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On H-spaces which are the homotopy fibres of self-maps of spheres

By

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§0. Introduction

Let p be an odd prime, $X(n, j)$ a 1-connected H-space satisfying

$$\tilde{H}_k(X(n,j); \boldsymbol{Z}) = \begin{cases} \boldsymbol{Z}/p^j\boldsymbol{Z} & (k\equiv 0 \bmod 2n,\ k>0) \\ 0 & (\text{otherwise}) \end{cases}$$

where n, j are integers and $n>1$. The purpose of this paper is to show

Theorem A. *There exist $a\in \mathrm{H}_{2n}X(n,j)$ and $b\in \mathrm{H}_{2n+1}X(n,j)$ such that as an algebra*

$$H_*X(n,j)\cong S(a)\otimes E(b)$$

where H_* denotes the mod p *homology, S the symmetric algebra, and E the exterior algebra.*

Let $S^{2n+1}\{p^j\}$ be the homotopy fibre of the self-map of S^{2n+1} of degree p^j. It is homotopy equivalent to that of the self-map of $S^{2n+1}_{(p)}$ of degree p^j. Since $S^{2n+1}_{(p)}$ is a homotopy commutative H-space (cf. [1]), $S^{2n+1}\{p^j\}$ is an H-space. An easy calculation with the Wang exact sequence shows $\mathrm{H}_*(S^{2n+1}\{p^j\}; \boldsymbol{Z})\cong \mathrm{H}_*(X(n,j); \boldsymbol{Z})$. Therefore $\mathrm{H}_*S^{2n+1}\{p^j\}\cong S(a)\otimes \mathrm{E}(b)$ for any H-structure of $S^{2n+1}\{p^j\}$ by Theorem A and the case $\lambda_i=0$ in Lemma IV of [4] does not occur. This simplifies [4] very much. Moreover in §2 we have

Theorem B. *If $X(n,j)$ is a loop space then*

(1) *$H^*BX(n,j)\cong S(x)\otimes E(y)$ where $|x|=2n+2$ and $|y|=2n+1$.*

(2) *There exists a weak homotopy equivalence of $S^{2n+1}\{p^j\}$ to $d(n,j)$ and n divides $p-1$.*

Throughout this paper H_* stands for the mod p homology, S the symmetric algebra, and E the exterior algebra.

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§1. Proof of Theorem A

First recall that

$$\tilde{H}_k X(n,j) = \begin{cases} \boldsymbol{Z}/p & (k \equiv 0, 1 \bmod 2n,\ k > 1) \\ 0 & (\text{otherwise}), \end{cases}$$

the l-th Bockstein homomorphism $\beta_l^* = 0$ for $l < j$ and $\beta_j^*\colon H_{2nk+1}X(n,j) \to H_{2nk}X(n,j)$ is an isomorphism for each $k > 0$. For simplicity we denote $H_k X(n,j)$ by H_k. Let a be a generator of H_{2n} and $b = (\beta_j^*)^{-1}(a)$, which spans H_{2n+1}. Using the fact that β_j^* is a derivation (*cf.* [3]), we have $\beta_j^*(ab) = a^2 = \beta_j^*(ba)$. Since β_j^* is an isomorphism, it follows that $ab = ba$. $H_{4n+2} = 0$ by the dimensional reason and so $b^2 = 0$ (note that $n > 1$). Therefore a and b generate a commutative Hopf subalgebra of H_*. If $a^k \neq 0$, then $a^k b \neq 0$ and so is $a^{k+1} = \beta_j^*(a^k b)$. Since $\dim H_{2nk} = \dim H_{2nk+1} = 1$, a^k and $a^{k-1}b$ generate H_{2nk} and H_{2nk+1} respectively for $k > 0$. Therefore $H_* X(n,j) \cong S(a) \otimes E(b)$.

§2. Proof of Theorem B

First $\tilde{H}_k(BX(n,j); \boldsymbol{Z})$ is a finite p-group for any k by Serre's $\mathcal{C}$-theory. The Rothenberg-Steenrod spectral sequence [7] happens to collapse and so $H_* BX(n,j) \cong S(x) \otimes E(y)$ where H^* means the mod p cohomology, $|x| = 2n+2$, $|y| = 2n+1$ and $\beta_j y = x$. Clearly $BX(n,j)$ is $2n$-connected and $\pi_{2n+1} BX(n,j) = \boldsymbol{Z}/p^j \boldsymbol{Z}$. Let $\pi\colon S^{2n+1} \to BX(n,j)$ be a generator of $\pi_{2n+1} BX(n,j)$, X is its homotopy fibre and $i\colon X \to S^{2n+1}$ the canonical map. Then the comparison theorem of the Serre spectral sequence in the mod p cohomology shows

$$H^* X = \begin{cases} \boldsymbol{Z}/p\boldsymbol{Z} & (* = 0, 2n+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Thus we have $H^*(X; \boldsymbol{Z}) \cong H^*(S^{2n+1}; \boldsymbol{Z})$. Because X is 1-connected, we have a weak equivalence $f\colon S^{2n+1} \to X$. It follows from the homotopy exact sequence that the degree of $i \circ f$ is p^j. Therefore the desired weak equivalence is obtained.

Using the Adem relations, if $j = 1$ we have n devides $p-1$ from the form of the mod p cohomology of the classifying space ([2], [4]). If $j > 1$, since β acts trivially on $H_* BX(n,j)$ and $\mathcal{P}^{n+1} x \neq 0$ we must have $\mathcal{P}^1$ acting non-trivially (by considering secondary operations in [6], [8] or [9]). Thus the degree of $\mathcal{P}^1$ has to be divided by $2n+2$. Therefore the proof of the latter half of (B) is essentially same as that of [4].

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