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Necessary and sufficient conditions for the local solvability of the Mizohata equations

By

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§ 1. Introduction.

As is well known, there exists a suitable $C^\infty(R^2)$ function $f(x_1, x_2)$ such that the Mizohata equation

$$(1.1) \qquad M_n u(x_1, x_2) \equiv \frac{\partial u}{\partial x_1} + i x_1^{2n+1} \frac{\partial u}{\partial x_2} = f(x_1, x_2),$$

where n is a non-negative integer, does not have a distribution solution in any neighborhood of the origin. But it seems the necessary and sufficient conditions on $f(x_1, x_2)$ for (1.1) to have a local solution are not yet known except for those of the micro-local solvability (see Sato-Kawai-Kashiwara [6] and Hörmander [2]).

In this article, we are concerned with the necessary and sufficient conditions on $f(x_1, x_2)$ for (1.1) to have a C^1 solution in a neighborhood of the origin.

Definition. We say a function $f(x_1, x_2)$ is the admissible data for the local solvability of (1.1) at the origin when (1.1) has a C^1 solution in a neighborhood of the origin.

Let Ω and $\mathcal{J}$ denote respectively an open neighborhood of the origin in R^2 and an open interval $(-r, r)$. Throughout this article m denotes $2n+2$. Now, our main result is stated thus:

Theorem A. *Assume that $f(x_1, x_2) \in C^0(\Omega)$ and $\partial_{x_2} f(x_1, x_2)$ is Hölder continuous in Ω. Let $f^\sharp(x_1, x_2)$ denote the function defined in Ω by*

$$\int_{-x_1}^{x_1} \partial_{x_2} f(t, x_2) dt .$$

Then, $f(x_1, x_2)$ is the admissible data for the local solvability of (1.1) *at the origin if and only if there exists a positive constant δ such that the function $A_m^\sharp f(x_2)$ defined in R^1 by*

$$\int_{-\delta}^{\delta} \int_0^{\delta} \frac{f^\sharp((my_1)^{1/m}, y_2)}{y_1 + i(y_2 - x_2)} dy_1 dy_2$$

Communicated by Prof. S. Mizohata May 22 1987 *is analytic in* $(-\delta, \delta)$.

According to this, from the integration by parts, for whatever function $f(x_2)$ such that $f''(x_2)$ is Hölder continuous in $\mathcal{J}$, $f(x_1, x_2) \equiv f(x_2)+ix_1^{2n+2}f'(x_2)$ is the admissible data for the local solvavility of (1.1) at the origin. On the other hand, applying the same theorem to $f(x_1, x_2)\equiv f(x_2)$, we get the following

Proposition B. *Assume that* $f(x_2)\in C^2(\mathcal{J})$. *Then,* $f(x_2)$ *is the admissible data for the local solvability of* (1.1) *at the origin if and only if* $f(x_2)$ *is analytic at* $x_2=0$.

Next, let us introduce the function $\mathcal{A}_m^\sharp f(x_2)$ defined in R^1 by

$$\int_{-\infty}^{\infty} dy_2 \int_0^{\infty} \frac{y_1^{m-1}f^\sharp(y_1, y_2)}{y_1^m/m+i(y_2-x_2)} dy_1$$

provided that $f(x_1, x_2)\in C_0^2(\Omega)$. Then, Theorem A can be restated thus:

Theorem A′. *Assume that* $f(x_1, x_2)\in C_0^2(\Omega)$. *Then,* $f(x_1, x_2)$ *is the admissible data for the local solvability of* (1.1) *at the origin if and only if* $\mathcal{A}_m^\sharp f(x_2)$ *is analytic at* $x_2=0$.

Treves [9] showed that $f(x_1, x_2)\in C_0^\infty(\Omega)$ is the admissible data for the local solvability of $M_0u(x_1, x_2)=f(x_1, x_2)$ at the origin when the function defined in R^1 by

$$\iint \frac{f(y_1, y_2)}{y_1^2/2+i(y_2-x_2)} dy_1 dy_2$$

is analytic at $x_2=0$. We find that his sufficient condition is necessary. Namely, we get the following

Theorem C. *Assume that* $f(x_1, x_2)\in C_0^2(\Omega)$. *The following conditions are equivalent.*

(i) $f(x_1, x_2)$ *is the admissible data for the local solvability of* (1.1) *at the origin.*

(ii) $\mathcal{A}_m^\sharp f(x_2)$ *is analytic at* $x_2=0$.

(iii) $A_m f(x_2)$ *is analytic at* $x_2=0$.

(iv) $Q_+ f(x_1, x_2)$ *is real analytic at the origin.*

where

$$A_m f(x_2) \equiv \iint \frac{f(y_1, y_2)}{y_1^m/m+i(y_2-x_2)} dy_1 dy_2$$

and

$$Q_+ f(x_1, x_2) \equiv \frac{1}{2\pi\Gamma(1+1/m)} \int_0^\infty \xi^{1/m} d\xi \iint \exp(-Q(x, y)\xi) f(y_1, y_2) dy_1 dy_2$$

where $Q(x, y)\equiv(x_1^m+y_1^m)/m+i(y_2-x_2)$. ($Q_+ f(x_1, x_2)$ *was introduced by Hörmander* [2; *Proposition* 26.3].)

Theorem A is proved in § 2, very elementally; Proposition B is proved in § 3;

Theorem C is proved in § 4. Finally, in § 5, we notice the problem of existence of C^1 solutions of $Lu=0$ such that grad $u\neq 0$, where L denote smooth complex vector fields in R^2. As an application, it is presented the necessary conditions for certain Mizohata type equations to have such a solution. This concerns with L. Nirenberg [5], F. Treves [8], and J. Sjöstrand [7].

§ 2. Proof of Theorem A.

Hereafter we say shortly $f(x_1, x_2)$ is the admissible data when it is the admissible data for the local solvability of (1.1) at the origin. First we remark this:

$f(x_1, x_2)$ is the admissible data when and only when the function $x_1^{2n+1}\int_0^{x_1}\partial_{x_2}f(t, x_2)dt$ is so.

Because: $v\equiv i(u-\int_0^{x_1}f(t, x_2)dt)$ for any C^1 solution u of (1.1) is a C^1 solution of

$$M_n v(x_1, x_2) = x_1^{2n+1}\int_0^{x_1}\partial_{x_2}f(t, x_2)dt\,. \tag{2.1}$$

Conversely, $u\equiv -iv+\int_0^{x_1}f(t, x_2)dt$ for any C^1 solution v of (2.1) is a C^1 solution of (1.1).

Next we see the following

Lemma 2.1. *Assume that $g(x_1, x_2)$ is Hölder continuous in Ω and even in x_1. Then, $x_1^{2n+1}g(x_1, x_2)$ is the admissible data.*

From this and the above remark, we get the following:

Lemma 2.2. *$f(x_1, x_2)$ is the admissible data if and only if $x_1^{2n+1}f^\sharp(x_1, x_2)$ is the admissible data.*

Proposition 2.3. *Under the same assumption as Theorem A, furthermore, assume that $f(x_1, x_2)$ is odd in x_1. Then, $f(x_1, x_2)$ is the admissible data.*

We omit the proof of Lemma 2.1, since it will be clear in the following arguments. Now, let us assume that $f(x_1, x_2)$ is the admissible data. Then, from Lemma 2.2, the equation $M_n u(x_1, x_2)=x_1^{2n+1}f^\sharp(x_1, x_2)$ has a C^1 solution u in a neighborhood of the origin. Let u_0 be the odd part of u with respect to x_1. Then, it holds that

$$\frac{\partial u_0}{\partial x_1}+ix_1^{2n+1}\frac{\partial u_0}{\partial x_2} = x_1^{2n+1}f^\sharp(x_1, x_2) \tag{2.2}$$

in a neighborhood of the origin. Hence, there is a suitable positive constant δ such that the function U defined in $\bar{\omega}$ by $U(x_1, x_2)=u_0((mx_1)^{1/m}, x_2)$ is $C^0(\bar{\omega})\cap C^1(\omega)$ and

$$\frac{\partial U}{\partial x_1}+i\frac{\partial U}{\partial x_2} = f^\sharp((mx_1)^{1/m}, x_2) \tag{2.3}$$

in ω where $\omega=\{(x_1, x_2); 0<x_1<\delta, |x_2|<\delta\}$. Notice that $f^\sharp((mx_1)^{1/m}, x_2)$ is Hölder continuous in $\bar{\omega}$. Making use of Stokes theorem, from (2.3) we have the following:

$$(2.4)\quad U(x_1, x_2) = \frac{1}{2\pi i}\left(\int_0^\delta \frac{U(y_1, -\delta)}{y_1-i\delta-(x_1+ix_2)}dy_1+i\int_{-\delta}^\delta \frac{U(\delta, y_2)}{\delta+iy_2-(x_1+ix_2)}dy_2 + \int_\delta^0 \frac{U(y_1, \delta)}{y_1+i\delta-(x_1+ix_2)}dy_1\right) - \frac{1}{2\pi}\int_{-\delta}^\delta\int_0^\delta \frac{f^\sharp((my_1)^{1/m}, y_2)}{y_1+iy_2-(x_1+ix_2)}dy_1dy_2$$

in ω. Since $U(0, x_2)=0$, it follows that

$$A_m^\sharp f(x_2) = \int_{-\delta}^\delta \frac{U(\delta, y_2)}{\delta+i(y_2-x_2)}dy_2 - i\int_0^\delta\left(\frac{U(y_1, -\delta)}{y_1-i(\delta+x_2)} - \frac{U(y_1, \delta)}{y_1+i(\delta-x_2)}\right)dy_1$$

in $(-\delta, \delta)$. Whence it follows that $A_m^\sharp f(x_2)$ is analytic in $(-\delta, \delta)$.

Conversely, let us assume that for some positive constant δ $A_m f(x_2)$ is analytic in $(-\delta, \delta)$. Then, there exists a holomorphic function $h(z)$ in a domain $\{z=x_1+ix_2; |x_1|<\rho, |x_2|<\rho\}$ such that $h(z)|_{x_1=0}=A_m^\sharp f(x_2)$ $(\rho\leqq\delta)$. By the way, we see the folowing

Lemma 2.4. *Let $v=v(x_1, x_2)$ denote the function defined in $\bar{\omega}$ by*

$$\frac{-1}{2\pi}\int_{-\delta}^\delta\int_0^\delta \frac{f^\sharp((my_1)^{1/m}, y_2)}{y_1+iy_2-(x_1+ix_2)}dy_1dy_2$$

Then, it holds that $v\in C^0(\bar{\omega})\cap C^1(\omega)$ and

$$\frac{\partial v}{\partial x_1}+i\frac{\partial v}{\partial x_2} = f^\sharp((mx_1)^{1/m}, x_2) \qquad \text{in} \quad \omega\,.$$

This follows from the well known theorem concerning the Beltrami equation (see, for instance, R. Courant-D. Hilbert [1]). Now, set $V=v+h(z)/(2\pi)$. In view of $V(0, x_2)=0$, we see that $V\in C^1(\omega_1^*)$ where $\omega_1^*=\{(x_1, x_2); 0\leqq x_1<\rho, |x_2|<\rho\}$ and

$$\frac{\partial V}{\partial x_1}+i\frac{\partial V}{\partial x_2} = f^\sharp((mx_1)^{1/m}, x_2) \qquad \text{in} \quad \omega_1^*.$$

Finally, let us define the function u in a neighborhood $\mathcal{D}$ of the origin in the following manner:

$$u(x_1, x_2) = \begin{cases} V(x_1^m/m, x_2) & \text{if} \quad x_2\geqq 0 \\ -V(x_1^m/m, x_2) & \text{if} \quad x_1<0 \end{cases}$$

where $\mathcal{D}=\{(x_1, x_2); x_1^m<m\rho, |x_2|<\rho\}$. Since $V(0, x_2)=0$, we see that $u\in C^1(\mathcal{D})$. It is evident that $M_n u(x_1, x_2)=x_1^{2n+1}f^\sharp(x_1, x_2)$ in $\mathcal{D}$. Therefore, from Lemma 2.2, $f(x_1, x_2)$ is the admissible data. Q.E.D.

Remark 2.1. In the above arguments, the following too has been proved.

Proposition 2.5. *Assume that $g(x_1, x_2)$ is Hölder continuous in Ω. Then,*

$x_1^{2n+1}g(x_1, x_2)$ is the admissible data if and only if there exists a positive constant δ such that the function defined by

$$\int_{-\delta}^{\delta}\int_0^{\delta}\frac{g((my_1)^{1/m}, y_2)-g(-(my_1)^{1/m}, y_2)}{y_1+i(y_2-x_2)}dy_1dy_2$$

is analytic in $(-\delta, \delta)$.

Remark 2.2. The assumption of Hölder continuity of $\partial_{x_2}f(x_1, x_2)$ is used only in the proof of the sufficiency. The condition of the continuity of it suffices for the proof of the necessity.

§ 3. Proof of Proposition B.

The sufficiency follows from Cauchy-Kowalewskaja theorem. Thus we shall prove the necessity. Take a $C_0^\infty(\mathcal{J})$ function $\alpha(x_2)$ such that $\alpha(x_2)=1$ in $[-r/2, r/2]$. Then, $f(x_2)$ is the admissible data if and only if $\alpha(x_2)f(x_2)$ is so. Hence, hereafter we can assume that $f(x_2)\in C_0^2(\mathcal{J})$ and, from Theorem A, for some positive constant δ $(<r/2)$.

$$\int_{-\delta}^{\delta}\int_0^{\delta}\frac{y_1^{1/m}f'(y_2)}{y_1+i(y_2-x_2)}dy_2dy_1 \quad \text{is analytic in } (-\delta, \delta)\,.$$

Lemma 3.1. *Let $|y_2-x_2|<\delta/2$. Then, it holds that*

$$\int_0^{\delta}\frac{y_1^{1/m}}{y_1+i(y_2-x_2)}dy_1 = \begin{cases} c_m(y_2-x_2)^{1/m}+S_m(x_2, y_2) & \textit{for} \quad 0\leqq y_2-x_2<\delta/2 \\ \bar{c}_m(x_2-y_2)^{1/m}+S_m(x_2, y_2) & \textit{for} \quad 0\leqq x_2-y_2>\delta/2 \end{cases}$$

where

$$c_m = \int_0^p\frac{t^{1/m}}{t+i}dt-\sum_{n=0}^{\infty}\frac{(-i)^n p^{1/m-n}}{1/m-n} \quad (p\text{: a constant, } 1<p\leqq 2),\ c_m\neq 0$$

$$S_m(x_2, y_2) = \delta^{1/m}\sum_{n=0}^{\infty}\frac{(-i)^n}{1/m-n}\left(\frac{y_2-x_2}{\delta}\right)^n$$

Proof. It is trivial when $x_2=y_2$. Let $0<y_2-x_2<\delta/2$. Then,

$$\int_0^{\delta}\frac{y_1^{1/m}}{y_1+i(y_2-x_2)}dy_1 = (y_2-x_2)^{1/m}\int_0^{\delta/(y_2-x_2)}\frac{t^{1/m}}{t+i}dt$$

$$= (y_2-x_2)^{1/m}\left(\int_0^p\frac{t^{1/m}}{t+i}dt+\int_p^{\delta(y_2-x_2)}\frac{t^{1/m}}{t+i}dt\right)$$

$$= (y_2-x_2)^{1/m}\left(\int_0^p\frac{t^{1/m}}{t+i}dt+\sum_{n=0}^{\infty}\int_p^{\delta(y_2-x_2)}t^{1/m-1}(-i/t)^n dt\right)$$

$= c_m(y_2-x_2)^{1/m}+S_m(x_1, y_2)$ (We take a constant p such that $1<p\leqq 2$, $c_m\neq 0$).

In case of $0<x_2-y_2<\delta/2$,

$$\int_0^\delta \frac{y_1^{1/m}}{y_1+i(y_2-x_2)} dy_1 = (x_2-y_2)^{1/m} \left(\int_0^p \frac{t^{1/m}}{t-i} dt + \int_p^{\delta(x_2-y_2)} \frac{t^{1/m}}{t-i} dt\right)$$
$$= \bar{c}_m (x_2-y_2)^{1/m} + S_m(x_2, y_2) .$$

From this, making use of the integration by parts, we get the following

Lemma 3.2.

$$c_m \int_{-\delta/4}^{x_2} f(y_2)(x_2-y_2)^a dy_2 - \bar{c}_m \int_{x_2}^{\delta/4} f(y_2)(y_2-x_2)^a dy_2$$

is analytic in $(-\delta/4, \delta/4)$ *where* $a=1/m-1$.

Thus we get the following

Lemma 3.3.

$$\int \exp(ix_2\xi)\hat{f}(\xi)\{(c_m-\bar{c}_m)\cos(\pi/(2m)) - i(c_m+\bar{c}_m)\operatorname{sgn}\xi \sin(\pi/(2m))\} |\xi|^{-1/m} d\xi$$

is analytic in $(-\delta/4, \delta/4)$.

Proof. Considering that $f(x_2) \in C_0^2(\mathcal{J})$, we have

$$c_m \int_{-\infty}^{x_2} f(y_2)(x_2-y_2)^a dy_2 - \bar{c}_m \int_{x_2}^{\infty} f(y_2)(y_2-x_2)^a dy_2$$
$$= c_m \int f(y_2) H(x_2-y_2) |x_2-y_2|^a dy_2 - \bar{c}_m \int f(y_2)(1-H(x_2-y_2)) |x_2-y_2|^a dy_2$$

($H(x)$ denotes the Heaviside function.)

$$= (c_m+\bar{c}_m)(f*H(x)|x|^a)(x_2) - \bar{c}_m (f*|x|^a)(x_2)$$
$$= \frac{c_m+\bar{c}_m}{2\pi} \int e^{ix_2\xi} \widehat{f*H(x)|x|^a} d\xi - \frac{\bar{c}_m}{2\pi} \int e^{ix_2\xi} \widehat{f*|x|^a} d\xi$$
$$= \frac{c_m+\bar{c}_m}{2\pi} \int e^{ix_2\xi} \hat{f}(\xi) \exp\{-\pi i/(2m) \operatorname{sgn}\xi\} a! |\xi|^{-1/m} d\xi$$
$$- \frac{\bar{c}_m}{2\pi} \int e^{ix_2\xi} \hat{f}(\xi) 2\cos(\pi/(2m)) a! |\xi|^{-1/m} d\xi \quad (a! \equiv \Gamma(a+1))$$
$$= \frac{a!}{2\pi} \int e^{ix_2\xi} \hat{f}(\xi) \{(c_m-\bar{c}_m)\cos(\pi/(2m)) - i(c_m+\bar{c}_m)\operatorname{sgn}\xi \sin(\pi/(2m))\} |\xi|^{-1/m} d\xi .$$

On the other hand, it is evident from Lemma 3.2 that the above first term is analytic in $(-\delta/4, \delta/4)$. Thus the lemma is proved.

Now, set $Q(\xi) = \{(c_m-\bar{c}_m)\cos(\pi/(2m)) - i(c_m+\bar{c}_m)\operatorname{sgn}\xi \sin(\pi/(2m))\} |\xi|^{-1/m}$. Let $\beta(x)$ be a $C^\infty(R^1)$ function such that $\beta(x)=0$ in $[-\delta/2, \delta/2]$ and $\beta(x)=1$ outside of $[-\delta, \delta]$. Then, set $p(\xi)=\beta(\xi)Q(\xi)$. We see $p(\xi) \in S_{1,0}^{-1/m}$. Denoting by P the pseudo-differential operator whose symbol is $p(\xi)$, we see that P is elliptic. From Lemma 3.3 it follows that $Pf(x_2) = \frac{1}{2\pi} \int e^{ix_2\xi} p(\xi) \hat{f}(\xi) d\xi$ is analytic in $(-\delta/4, \delta/4)$.

Thus we can conclude that $f(x_2)$ is analytic in $(-\delta/4, \delta/4)$ because of the analytic-hypoellipticity of P. Q.E.D.

Remark 3.1. Let $f_k''(x_2)$ be Hölder continuous in $\mathcal{I}(k=0, 1, \cdots, N)$. Then, $\sum_{k=0}^{N}\left\{x_1^{km} f_k(x_2)+\frac{i}{km+1} x_1^{(k+1)m} f_k'(x_2)\right\}$ is the admissible data.

§ 4. Proof of Theorem C.

Let $f(x_1, x_2)\in C_0^2(\Omega)$. We get the following in relation to $\mathcal{A}_m^{\sharp} f(x_2)$ and $A_m f(x_2)$.

Lemma 4.1. $\mathcal{A}_m^{\sharp} f(x_2)=iA_m f(x_2)$.

Proof. Denote by $F(y_1, y_2; x_2)$ $1/(y_1^m/m+i(y_2-x_2))$. Then, it follows from Fubini theorem and the integration by parts that

$$\begin{aligned}\mathcal{A}_m^{\sharp} f(x_2) &= \int_0^{\infty} y_1^{m-1}dy_1 \int_{-\infty}^{\infty}\frac{\partial}{\partial y_2}\left(\int_{-y_1}^{y_1} f(t, y_2)dt\right)F(y_1, y_2; x_2)dy_2 \\ &= i\int_0^{\infty} y_1^{m-1}dy_1 \int_{-\infty}^{\infty}\left(\int_{-y_1}^{y_1} f(t, y_2)dt\right)(F(y_1, y_2; x_2))^2 dy_2 \\ &= -i\int_{-\infty}^{\infty} dy_2 \int_0^{\infty}\left(\frac{\partial}{\partial y_1}F(y_1, y_2; x_2)\int_{-y_1}^{y_1} f(t, y_2)dt\right)dy_1 \\ &= i\int_{-\infty}^{\infty} dy_2 \int_0^{\infty}(f(y_1, y_2)+f(-y_1, y_2))F(y_1, y_2; x_2)dy_1 = iA_m f(x_2)\,.\end{aligned}$$

Next, we note that $Q_+f(x_1, x_2)$ is continuous in R^2. In relation to $Q_+f(x_1, x_2)$ and $A_m f(x_2)$, we get the following

Lemma 4.2.

$$\int_{-\infty}^{\infty} Q_+f(x_1, x_2)dx_1 = m^{1/m}/\pi\, A_m f(x_2)\,.$$

Proof. In view of $Q_+f(x_1, x_2)=Q_+f(-x_1, x_2)$, making use of Fubini theorem, we have

$$\begin{aligned}&\int_{-\infty}^{\infty} Q_+f(x_1, x_2)dx_1 = \lim_{c\to 0_+} 2\int_c^{\infty} Q_+f(x_1, x_2)dx_1 \\ &= \lim_{c\to 0_+} k_m \iint f(y_1, y_2)dy_1dy_2 \int_0^{\infty} e^{-F(y_1,y_2\,;\,x_2)\xi}d\xi \int_c^{\infty} e^{-x_1^m\xi/m}\,\xi^{1/m}dx_1 \\ &(k_m\equiv 1/(\pi\Gamma(1+1/m))) \\ &= m^{1/m-1}k_m \lim_{c\to 0_+}\iint f(y_1, y_2)dy_1dy_2 \int_0^{\infty} e^{-F(y_1,y_2\,;\,x_2)\xi}d\xi \int_{c^m\xi/m}^{\infty} e^{-t}\,t^{1/m}\,dt \\ &= m^{1/m-1}k_m \lim_{c\to 0_+}\iint f(y_1, y_2)dy_1dy_2 \int_0^{\infty} e^{-F(y_1,y_2\,;\,x_2)\xi}(\Gamma(1/m)-\int_0^{c^m\xi/m} e^{-t}\,t^{1/m-1}dt)d\xi \\ &= m^{1/m}/\pi\, A_m f(x_2)-m^{1/m-1}k_m \lim_{c\to 0_+}\iint f(y_1, y_1)dy_1dy_2\int_0^{\infty} e^{F(y_1,y_2\,;\,x_2)\xi}d\xi\int_0^{c^m\xi/m} e^{-t}t^{1/m-1}dt \\ &= m^{1/m}/\pi\, A_m f(x_2)\,.\end{aligned}$$

On the other hand, we get the following

Lemma 4.3.

$$Q_+f(x_1, x_2) = \frac{1}{2\pi i}\iint y_1^{m-1}\Big\{\int_0^{y_1}\frac{\partial}{\partial y_2}f(t, y_2)dt\Big\}Q(x, y)^{-1-1/m}dy_1dy_2\,.$$

Proof. For simplicity, we set $Q = Q(x, y)$ $(=(x_1^m+y_1^m)/m+i(y_2-x_2))$. Making use of Fubini theorem and the integration by parts, we have:

$$\iint e^{-Q\xi}\Big\{y_1^{m-1}\int_0^{y_1}\frac{\partial}{\partial y_2}f(t, y_2)dt\Big\}dy_1dy_2 = \int y_1^{m-1}dy_1\int\Big\{\int_0^{y_1}e^{-Q\xi}\frac{\partial}{\partial y_2}f(t, y_2)dt\Big\}dy_2$$

$$= i\xi\int y_1^{m-1}dy_1\int\{e^{-Q\xi}\int_0^{y_1}f(t, y_2)dt\}dy_2 = -i\int dy_2\int\Big(\frac{\partial}{\partial y_1}(e^{-Q\xi})\int_0^{y_1}f(t, y_2)dt\Big)dy_1$$

$$= i\iint e^{-Q\xi}f(y_1, y_2)dy_1dy_2\,.$$

Whence it follows that

$$Q_+f(x_1, x_2) = \frac{1}{2\pi i\Gamma(1+1/m)}\int_0^\infty \xi^{1/m}d\xi\iint e^{-Q\xi}\Big(y_1^{m-1}\int_0^{y_1}\frac{\partial}{\partial y_2}f(t, y_2)dt\Big)dy_1dy_2$$

$$= \frac{1}{2\pi i}\iint y_1^{m-1}\int_0^{y_1}\frac{\partial}{\partial y_2}f(t, y_2)dt\,Q^{-1-1/m}dy_1dy_2\,.$$

Now, it is evident from Theorem A′ and Lemma 4.1 that (i)$\rightleftarrows$(ii)$\rightleftarrows$(iii). Since $Q_+f(x_1, x_2)$ is real analytic at $x_1\neq 0$, (iv) implies (i) because of Lemma 4.2. Thus we have only to prove that (i) implies (iv). That is as follows: as is remarked in §2, (i) implies that there is a C^1 solution u of the Mizohata equation $M_nu(x_1, x_2)= x_1^{2n+1}\int_0^{x_1}\frac{\partial}{\partial x_2}f(t, x_2)dt$ in a neighborhood ω of the origin. We can take $\omega=(-\delta, \delta)\times(-\delta, \delta)$ where δ is a positive constant. Then, it follows from Lemma 4.3 that

$$2\pi iQ_+f(x_1, x_2) = \iint_{R^2\backslash\omega}y_1^{m-1}\int_0^{x_1}\frac{\partial}{\partial y_2}f(t, y_2)dt\,Q^{-1-1/m}dy_1dy_2+$$

$$+\iint_\omega M_nu(y_1, y_2)Q^{-1-1/m}dy_1dy_2\,.$$

The first term of the righthand side is real analytic in ω. Making use of Fubini theorem and the integration by parts, we see that the second term of the righthand side is expressed thus:

$$\int_{-\delta}^{\delta}\{u(\delta, y_2)Q(x;\, \delta, y_2)-u(-\delta, y_2)Q(x;\, -\delta, y_2)\}dy_2+$$

$$i\int_{-\delta}^{\delta}y_1^{m-1}\{u(y_1, \delta)Q(x;\, y_1, \delta)-u(y_1, -\delta)Q(x;\, y_1, -\delta)\}dy_1$$

where $Q(x;\, y_1, y_2)$ denotes $1/((x_1^m+y_1^m)/m+i(y_2-x_2))^{1/m}$.

Whence it follows that $Q_+ f(x_1, x_2)$ is real analytic in ω.

Remark 4.1. Notice that

$$\iint \left| y_1^{m-1} \int_0^{y_1} \frac{\partial}{\partial y_2} f(t, y_2) dt\, Q^{-1-1/m} \right| dy_1 dy_2 < \infty \qquad \text{for any} \quad (x_1, x_2) \in R^2 .$$

§ 5. An application.

In this section, we notice the problem of the existence of C^1 solutions such that grad $u \neq 0$ to the equation of the form

$$Mu(x, y) \equiv \frac{\partial u}{\partial x} + ia(x, y) \frac{\partial u}{\partial y} = 0 \tag{5.1}$$

where $a(x, y)$ is assumed to be realvalued and $C^\infty(R^2)$. When $a(x, y)$ is real analytic, (5.1) has a real analytic solution such that grad $u \neq 0$. When it is C^∞, for instance, if it is non-negative (non-positive) in a neighborhood of the origin, (5.1) has such a solution in a neighborhood of the origin (H. Ninomiya [4]). But, L. Nirenberg [5] constructed the example which admits only constant C^1 solutions in any small neighborhood of the origin. That is one of the Mizohata type equations; here we call the equation (5.1) Mizohata type when $a(x, y)$ is of the form

$$a(x, y) = x^{2n+1} b(x, y) \tag{5.2}$$

where n denotes a non-negative integer and $b(x, y)$ is a nonvanishing realvalued C^∞ function. His example is of the form:

$$\frac{\partial u}{\partial x} + ix(1 + x\phi(x, y)) \frac{\partial u}{\partial y} = 0 \tag{5.3}$$

where $\phi(x, y)$ is a suitably choosen realvalued C_0^∞ function which is even in x. Hereafter, let (5.1) be Mizohata type. In relation to (5.3), we shall set

$$b_e(x, y) = \text{the even part of } b(x, y) \text{ in } x$$
$$b_o(x, y) = \text{the odd part of } b(x, y) \text{ in } x .$$

Notice that $b_e(x, y) \neq 0$. First, we see the following

Proposition 5.1. *Assume that* $b_o(x, y) \equiv 0$. *Then,* (5.1) *has a* C^1 *solution* u, *such that grad* $u \neq 0$, *in a neighborhood of the origin.*

The proof is omitted (see [4]). From this, we see that it is the very problem only when $b_o(x, y) \not\equiv 0$ in any small neighborhood of the origin. Then, as an application of Proposition 2.5, we get the following

Proposition 5.2. *Assume that* $b_e(x, y)$ *is real analytic. In order that* (5.1) *has a* C^1 *solution* u, *such that grad* $u \neq 0$, *in a neighborhood of the origin, it is necessary* *that there exist some positive constant* δ *and some nonvanishing* $C^0(\omega)$ (ω: *a neighborhood of the origin*) *function* $\Phi(x, y)$ *which is even in* x *such that*

$$\int_{-\delta}^{\delta}\int_{0}^{\delta}\frac{\Phi(\Psi(y_1, y_2))b_o(\Psi(y_1, y_2))}{y_1+i(y_2-x)}dy_1dy_2$$

is analytic in $(-\delta, \delta)$; Ψ *is a suitably choosen analytic diffeomorphism from a neighborhood of the origin onto one.*

The proof is not difficult. Since the idea is same, in relation to Nirenberg's equation (5.3), we shall prove the following

Proposition 5.3. *Assume that* $a(x, y)=x(1+x\alpha(x, y))$ *where* $\alpha(x, y)$ *is a realvalued* C^∞ *function which is even in* x. *Then, if* (5.1) *has a* C^1 *solution with grad* $u\neq 0$ *in a neighborhood of the origin, there exist some positive constant* δ *and some nonvanishing continuous function* $\Phi(x, y)$ *which is even in* x *such that the function defined by*

$$\int_{-\delta}^{\delta}\int_{0}^{\delta}\frac{\Phi((2y_1)^{1/2}, y_2)\alpha((2y_1)^{1/2}, y_2)y_1^{1/2}}{y_1+i(y_2-x)}dy_1dy_2$$

is analytic in $(-\delta, \delta)$.

Proof. Let u be such a solution. Set u_e=the even part of u in x and u_o=the odd part of u in x. Then, we have

$$M_0u_o(x, y)\equiv\frac{\partial u_o}{\partial x}+ix\frac{\partial u_o}{\partial y}=x\left(-ix\alpha(x, y)\frac{\partial u_e}{\partial y}\right). \tag{5.4}$$

Set $\Phi(x, y)=-\frac{\partial u_e}{\partial y}$. Then, $\Phi(x, y)$ is nonvanishing and continuous in a neighborhood of the origin, and even in x. Therefore, we get the conclusion by virtue of Proposition 2.5 (and Remark 2.2).

Remark 5.1. Under the assumption that $b_e(x, y)$ is real analytic, the necessary condition for (5.1) to have a nonconstant C^1 solution in a neighborhood of the origin can be derived by the above method. That is the same as the above propositions except that the function $\Phi(x, y)$ is not identically null in place of the condition that it is nonvanishing. Then, we can verify that the Nirenberg's equation (5.3) admits only constant C^1 solutions in any small neighborhood of the origin.

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