 \leq i \leq \lambda)$.

(2) X is trivial over $Y \setminus \bigcup_{i=1}^{\lambda} D_i$.

Step 4. The canonical bundle of X is as follows.

$$K_X \underset{\boldsymbol{Q}}{\sim} f^*\Big(K_Y + \sum_{i=1}^{\lambda} \frac{m_i - 1}{m_i} D_i\Big) + \sum_{i=1}^{\lambda} \frac{1}{m_i} E_i.$$

So, for any positive integer m, we have

$$|mK_X| = f^*\Big(mK_Y + \sum_{i=1}^{\lambda} \Big[\frac{m(m_i - 1)}{m_i}\Big] D_i\Big) + (\text{fixed components})$$
$$+ (\textit{effective} \text{ exceptional divisors}),$$

where $[n]$ denotes the greatest integer which does not exceed n.

Therefore $f_*(K_{X/Y}^{\otimes m})$ is invertible for any positive integer m. q.e.d.

Remark 5.6.

(1) If we blow up Y along the intersections of D_i's and perform logarithmic trans-

formations along the strict transform of D_i's as in [3], $f^*(K_{X/Y}^{\otimes m})$ is not necessarily invertible for a positive integer m.
(2) If D_i's do not have normal crossings, $f_*(K_{X/Y}^{\otimes m})$ is not necessarily invertible.
(3) If the arrangements of Δ_i's in step 1 are not general, W in step 1 is not smooth. So we have to resolve singularities and $f^*(K_{X/Y}^{\otimes m})$ is not invertible.

In these cases, for calculation of $\dim|mK_X|$, we have to consider the base point conditions.

Proposition 5.7. *If there exist at least two indices i and j such that $\alpha_i>0$ and $\alpha_j<0$, then $X(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ cannot be bimeromorphic to a subvariety of a Hopf manifold.*

To prove this proposition, we need the following lemma. Here, we use the same notation as in Proposition 5.2. We define a line bundle $\pi\colon L\to\prod_{i=1}^{\lambda} C_i$ as follows.

$$L:=\bigotimes_{i=1}^{\lambda} p_i^*\mathcal{O}_{C_i}(\alpha_i m_1\cdots\check{m}_i\cdots m_\lambda P_i)\,,$$

where $p_i\colon \prod_{i=1}^{\lambda} C_i\to C_i$ is a projection.

Put $\rho=\exp(2\pi\sqrt{-1}\tau)$, where E is a smooth elliptic curve with the period $(1, \tau)$, $\mathrm{Im}(\tau)>0$. Consider a $\boldsymbol{C}^*$-bundle $L^*=L\backslash\{0\text{-section}\}$ on $\prod_{i=1}^{\lambda} C_i$. $\langle\rho\rangle$ acts on each fiber of L^*, so put $Y(\alpha_1, \cdots, \alpha_\lambda)=L^*/\langle\rho\rangle$.

Then the canonical projection $\pi\colon L^*\to\prod_{i=1}^{\lambda} C_i$ induces a projection

$$h\colon Y(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)\to\prod_{i=1}^{\lambda} C_i$$

and $Y(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ has a structure of an elliptic bundle over $\prod_{i=1}^{\lambda} C_i$.

Lemma 5.8. *There exists a finite abelian covering ϕ from $X(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ onto $Y(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ such that the following diagram commutes.*

$$\begin{tikzcd} X(\alpha_1, \alpha_2, \cdots, \alpha_\lambda) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & Y(\alpha_1, \alpha_2, \cdots, \alpha_\lambda) \arrow[dl, "h"] \\ & \prod_{i=1}^{\lambda} C_i & \end{tikzcd}$$

ϕ is a finite unramified covering on each fiber of f.

Proof. First, we show that for any $a\in\boldsymbol{Z}_2^{\oplus\lambda}$, there exists a finite abelian covering $\phi_a\colon X_a\to U_a\times E$ such that the following diagram commutes.

$$\begin{tikzcd} \phi_a\colon X_a \arrow[r] \arrow[dr] & U_a\times E \arrow[d] \\ & U_a \end{tikzcd}$$

In fact, define ϕ_a $(a\in\boldsymbol{Z}_2^{\oplus\lambda})$ as follows.

$$\begin{array}{ccc}
\phi_0\colon X_0 & \longrightarrow & U_0\times E \\
\text{\rotatebox{90}{$\Subset$}} & & \text{\rotatebox{90}{$\Subset$}} \\
\overline{(t_1, t_2, \cdots, t_\lambda, [\zeta])} & \mapsto & (t_1^{m_1}, t_2^{m_2}, \cdots, t_\lambda^{m_\lambda}, [m_1 m_2 \cdots m_\lambda \zeta])
\end{array}$$

$$\begin{array}{ccc}
\phi_a\colon X_a & \longrightarrow & U_a\times E \qquad \text{for} \quad a\neq 0 \\
\text{\rotatebox{90}{$\Subset$}} & & \text{\rotatebox{90}{$\Subset$}} \\
(t_{j_1}, \cdots, t_{j_{\lambda-p}}, z_{i_1}, \cdots, z_{i_p}, [\eta_a]) & \mapsto & (t_{j_1}^{m_{j_1}}, \cdots, t_{j_{\lambda-p}}^{m_{j_{\lambda-p}}}, z_{i_1}, \cdots, z_{i_p}, [m_1 m_2 \cdots m_\lambda \eta_a])\,,
\end{array}$$

where $a=(\cdots, \underset{i_1}{\underbrace{1}}, \cdots, \underset{i_2}{\underbrace{1}}, \cdots, \underset{i_p}{\underbrace{1}}, \cdots)$,

$$\{j_1, j_2, \cdots, j_{\lambda-p}\} = \{1, 2, \cdots, \lambda\}\setminus\{i_1, i_2, \cdots, i_p\}\,,$$

$$\text{and} \quad [\eta_a] = \left[\zeta - \sum_{\nu=1}^{p} \frac{\alpha_{i_\nu}}{2\pi\sqrt{-1}} \log(t_{i_\nu})\right].$$

We can easily check that ϕ_a's are compatible with the patching of X_a's and $U_a\times E$'s and define a finite abelian covering ϕ from $X(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ onto $Y(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$. l.e.d.

Lemma 5.9. *If there exists at least two indices i and j such that $\alpha_i>0$ and $\alpha_j<0$, then $Y(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ cannot be bimeromorphic to a subvariety of a Hopf manifold.*

Proof. We may assume that $\alpha_1>0$ and $\alpha_2<0$. Take one point Q_i on each C_i arbitrarily ($3\leq i\leq\lambda$) and put $V:=C_1\times C_2\times Q_3\times\cdots\times Q_\lambda$. Then $Z:=Y(\alpha_1, \alpha_2,\cdots, \alpha_\lambda)|_V\to V$ is an elliptic bundle over V and from the construction of $Y(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$, Z can be expressed as follows.

$$Z \xrightarrow{\sim} L|_V\setminus\{0\text{-section}/\langle\rho\rangle, \text{ where } L|_V \xrightarrow{\sim} \bigotimes_{i=1}^{2} p_i^* O_{C_i}(\alpha_i m_1 \cdots \check{m}_i \cdots m_i p_\lambda)$$

Let D be a smooth curve on V. By a theorem (11.9) in Kodaira [11], the following conditions are equivalent.

(1) The elliptic bundle $Z|_D\to D$ has a multi-section.
(2) $Z|_D$ is algebraic.
(3) $\deg(L|_D)=0$.

Let r_1, r_2 be a positive integer such that $[r_1P_1\times C_2+r_2C_2\times P_2]\in\mathrm{Pic}(V)$ is very ample. In particulae, let D be a general member of $|r_1P_1\times C_2+r_2C_2\times P_2|$. Since we have $\alpha_1>0$, $\alpha_2<0$ and $\deg(L|_D)=r_1\alpha_2 m_1 m_3\cdots m_\lambda+r_2\alpha_1 m_2 m_3\cdots m_\lambda$, we can choose r_1 and r_2 sufficiently positive such that $\deg(L|_D)=0$ and $\pi(D)>1$. Then from the above remark, there exists a smooth curve of genus greater than 1 on $Z|_D$. Therefore, for any point y on $Y(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$, there exists a smooth curve C of genus greater than 1, which passes through y and $\dim h(C)=1$. However, by Kato [8], any irreducible curve in a Hopf manifold is a smooth elliptic curve. Therefore, $Y(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ can never be bimeromorphic to a subvariety of a Hopf manifold. q.e.d.

Proof of Proposition 5.7. From Lemma 5.8 and 5.9, it follows that for any point x on $X(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$, there exists a smooth curve C of genus greater than 1, which passes through x and $\dim f(C)=1$. Then from Kato's theorem [8], $X(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ cannot be bimeromorphic to a subvariety of a Hopf manifold.

q.e.d.

Proposition 5.10. *Let C_i $(1 \leq i \leq \lambda)$ be a smooth curve and take one point P_i on each C_i. For an arbitrary integer $\lambda \geq 2$, let $(m_1, m_2, \cdots, m_\lambda)$ be a λ-tuple of integers with $m_i \geq 2$ for all i, and assum that except m_λ, any two of m_i's $(1 \leq i \leq \lambda-1)$ are relatively prime. Then there exists an elliptic fiber space X over $\prod_{i=1}^{\lambda} C_i$ which satisfies the same conditions as in Proposition* 5.4.

Proof. The idea is almost all the same as in Proposition 5.4. We use the same notation as in 5.4, and we shall slightly modify the arguments in step 1.

Let us consider analytic automorphisms of $\prod_{i=1}^{\lambda} \tilde{D}_i \times E$ defined by

$$g\colon (t_1, t_2, \cdots, t_{\lambda-1}, t_\lambda, [\zeta]) \mapsto \left(e_{m_1} t_1, \cdots, e_{m_{\lambda-1}} t_{\lambda-1}, t_\lambda, \left[\zeta + \frac{1}{m_1 \cdots m_{\lambda-1}}\right]\right)$$

$$h\colon (t_1, \cdots, t_{\lambda-1}, t_\lambda, [\zeta]) \mapsto \left(t_2, \cdots, t_{\lambda-1}, e_{m_\lambda} t_\lambda, \left[\zeta + \frac{\tau}{m_\lambda}\right)\right].$$

The automorphism groups generated by g and h acts on $\tilde{D} \times E$ freely and properly discontinuously, so the quotient space $X_0 := \tilde{D} \times E / \langle g, h \rangle$ is smooth. By a natural holomorphic mapping

$$\begin{array}{cccc} f\colon & X_0 & \longrightarrow & D_1 \times D_2 \times \cdots \times D_\lambda := D\,, \\ & \Cup & & \Cup \\ & \overline{(t_1, t_2, \cdots, t_\lambda, [\zeta])} & \mapsto & (t_1^{m_1}, t_2^{m_2}, \cdots, t_\lambda^{m_\lambda}) \end{array}$$

X_0 is an elliptic fiber space over D. Clearly X_0 has multiple fibers of multiplicity m_i along each $\{z_i=0\}$. There is an isomorphism

$$\begin{array}{ccc} X_0\big|_{D_1^* \times D_2^* \times \cdots \times D_\lambda^*} & \xrightarrow{\sim} & D_1^* \times D_2^* \times \cdots \times D_\lambda^* \times E \\ \Cup & & \Cup \\ \overline{(t_1, t_2, \cdots, t_\lambda, [\zeta])} & \mapsto & (t_1^{m_1}, t_2^{m_2}, \cdots, t_\lambda^{m_\lambda}, \\ & & \left[\zeta - \sum_{i=1}^{\lambda-1} \frac{\alpha_i}{2\pi\sqrt{-1}} \log(t_i) - \frac{\tau}{2\pi\sqrt{-1}} \log(t_\lambda)\right]), \end{array}$$

where α_i $(1 \leq i \leq \lambda-1) \in \boldsymbol{Z}$ are defined as follows. By the assumption, there exists $\alpha_i \in \boldsymbol{Z}$ such that

$$(**) \qquad \alpha_i m_1 m_2 \cdots \check{m}_i \cdots m_{\lambda-1} \equiv 1 \quad \mod m_i \quad \text{for each } i\,.$$

Take such α_i's and fix them.

From now on, we can apply the same arguments as in Step 2, 3, 4 in Proposition 5.4, so we omit the proof.

q.e.d.

Remark 5.11. If we assume in the above proposition that *except* m_λ and $m_{\lambda-1}$, any two of m_i's $(1\leq i\leq\lambda-2)$ are relatively prime, the same result holds. However, in this case, the elliptic fiber space $X\to\prod_{1=i}^{\lambda}C_i$ is *not flat.*
In fact, we can slightly modify the arguments in Step 1 in the proof of 3.2.

§ 6. Some examples.

If S is an analytic surface with $\kappa(S)=0$, by the classification theory of surfaces we have $P_{12}(S)=1$ and 12 is the best possible number. Now we shall construct similar examples for elliptic fiber spaces with $\kappa=0$, as an application of Theorem 5.1. Our result is the following.

Example 6.1. Let $\{a_n\}_{n=1,2,\cdots}$ be a sequence of positive integers defined as follows. $a_1=2$, $a_{n+1}=a_1a_2\cdots a_n+1$. And let $\{c_n\}_{n=1,2,\cdots}$ be a sequence of positive integers defined as follows.

$$c_n=a_{n+2}-1\,.$$

Then for every positive integer n, there exists an elliptic fiber space $Y^{(n+1)}\to\boldsymbol{P}^n$ over $\boldsymbol{P}^n$ which satisfies the following conditions.
(1) $\kappa(Y^{(n+1)})=0$.
(2) $m=c_n$ is the smallest integer such that $P_m(Y^{(n+1)})=1$.
Moreover $Y^{(n+1)}$ is not in the class $\mathcal{C}$.

Examples. We write down the first few terms of $\{a_n\}$ and $\{c_n\}$.

n	1	2	3	4	5	6
a_n	2	3	7	43	1807	3263443
c_n	6	42	1806	3263442	$\sim10^{13}$	$\sim10^{26}$

To prove Example 6.1, we need the following lemma.

Lemma 6.2. *Let $\{a_n\}$ and $\{c_n\}$ be sequences of positive integers defined as in* (6.1). *Then for every positive integer k, we have*

$$-k(n+1)+\sum_{i=1}^{n+1}\left[\frac{k(a_i-1)}{a_i}\right]+\left[\frac{k(c_n-1)}{c_n}\right]\leq 0$$

and the equality holds if and only if c_n divides k.

(The proof is the same as in 4.5, so we omit it.)

Proof of Example 6.1. Let H_i's $(1\leq i\leq n+2)$ be $(n+2)$ hyperplanes on $\boldsymbol{P}^n$ which are in a general position. And let $\{a_n\}_{n=1,2,\cdots}$ and $\{c_n\}_{n=1,2,\cdots}$ be sequences of positive integers defined as in Example 6.1. Take $(a_1, a_2, \cdots, a_{n+1}, c_n)$, a $(n+2)$-

tuple of positive integers. From the construction of $\{a_n\}$, any two of a_n's are relatively prime. Hence it follows from Proposition 5.8 that there exists an elliptic fiber space f: $Y^{(n+1)} \to \boldsymbol{P}^n$ over $\boldsymbol{P}^n$ with constant moduli which has multiple fibers of multiplicity a_i (resp. c_n) along each H_i $(1 \leq i \leq n+1)$. (resp. along H_{n+2})

The canonical bundle of $Y^{(n+1)}$ is as follows.

$$K_{Y^{(n+1)}} = f^*\Big(\mathcal{O}_{P^n}(-n-1) + \sum_{i=1}^{n+1} \frac{a_i - 1}{a} H_i + \frac{c_n - 1}{c_n} H_{n+2}\Big) + \sum_{i=1}^{n+1} \frac{1}{a_i} E_i + \frac{1}{c_n} E_{n+2}\,.$$

And for every positive integer k, we have

$$|kK_{Y^{(n+1)}}| = f^*\Big(\mathcal{O}_{P^n}(-f(n+1)) + \sum_{i=1}^{n+1}\Big[\frac{k(a_i - 1)}{a_i}\Big]H_i + \Big[\frac{k(c_n - 1)}{c_n}\Big]H_{n+2}\Big)$$
$$+(\text{fixed components}) + (\textit{effective} \text{ exceptional divisors}).$$

So it follows from Lemma 6.2 that for every positive integer k, we have $P_k(Y^{(n+1)}) \leq 1$ and the equality holds if and only if c_n divides k. Thus $Y^{(n+1)}$ has the desired properties q.e.d.

Remark 6.3. There is no algebraic elliptic fiber space over $\boldsymbol{P}^n$ with $\kappa = 0$ and with constant moduli which has multiple fibers of multiplicity a_i (resp. c_n) along each H_i $(1 \leq i \leq n+1)$. (resp. H_{n+2})

Proof. If such an algebraic elliptic fiber space $X \to \boldsymbol{P}^n$ exists, it follows from Theorem 3.5 that there exists a finite abelian covering $\tilde{\boldsymbol{P}}^n \to \boldsymbol{P}^n$ of $\boldsymbol{P}^n$ which branches at $\sum_{i=1}^{n+1} a_i H_i + c_n H_{n+2}$ with the Galois group $G \lesssim \boldsymbol{Z}/a_1 \oplus \boldsymbol{Z}/a_2 \oplus \cdots \oplus \boldsymbol{Z}/a_\lambda$, and X is bimeromorphic to the quotient space $\tilde{\boldsymbol{P}}^n \times E/G$. By a direct calculation, we see that $f^*(K_{X/P^n}{}^{\otimes m})$ is not invertible and $\kappa(X) = -\infty$. q.e.d.

Remark 6.4. We cannot apply Proposition 4.1 to our situation, since $(a_1, a_2, \cdots, a_{n+1}, c_n)$ does not satisfy the assumption in Proposition 4.1).

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References

[1] A. Fujiki, On automorphism groups of compact Kähler manifolds, Invent. Math., **44** (1978), 225–258.

[2] A. Fujiki, On resolution of cyclic quotient singularities, Publ. of. RIMS, **10** (1974), 293–328.

[3] Y Fujimoto, Logarithmic transformations on elliptic fiber spaces (to appear in J. of Kyoto Univ.)

[4] Y. Fujimoto, Pluricanonical mappings of elliptic fiber spaces, Proc. Japan Acad, **42** (1986), 308–310.

[5] T. Fujita, Zariski decomposition and the canonical rings of elliptic threefolds, J. Math. Soc. Japan., **38**–1 (1986), 19–37.

[6] S. Iitaka, Deformations of compact complex surfaces, ii. J. Math. Soc. Japan, **22** (1970), 247–261.

[7] T. Katsura and Ueno, On elliptic surfaces in characteristic p. Math. Ann., **272** (1985), 291–330.

[8] Ma. Kato, Some remarks on subvarieties of Hopf manifolds. Suriken—Kokyuroku, **240** (1975), 64–87.

[9] Ma, Kato, Complex structures on $S^1 \times S^5$, J. Math. Soc. Japan, **28** (1976), 550–576.

[10] Ma, Kato, On a characterization of submanifolds of Hopf manifolds. Iwanami Shoten Publ. Tokyo, (1977).

[11] K. Kodaira, On compact analytic surfaces, ii–iii, Ann of Math, **77** (1963) 563–626, (1963), 1–40.

[12] K. Kodaira, On the structure of compact complex analytic surface, II, Amer. J. Math., **84** (1964), 751–798.

[13] M. Namba, Lectures on branched coverings and algebraic functions. (preprint)

[14] K. Ueno, On compact analytic threefolds with non—trivial Albanese tori. (preprint)

[15] K. Ueno, On algebraic fiber spaces of abelian varieties, Math. Ann., **217** (1978), 1–22.