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On the non-cocomutativity of the mod 2 cohomology ring of certain finite H-spaces

By

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§0. Introduction

Since Heinz Hopf investigated the cohomology of a space equipped with a continuous multiplication, the study of cohomology of H-spaces has been developed by many authors. In the case where the coefficient group is $\boldsymbol{F}_2=\boldsymbol{Z}/2\boldsymbol{Z}$, the cohomology $H^*=H^*(\ ;\boldsymbol{F}_2)$ has the structure of a (not necessarily coassociative) Hopf algebra over the Steenrod algebra, that is, Hopf algebra of which the underlying algebra-coalgebra structure is a "left module algebra-quasicoalgebra" one in the sense of Milnor-Moore [9].

In the latest paper of J. P. Lin [5], he got a non-cocomutative theorem on the mod 2 cohomology of certain finite H-spaces.

Unfortunately his result is not applicable to a finite H-space X whose mod 2 cohomology ring is isomorphic to that of Spin(N). The purpose of this paper is to show

Theorem 1. *Let X be a* mod 2 *finite H-space satisfying $H^*(X)$ is isomorphic to $H^*(\mathrm{Spin}(N))$ as algebras for* $2^{n+1}+2\leq N\leq 2^{n+2}-4$ $(n\geq 3)$. *Then $H^*(X)$ is not primitively generated.*

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§1. Proof of the main theorem

On the mod 2 cohomology of mod 2 finite H-spaces the following lemmas are known.

Lemma 2 ([3]). *Let X be a* 1-*connected* mod 2 *finite H-space, then* $QH^{\mathrm{even}}(X)=0$.

Lemma 3 ([10]). *Let X be a* 1-*connected* mod 2 *finite H-space with $H^*(X)$ primitively generated. Then the following hold.*

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(a) *If* $\binom{2n-1}{2^k}=1 \bmod 2,\ 2^k<2n-1$ *then* $Sq^{2^k}PH^{2n-1}(X)=PH^{2n-1+2^k}(X)$.

(b) *If* $\binom{2n-1}{2^k}=0 \bmod 2,$ *then* $Sq^{2^k}PH^{2n-1}(X)=0$.

Lemma 4 ([1]). *Let X be a* mod 2 *H-space and P_2X be its projective plane. Then we have the following exact sequence with properties* (*a*) (*b*) *and* (*c*).

$$\cdots\to \tilde{H}^q(X)\xrightarrow{\bar{\Psi}}[\tilde{H}^*(X)\otimes\tilde{H}^*(X)]^q\xrightarrow{\lambda}\tilde{H}^{q+2}(P_2X)\xrightarrow{\iota}\tilde{H}^{q+1}(X)\cdots$$

(a) *Each homomorphism commutes with the A^*-action, where A^* denotes the* mod 2 *Steenrod algebra.*

(b) Im $\iota=PH^*$ *and if* $\iota(y_i)=x_i\neq 0$ $(i=1,2)$ *then* $\lambda(x_1\otimes x_2)=y_1y_2$.

(c) *Any threefold product vanishes in* $H^*(P_2X)$.

Let X be a 1-connected mod 2 finite H-psace. If $H^*(X)$ is primitively generated, $H^*(X)$ has a simple system of primitive generators $\{y_1,\cdots,y_s\}$, that is, $y_1,\cdots,y_s$ are primitive elements such that $\{y_1^{\varepsilon_1}\cdots y_s^{\varepsilon_s}\mid \varepsilon_i=0 \text{ or } 1\}$ is a basis of $H^*(X)$. Then we have next two lemmas.

Notation. (1) P: the linear subspace spanned by $\{y_1,y_2,\cdots,y_s\}$.

(2) C: the linear subspace spanned by

$$\{y_{i(1)}\,y_{i(2)}\cdots y_{i(t)};\ i(1)<i(2)<\cdots<i(t),\ t\geq 2\}\ .$$

(3) $H=\tilde{H}^*(X)$, $H_j=H^j(X)$, $H_j^k=(\underbrace{\mathrm{H}\cdot\mathrm{H}\cdots\mathrm{H}}_{(k\ \text{times})})\cap H_j$, $P_j=P\cap H_j$, $C_j=C\cap H_j$

Note that $P\oplus C=H$ and $P_j\cdot P_k\subset C_{j+k}$ if $k\neq j$.

Lemma 5. *For $k,\ l\geq 1$ the following hold.*

(a) $Sq^{2k+1}\,\mathrm{H}_{2l}\subset C_{2k+2k+1}$.

(b) $Sq^1\,C_{4k-3}\subset C_{4l-2}$.

(c) $Sq^2\,\mathrm{H}_{8l-2}\subset C_{8l-2}$.

(d) $Sq^4\,\mathrm{H}_{16l-6}\subset C_{16l-2}$.

Proof. (a): $\mathrm{S}q^{2k+1}\ \mathrm{H}_{2l}=Sq^{2k+1}\ \mathrm{H}_{2l}^2\subset\mathrm{H}_2^{2k+2l+1}=C_{2k+2l+1}$. (b): $C_{4l-3}=\sum P_{2l-1-2l}\cdot P_{2l-2+2l}+\mathrm{H}_3^{4j-3}$. Lemma 3 says $Sq^1\,C_{4l-3}=\sum P_{2l-2t}\cdot P_{2l-2+2i}+\mathrm{H}_{4l-2}^3$. But $\mathrm{H}_{4l-2}^3=C_{4l-2}$ and $P_{2l-2i}\cdot P_{2l-2+2i}\subset C_{4l-2}$. Thus we get (b). (c) and (d) can be proved in a similar way. Q.E.D.

Notation. (1) For $\alpha\in\mathrm{H}^n(X\wedge X)$ we put $a=\sum_i a(i,n-i)$ where $a(i,n-i)\in \mathrm{H}_i\otimes\mathrm{H}_{n-i}$ (note the preceding notations).

(2) $\mathrm{C}(i,n-1)=\mathrm{C}_i\otimes\mathrm{H}_{n-i}+\mathrm{H}_i\otimes\mathrm{C}_{n-i}$.

Lemma 6. *If* $a\in\mathrm{H}^{32n-5}(X\wedge X)$ *satisfies* $a(i,32n-5-i)\in\mathrm{C}(i,32n-5-i)$ *for* $i=16n-2,\ 16n-3$, *then the following hold.*

(a) $(\mathrm{S}q^1\,a)(16n-2,16n-2)\in\mathrm{C}(16n-2,16n-2)$.

(b) $(\mathrm{S}q^4\,a)(i,32n-5-i)\in\mathrm{C}(i,32n-5-i)$ $(i=16n-2,\ 16n-3)$.

The proof is given by a calculation using the preceding lemmas.

Now Theorem 1 is a corollary to the next theorem.

Theorem 7. *Let X be a* 1*-connected* mod 2 *finite H-space satisfying the following conditions, where $n\geq 3$.*

(a) $H^*(X)=\boldsymbol{F}_2[x]/(x^4)\otimes R$ *as an algebra, and* $x\in PH^{2^n-1}$.

(b) $QR^{2^n-1}=0$, $\dim QR^{2^{n+1}-3}=\dim QR^{2^{n+1}+1}$.

Then $H^*(X)$ *is not primitively generated.*

The proof of Theorem 1. Suppose $H^*(X)$ is primitively generated. We only have to show there is a primitive element x in $H^{2^n-1}(X)$ whose height is four. Now let x be the non zero primitive element. Anyway there is an indecomposable element y in $H^{2^n-1}(X)$ such that $y^2\neq 0$ and $y^4=0$.(*) Since $H^*(X)$ is a polynomial algebra for $*<2^{n+1}-1$, $x^2\neq 0$. If $x^4\neq 0$, $(y-x)^4=x^4\neq 0$ and is primitive. On the other hand $(y-x)$ is decomposable and therefore its fourth power cannot be primitive. This is a contradiction. Thus $x^4=0$.

The proof of Theorem 7. If $H^*(X)$ is primitively generated, we have a contradiction as follows. Fix a simple system of primitive generators $\{y_1, \cdots, y_s\}$ and use the above notations. Let y be an element in $\iota^{-1}(x^2)$, where ι: $H^{2^{n+1}-1}(P_2X)\to H^{2^{n+1}-2}(X)$ (*see* Lemma 4). Then $y^2=\lambda(x^2\otimes x^2)\neq 0$. By the assumption there exists $u\in H^{2^{n+2}-5}(X\wedge X)$ such that $\lambda(u)=Sq^{2^{n+1}-2}y$. Lemma 6 says that if $u(i, 2^{n+2}-5-i)\in C(i, 2^{n+2}-5-i)$ $(i=2^{n+1}-3, 2^{n+1}-2)$, $(Sq^1 u)(2^{n+1}-2, 2^{n+1}-2)\in C(2^{n+1}-2, 2^{n+1}-2)$. Then $x^2\otimes x^2+Sq^1 u=x^2\otimes x^2 \mod C(2^{n+1}-2, 2^{n+1}-2)\oplus(\bigoplus_{j\neq 2^{n+1}-2} H_j\otimes H_k)$. But the left hand side is in Ker λ, hence $x^2\otimes x^2+Sq^1 u\in \mathrm{Im}\,\bar{\psi}$. Thus $x^2\otimes x^2\in \mathrm{Im}\,\bar{\psi}+C(2^{n+1}-2, 2^{n+1}-2)\oplus(\bigoplus_{j\neq 2^{n+1}-2} H_j\otimes H_k)$, as is false. Therefore

(1) there exists $x'\in P_{2^{n+1}-3}$, and $u\in H^{2^{n+2}-5}(X\wedge X)$ such that $x'\neq 0$, $Sq^{2^{n+1}-2}y=\lambda(x^2\otimes x'+u)$, and $u(i, 2^{n+3}-5-i)\in C(i, 2^{n+2}-5-i)$ $(i=2^{n+1}-3, 2^{n+1}-2)$.

Note that $Sq^4 x'\neq 0$ because of Lemma 3 and the assumption. Moreover we get $Sq^4 Sq^{2^{n+1}-2}y\neq 0$ from this by analogous computations.

(2) $Sq^4 Sq^{2^{n+1}-2}y\neq 0$.

But by the Adem relations and for dimensional reason, $Sq^4 Sq^{2^{n+1}-2}y$ is equal to $Sq^{2^{n+2}}Sq^2 y$, and $Sq^2 y$ is an image of some w by λ. The dimension of w is $2^{n+1}-1$, hence $Sq^{2^{n+2}}Sq^2 y=0$, as contradicts (2). *Q.E.D.*

Added in proof: *In the proof of Theorem* 1 *we assumed* $PH^{2^n-1}\neq 0$. *This is shown in a forthcoming paper to appear in Publ. RIMS Kyoto Univ.*

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