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Lefschetz operators and the existence of projective equations

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§0. Introduction

To clarify our position, let us consider an arithmetically normal embedding $j: X \hookrightarrow \boldsymbol{P}^N(\boldsymbol{C})=P$ of a projective manifold X. In $[U-2]$, we introduced a map $\bar{\delta}_{LFT}: H^0(X, \Omega_P^1|_X(*)) \to H^1(X, N_{X/P}^\vee \otimes \Omega_P^1|_X(*))$, which brought us a foundation, or the Main Theorem in studying the embedding j from the viewpoint of the normal bundle. Nevertheless, it still remains difficult to see geometric phenomena relative to arithmetically normal embeddings and their normal bundles. It may be one reason of the difficulty that the map $\bar{\delta}_{LFT}$ has something hard to control though it has fine properties.

This article aims to provide us with one of the tools instead of the map $\bar{\delta}_{LFT}$, namely, a Lefschetz operator acting on the cohomologies $H^q(X, \Omega_X^p \otimes N_{X/P}^\vee(*))$ (cf. (2.2) Corollary). As a consequence, it enables us to see directly the existence of a projective equation corresponding to a special direct summand of the normal bundle, and also gives us a relative version of (3.7) Corollary of $[U-2]$ (cf. (3.1) Theorem).

The essential point of our argument is to overcome the difficulty of the existence of non-vanishing obstruction spaces such as $H^1(P, I_X^2(*))$, $H^1(X, S^t(N_{X/P}^\vee)(*))$ $(t \geqq 1)$. For the sake of providing the Lefschetz operator with the power for breaking through the difficulty, an investigation will be done for the difference $\rho := (-\bar{\delta}_{LFT}\, \bar{d}_I) - (m\, \bar{\delta}_{EN}): H^0(X, N_{X/P}^\vee(m)) \to H^1(X, \Omega_P^1|_X(m) \otimes N_{X/P}^\vee)$ of the two maps introduced by $[U-2]$. This map ρ arised from the non-commutativity of the diagram(*) appeared in $[U-2]$, or from the non-linearity of our infinitesimal lifting problem. As we shall see in (3.4) Corollary, it also measures the gap between X and the ambient space $\boldsymbol{P}^N(\boldsymbol{C})$.

Throughout this paper, we still use the notation and the convention employed in $[U-2]$. Moreover, this time, we restrict ourselves to the case that the base field k is the complex number field $\boldsymbol{C}$ and X is a non-singular projective variety, otherwise mentioned explicitly.

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§1. Lefschetz operators

In this section, we shall study the elementary properties of a Lefschetz operator acting on cohomology groups $H^q(X, \Omega_X^p \otimes F)$ for an O_X-module F. First we give three definitions relating to Lefschetz operators.

(1.1) Definition. *Let $j\colon X \hookrightarrow \boldsymbol{P}^N(\boldsymbol{C}) = P$ be an embedding of a projective manifold X, and $\omega \in H^1(X, \Omega_X^1)$ the Hodge-Kähler class induced by the embedding j.*

(i) *For an O_X-module F, the class ω defines a Lefschetz operator:*

$$\begin{array}{ccc} L\colon H^q(X, \Omega_X^p \otimes F) & \to & H^{q+1}(X, \Omega_X^{p+1} \otimes F) \quad (p, q \in \boldsymbol{N} \cup \{0\}), \\ \Cup & & \Cup \\ \phi & \longrightarrow & \omega \wedge \phi \end{array}$$

where $\omega \wedge \phi$ is defined as follows.

$$\begin{array}{ccc} H^1(X, \Omega_X^1) \otimes_C H^q(X, \Omega_X^p \otimes F) & \xrightarrow{\text{Čech tensoring}} & H^{q+1}(X, \Omega_X^1 \otimes \Omega_X^p \otimes F) \\ \Cup & & \Cup \\ (\omega, \phi) & \longrightarrow & \omega \otimes \phi \end{array}$$

$$\begin{array}{cc} \xrightarrow{\text{wedge product}} & H^{q+1}(X, \Omega_X^{p+1} \otimes F) \\ & \Cup \\ \longrightarrow & \omega \wedge \phi \end{array}$$

(ii) *In the situation above, let us consider the following three exact sequences of O_P-modules.*

(1.1.1)

$$0 \to I^2 \xrightarrow{\alpha_{SQ}} I \xrightarrow{\beta_{SQ}} I/I^2 \to 0 \cdots (SQ)$$

$$0 \to I^3 \xrightarrow{\alpha_{CB}} I^2 \xrightarrow{\beta_{CB}} I^2/I^3 \to 0 \cdots (CB)$$

$$0 \to I/I^2 \otimes I/I^2 \xrightarrow{\alpha_{NF} = 1 \otimes \bar{d}_I} I/I^2 \otimes \Omega_P^1|_X \xrightarrow{\beta_{NF}} I/I^2 \otimes \Omega_X^1 \to 0 \cdots (NF),$$

where $I\colon = I_X$ denotes the sheaf of ideals which defines $j(X)$ in P, and $\Omega_P^1|_X = \Omega_P^1 \otimes O_X$. Then we get four maps δ_{SQ}, γ, α_{NF}, and β_{NF} as follows.

$$\delta_{SQ}\colon H^0(X, I/I^2(m)) \to H^1(P, I^2(m))$$

$$\begin{aligned} \gamma\colon & H^1(P, I^2(m)) \xrightarrow{\beta_{CB}} H^1(X, I^2/I^3(m)) \underset{\leftarrow}{\overset{\mu}{\sim}} \\ & H^1(X, S^2(I/I^2)(m)) \underset{\eta}{\hookrightarrow} H^1(X, I/I^2 \otimes I/I^2(m)), \end{aligned}$$

where the map μ is induced by the multiplication, and the map η is given by sending $v \otimes w$ to $(1/2)(v \otimes w + w \otimes v)$ for local sections v and w of I/I^2.

$$\alpha_{NF}\colon H^1(X, I/I^2 \otimes I/I^2(m)) \to H^1(X, I/I^2 \otimes \Omega_P^1|_X(m))$$
$$\beta_{NF}\colon H^1(X, I/I^2 \otimes \Omega_P^1|_X(m)) \to H^1(X, I/I^2 \otimes \Omega_X^1(m))$$

(iii) *Moreover, an exact commutative diagram of O_X-modules:*

(1.1.2)
$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & O_X & = & O_X & & \\
 & & \bar{\beta}_E \uparrow & & \uparrow \beta_{LS} & & \\
0 \to I/I^2 & \longrightarrow & \bigoplus_{s=0}^{N} O_X(-1)\, e_s & \longrightarrow & \Pi & \to 0 \\
\| & & \bar{\alpha}_E \uparrow & & \uparrow \alpha_{LS} & & \\
0 \to I/I^2 & \xrightarrow{\bar{d}_I} & \Omega_P^1|_X & \longrightarrow & \Omega_X^1 & \to 0 \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

is obtained after putting an O_X-module Π to be the cokernel of the map $\bar{\alpha}_E \cdot \bar{d}_I$.

The relation between the O_X-module Π and Lefschetz operators is given by the following lemma.

(1.2) Lemma. *Let $j\colon X \hookrightarrow \boldsymbol{P}^N(\boldsymbol{C}) = P$ be an embedding of a projective manifold X, and F an O_X-module. The exact sequence:*

$$0 \to \Omega_X^1 \otimes F \xrightarrow{\tilde{\alpha}_{LS}} \Pi \otimes F \xrightarrow{\tilde{\beta}_{LS}} F \to 0 \cdots (\widetilde{LS})$$

defined by the diagram (1.1.2) *with tensoring F induces a map:*

$$\tilde{\delta}_{LS}\colon H^0(X, F) \to H^1(X, \Omega_X^1 \otimes F)\,.$$

Then, on $H^0(X, F)$, the Lefschetz operator L coincides with $\tilde{\delta}_{LS}$ up to multiplying by a non-zero constant.

Proof. This can be proved even by a dierct computation. For simplicity, however, we shall proceed as follows. Let us consider the commutative diagram:

$$\begin{array}{ccccccc}
0 \to \Omega_P^1 & \xrightarrow{\alpha_E} & \bigoplus_{s=0}^{N} O_P(-1)\, e_s & \xrightarrow{\beta_E} & O_P \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \Omega_P^1|_X & \xrightarrow{\bar{\alpha}_E} & \bigoplus_{s=0}^{N} O_X(-1)\, e_s & \xrightarrow{\bar{\beta}_E} & O_X \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \Omega_X^1 & \xrightarrow{\alpha_{LS}} & \Pi & \xrightarrow{\beta_{LS}} & O_X \to 0\,.
\end{array}$$

Taking their cohomology groups, we have:

$$\begin{array}{ccccccc} 0=\bigoplus_{s=0}^{N} H^0(O_P(-1))\,e_s & \to & H^0(O_P) & \overset{\delta_E}{\cong} & H^1(\Omega_P^1) & \to & \bigoplus_{s=0}^{N} H^1(O_P(-1))\,e_s=0 \\ & & \downarrow & & \downarrow & & \\ & & H^0(O_X) & \to & H^1(\Omega_P^1|_X) & & \\ & & \downarrow & & \downarrow & & \\ & & H^0(O_X) & \underset{\delta_{LS}}{\to} & H^1(\Omega_X^1)\,. & & \end{array}$$

Because dim $H^1(P, \Omega_P^1)=1$, $\{\delta_E(1)\}$ forms a $\boldsymbol{C}$-basis of $H^1(P, \Omega_P^1)$. Hence, the class δ_E (1) coincides with the Hodge-Kähler class $c_1(O_P(1))$ up to multiplying by a non-zero constant. Since $\delta_{LS}(1)$ is the image of $\delta_E(1)$, the class $\delta_{LS}(1)$ is equal to the Hodge-Kähler class $\omega=j^*c_1(O_P(1))$ induced by the embedding j through multiplication by a suitable non-zero constant. After tensored by F, for an arbitrary global section $\sigma\in H^0(X, F)$, we get a commutative diagram:

$$\begin{array}{ccccccc} 0\to\Omega_X^1 & \xrightarrow{\alpha_{LS}} & \Pi & \xrightarrow{\beta_{LS}} & O_X\to 0 \\ \downarrow\otimes\sigma & & \downarrow\otimes\sigma & & \downarrow\otimes\sigma \\ 0\to\Omega_X^1\otimes F & \xrightarrow[\tilde{\alpha}_{LS}]{} & \Pi\otimes F & \xrightarrow[\tilde{\beta}_{LS}]{} & F\to 0\,. \end{array}$$

Hence, we obtain:

$$\begin{array}{ccc} H^0(X, O_X) & \xrightarrow{\delta_{LS}} & H^1(X, \Omega_X^1) \\ \downarrow\sigma & & \downarrow\otimes\sigma \\ H^0(X, F) & \xrightarrow[\tilde{\delta}_{LS}]{} & H^1(X, \Omega_X^1\otimes F)\,. \end{array}$$

Thus, $\tilde{\delta}_{LS}(\sigma)=\delta_{LS}(1)\otimes\sigma=$(non-zero constant) $L(\sigma)$. Q.E.D.

(1.3) Remark. (i) In this paper, there is no effect of multiplication by a non-zero constant on the proofs below. Hence, on $H^0(X, F)$, we shall identify the Lefschetz operator L with the connecting homomorphism $\tilde{\delta}_{LS}$ in the sequel.
(ii) The exact sequence (LS): $0\to\Omega_X^1\to\Pi\to O_X\to 0$ induces an exact sequence of O_X-modules $(p\text{-}LS)$: $0\to\Omega_X^p\to\Lambda^p\,\Pi\to\Omega_X^{p-1}\to 0$ by taking the p-th exterior product of Π. After tensoring F, and taking their cohomology groups, we get

$$\cdots\to H^{p-1}(X, \Lambda^p\,\Pi\otimes F)\xrightarrow{\tilde{\beta}_{p-LS}} H^{p-1}(X, \Omega_X^{p-1}\otimes F)$$
$$\xrightarrow{\tilde{\delta}_{p-LS}} H^p(X, \Omega_X^p\otimes F)\xrightarrow{\tilde{\alpha}_{p-LS}}\cdots\,.$$

Then, $\tilde{\delta}_{p-LS}$ also coincides with the Lefschetz operator L on $H^{p-1}(X, \Omega_X^{p-1}\otimes F)$.

§2. The relation among $(-\bar{\delta}_{LFT}\cdot\bar{d}_I)$, $(m\,\bar{\delta}_{EN})$, and L

In [U-2], we introduced three maps $\bar{\delta}_{LFT}$: $H^0(X, \Omega_P^1|_X(m))\to H^1(X, N_{X/P}^\vee\otimes$

$\Omega_P^1|_X(m))$, $\bar{d}_I$: $H^0(X, N_{X/P}^\vee(m)) \to H^0(X, \Omega_P^1|_X(m))$, and $\bar{\delta}_{EN}$: $H^0(X, N_{X/P}^\vee(m)) \to H^1(X, N_{X/P}^\vee \otimes \Omega_P^1|_X(m))$ for a given embedding j: $X \hookrightarrow \boldsymbol{P}^N(\boldsymbol{C}) = P$ of a projective manifold X. These maps were defined by the following exact sequences of O_P-modules, respectively.

$$(2.0.1)\quad 0 \to N_{X/P}^\vee \otimes \Omega_P^1|_X(m) \xrightarrow{\alpha_{LFT}} \Omega_P^1(m) \otimes O_P/I^2 \xrightarrow{\bar{\beta}_{LFT}} \Omega_P^1|_X(m) \to 0 \cdots (\overline{LFT})$$

$$0 \to N_{X/P}^\vee(m) \xrightarrow{\bar{d}_I} \Omega_P^1|_X(m) \to \Omega_X^1(m) \to 0$$

$$0 \to N_{X/P}^\vee \otimes \Omega_P^1|_X(m) \xrightarrow{\alpha_{EN}} \bigoplus_{s=0}^{N} N_{X/P}^\vee(m-1)\, e_s \xrightarrow{\bar{\beta}_{EN}} N_{X/P}^\vee(m) \to 0 \cdots (\overline{EN}),$$

where $N_{X/P}^\vee = I/I^2$ and the last sequence $(\overline{EN})$ is obtained by the Euler sequence of Ω_P^1 with tensoring $N_{X/P}^\vee(m)$.

The precise relation among $(-\bar{\delta}_{LFT} \cdot \bar{d}_I)$, $(m\, \bar{\delta}_{FN})$, and the Lefschetz operator L is given by the theorem below with using the notation of (1.1) Definition.

(2.1) Theorem. *Let j: $X \hookrightarrow \boldsymbol{P}^N(\boldsymbol{C}) = P$ be an embedding of a projective manifold X. Then, as maps from $H^0(X, N_{X/P}^\vee(m))$ to $H^1(X, N_{X/P}^\vee \otimes \Omega_P^1|_X(m))$, the following relation holds.*

$$(2.1.1)\qquad (-\bar{\delta}_{LFT} \cdot \bar{d}_I) - m\, \bar{\delta}_{EN} + 2\alpha_{NF} \cdot \gamma \cdot \delta_{SQ} = 0$$

Hence, as maps from $H^0(X, N_{X/P}^\vee(m))$ to $H^1(X, N_{X/P}^\vee(m) \otimes \Omega_X^1)$,

$$(2.1.2)\qquad mL = m \cdot \beta_{NF} \cdot \bar{\delta}_{EN} = -\beta_{NF} \cdot \bar{\delta}_{LFT} \cdot \bar{d}_I$$

Proof. Let us take a standard open affine covering $\mathfrak{U} = \{U_s \mid s = 0, 1, \cdots N\}$ of $\boldsymbol{P}^N(\boldsymbol{C}) = P$ defined by a system of homogeneous coordinates as in the proof of the Main Theorem in [U-2]. Then, we choose a sufficiently fine refinement $\mathfrak{V} = \{V_a \mid a \in A\}$ of $\mathfrak{U}$ and a refinement map u: $A \to \{0, 1, \cdots N\}$ such that for each $a \in A$, $V_a \subseteq U_{u(a)}$, and we can find a system of minimal generators $\{h_{a1}, \cdots h_{ar}\}$ of $I = I_X$ $(r = N - \dim(X), h_{aj} \in \Gamma(V_a, I))$ on the open affine set V_a. Now we take an arbitrary global section $\sigma \in H^0(X, N_{X/P}^\vee(m))$. The section σ is represented by a Čech cocycle:

$$\{(V_a, \bar{f}_{u(a)} \otimes Z_{u(a)}^m)\} \in C^0(\mathfrak{V}, N_{X/P}^\vee(m)),$$

where f_s is an element of $\Gamma(U_s, I)$ and $\bar{f}_s$ denotes its equivalence class modulo I^2. The cocycle condition of the Čech cocycle is described as follows in terms of $\{f_s\}$.

$$(2.1.3)\qquad f_s - (Z_t/Z_s)^m \cdot f_t = g_{st} \quad \text{in} \quad \Gamma(U_s \cap U_t, I)$$
$$(\exists\, g_{st} \in \Gamma(U_s \cap U_t, I^2))$$

By the choice of $\mathfrak{V}$, we can find local sections $p_{abij} \in \Gamma(V_a \cap V_b, O_P)$ $(i, j = 1, \cdots, r)$ which satisfy the conditions:

$$(2.1.4)\qquad g_{u(a)\,u(b)} \mid V_a \cap V_b = \sum_{i,j=1}^{r} p_{abij}\, h_{ai}\, h_{aj}$$
$$p_{abij} = p_{abji}\,.$$

First we calculate $m\cdot\bar{\delta}_{EN}(\sigma)$ as follows. (cf. [U-2])

$$\bar{\beta}_{EN}^{-1}(\sigma)=\{(V_a, \bar{f}_{u(a)}\otimes Z_{u(a)}^{m-1}\, e_{u(a)})\}\in C^0(\mathfrak{V}, \bigoplus_{s=0}^{N} N_{X/P}^{\vee}(m-1)\, e_s)$$

Then, $\delta^{\vee}\cdot\bar{\beta}_{EN}^{-1}(\sigma)=\{(V_a\cap V_b, (\delta^{\vee}\bar{\beta}_{EN}^{-1}(\sigma))_{ab})\}\in C^1(\mathfrak{V}, \bigoplus_{s=0}^{N} N_{X/P}^{\vee}(m-1)\, e_s)$,

where $\delta^{\vee}$ denotes Čech derivation, and

$$(\delta^{\vee}\bar{\beta}_{EN}^{-1}(\sigma))_{ab}=\bar{f}_{u(b)}\otimes Z_{u(b)}^{m-1}\, e_{u(b)}-\bar{f}_{u(a)}\otimes Z_{u(a)}^{m-1}\, e_{u(a)}\,,$$

with using (2.1.3) modulo I^2,

$$\begin{aligned}&=(Z_{u(a)}/Z_{u(b)})^{m}\cdot\bar{f}_{u(a)}\otimes Z_{u(b)}^{m-1}\, e_{u(b)}-\bar{f}_{u(a)}\otimes Z_{u(a)}^{m-1}\, e_{u(a)}\\ &=(Z_{u(a)}/Z_{u(b)})\cdot\bar{f}_{u(a)}\otimes Z_{u(a)}^{m}\\ &\quad\times\{1/Z_{u(a)}\, e_{u(b)}-(Z_{u(b)}/Z_{u(a)})\cdot 1/Z_{u(a)}\, e_{u(a)}\}\,.\end{aligned}$$

The above are computed in $\Gamma(V_a\cap V_b, \bigoplus_{s=0}^{N} N_{X/P}^{\vee}(m-1)\, e_s)$. Then,

$$\begin{aligned}(\alpha_{EN}^{-1}\cdot\delta^{\vee}\cdot\bar{\beta}_{EN}^{-1}(\sigma))_{ab}&=(Z_{u(a)}/Z_{u(b)})\cdot\bar{f}_{u(a)}\otimes Z_{u(a)}^{m}\otimes d_{EX}(Z_{u(b)}/Z_{u(a)})\\ &\in\Gamma(V_a\cap V_b, N_{X/P}^{\vee}(m)\otimes\Omega_P^1|_X)\,.\end{aligned}$$

Thus we have

(2.1.5) $m\,\bar{\delta}_{EN}(\sigma)=$ the class of

$$\{(V_a\cap V_b, m(Z_{u(a)}/Z_{u(b)})\cdot\bar{f}_{u(a)}\otimes d_{EX}(Z_{u(b)}/Z_{u(a)})\otimes Z_{u(a)}^{m})\}\in H^1(X, N_{X/P}^{\vee}(m)\otimes\Omega_P^1|_X)\,.$$

Next we shall calculate $-\bar{\delta}_{LFT}\cdot\bar{d}_I(\sigma)$.

$\bar{d}_I(\sigma)=$ the class of $\{(V_a, (d_{EX}f_{u(a)})|_X\otimes Z_{u(a)}^{m})\}\in H^0(X, \Omega_P^1|_X(m))$.

$$\bar{\beta}_{LFT}^{-1}\cdot\bar{d}_I(\sigma)=\{(V_a, d_{EX}f_{u(a)}\otimes Z_{u(a)}^{m})\}\in C^0(\mathfrak{V}, \Omega_P^1(m)\otimes O_P/I^2)$$

$$(\delta^{\vee}\,\bar{\beta}_{LFT}^{-1}\,\bar{d}_I(\sigma))_{ab}=d_{EX}f_{u(b)}\otimes Z_{u(b)}^{m}-d_{EX}f_{u(a)}\otimes Z_{u(a)}^{m}\,,$$

with applying (2.1.3),

$$\begin{aligned}&=d_{EX}\{(Z_{u(b)}/Z_{u(a)})^{-m}(f_{u(a)}-g_{u(a)\,u(b)})\}\otimes Z_{u(b)}^{m}-d_{EX}f_{u(a)}\otimes Z_{u(a)}^{m}\\ &=d_{EX}f_{u(a)}\otimes Z_{u(a)}^{m}-d_{EX}\,g_{u(a)\,u(b)}\otimes Z_{u(a)}^{m}\\ &\quad-m(Z_{u(b)}/Z_{u(a)})^{-m-1}(f_{u(a)}-g_{u(a)\,u(b)})\,d_{EX}(Z_{u(b)}/Z_{u(a)})\otimes Z_{u(b)}^{m}-d_{EX}f_{u(a)}\otimes Z_{u(a)}^{m}\\ &=-d_{EX}\,g_{u(a)\,u(b)}\otimes Z_{u(a)}^{m}-m(Z_{u(a)}/Z_{u(b)})f_{u(a)}\,d_{EX}(Z_{u(b)}/Z_{u(a)})\otimes Z_{u(a)}^{m}\,.\end{aligned}$$

The above are computed in $\Gamma(V_a\cap V_b, \Omega_P^1(m)\otimes O_P/I^2)$. Hence, the expression (2.1.4) shows that

$$\begin{aligned}(\alpha_{LFT}^{-1}\,\delta^{\vee}\,\bar{\beta}_{LFT}^{-1}\,\bar{d}_I(\sigma))_{ab}&=-d_{EX}\,(\sum_{i,j=1}^{r}p_{abij}\,h_{ai}\,h_{aj})\otimes Z_{u(a)}^{m}\\ &\quad-m(Z_{u(a)}/Z_{u(b)})f_{u(a)}\,d_{EX}(Z_{u(b)}/Z_{u(a)})\otimes Z_{u(a)}^{m}\\ &=-2\sum_{i,j=1}^{r}p_{abij}\,\bar{h}_{ai}\,d_{EX}\,h_{aj}\otimes Z_{u(a)}^{m}\\ &\quad-m(Z_{u(a)}/Z_{u(b)})\,\bar{f}_{u(a)}\,d_{EX}(Z_{u(b)}/Z_{u(a)})\otimes Z_{u(a)}^{m}\,.\end{aligned}$$

The above are computed in $\Gamma(V_a \cap V_b,\ N^\vee_{X/P}(m) \otimes \Omega^1_P|_X)$. That means

$$(2.1.6)\quad -\bar{\delta}_{LFT} \cdot \bar{d}_I(\sigma)$$
$$= \text{the class of}\quad \{(V_a \cap V_b,\ 2\sum_{i,j=1}^{r} p_{abij}\ \bar{h}_{ai} \otimes d_{EX}\ h_{aj} \otimes Z^m_{u(a)}$$
$$+m(Z_{u(a)}/Z_{u(b)})\ \bar{f}_{u(a)} \otimes d_{EX}(Z_{u(b)}/Z_{u(a)}) \otimes Z^m_{u(a)})\}$$
$$\text{in}\quad H^1(X,\ N^\vee_{X/P}(m) \otimes \Omega^1_P|_X)\,.$$

Finally, we calculate $\alpha_{NF} \cdot \gamma \cdot \delta_{SQ}(\sigma)$ as follows.

$$\begin{aligned}(\delta^\vee\ \beta^{-1}_{SQ}(\sigma))_{ab} &= f_{u(b)} \otimes Z^m_{u(b)} - f_{u(a)} \otimes Z^m_{u(a)} \\ &= (Z_{u(a)}/Z_{u(b)})^m (f_{u(a)} - g_{u(a)\,u(b)}) \otimes Z^m_{u(b)} - f_{u(a)} \otimes Z^m_{u(a)} \\ &= -g_{u(a)\,u(b)} \otimes Z^m_{u(a)}\end{aligned}$$

The above are calculated in $\Gamma(V_a \cap V_b,\ I(m))$. Then, with using the expression (2.1.4), we see that

$$\mu^{-1} \cdot \beta_{CB} \cdot \delta_{SQ}(\sigma) = \text{the class of}\quad \{(V_a \cap V_b,\ -\sum_{i,j=1}^{r} p_{abij} \cdot h_{ai} \otimes h_{aj} \otimes Z^m_{u(a)})\}$$
$$\text{in}\quad H^1(X,\ S^2(N^\vee_{X/P})\,(m))\,.$$
$$\gamma\ \delta_{SQ}(\sigma) = \text{the class of}\quad \{(V_a \cap V_b,\ -\sum_{i,j=1}^{r} p_{abij} h_{ai} \otimes h_{aj} \otimes Z^m_{u(a)})\}$$
$$\text{in}\quad H^1(X,\ N^\vee_{X/P} \otimes N^\vee_{X/P}(m))\,.$$

Thus we obtain

$$(2.1.7)\quad \alpha_{NF} \cdot \gamma \cdot \delta_{SQ}(\sigma) = \text{the class of}\quad \{(V_a \cap V_b,\ -\sum_{i,j=1}^{r} p_{abij}\ \bar{h}_{ai} \otimes d_{EX}\ h_{aj} \otimes Z^m_{u(a)})\}$$
$$\text{in}\quad H^1(X,\ N^\vee_{X/P}(m) \otimes \Omega^1_P|_X)\,.$$

Hence, by (2.1.5), (2.1.6) and (2.1.7), we can show that

$$((-\bar{\delta}_{LFT}\ \bar{d}_I) - m\ \bar{\delta}_{EN} + 2\alpha_{NF} \cdot \gamma \cdot \delta_{SQ})\,(\sigma) = 0$$

for an arbitrary section $\sigma \in H^0(X,\ N^\vee_{X/P}(m))$, namely, (2.1.1). Then, it is easy to see that

$$(2.1.8)\qquad m\ \beta_{NF} \cdot \bar{\delta}_{EN} = -\beta_{NF} \cdot \bar{\delta}_{LFT} \cdot \bar{d}_I\,.$$

On the other hand, we have the following diagram after tensoring $N^\vee(m) = N^\vee_{X/P}(m)$ to the diagram (1.1.2).

$$(2.1.9)\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & N^\vee(m) \otimes N^\vee & = & N^\vee(m) \otimes N^\vee & & & & \\ & & \downarrow \alpha_{NF} & & \downarrow & & & & \\ 0 \to & & N^\vee(m) \otimes \Omega^1_P|_X & \xrightarrow{\ \bar{\alpha}_{EN}\ } & \bigoplus_{s=0}^{N} N^\vee(m-1)\, e_s & \xrightarrow{\ \bar{\beta}_{EN}\ } & N^\vee(m) & \to 0 & \\ & & \downarrow \beta_{NF} & & \downarrow & & \| & & \\ 0 \to & & N^\vee(m) \otimes \Omega^1_X & \xrightarrow{\ \tilde{\alpha}_{LS}\ } & N^\vee(m) \otimes \Pi & \xrightarrow{\ \tilde{\beta}_{LS}\ } & N^\vee(m) & \to 0 & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Taking their cohomology groups,

$$\begin{array}{ccc} H^0(X, N^\vee(m)) & \xrightarrow{\bar{\delta}_{EN}} & H^1(X, N^\vee(m)\otimes\Omega^1_P|_X) \\ \| & & \downarrow \beta_{NF} \\ H^0(X, N^\vee(m)) & \xrightarrow[L=\bar{\delta}_{LS}]{} & H^1(X, N^\vee(m)\otimes\Omega^1_X)\,, \end{array}$$

which means $L=\beta_{NF}\cdot\bar{\delta}_{EN}$. Hence, by the formula (2.1.8), we obtain $mL=m\,\beta_{NF}\cdot\bar{\delta}_{EN}=-\beta_{NF}\cdot\bar{\delta}_{LFT}\cdot\bar{d}_I$. Q.E.D.

Through (2.1) Theorem, we can understand the fundamental role of Lefschetz operators in the study of arithmetically normal embeddings from the view point of the (co-)normal bundles as follows.

(2.2) Corollary. *Let j: $X\hookrightarrow \boldsymbol{P}^N(\boldsymbol{C})=P$ be an arithmetically normal embedding of a projective manifold X of positive dimension. Assume that* $\dim_{\boldsymbol{C}}\operatorname{Im}(L\colon H^0(X, N^\vee(m))\to H^1(X, N^\vee(m)\otimes\Omega^1_X))=s$. *Then, we can find homogeneous polynomials $F_1, \cdots, F_s$ in degree m such that $\{F_1, \cdots, F_s\}$ is a sub-S.P.E of $j(X)$ in degree m* (cf. [*U*-2]), *namely, there exists a system of minimal generators for the homogeneous ideal of $j(X)$ which has $\{F_1, \cdots, F_s\}$ as a part in degree m. Moreover, for a section $\phi\in H^0(X, N^\vee(m))$ with $L(\phi)\neq 0$, we can find a homogeneous defining equation F of $j(X)$ in degree m which satisfies $L(\bar{F})=L(\phi)$, where $\bar{F}$ denotes the equivalence class in the space $H^0(X, N^\vee(m))$ of the homogeneous polynomial F.*

Proof. Let $\{G_1, \cdots, G_t\}$ be an S.P.E. of $j(X)$ in degree m. Then, as we saw in (3.1) Corollary of [U-2], $\{\bar{\delta}_{LFT}\,\bar{d}_I(\bar{G}_1), \cdots, \bar{\delta}_{LFT}\,\bar{d}_I(\bar{G}_t)\}$ forms a $\boldsymbol{C}$-basis of $\operatorname{Im}(\bar{\delta}_{LFT}\,\bar{d}_I\colon H^0(X, N^\vee(m))\to H^1(X, N^\vee\otimes\Omega^1_P|_X(m)))$. Hence, by the formula (2.1.2), $\{L(\bar{G}_1), \cdots, L(\bar{G}_t)\}$ generates the vector space $\operatorname{Im}(L\colon H^0(X, N^\vee(m))\to H^1(X, N^\vee(m)\otimes\Omega^1_X)$. ||

§3. Applications

By (2.2) Corollary above, we can get a partial generalization of (3.7) Corollary in [U-2] as follows.

(3.1) Theorem. *Let X be a closed submaniofld of a projective manifold W with codimension r, and i: $W\hookrightarrow \boldsymbol{P}^N(\boldsymbol{C})=P$ an embedding which induces an arithmetically normal embedding j: $X\hookrightarrow \boldsymbol{P}^N(\boldsymbol{C})=P$.*

$$\begin{array}{ccc} X & \xhookrightarrow{\quad j \quad} & \boldsymbol{P}^N(\boldsymbol{C})=P \\ & {\scriptstyle inclusion}\searrow \quad \nearrow{\scriptstyle i} & \\ & W & \end{array}$$

Assume that X is of positive dimension. Then, the following three conditions are *equivalent.*

(i) *The exact sequence of the normal bundles:*

(3.1.1) $$0 \to N_{X/W} \to N_{X/P} \to N_{W/P}|_X \to 0$$

splits, and $N_{X/W} \simeq O_X(m_1) \oplus \cdots \oplus O_X(m_r)$ $(O_X(a):=j^*O_P(a))$.

(ii) *We can find hypersurfaces* $S_1, \cdots, S_r$ *of degree* $m_1, \cdots, m_r$ *respectively which satisfy the condition:*

$$j(X) = i(W) \cap S_1 \cap \cdots \cap S_r \quad (\textit{transversal}).$$

(iii) *There exist homogeneous polynomials* $F_1, \cdots, F_r$ *of degree* $m_1, \cdots, m_r$ *respectively such that the set-theoretic union of any S.P.E. of* $i(W)$ *and* $\{F_1, \cdots, F_r\}$ *forms an S.P.E. of* $j(X)$.

Proof. Obviously (iii) implies (ii), and (ii) does (i). First, we prove that (ii) implies (iii). Let us denote $i(W)$ by W_0 and $i(W) \cap S_1 \cap \cdots \cap S_k$ by W_k $(k=1\cdots r)$. Then, we see that every W_k is an integral scheme, which needs a little more than the usual argument on regular sequences in a local ring. In fact, if $W_{k(0)}$ has a component whose *codimension* in $i(W)$ is smaller than $k(0)$, then, using the ampleness of $S_{k(0)+1}$, we can show that $W_{k(0)+1}$ also has a component whose codimension in $i(W)$ is smaller than $k(0)+1$. Then, by an induction on $k(0)$, this contradicts the assumption on $j(X)$. Hence, every W_k is an equidimensional locally complete intersection subscheme of $i(W)$. By the similar argument, we see that every component of W_k includes $j(X)$. Since W_k has no embedded point, if $W_{k(0)}$ has a nilpotent element in somewhere, then $W_{k(0)}$ has a nilpotent element at the generic point of $W_r=j(X)$. The facts that the codimension of W_r in $W_{k(0)}$ equals $r-k(0)$ and W_r is defined by $r-k(0)$ elements in $W_{k(0)}$ imply that $W_{k(0)}$ is regular at the generic point of W_r (because W_r is regular). Hence, for every k, W_k is a reduced irreducible Cohen-Macaulay scheme. Taking notice of the facts above and applying the following lemma, we can show that the condition (iii) holds.

(3.2) Lemma. (*Mori-Fujita*) *Let X be an integral closed subscheme of* $\boldsymbol{P}^N(\boldsymbol{C})=P$, *and S a hypersurface of degree=d defined by a homogeneous polynomial F. Assume that* $D:=X \cap S$ *(properly intersecing) satisfies the arithmetic* D_2*-condition, namely, the depth of the local ring at the vertex of its affine cone is greater than or equal to* 2. *Then X also satisfies the arithmetic* D_2*-condition, and the set-theoretic union of any S.P.E. of X and* $\{F\}$ *is an S.P.E. of D.*

Proof. It needs a little more than the usual argument on the depth of a local ring. Let us consider the exact commutative diagram:

(3.2.1) $$\begin{array}{ccccccccc} 0 \to & H^0(O_X(m-d)) & \xrightarrow{\alpha_X^0} & H^0(O_X(m)) & \xrightarrow{\beta_X^0} & H^0(O_D(m)) & \to 0 \\ & \uparrow \varepsilon' & & \uparrow \varepsilon & & \uparrow \varepsilon'' & \\ 0 \to & H^0(O_P(m-d)) & \xrightarrow[\alpha_P]{} & H^0(O_P(m)) & \xrightarrow[\beta_P]{} & H^0(O_S(m)) & \to 0, \end{array}$$

where β_P is obviously surjective, and the surjectivity of β_X^0 and ε'' are implied by the assumption on D. Since for every sufficiently small m, ε' is surjective, we may assume that ε' is surjective as an induction hypothesis. Hence we see that ε is surjective, which means that X is also an arithmetically D_2-subscheme of P. As for an S.P.E. of X, first we consider the exact sequence:

$$0 \to H^1(O_X(m-d)) \xrightarrow{\alpha_X^1} H^1(O_X(m)) \xrightarrow{\beta_X^1} H^1(O_D(m)),$$

where the injectivity of α_X^1 is induced by the surjectivity of β_X^0 in the diagram (3.2.1). Then, $\dim H^1(O_X(m)) \leqq \dim H^1(O_X(m+d)) \leqq \dim H^1(O_X(m+2d)) \leqq \cdots$. Thus, by Serre's vanishing theorem, $H^1(O_X(m))$ is zero for any integer m. Then, we consider the following exact sequence induced by the Euler sequence.

$$\begin{aligned} 0 &\to H^0(\Omega_P^1|_X(m)) \xrightarrow{\alpha_E^0} \bigoplus^{N+1} H^0(O_X(m-1)) \xrightarrow{\bar\beta_E^0} H^0(O_X(m)) \\ &\xrightarrow{\bar\delta_E^0} H^1(\Omega_P^1|_X(m)) \to \bigoplus^{N+1} H^1(O_X(m-1)) = 0 \end{aligned}$$

The surjectivity of the map $\bar\delta_E^0$ implies

(3.2.2) $$H^1(\Omega_P^1|_X(m)) \simeq \begin{cases} C & \text{if } m=0 \\ 0 & \text{otherwise}. \end{cases}$$

Tesoring an exact sequence: $0 \to I_X \to I_D \to O_X(-d) \to 0$ to the Euler sequence, we get an exact commutative diagram (for δ_{EN}, see [U-2]):

(3.2.3)
$$\begin{array}{ccccc} H^1(I_X \otimes \Omega_P^1(m)) & \xrightarrow{\alpha_{XD}} & H^1(I_D \otimes \Omega_P^1(m)) & \xrightarrow{\beta_{XD}} & H^1(\Omega_P^1|_X(m-d)) \\ \uparrow \delta_{EN}(X) & & \uparrow \delta_{EN}(D) & & \uparrow \bar\delta_E^0 \\ H^0(I_X(m)) & \longrightarrow & H^0(I_D(m)) & \longrightarrow & H^0(O_X(m-d)). \end{array}$$

Since X and D satisfy the arithmetic D_2-condition, $\delta_{EN}(X)$ and $\delta_{EN}(D)$ are surjective. Since each basis of Im (δ_{EN}) corresponds to an S.P.E. (cf. [U-2]), using (3.2.2) and $F \in H^0(I_D(d))$, we see that β_{XD} is surjective and the union of any S.P.E. $\{G_1, \cdots, G_s\}$ of X and $\{F\}$ generates $H^0(I_D(*))$. Let us show the minimality of $\{G_1, \cdots, G_s, F\}$. We may assume that $\deg G_1 \leqq \deg G_2 \leqq \cdots < \deg G_{t(0)} = \cdots = \deg G_{t(1)} = \deg F < \deg G_{t(1)+1} \leqq \cdots \leqq \deg G_s$. If $\{G_1 \cdots G_s, F\}$ is not minimal, then there exist an integer $k (\geqq t(0))$ and homogeneous polynomials $H, H_1, \cdots, H_{k-1}$ such that

$$G_k = \sum_{i=1}^{k-1} G_i H_i + H \cdot F, \quad \text{or} \quad H \cdot F = G_k - \sum_{i=1}^{k-1} G_i H_i \in H^0(I_X(*)).$$

By our assumption, F is not contained in $H^0(I_X(*))$. Since $H^0(I_X(*))$ is a prime ideal, $H \in H^0(I_X(*))$. Hence, using $\deg H = \deg G_k - \deg F < \deg G_k$, we see that G_k is represented by $G_1, \cdots, G_{k-1}$, which contradicts the minimality of $\{G_1, \cdots, G_s\}$. ||

Now let us go back to our proof of (3.1) Theorem and show that (i) implies (ii). We may assume $m_1 = m_2 = \cdots = m_{r(1)} < m_{r(1)+1} = \cdots = m_{r(2)} < \cdots < m_{r(s-1)+1} = \cdots = m_{r(s)}$ $(r(s) = r)$. Then we shall consider the diagram:

$$\begin{array}{llll} H^0(N_X^\vee(m)) \simeq H^0(O_X(m-m_1)) \oplus & \cdots \oplus & H^0(O_X(m-m_r)) \oplus & H^0(N_{W/P}^\vee(m)\,|\,_X) \\ \quad\downarrow L \qquad\qquad \downarrow L & & \quad \downarrow L & \quad\downarrow L \\ H^1(N_X^\vee(m)\otimes \Omega_X^1) \simeq H^1(\Omega_X^1(m-m_1)) & \oplus\cdots\oplus & H^1(\Omega_X^1(m-m_r)) & \\ (N_X^\vee := N_{X/P}^\vee) & & & \oplus\, H^1(\Omega_X^1 \otimes N_{W/P}^\vee(m)\,|\,_X)\,, \end{array}$$

where, by the definition of Lefschetz operators, the action of the operator L on $H^0(N_X^\vee(m))$ can be separated into each operation L on its own direct summand of $N_X^\vee(m)$. In case of $m=m_1$, we choose $r(1)$ sections $\sigma_1, \cdots, \sigma_{r(1)} \in H^0(N_X^\vee(m))$ such that $\sigma_i=(0, \cdots, 0, \underset{\widehat{i}}{1}, 0, \cdots, 0, 0)$ (zero in the last column corresponds to the component of $H^0(N_{W/P}^\vee(m)\,|\,_X)$. Obviously, $L(\sigma_i)$ is not zero. By (2.2) Corollary, we can find homogeneous polynomials $F_1, \cdots, F_{r(1)} \in H^0(P, I_X(m_1))$ such that $L\cdot(\bar{F}_i)=L(\sigma_i)$, where $\bar{F}_i$ denotes the equivalence class in $H^0(X, N_X^\vee(m_1))$ of the section F_i. Since $m-m_j<0$ for any $j \geqq r(1)+1$, $\bar{F}_i$ is of the type $(0, \cdots, 0, \underset{\widehat{i}}{1}, 0, \cdots, 0, *)$. Next, in case of $m=m_{r(1)+1}$, the same method brings us homogeneous polynomials $F_{r(1)+1}, \cdots, F_{r(2)}$ such that $\bar{F}_j=(\underset{\widehat{r(1)}}{*\cdots*}, 0, \cdots, 0, \underset{\widehat{j}}{1}, 0, \cdots, 0, *)$ for any j with $r(1)+1 \leqq j \leqq r(2)$. Using the arithmetic D_2-condition on $j(X)$ and $F_1, \cdots, F_{r(1)}$, we may assume that $\bar{F}_j=(\underset{\widehat{r(1)}}{0, \cdots, 0}, 0, \cdots, 0, \underset{\widehat{j}}{1}, 0, \cdots, 0, *)$. By the same way, we can finally find homogeneous polynomials $F_1, \cdots, F_r$ such that $\bar{F}_i=(0, \cdots, 0, \underset{\widehat{i}}{1}, 0, \cdots, 0, *)$. Then, $F_1= \cdots =F_r=0$ defines a closed subscheme Y of $i(W)$. By the choice of $F_1 \cdots F_r$, Y coincides with $j(X)$ on each point of $j(X)$. Thus, it is sufficient to show that Y is connected even in the case that Y is not equidimensional. We put T to be $\boldsymbol{P}^{M(1)}(\boldsymbol{C}) \times \cdots \times \boldsymbol{P}^{M(r)}(\boldsymbol{C})$, where $M(k):={}_{N+m_k}C_N-1$. Then $\boldsymbol{P}^{M(k)}(\boldsymbol{C})$ parametrizes the family of hypersurfaces of degre $=m_k$ in $\boldsymbol{P}^N(\boldsymbol{C})$. Then we define a closed set $\mathcal{X}$ of $W \times T$ as follows.

$$\mathcal{X} := \{(x, t(1), \cdots, t(r)) \in W \times T \,|\, x \in S_{t(k)}\ (k = 1 \cdots r)\} \subset W \times T$$

We give the reduced structure to $\mathcal{X}$, and define $f\colon \mathcal{X} \to W$ and $g\colon \mathcal{X} \to T$ to be the restrictions of the first projection and the second projection of $W \times T$, respectively. Let us consider $f\colon \mathcal{X} \to W$. For every closed point $x \in W$, $f^{-1}(x)$ is isomorphic to $\boldsymbol{P}^{M(1)-1}(\boldsymbol{C}) \times \cdots \times \boldsymbol{P}^{M(r)-1}(\boldsymbol{C})$, which means that $\dim f^{-1}(x)$ is independent of x and $f^{-1}(x)$ is irreducible. Hence $\mathcal{X}$ is an integral scheme. Then we study the morphism $g\colon \mathcal{X} \to T$. Since r is smaller than dim W, g is a dominant proper morphism. If we generally take hypersurfaces $S_1, \cdots, S_r$ of degree $m_1, \cdots, m_r$, respectively, then $W \cap S_1 \cap \cdots \cap S_r$ is integral by Bertini's theorem. Hence, the function field of T is algebraically closed in the function field of $\mathcal{X}$. Moreover, T is normal, which implies $g_* O_{\mathcal{X}} = O_T$. Then $g^{-1}(t)=i(W) \cap S_1 \cap \cdots \cap S_r$ is connected for any $t=(t_1, \cdots, t_r) \in T$. Q.E.D.

(3.3) Remark. (i) As for the condition (i) of (3.1) Theorem, we should make a remark that, even in the case of $r=1$, the splitting of the sequence (3.1.1) is essential. Roughly speaking, $N_{X/W} \simeq O_X(m)$ does not always imply the ampleness of X in W.

(ii) (3.2) Lemma can be also shown by a slight modification of Mori-Fujita's argument in [*F*].

In the final place, we shall study the situation where $-\bar{\delta}_{LFT}\,\bar{d}_I$ coincides with $m\,\bar{\delta}_{EN}$ for any integer m.

(3.4) Corollary. *Let $j: X \hookrightarrow \boldsymbol{P}^N(\boldsymbol{C})=P$ be an arithmetically normal embedding of a projective manifold X of positive dimension. Take an integer $m(0)$. Then, the following three conditions are equivalent.*

(i) $-\bar{\delta}_{LFT}\cdot\bar{d}_I = m\,\bar{\delta}_{EN}\colon H^0(N_X^\vee(m)) \to H^1(N_X^\vee\otimes\Omega_P^1|_X(m))$ $(\forall\ m\in\boldsymbol{Z}\ \&\ m\leqq m(0))$.

(ii) $\beta_{SQ}\colon H^0(P, I_X(m)) \to H^0(X, N_X^\vee(m))$ *is surjective* $(\forall\ m\in\boldsymbol{Z}\ \&\ m\leqq m(0))$.

(iii) $H^1(P, I_X^2(m)) = 0$ $(\forall\ m\in\boldsymbol{Z}\ \&\ m\leqq m(0))$.

Moreover, if $H^1(X, S^2(N_X^\vee)(m))=0$ $(\forall\ m\in\boldsymbol{Z}\ \&\ m\leqq m(0))$ or $\alpha_{CB}\colon H^1(P, I_X^3(m))\to H^1(P, I_X^2(m))$ is surjective $(\forall\ m\in\boldsymbol{Z}\ \&\ m\leqq m(0))$, then (i), (ii), *and* (iii) *hold.*

Proof. By the sequence (SQ) of (1.1.1), it is obvious that the conditions (ii) and (iii) are equivalent. Assume one of the three conditions: (a) the condition (ii) above; (b) $H^1(X, S^2(N_X^\vee)(m))=0$ $(\forall\ m\in\boldsymbol{Z}\ \&\ m\leqq m(0))$; (c) $H^1(P, I_X^3(m))\to H^1(P, I_X^2(m))$ is surjective $(\forall\ m\in\boldsymbol{Z}\ \&\ m\leqq m(0))$. Then the condition (i) holds by (2.1) Theorem. Hence, we have only to show that (i) implies (ii). Let us suppose that the condition (i) is affirmative. In case of $m(0)\leqq 0$, then $H^0(X, N_X^\vee(m))=0$, which means (ii) holding. To apply an inductive argument on $m(0)$, we may assume that $m(0)>0$ and the condition (ii) holds for $m(0)-1$. To simplify our notation, we shall denote the integer $m(0)$ by m in the sequel. Now we study the diagram below, whose commutativity is guaranteed by the condition (i).

$$\begin{array}{ccccc}
\bigoplus_{s=0}^{N} H^0(I(m-1))\,e_s & \xrightarrow{r_{m-1}} & \bigoplus_{s=0}^{N} H^0(N_X^\vee(m-1))\,e_s & & \\
\downarrow \beta_{EN} & & \downarrow \bar{\beta}_{EN} & & \\
H^0(I(m)) & \xrightarrow{r_m} & H^0(N_X^\vee(m)) & \xrightarrow{\bar{d}_I} & H^0(\Omega_P^1|_X(m)) \\
\downarrow m\,\delta_{EN} & & \downarrow m\,\bar{\delta}_{EN} & \swarrow \bar{\delta}_{LFT} & \\
H^1(I(m)\otimes\Omega_P^1) & \xrightarrow{r'_m} & H^1(N_X^\vee(m)\otimes\Omega_P^1|_X), & &
\end{array}$$

where r_{m-1} is surjective by the induction hypothesis. Let us take an arbitrary section $\phi\in H^0(X, N_X^\vee(m))$. By (2.3) Lemma of [U-2], we can find sections $F\in H^0(P, I_X(m))$ and $\sigma\in H^0(P, \Omega_P^1(m))$ such that

$$\bar{d}_I(\phi) = \bar{d}_I\cdot r_m(F)+\sigma\,|_X \quad \text{in} \quad H^0(X, \Omega_P^1|_X(m))\,.$$

Then, the condition (i) shows that

$$\bar{\delta}_{EN}(\phi-r_m(F)) = -(1/m)\,\bar{\delta}_{LFT}\,\bar{d}_I(\phi-r_m(F)) = -(1/m)\,\bar{\delta}_{LFT}(\sigma\,|_X) = 0\,.$$

By the surjectivity of r_{m-1}, we can find $G_0, \cdots, G_N\in H^0(P, I_X(m-1))$ such that $\phi-r_m(F)=\sum_{i=0}^{N} r_{m-1}(G_i)\otimes Z_i=r_m(\sum_{i=0}^{N} G_i\otimes Z_i)$, which implies that $\phi=r_m(F+\sum_{i=0}^{N} G_i\otimes Z_i)$. Thus we obtain the surjectivity of r_m as we required. Q.E.D.

(3.5) Remark. For example, the condition (ii) of (3.4) Corollary is satisfied if $j(X)$ is a complete intersection.

As an application of (3.4) Corollary, we shall calculate $h^0(P, O_P(2H)\otimes O_P/I_X^3)$ of a twisted cubic curve $j\colon X=\boldsymbol{P}^1(\boldsymbol{C})\hookrightarrow \boldsymbol{P}^3(\boldsymbol{C})=P$, where $O_P(H)$ denotes the tautological line bundle of $\boldsymbol{P}^3(\boldsymbol{C})$ (cf. (4.2) Example (XVII) of [U-2]).

(3.6) Example. First we raise four facts which are easy to see by direct calculations of familiar exact sequences.

(I) $H^0(P, O_P(2H))\subseteq H^0(P, O_P(2H)\otimes O_P/I_X^3)$.

(II) $h^0(P, O_P(2H)) = 10$.

(III) $h^1(P, I_X^2(2H)) = 1$, $H^1(P, I_X^3(2H))\subseteq H^1(P, I_X^2(2H))$.

(IV) $h^0(P, I_X(2H)) = 3$, $h^0(X, N_X^\vee(2H)) = 4$.

Now our claim is:

Claim $h^0(P, O_P(2H)\otimes O_P/I_X^3)=10.$

By (I) and (II), it is equivalent to see that $h^1(P, I_X^3(2H))=0$. Let us assume $h^1(P, I_X^3(2H))>0$. Then, (III) shows that $\alpha_{CB}\colon H^1(P, I_X^3(2H))\to H^1(P, I_X^2(2H))$ is surjective, or equivalently,

$$-\bar{\delta}_{LFT}\cdot\bar{d}_I = 2\bar{\delta}_{EN}\colon H^0(X, N_X^\vee(2H)) \to H^1(X, N_X^\vee\otimes\Omega_P^1|_X(2H)) .$$

For $m\leqq 1$, $H^0(X, N_X^\vee(mH))=0$. Hence, putting $m(0)=2$ in (i) of (3.4) Corollary, we obtain that $\beta_{SQ}\colon H^0(P, I_X(2H))\to H^0(X, N_X^\vee(2H))$ is surjective. This contradicts (IV).

§4. Problems

Based on the results above, we shall raise two problems as our working hypotheses in studying the mutual relation between arithmetically normal embeddings and their normal bundles.

(4.1) Problem. Let $j\colon X\hookrightarrow \boldsymbol{P}^N(\boldsymbol{C})=P$ be an arithmetically normal embedding of a projective manifold X of positive dimension. Assume that the normal bundle $N_{X/P}$ splits into a direct sum $F\oplus O_X(m_1)\oplus\cdots\oplus O_X(m_t)$, where $O_X(m_i)$ denotes an extendable line bundle. Do there exist hypersurfaces $S_1, \cdots, S_t$ of degree $m_1, \cdots, m_t$, respectively and a closed subvariety W of P which satisfy $j(X)=W\cap S_1\cap\cdots\cap S_t$ (transversal) ?

(4.2) Remark. By the result of (3.1) Theorem, the existence of W is almost crucial. If rank $F=1$, then (4.1) Problem is slightly affirmative. (2.2) Corollary and Lefschetz's theorem on Picard groups give the required result except the case dim $X=1$ or $S_1\cap\cdots\cap S_t$ is not equidimensional. In case of dim $X=1$ and $S_1\cap\cdots\cap S_t$ is a smooth surface, this is shown by Harris and Hulek [H].

To explain the next problem, we give three definitions.

(4.3) Definition. Let j: $X \hookrightarrow \boldsymbol{P}^N(\boldsymbol{C})=P$ be an embedding of a projective manifold X.

(i) A closed subvariety W of P is called an *intermediate ambient space* (*I.A.S.*) of $j(X)$ if W includes $j(X)$ and is non-singular along $j(X)$.

(ii) Let F be a subbundle of the conormal bundle $N^\vee_{X/P}$. We define a closed subscheme $(X|F)(1)$ of P to be $(|j(X)|, O_{X(1)}/F)$, where $X(1):=(|j(X)|, O_P/I^2_X)$. Then, we say that the subbundle F is *relatively infinitesimally liftable* (*R.I.L.*) if F $(\subseteq \Omega^1_P|_X)$ can be lifted to a subbundle of $\Omega^1_P \otimes O_{(X|F)(1)}$.

(iii) Let F be a subbundle of the conormal bundle $N^\vee_{X/P}$, and W an I.A.S. of $j(X)$. Then, it is said that W corresponds to F if $N^\vee_{W/P} \otimes O_X$ coincides with F as a subbundle of $N^\vee_{X/P}$.

Now the second our problem is stated as follows.

(4.4) Problem. Let j: $X \hookrightarrow \boldsymbol{P}^N(\boldsymbol{C})=P$ be an arithmetically normal embedding of a projective manifold X, and F a subbundle of the conormal bundle $N^\vee_{X/P}$ with the relatively infinitesimal liftability. Then, does there exist an I.A.S. which corresponds to the subbundle F?

(4.5) Remark. The converse of (4.4) Problem is affirmative. In fact, if $F=N^\vee_{W/P}|_X$, then $(X|F)(1)=(X/W)(1):=(|j(X)|, O_W/I^2_{X/W})$ as a closed subscheme of P. On the other hand, we have an exact sequence of locally free sheaves (on a neighborhood of $j(X)$ in W): $0 \to N^\vee_{W/P} \to \Omega^1_P|_W \to \Omega^1_W \to 0$. Hence, $0 \to N^\vee_{W/P} \otimes O_{(X|F)(1)} \to \Omega^1_P|_W \otimes O_{(X|F)(1)}$ is a lifting of $0 \to F \to \Omega^1_P|_X$.

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