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The inverse problem of variation calculus in two-dimensional Finsler space

By

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In 1980 S. Hōjō [4] gave five Finsler metrics in R^2-O all the geodesics of which are logarithmic spirals with the pole O, and he showed a certain sufficient condition for a Finsler space to have such special geodesics. To find not only sufficient but also necessary condition was the motive of the present paper, and then the author's attention was turned to the projectively flatness in two-dimensional case. Thus the main result is stated as Theorem 2.

§1. Differential equations of extremals

It is well-known [2, p. 22] that every function $y=f(x)$ which minimizes or maximizes a definite integral

(1.1)
$$J=\int_{x_1}^{x_2} F(x, y, z)\, dx\,, \quad z=y'\,,$$

must satisfy the differential equation

(1.2)
$$[F]\colon F_y-\frac{dF_z}{dx}=F_y-F_{xz}-F_{yz}\, z-F_{zz}\, z'=0\,.$$

This is referred to as *Euler's equation* and the curve $f=f(x)$ in R^2 is called an *extremal* for the integral (1.1).

If we deal with the same problem in a parametric form

(1.3)
$$J=\int_{t_1}^{t_2} L(x, y, p, q)\, dt\,, \quad p=\dot{x}, \quad q=\dot{y}\,,$$

then every set of functions $\{x=\phi(t), y=\psi(t)\}$ as above must satisfy the differential equations

(1.4)
$$[L]_1\colon =L_x-\frac{dL_p}{dt}=0\,, \qquad [L]_2\colon =L_y-\frac{dL_q}{dt}=0\,.$$

The curve $\{x=\phi(t), y=\psi(t)\}$ in R^2 is called a *geodesics* of the two-dimensional Finsler space $F^2=(R^2, L)$.

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From (1.1) and (1.3) we have

$$L(x, y, p, q) = F\left(x, y, \frac{q}{p}\right) p, \tag{1.5}$$

where $p=dx/dt$ is supposed to be positive. Since the fundamental function $L(x, y, p, q)$ of the F^2 is (1) p-homogeneous [8, p. 83], we have so-called *Weierstrass' invariant* $W(x, y, p, q)$ such that

$$L_{pp} = Wq^2, \quad L_{pq} = -Wpq, \quad L_{qq} = Wp^2. \tag{1.6}$$

The W is obviously (-3) p-homogeneous and we have

$$W = \frac{F_{zz}}{p^3}. \tag{1.7}$$

Then two equations of (1.4) reduce to the single equation

$$[W]: = L_{xq} - L_{yp} + (p\dot{q} - q\dot{p})\, W = 0, \tag{1.8}$$

called *Weierstrass' form* of Euler's equation [2, p. 123], because we get the relations

$$[L]_1 = [W]\, q, \quad [L]_2 = [W]\, p. \tag{1.9}$$

§2. Darboux's solution of the inverse problem

The so-called inverse problem of variation calculus is formulated as follows [2, p. 30]: Given a doubly infinite system of curves (functions) $y=f(x; a, b)$ with two parameters (a, b) to determine a function $F(x, y, y')$ so that the given system of curves shall be the extremals for the integral (1.1).

Remark. So far as the author knows, we have two papers ([5], [6]) on the inverse problem among literatures of Finsler geometry. Our problem may be the same with theirs in essential, but the latter belongs rather to the category of metrizability problem ([10], [11]).

Our inverse problem was already solved by Darboux [3, Nos. 604, 605] and has always an infinitude of solutions which can be obtained by quadratures as follows:

First of all, from

$$y = f(x; a, b), \quad z = f_x(x; a, b) \tag{2.1}$$

the parameters a and b are solved as functions of variables (x, y, z):

$$a = A(x, y, z), \quad b = B(x, y, z). \tag{2.2}$$

Then we have

$$z' = f_{xx}(x; A, B) =: \bar{Z}(x, y, z). \tag{2.3}$$

Therefore our problem is that (1.2) is equivalent to (2.3), that is, to find $F(x, y, z)$ such that

$$F_y - F_{xz} - F_{yz}\, z - F_{zz}\, \bar{Z} = 0 \tag{2.4}$$

should be an identity in (x, y, z).

Secondly, putting $G:=F_{zz}$, differentiation of (2.4) by z yields

$$G_x + G_y\, z + G_z\, \bar{Z} + G\bar{Z}_z = 0\,. \tag{2.5}$$

Thus we must solve the differential equation (2.5) for G. It is well-known that it is equivalent to solving

$$dx = \frac{dy}{z} = \frac{dz}{\bar{Z}} = -\frac{dG}{G\bar{Z}_z}\,. \tag{2.6}$$

It is observed that the first two equations of (2.6) give the general integrals (2.1), and then the third reduces to $dG/G = -\bar{Z}_z(x, f, f_x)\, dx$. Thus, putting

$$U(x; a, b) := \exp \int \bar{Z}_z(x, f, f_x)\, dx\,, \tag{2.7}$$

we get another general integral $G = c/U(x; a, b)$ with an integral constant c. By solving those integral constants a, b, c we get (2.2) and $c = GV$ where

$$V(x, y, z) := U(x; A, B)\,. \tag{2.8}$$

Therefore (2.5) shows the existence of a functional relation among A, B and GV, which is written in

$$G(x, y, z) = H(A, B)/V\,, \tag{2.9}$$

where $H(\xi, \eta)$ is an arbitrary function of (ξ, η).

Consequently we obtain the F in the form

$$F(x, y, z) = F^*(x, y, z) + C(x, y) + zD(x, y)\,, \tag{2.10}$$

where we take an indefinite integral

$$F^*(x, y, z) = \iint G(x, y, z)\,(dz)^2 \tag{2.11}$$

and $C(x, y)$ and $D(x, y)$ are functions which should be determined such that F of (2.10) satisfies (2.4), that is,

$$D_x - C_y = F^*_y - F^*_{xz} - F^*_{yz}\, z - F^*_{zz}\, \bar{Z}\,. \tag{2.12}$$

It will be obvious from (2.4) and (2.5) that the right-hand side of (2.12) does not contain z.

Summarizing up all the above, we get

Theorem (Darboux). *The solution of our inverse problem is given by* (2.10) *together with* (2.12).

Example 1. We consider the family of *straight lines* $y=ax+b$ in R^2. Then we have $z=a$, $\bar{Z}=0$, $A=z$, $B=y-zx$ and $U=V=1$, and so $G(x, y, z)=H(z, y-zx)$. Since for any function $f(u)$ we have the formula

$$\iint f(u)\,(du)^2 = \int_0^u (u-t) f(t)\, dt\,, \tag{2.13}$$

we now get $F^*(x, y, z) = \int_0^z (z-t)\, H(t, y-tx)\, dt$. Then (2.12) becomes $D_x - C_y = 0$, and so we have a function $E(x, y)$ satisfying $C=E_x$ and $D=E_y$. Consequently we obtain Darboux's result ([3, No. 606], [2, p. 32]):

$$F(x, y, z) = \int_0^z (z-t)\, H(t, y-tx)\, dt + E_x + zE_y\,. \tag{2.14}$$

The functions $H(\xi, \eta)$ and $E(x, y)$ are arbitrary. See §4.

Example 2. The problem to find a *surface of revolution having the minimum area* leads us to the integral $J=\int y\sqrt{1+(y')^2}\, dx$, and the extremals for it are *catenaries* $y=a\cosh(x-b)/a$. For this family of curves we have the A and B of (2.2):

$$A = \frac{y}{\sqrt{v}}\,, \quad B = x - A\log(z+\sqrt{v})\,, \quad v = 1+z^2\,. \tag{2.15}$$

In case of the above surface of revolution we have specially $H(\xi, \eta)=\xi$, $C=y$ and $D=0$.

§3. The inverse problem in a parametric form

We shall deal with the inverse problem in a parametric form (1.3); this is just regarded as a problem on Finsler metrics of dimension two. In this case a doubly infinite system of curves with two parameters (a, b) is given as

$$x = \phi(t; a, b)\,, \quad y = \psi(t; a, b)\,, \tag{3.1}$$

and we have $p=\dot{\phi}(t; a, b)$ and $q=\dot{\psi}(t; a, b)$, where the dot stands for $\partial/\partial t$.

Since we are concerned with a problem of Finsler geometry, the homogeneity of functions in (p, q) is important. In order to get the homogeneity of functions which will appear later on, we take an auxiliary parameter c such that $x=\phi(ct; a, b)$ and $y=\psi(ct; a, b)$. It is remarked that the c should be restricted to be positive, because the orientation of a curve is essential in Finsler geometry.

First of all, solving a, b and ct from

$$x = \phi(ct; a, b)\,, \quad y = \psi(ct; a, b)\,, \quad \frac{p}{c} = \dot{\phi}(ct; a, b)\,, \tag{3.2}$$

for instance, we get

$$a = \alpha^*\left(x, y, \frac{p}{c}\right), \quad b = \beta^*\left(x, y, \frac{p}{c}\right), \quad ct = \tau^*\left(x, y, \frac{p}{c}\right). \tag{3.3}$$

Then we have

$$\frac{q}{c} = \dot{\psi}(\tau^*; \alpha^*, \beta^*) = : \Psi\left(x, y, \frac{p}{c}\right), \tag{3.4}$$

from which c will be solved as

$$c = r(x, y, p, q). \tag{3.5}$$

Therefore (3.3) yields

$$\begin{aligned} a &= \alpha^*\left(x, y, \frac{p}{r}\right) = : \alpha(x, y, p, q), \\ b &= \beta^*\left(x, y, \frac{p}{r}\right) = : \beta(x, y, p, q), \\ t &= \tau^*\left(x, y, \frac{p}{r}\right)\frac{1}{r} = : \tau(x, y, p, q). \end{aligned} \tag{3.6}$$

In (3.4) we observe $(kq)/(kc) = \Psi(x, y, (kp)/(kc))$ for any positive k, which implies $kc = r(x, y, kp, kq)$ for (3.5), that is, $r(x, y, p, q)$ is (1) p-homogeneous in (p, q). In consequence α, β and τ of (3.6) are (0), (0) and (-1) p-homogeneous.

Next we have

$$\begin{aligned} \dot{p} &= \ddot{\phi}(r\tau; \alpha, \beta)\, r^2 = : P(x, y, p, q), \\ \dot{q} &= \ddot{\psi}(r\tau; \alpha, \beta)\, r^2 = : Q(x, y, p, q), \end{aligned} \tag{3.7}$$

both of which are (2) p-homogeneous obviously. Therefore our problem is to find $L(x, y, p, q)$ such that (1.8), that is,

$$[W]^* : = L_{xq} - L_{yp} + RW = 0, \quad (R = pQ - qP), \tag{3.8}$$

should be an identity in (x, y, p, q). It is remarked that the function R is (3) p-homogeneous, and so every term of $[W]^*$ is (0) p-homogeneous.

Differentiate $[W]^*$ by p or q. Then (1.6) shows that $[W]^*_p$, for instance, $= -(W_x p + W_y q) q + R_p W + R W_p$. On account of the homogeneity we have $-3W = W_p p + W_q q$ and $3R = R_p p + R_q q$, which lead us to

$$[W]^*_p = -(W_x p + W_y q - W_p R_q/3 + W_q R_p/3)\, q.$$

Next we have $Q_p p = 2Q - Q_q q$ and $P_q q = 2P - P_p p$ from the homogeneity, and so we have $R_p = 3Q - (P_p + Q_q) q$ and $R_q = -3P + (P_p + Q_q) p$. Therefore we obtain

$$[W]^*_p = -q[W]_0, \quad [W]^*_q = p[W]_0, \tag{3.9}$$

where we put

$$[W]_0 = W_x p + W_y q + W_p P + W_q Q + (P_p + Q_q) W. \tag{3.9$'$}$$

The differential equation $[W]_0 = 0$ for W, as thus obtained, is to correspond to (2.5) in the non-parametric form and to solve this is equivalent to solving

(3.10)
$$\frac{dx}{p}=\frac{dy}{q}=\frac{dp}{P}=\frac{dq}{Q}=-\frac{dW}{(P_p+Q_q)\,W}.$$

If we equate the common ratio of (3.10) to dt, then we get $dx/dt=p$, $dy/dt=q$, $dp/dt=P$ and $dq/dt=Q$, which yield four general integrals

(3.11)
$$\begin{aligned} x&=\phi(c(t+t_0);a,b)\,, & y&=\psi(c(t+t_0);a,b)\,,\\ p&=\dot{\phi}(c(t+t_0);a,b)\,, & q&=\dot{\psi}(c(t+t_0);a,b)\,,\end{aligned}$$

with fourth integral constant t_0. Then (3.10) reduces to $dW/W=-\{P_p(\phi,\psi,\dot{\phi},\dot{\psi})+Q_q(\phi,\psi,\dot{\phi},\dot{\psi})\}\,dt$ and putting

(3.12)
$$U(c(t+t_0),a,b):=\exp\int\{P_p(\phi,\psi,\dot{\phi},\dot{\psi})+Q_q(\phi,\psi,\dot{\phi},\dot{\psi})\}\,dt\,,$$

we then get the fifth general integral

(3.13)
$$W=\frac{d}{U},$$

with an integral constant d.

We are now in a position to solve the integral constants a, b, c, t_0 and d from (3.11) and (3.13). First (3.5) and (3.6) show $a=\alpha$, $b=\beta$, $c=\gamma$, $t_0=\tau-t$. Thus, if we put

(3.14)
$$V(x,y,p,q):=U(\gamma\tau;\alpha,\beta)\,,$$

then we get $d=WV$ from (3.13). Consequently the differential equation $[W]_0=0$ shows the existence of a functional relation among α, β, γ, $\tau-t$ and WV.

We must pay attention to the following two requests. First our W should not depend on t explicitly, and so we have a relation of the form $WV=H^*(\alpha,\beta,\gamma)$ where H^* is an arbitrary function. Secondly our W must be (-3) p-homogeneous. Since $V(x,y,p,q)$ is (0) p-homogeneous, the second request is that $(W/k^3)\,V=H^*(\alpha,\beta,k\gamma)$ for any positive number k, and so we obtain finally

(3.15)
$$W=\frac{H(\alpha,\beta)}{V\gamma^3},$$

where $H(\xi,\eta)$ is an arbitrary function.

Now we shall return to (1.6) to get the $L(x,y,p,q)$. Take indefinite integrals $\iint W(dp)^2$ and $\iint W(dq)^2$ and put

(3.16)
$$L_1^*:=q^2\iint W(dp)^2\,,\quad L_2^*:=p^2\iint W(dq)^2\,.$$

Then we get L in the form $L=L_1^*+pC^*+D^*$, for instance, where C^* and D^* are functions of (x,y,q). Paying attention to the homogeneity again, we observe that C^* and D^* must be (0) and (1) p-homogeneous in q respectively, and consequently we may write $C^*=C(x,y)$ and $D^*=qD(x,y)$. Therefore we obtain

$$(3.17) \qquad L = L_i^* + pC(x, y) + qD(x, y), \quad (i = 1 \text{ or } 2).$$

Substituting from (3.17) into (1.8), we have

$$(3.18) \qquad C_y - D_x = [W_i]^* := (L_i^*)_{xq} - (L_i^*)_{yp} + RW.$$

The $[W_i]^*$ does not depend on p and q, as will be obvious from (3.9′).
Summarizing up all the above, we have

Theorem 1. *All the Finsler metrics $L(x, y, p, q)$ in R^2 the geodesics of which are given by* (3.1) *are written in the form* (3.17) *together with* (3.18), *where $L_i^*(i=1, 2)$, W, V and U are given by* (3.16), (3.15), (3.14) *and* (3.12) *respectively.*

§4. Some examples

We shall again consider Example 1 in §2. Putting $\phi(ct; a, b) = ct$, $\psi(ct; a, b) = act + b$, we have $\alpha = q/p$, $\beta = y - qx/p$, $\gamma = p$, $\tau = x/p$, $P = Q = R = 0$, $U = V = 1$, and so $W = H(q/p, y - qx/p)/p^3$. Then $L_2^* = p^{-1} \iint H(q/p, y - qx/p)\,(dq)^2 = p \iint H(z, y - zx)\,(dz)^2$ where $z = q/p$, which yields the previous result (2.14). In terms of Finsler geometry is the result stated as follows:

Theorem 2. *All the projectively flat Finsler metrics in R^2 are written in the form*

$$L(x, y, \dot{x}, \dot{y}) = \dot{x} \int_0^z (z-t)\, H(t, y-tx)\, dt + \dot{x}E_x + \dot{y}E_y,$$

where $z = \dot{x}/\dot{y}$ and $H(\xi, \eta)$ and $E(x, y)$ are arbitrary functions. This (x, y) is a rectilinear coordinate system [7].

Remark. As to the additional terms $\dot{x}E_x + \dot{y}E_y$, see [4, the second remark in p. 213] and [9].

Remark. Berwald [1] gave a table of all the projectively flat Finsler metrics of dimension two the main scalar of which is a function of position alone. See [8, §28].

Example 3. We are concerned with our original problem in which the geodesics are *logarithmic spirals* $r = \exp(a\theta + b)$ in $R^2 - O\,(x = r \cos\theta,\ y = r \sin\theta)$. Since we get $\log r = a\theta + b$ similar to the case of Example 1 ($x = \theta$, $y = \log r$), it is easy to show that the fundamental function L is given by

$$(4.1) \qquad L(\theta, r, \dot{\theta}, \dot{r}) = \dot{\theta} \int_0^z (z-t)\, H(t, \log r - t\theta)\, dt + \dot{\theta}E_\theta + \dot{r}E_r,$$

where $H(\xi, \eta)$ and $E(\theta, r)$ are arbitrary functions and $z = \dot{r}/r\dot{\theta}$.

Therefore we found all the two-dimensional Finsler spaces in $R^2 - O$ having such geodesics. These are *projectively flat,* and $(\theta, \log r)$ is a *rectilinear coordinate system.*

Example 4. The famous problem of *Brachistochrone,* i.e., to determine for a heavy particle the curve of quickest descent in a vertical plane between two given points [2, p. 126] leads us to the Riemannian metric $L(x, y, p, q)=\sqrt{p^2+q^2}/\sqrt{y}$ and the geodesics are *cycloids* $\{x=a(t-\sin t)+b, y=a(1-\cos t)\}$. From (3.15) it follows that this family of cycloids gives rise to Weierstrass' invariant $W=(y/p^3)H(\xi, \eta)$ where $\xi=y(p^2+q^2)/2p^2$ and $\eta=x-\xi \operatorname{Arctan}(2pq/(q^2-p^2))+yq/p$. For the Brachistchrone we have specially $H(\xi, \eta)=(2\xi)^{-3/2}$.

Example 5. We finally consider the family of *semicircles* $(x-a)^2+y^2=b^2 (b>0)$ in $R^2_+=\{(x, y) \mid y>0\}$. Then we have $\phi(t; a, b)=a+b \cos t$, $\psi(t; a, b)=b \sin t$ and (3.15) gives Weierstrass' invariant $W=(y^2/p^3)H(x+yq/p, y\sqrt{p^2+q^2}/p)$. It is well-known that the family of semicircles are geodesics of the Riemannian metric $ds^2=(dx^2+dy^2)/y^2$ of *constant curvature* -1, and that this metric is projectively flat. Therefore the above W gives rise to Finsler metrics which are *projectively flat,* but this (x, y) is *not* a rectilinear coordinate system, contrary to the case of Theorem 2.

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